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Biharmonic Maps from Tori into a 2-Sphere*

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Abstract Biharmonic maps are generalizations of harmonic maps. A well-known result on harmonic maps between surfaces shows that there exists no harmonic map from a torus into a sphere (whatever the metrics chosen) in the homotopy class of maps of Brower degree ± 1. It would be interesting to know if there exists any biharmonic map in that homotopy class of maps. The authors obtain some classifications on biharmonic maps from a torus into a sphere, where the torus is provided with a flat or a class of non-flat metrics whilst the sphere is provided with the standard metric. The results in this paper show that there exists no proper biharmonic maps of degree ± 1 in a large family of maps from a torus into a sphere.

Keywords Biharmonic maps, Biharmonic tori, Harmonic maps, Gauss maps, Maps into a sphere
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1 Introduction

All objects including manifolds, tensor fields, and maps studied in this paper are supposed to be smooth.

Harmonic maps are maps $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds that minimize the energy functional

$$E(\varphi) = \frac{1}{2}\int_\Omega |\mathrm{d}\varphi|^2 v_g,$$

where Ω is a compact domain of M. Analytically, a harmonic map is a solution of a system of 2nd order PDEs

$$\tau(\varphi) \equiv \mathrm{Tr}_g \nabla\,\mathrm{d}\varphi = g^{ij}(\varphi^\sigma_{ij} - \Gamma^k_{ij}\varphi^\sigma_k + \overline{\Gamma}^\sigma_{\alpha\beta}\varphi^\alpha_i\varphi^\beta_j)\frac{\partial}{\partial y^\sigma} = 0, \tag{1.1}$$

where $\tau(\varphi)$ is called the tension field of the map φ.

Biharmonic maps are generalizations of harmonic maps, which are maps $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds that are critical points of the bienergy functional defined

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by

$$E_2(\varphi) = \frac{1}{2}\int_{\Omega} |\tau(\varphi)|^2 v_g,$$

where Ω is a compact domain of M. Biharmonic map equation is a system of 4-th order nonlinear PDEs (see [24])

$$\tau_2(\varphi) := \mathrm{Trace}_g(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0, \tag{1.2}$$

where R^N denotes the curvature operator of (N, h) defined by

$$R^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

As a harmonic map is always a biharmonic map, we call a biharmonic map that is not harmonic a proper biharmonic map.

Since 2000, the study of biharmonic maps has been attracting growing interest of many mathematicians and it has become a popular topic of research with many interesting results. For some recent geometric study of general biharmonic maps, we refer the readers to [3, 5, 8, 28, 32, 35–36, 39, 43, 46, 50] and the references therein. For some recent progress on biharmonic submanifolds (i.e., submanifolds whose defining isometric immersions are biharmonic maps), see a recent survey [1, 7, 9, 11–19, 22–23, 25–26, 30–31, 34, 38, 41, 44–45, 48, 51] and the references therein. For biharmonic conformal immersions and submersions, see [4, 29, 37, 40, 42, 49] and the references therein.

For harmonic maps between surfaces, a very interesting result proved by Eells and Wood in [21] states that there exists no harmonic map from a torus T^2 into a sphere S^2 (whatever the metrics chosen) in the homotopy class of maps of Brower degree ± 1. It would be interesting to know if there exists any proper biharmonic map from a torus T^2 into a sphere S^2 (whatever the metrics chosen) in the homotopy class of maps of Brower degree ± 1. Motivated by this, we study biharmonic maps from a torus into a sphere in this paper. We are able to obtain some classifications of proper biharmonic maps in a large family of maps from T^2 into S^2 which include the Gauss map of the torus $T^2 \longrightarrow \mathbb{R}^3$ and the compositions $T^2 \longrightarrow S^3 \longrightarrow S^2$ of some immersions of T^2 into S^3 followed by the Hopf fibration. Here, the torus is provided with a flat or a class of non-flat metrics whilst the sphere is provided with the standard metric (see Theorem 3.1 and Theorem 4.1).

2 Constructions of Maps from a Torus into a 2-Sphere

In order to study biharmonic maps from a torus T^2 into a sphere S^2, we need to have a good source of maps from a torus into a sphere. In this section, we present three ways to construct maps from T^2 into S^2.

(1) Construction via Hopf fibration. For any map $f : T^2 \longrightarrow S^3$, we have a map from torus into 2-sphere, $H \circ f : T^2 \longrightarrow S^2$, where $H : S^3 \longrightarrow S^2$ is the Hopf fibration. Here, we view

the Hopf fibration as the restriction of the Hopf construction of the standard multiplication of complex numbers, i.e., $H:\mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{R}\times\mathbb{C}$ with $H(z,w)=(|z|^2-|w|^2,2z\overline{w})$.

(2) Construction via radial projection. For any map $f:T^2\longrightarrow\mathbb{R}^3\setminus\{0\}$, we have a map from torus into 2-sphere, $P\circ f:T^2\longrightarrow S^2$, where $P:\mathbb{R}^3\setminus\{0\}\longrightarrow S^2$ with $P(x)=\frac{x}{|x|}$ is the radial projection from $\mathbb{R}^3\setminus\{0\}$ onto S^2.

(3) Construction via Gauss map of a torus. It is well known that if $f:T^2\longrightarrow\mathbb{R}^3$ is an immersion of a torus into $\mathbb{R}^3$, then the Gauss map gives a map from the torus into a 2-sphere defined by $G:T^2\longrightarrow S^2$, $G(x)=n(x)$ with $n(x)$ being the unit normal vector at the point $x\in T^2$.

Example 2.1 Let $f:T^2\longrightarrow S^3$ with $f(x,y)=(\cos(kx)\mathrm{e}^{\mathrm{i}my},\ \sin(kx)\mathrm{e}^{\mathrm{i}ny})$ be a family of immersions studied by Lawson [27] . Then, the composition $\varphi=H\circ f:T^2\longrightarrow S^2$ gives a family of maps from a torus into a 2-sphere defined by

$$\begin{aligned}\varphi(x,y)&=(\cos^2(kx)-\sin^2(kx),\ 2\cos(kx)\sin(kx)\mathrm{e}^{\mathrm{i}(m-n)y})\\&=(\cos(2kx),\ \sin(2kx)\mathrm{e}^{\mathrm{i}(m-n)y}).\end{aligned}\tag{2.1}$$

If we use geodesic polar coordinates (ρ,ϕ) on the 2-sphere, then this family of maps can be represented as

$$\varphi=H\circ f:T^2\longrightarrow(S^2,\mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2),\quad\varphi(x,y)=(\rho(x,y),\phi(x,y))$$

with

$$\begin{cases}\rho(x,y)=2kx,\\\phi(x,y)=(m-n)y.\end{cases}\tag{2.2}$$

Example 2.2 Let $f:S^1\times S^1\longrightarrow\mathbb{R}^4$ be a family of immersions of flat tori into $\mathbb{R}^4$ defined by $f(x,y)=(A\mathrm{e}^{\mathrm{i}x},B\mathrm{e}^{\mathrm{i}y})$ with constants A,B satisfying $A^2+B^2\neq0$. Postcomposing this map with the radial projection $P:\mathbb{R}^4\longrightarrow S^3$, $P(x)=\frac{x}{|x|}$, we have a family of map $F=P\circ f:S^1\times S^1\longrightarrow S^3$ with $F(x,y)=\big(\frac{A}{\sqrt{A^2+B^2}}\mathrm{e}^{\mathrm{i}x},\frac{B}{\sqrt{A^2+B^2}}\mathrm{e}^{\mathrm{i}y}\big)$. If we denote $\frac{A}{\sqrt{A^2+B^2}}=\cos s$, then $\frac{B}{\sqrt{A^2+B^2}}=\sin s$, then the family of the maps can be written as $F_s:T^2\longrightarrow S^3$ with $F_s(x,y)=(\mathrm{e}^{\mathrm{i}x}\cos s,\mathrm{e}^{\mathrm{i}y}\sin s)$. Applying construction via Hopf fibration, we have a map from T^2 into S^2:

$$\varphi=H\circ f:T^2\longrightarrow(S^2,\mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2),\quad\varphi(x,y)=(\rho(x,y),\phi(x,y))$$

with

$$\begin{cases}\rho(x,y)=2s,\quad s=\text{constant},\\\phi(x,y)=x-y,\end{cases}\tag{2.3}$$

where (ρ,ϕ) are the geodesic polar coordinates on S^2.

Remark 2.1 As it was observed in [6, Example 3.3.18] that except $s=0,\frac{\pi}{2}$, the family of maps F_s are embeddings of tori into S^3 and all these maps are harmonic maps with constant energy density called eigenmaps.

Example 2.3 Let $f: S^1 \times S^1 \longrightarrow S^3$ with $f(x,y) = (\sqrt{1-\beta^2(x)}\ \mathrm{e}^{\mathrm{i}x}, \beta(x)\mathrm{e}^{\mathrm{i}y})$, where $\beta(x)$ is a smooth function taking value between 0 and 1, be the family of immersions of flat tori into S^3 studied by Brendle [10]. Introducing the new variable by denoting $\cos\alpha(x) = \sqrt{1-\beta^2(x)}$, we have $f(x,y) = (\mathrm{e}^{\mathrm{i}x}\cos\alpha(x), \mathrm{e}^{\mathrm{i}y}\sin\alpha(x))$. Postcomposing this map with the Hopf fibration $H: S^3 \longrightarrow S^2$, we have a family of map $F = H \circ f: S^1 \times S^1 \longrightarrow S^3$ with $F(x,y) = (\cos 2\alpha(x), \mathrm{e}^{\mathrm{i}(x-y)}\sin 2\alpha(x))$. If we use the geodesic polar coordinates (ρ, ϕ) on the 2-sphere, then, this family of maps can be represented as

$$\varphi = H \circ f : T^2 \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho\mathrm{d}\phi^2), \varphi(x,y) = (\rho(x,y), \phi(x,y))$$

with

$$\begin{cases} \rho(x,y) = 2\alpha(x), \\ \phi(x,y) = x - y. \end{cases} \tag{2.4}$$

Example 2.4 Let $X: T^2 \longrightarrow \mathbb{R}^3$ be the standard embedding $X(r,\theta) = (a\sin r,\ (b + a\cos r)\cos\theta,\ (b + a\cos r)\sin\theta)$ of a torus into $\mathbb{R}^3$. A straightforward computation gives the induced metric and the unit normal vector field of the torus to be

$$g_T = a^2\mathrm{d}r^2 + (b + a\cos r)^2\mathrm{d}\theta^2$$

and

$$n = \frac{X_r \times X_\theta}{|X_r \times X_\theta|} = (\sin r,\ \cos r\cos\theta,\ \cos r\sin\theta),$$

respectively. If we use the geodesic polar coordinates (ρ, ϕ) on the unit sphere so that a generic point $(x,y,z) \in S^2$ is represented as $(x,y,z) = (\sin\rho, \mathrm{e}^{\mathrm{i}\phi}\cos\rho)$, then, the Gauss map of the torus can be written as

$$\begin{aligned} &\varphi : (T^2, a^2\mathrm{d}r^2 + (b + a\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \cos^2\rho\mathrm{d}\phi^2), \\ &\varphi(r,\theta) = (\rho(r,\theta), \phi(r,\theta)) \end{aligned}$$

with

$$\begin{cases} \rho(r,\theta) = r, \\ \phi(r,\theta) = \theta. \end{cases} \tag{2.5}$$

Example 2.5 Let $X: T^2 \longrightarrow \mathbb{R}^3$ be the standard embedding $X(r,\theta) = (a\sin r,\ (b + a\cos r)\cos\theta,\ (b + a\cos r)\sin\theta)$ of a torus into $\mathbb{R}^3$. Using the construction via radial projection $P: \mathbb{R}^3 \setminus \{0\} \longrightarrow S^2$ with $P(x) = \frac{x}{|x|}$, we have a family of maps from tori into a 2-sphere given by $\varphi = P \circ X : T^2 \longrightarrow S^2$ with

$$\varphi(r,\theta) = (\cos\alpha(r),\ \sin\alpha(r)\cos\theta,\ \sin\alpha(r)\sin\theta), \tag{2.6}$$

where

$$\arccos\alpha(r) = \frac{a\sin r}{\sqrt{a^2 + b^2 + 2ab\cos r}}.$$

Again, with respect to the geodesic polar coordinates (ρ, ϕ) on the unit sphere so that a generic point $(x, y, z) \in S^2$ is represented as $(x, y, z) = (\cos\rho, \mathrm{e}^{\mathrm{i}\phi}\sin\rho)$, then, this family of maps from tori into S^2 can be written as

$$\varphi : (T^2, a^2\mathrm{d}r^2 + (b + a\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho\mathrm{d}\phi^2),$$
$$\varphi(r, \theta) = (\rho(r, \theta), \phi(r, \theta))$$

with

$$\begin{cases} \rho(r, \theta) = \alpha(r), \\ \phi(r, \theta) = \theta, \end{cases} \tag{2.7}$$

which are rotationally symmetric maps.

The 6 families of maps given in Examples 1–6 lead us to study the biharmonicity of the following family of maps

$$\begin{aligned} &\varphi : T^2 \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho\mathrm{d}\phi^2), \\ &\varphi(r, \theta) = (ar + b\theta + c, mr + n\theta + l). \end{aligned} \tag{2.8}$$

Clearly, the family of maps defined by (2.8) includes families of maps defined in (2.2)–(2.3), (2.5), and parts of the families given in (2.4) and (2.7). Our main results in this paper include a complete classifications of proper biharmonic maps in the family of maps defined by (2.8), where the torus is provided with a flat or a non-flat metric whilst the sphere is provided with the standard metric (see Theorem 3.1 and Theorem 4.1).

3 Biharmonic Maps from a Flat Torus into a 2-Sphere

Lemma 3.1 (see [43]) *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a map between Riemannian manifolds with $\varphi(x^1, \cdots, x^m) = (\varphi^1(x), \cdots, \varphi^n(x))$ with respect to local coordinates (x^i) in M and (y^α) in N. Then, φ is biharmonic if and only if it is a solution of the following system of PDE's:*

$$\begin{aligned} &\Delta\tau^\sigma + 2g(\nabla\tau^\alpha, \nabla\varphi^\beta)\overline{\Gamma}^\sigma_{\alpha\beta} + \tau^\alpha\Delta\varphi^\beta\overline{\Gamma}^\sigma_{\alpha\beta} + \tau^\alpha g(\nabla\varphi^\beta, \nabla\varphi^\rho)(\partial_\rho\overline{\Gamma}^\sigma_{\alpha\beta} + \overline{\Gamma}^\nu_{\alpha\beta}\overline{\Gamma}^\sigma_{\nu\rho}) \\ &- \tau^\nu g(\nabla\varphi^\alpha, \nabla\varphi^\beta)\overline{R}^\sigma_{\beta\,\alpha\nu} = 0, \quad \sigma = 1, 2, \cdots, n, \end{aligned} \tag{3.1}$$

where $\tau^1, \cdots, \tau^n$ are components of the tension field of the map φ, ∇, Δ denote the gradient and the Laplace operators defined by the metric g, and $\overline{\Gamma}^\sigma_{\alpha\beta}$ and $\overline{R}^\sigma_{\beta\,\alpha\nu}$ are the components of the connection and the curvature of the target manifold.

In order to prove our classification theorem, we need the following lemma.

Lemma 3.2 *Let* $\varphi : (M^2, \mathrm{d}r^2 + \sigma^2(r)\mathrm{d}\theta^2) \longrightarrow (N^2, \mathrm{d}\rho^2 + \lambda^2(\rho)\mathrm{d}\phi^2)$ *with* $\varphi(r,\theta) = (ar + b\theta + c, mr + n\theta + l)$ *. Then,* φ *is biharmonic if and only if it solves the system*

$$\begin{cases} \frac{x_{\theta\theta}}{\sigma^2} + x_{rr} + \frac{\sigma'}{\sigma}x_r - \Big(m^2 + \frac{n^2}{\sigma^2}\Big)(\lambda\lambda'(\rho))'(\rho)x - \Big(2my_r + \frac{2n}{\sigma^2}y_\theta + y^2\Big)\lambda\lambda'(\rho) = 0, \\ \frac{y_{\theta\theta}}{\sigma^2} + y_{rr} + \frac{\sigma'}{\sigma}y_r + 2\Big(ay_r + mx_r + \frac{by_\theta}{\sigma^2} + \frac{nx_\theta}{\sigma^2}\Big)\frac{\lambda'(\rho)}{\lambda} \\ +2\Big(\frac{m\sigma'\lambda'(\rho)}{\sigma\lambda} + \Big(am + \frac{bn}{\sigma^2}\Big)\frac{(\lambda\lambda'(\rho))'(\rho)}{\lambda^2}\Big)x = 0, \\ x = \tau^1 = a\frac{\sigma'}{\sigma} - \Big(m^2 + \frac{n^2}{\sigma^2}\Big)\lambda\lambda'(\rho), \\ y = \tau^2 = m\frac{\sigma'}{\sigma} + 2\Big(am + \frac{bn}{\sigma^2}\Big)\frac{\lambda'(\rho)}{\lambda}. \end{cases} \tag{3.2}$$

Proof One can easily compute the connection coefficients of the domain and the target surfaces to get

$$\Gamma_{11}^1 = 0, \quad \Gamma_{12}^1 = 0, \quad \Gamma_{22}^1 = -\sigma\sigma', \quad \Gamma_{11}^2 = 0, \quad \Gamma_{12}^2 = \frac{\sigma'}{\sigma}, \quad \Gamma_{22}^2 = 0 \tag{3.3}$$

and

$$\overline{\Gamma}_{11}^1 = 0, \quad \overline{\Gamma}_{12}^1 = 0, \quad \overline{\Gamma}_{22}^1 = -\lambda\lambda'(\rho), \quad \overline{\Gamma}_{11}^2 = 0, \quad \overline{\Gamma}_{12}^2 = \frac{\lambda'(\rho)}{\lambda}, \quad \overline{\Gamma}_{22}^2 = 0. \tag{3.4}$$

A further computation gives the components of the Riemannian curvature of the target surface as

$$\overline{R}_{221}^1 = -\overline{R}_{212}^1 = \lambda\lambda''(\rho), \quad \overline{R}_{112}^2 = -\overline{R}_{121}^2 = \frac{\lambda''(\rho)}{\lambda}, \quad \text{all other} \quad \overline{R}_{kij}^l = 0. \tag{3.5}$$

We compute the components of tension field of the map φ to have

$$\begin{aligned} \tau^1 &= g^{ij}(\varphi_{ij}^1 - \Gamma_{ij}^k\varphi_k^1 + \overline{\Gamma}_{\alpha\beta}^1\varphi_i^\alpha\varphi_j^\beta) = -\Gamma_{11}^k\varphi_k^1 - \frac{1}{\sigma^2}\Gamma_{22}^k\varphi_k^1 + g(\nabla\varphi^2, \nabla\varphi^2)\overline{\Gamma}_{22}^1 \\ &= -m^2\lambda\lambda'(\rho) + \frac{a\sigma\sigma' - n^2\lambda\lambda'(\rho)}{\sigma^2}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \tau^2 &= g^{ij}(\varphi_{ij}^2 - \Gamma_{ij}^k\varphi_k^2 + \overline{\Gamma}_{\alpha\beta}^2\varphi_i^\alpha\varphi_j^\beta) = -\Gamma_{11}^k\varphi_k^2 - \frac{1}{\sigma^2}\Gamma_{22}^k\varphi_k^2 + 2g(\nabla\varphi^1, \nabla\varphi^2)\overline{\Gamma}_{12}^2 \\ &= m\frac{\sigma'}{\sigma} + 2\Big(am + \frac{bn}{\sigma^2}\Big)\frac{\lambda'(\rho)}{\lambda}. \end{aligned} \tag{3.7}$$

Using notations $x = \tau^1$, $y = \tau^2$, $x_r = \frac{\partial x}{\partial r}$, $x_\theta = \frac{\partial x}{\partial\theta}$, $y_r = \frac{\partial y}{\partial r}$, $y_\theta = \frac{\partial y}{\partial\theta}$, $x_{r\theta} = \frac{\partial^2 x}{\partial r\partial\theta}$, $y_{r\theta} = \frac{\partial^2 y}{\partial r\partial\theta}$, $x_{rr} = \frac{\partial^2 x}{\partial r^2}$, $y_{rr} = \frac{\partial^2 y}{\partial r^2}$, $x_{\theta\theta} = \frac{\partial^2 x}{\partial\theta^2}$, $y_{\theta\theta} = \frac{\partial^2 y}{\partial\theta^2}$, we compute

$$\Delta\tau^1 = g^{ij}(\tau_{ij}^1 - \Gamma_{ij}^k\tau_k^1) = x_{rr} + \frac{\sigma\sigma' x_r + x_{\theta\theta}}{\sigma^2}, \tag{3.8}$$

$$\Delta\tau^2 = g^{ij}(\tau_{ij}^2 - \Gamma_{ij}^k\tau_k^2) = y_{rr} + \frac{\sigma\sigma' y_r + y_{\theta\theta}}{\sigma^2}, \tag{3.9}$$

$$2g(\nabla\tau^\alpha, \nabla\varphi^\beta)\overline{\Gamma}_{\alpha\beta}^1 = 2g(\nabla\tau^2, \nabla\varphi^2)\overline{\Gamma}_{22}^1 = -2\lambda\lambda'(\rho)\Big(my_r + \frac{n}{\sigma^2}y_\theta\Big), \tag{3.10}$$

$$2g(\nabla\tau^\alpha, \nabla\varphi^\beta)\overline{\Gamma}_{\alpha\beta}^2 = 2g(\nabla\tau^1, \nabla\varphi^2)\overline{\Gamma}_{12}^2 + 2g(\nabla\tau^2, \nabla\varphi^1)\overline{\Gamma}_{21}^2$$

$$= \frac{2(mx_r + \frac{n}{\sigma^2}x_\theta)\lambda'(\rho)}{\lambda} + \frac{2(ay_r + \frac{b}{\sigma^2}y_\theta)\lambda'(\rho)}{\lambda}, \tag{3.11}$$

$$\tau^\alpha \Delta\varphi^\beta \overline{\Gamma}^1_{\alpha\beta} = \tau^2 \Delta\varphi^2 \overline{\Gamma}^1_{22} = -\frac{m\sigma'\lambda\lambda'(\rho)}{\sigma} y, \tag{3.12}$$

$$\tau^\alpha \Delta\varphi^\beta \overline{\Gamma}^2_{\alpha\beta} = \tau^1 \Delta\varphi^2 \overline{\Gamma}^2_{12} + \tau^2 \Delta\varphi^1 \overline{\Gamma}^2_{21} = \frac{\lambda'(\rho)}{\lambda}\Big(\frac{m\sigma'}{\sigma}x + \frac{a\sigma'}{\sigma}y\Big), \tag{3.13}$$

$$g(\nabla\varphi^\beta, \nabla\varphi^\rho)\partial_\rho \overline{\Gamma}^1_{\alpha\beta} = \tau^\alpha g(\nabla\varphi^\beta, \nabla\overline{\Gamma}^1_{\alpha\beta}) = \tau^2 g(\nabla\varphi^2, \nabla\overline{\Gamma}^1_{22})$$
$$= -\Big(am + \frac{bn}{\sigma^2}\Big)(\lambda\lambda'(\rho))'(\rho)y, \tag{3.14}$$

$$g(\nabla\varphi^\beta, \nabla\varphi^\rho)\partial_\rho \overline{\Gamma}^2_{\alpha\beta} = \tau^\alpha g(\nabla\varphi^\beta, \nabla\overline{\Gamma}^2_{\alpha\beta})$$
$$= \tau^1 g(\nabla\varphi^2, \nabla\overline{\Gamma}^2_{12}) + \tau^2 g(\nabla\varphi^1, \nabla\overline{\Gamma}^2_{21})$$
$$= \Big(am + \frac{bn}{\sigma^2}\Big)\Big(\frac{\lambda'(\rho)}{\lambda}\Big)'(\rho)x + \Big(a^2 + \frac{b^2}{\sigma^2}\Big)\Big(\frac{\lambda'(\rho)}{\lambda}\Big)'(\rho)y, \tag{3.15}$$

$$\tau^\alpha g(\nabla\varphi^\beta, \nabla\varphi^\rho)\overline{\Gamma}^v_{\alpha\beta}\overline{\Gamma}^1_{v\rho} = \tau^1 g(\nabla\varphi^2, \nabla\varphi^2)\overline{\Gamma}^2_{12}\overline{\Gamma}^1_{22} + \tau^2 g(\nabla\varphi^1, \nabla\varphi^2)\overline{\Gamma}^2_{21}\overline{\Gamma}^1_{22}$$
$$= -\Big(m^2 + \frac{n^2}{\sigma^2}\Big)\lambda'^2(\rho)x - \Big(am + \frac{bn}{\sigma^2}\Big)\lambda'^2(\rho)y, \tag{3.16}$$

$$\tau^\alpha g(\nabla\varphi^\beta, \nabla\varphi^\rho)\overline{\Gamma}^v_{\alpha\beta}\overline{\Gamma}^2_{v\rho}$$
$$= \tau^1 g(\nabla\varphi^2, \nabla\varphi^1)\overline{\Gamma}^2_{12}\overline{\Gamma}^2_{21} + \tau^2 g(\nabla\varphi^1, \nabla\varphi^1)\overline{\Gamma}^2_{21}\overline{\Gamma}^2_{21} + \tau^2 g(\nabla\varphi^2, \nabla\varphi^2)\overline{\Gamma}^1_{22}\overline{\Gamma}^2_{12}$$
$$= \Big(am + \frac{bn}{\sigma^2}\Big)\Big(\frac{\lambda'(\rho)}{\lambda}\Big)^2 x + \Big(a^2 + \frac{b^2}{\sigma^2}\Big)\Big(\frac{\lambda'(\rho)}{\lambda}\Big)^2 y - \Big(m^2 + \frac{n^2}{\sigma^2}\Big)\lambda'^2(\rho)y, \tag{3.17}$$

$$-\tau^v g(\nabla\varphi^\alpha, \nabla\varphi^\beta)\overline{R}^1_{\beta\alpha v} = -\tau^1 g(\nabla\varphi^2, \nabla\varphi^2)\overline{R}^1_{221} - \tau^2 g(\nabla\varphi^1, \nabla\varphi^2)\overline{R}^1_{212}$$
$$= -\Big(m^2 + \frac{n^2}{\sigma^2}\Big)\lambda\lambda''(\rho)x + \Big(am + \frac{bn}{\sigma^2}\Big)\lambda\lambda''(\rho)y \tag{3.18}$$

and

$$-\tau^v g(\nabla\varphi^\alpha, \nabla\varphi^\beta)\overline{R}^2_{\beta\alpha v} = -\tau^1 g(\nabla\varphi^2, \nabla\varphi^1)\overline{R}^2_{121} - \tau^2 g(\nabla\varphi^1, \nabla\varphi^1)\overline{R}^2_{112}$$
$$= \Big(am + \frac{bn}{\sigma^2}\Big)\frac{\lambda''(\rho)}{\lambda}x - \Big(a^2 + \frac{b^2}{\sigma^2}\Big)\frac{\lambda''(\rho)}{\lambda}y. \tag{3.19}$$

Substituting (3.8)–(3.19) into (3.1), we conclude that φ is biharmonic if and only if it solves the system

$$\begin{cases} x_{rr} + \dfrac{x_{\theta\theta}}{\sigma^2} + \dfrac{\sigma'}{\sigma}x_r - \Big(m^2 + \dfrac{n^2}{\sigma^2}\Big)(\lambda\lambda'(\rho))'(\rho)x \\ -2m\lambda\lambda'(\rho)y_r - \dfrac{2n\lambda\lambda'(\rho)}{\sigma^2}y_\theta - \Big(\dfrac{m\sigma'\lambda\lambda'(\rho)}{\sigma} + 2\Big(am + \dfrac{bn}{\sigma^2}\Big)\lambda'^2(\rho)\Big)y = 0, \\ y_{rr} + \dfrac{y_{\theta\theta}}{\sigma^2} + \Big(\dfrac{\sigma'}{\sigma} + \dfrac{2a\lambda'(\rho)}{\lambda}\Big)y_r + \dfrac{2b\lambda'(\rho)}{\sigma^2\lambda}y_\theta + \Big(\dfrac{a\sigma'\lambda'(\rho)}{\sigma\lambda} - \Big(m^2 + \dfrac{n^2}{\sigma^2}\Big)\lambda'^2(\rho)\Big)y \\ +\dfrac{2m\lambda'(\rho)}{\lambda}x_r + \dfrac{2n\lambda'(\rho)}{\sigma^2\lambda}x_\theta + \Big(\dfrac{m\sigma'\lambda'(\rho)}{\sigma\lambda} + 2\Big(am + \dfrac{bn}{\sigma^2}\Big)\dfrac{\lambda''(\rho)}{\lambda}\Big)x = 0, \\ x = \tau^1 = a\dfrac{\sigma'}{\sigma} - \Big(m^2 + \dfrac{n^2}{\sigma^2}\Big)\lambda\lambda'(\rho), \\ y = \tau^2 = m\dfrac{\sigma'}{\sigma} + 2\Big(am + \dfrac{bn}{\sigma^2}\Big)\dfrac{\lambda'(\rho)}{\lambda}, \end{cases} \tag{3.20}$$

which is equivalent to the system (3.2). Thus, the lemma follows.

Now we are ready to prove the following theorem that gives a classification of all biharmonic maps in a large family that includes most examples mentioned in Section 2.

Theorem 3.1 *The map* $\varphi : (T^2, \mathrm{d}r^2 + \mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho \mathrm{d}\phi^2)$ *from a flat torus into a 2-sphere with* $\varphi(r, \theta) = (ar + b\theta + c, mr + n\theta + l)$ *is biharmonic if and only if one of the following cases happens:*

(A) $a = b = 0$ *and* $c = \frac{\pi}{2}$. *In this case, the map* $\varphi(r, \theta) = \left(\frac{\pi}{2}, mr + n\theta + l\right)$ *is actually a harmonic map,*

(B) $m = n = 0$. *In this case, the map* $\varphi(r, \theta) = (ar + b\theta + c, l)$ *is actually a harmonic map, or*

(C) $a = b = 0$, $m^2 + n^2 \neq 0$ *and* $c = \frac{\pi}{4}$, *or*, $c = \frac{3\pi}{4}$. *In this case, the map* $\varphi(r, \theta) = (c, mr + n\theta + l)$ *with* $c = \frac{\pi}{4}$, *or*, $c = \frac{3\pi}{4}$, *is a proper biharmonic map.*

Proof Applying Lemma 3.2 with $\sigma(r) = 1$, $\lambda(\rho) = \sin\rho$, we conclude that the map $\varphi : (S^1 \times S^1, \mathrm{d}r^2 + \mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho \mathrm{d}\phi^2)$ from a flat torus into a 2-sphere with $\varphi(r, \theta) = (ar + b\theta + c, mr + n\theta + l)$ is biharmonic if and only if

$$\begin{cases} x_{rr} + x_{\theta\theta} - (m^2 + n^2)(\cos 2\rho)x - 2m(\sin\rho\cos\rho)y_r \\ \ -2n(\sin\rho\cos\rho)y_\theta - 2(am + bn)(\cos^2\rho)\, y = 0, \\ y_{rr} + y_{\theta\theta} + 2a(\cot\rho)y_r + 2b(\cot\rho)y_\theta - (m^2 + n^2)(\cos^2\rho)\, y \\ +2m(\cot\rho)x_r + 2n(\cot\rho)x_\theta - 2(am + bn)x = 0, \\ x = \tau^1 = -\frac{1}{2}(m^2 + n^2)\sin(2\rho), \\ y = \tau^2 = 2(am + bn)\cot\rho. \end{cases} \tag{3.21}$$

Substituting the last two equations into the first two equations of (3.21), we see that the biharmonicity of the map φ becomes

$$\begin{cases} [4(m^2 + n^2)(a^2 + b^2) + (m^2 + n^2)^2\cos(2\rho) + 4(am + bn)^2]\sin\rho\cos\rho = 0, \\ 4(m^2 + n^2)(am + bn)\dfrac{\cos\rho}{\sin\rho}\cos(2\rho) = 0. \end{cases} \tag{3.22}$$

Noting that $0 < \rho = ar + b\theta + c < \pi$ (and hence) $\sin\rho \neq 0$, we conclude that Equation (3.22) is equivalent to

$$\begin{cases} [4(m^2 + n^2)(a^2 + b^2) + (m^2 + n^2)^2\cos(2\rho) + 4(am + bn)^2]\cos\rho = 0, \\ 4(m^2 + n^2)(am + bn)\cos\rho\cos(2\rho) = 0. \end{cases} \tag{3.23}$$

We solve Equation (3.23) by considering the following cases.

Case I $\cos\rho = 0$. This means that $\cos(ar + b\theta + c) = 0$ for any $r, \theta \in \mathbb{R}$. This, together with $0 < \rho = ar + b\theta + c < \pi$, implies that $a = b = 0$, $c = \frac{\pi}{2}$. Noting that the components of the tension field of the map φ are given by the last two equations of (3.21), we conclude that the solutions $\left(a = b = 0,\ c = \frac{\pi}{2},\ m, n, l \in \mathbb{R}\right)$ given in this case are actually harmonic maps. From this we obtain the case (A).

Case II $\cos\rho \neq 0$. In this case, the biharmonicity of the map φ is equivalent to

$$\begin{cases} 4(m^2+n^2)(a^2+b^2)+(m^2+n^2)^2\cos(2\rho)+4(am+bn)^2=0, \\ 4(m^2+n^2)(am+bn)\cos(2\rho)=0. \end{cases} \tag{3.24}$$

If $m=n=0$, then, $m=n=0,\ a,b,c,l\in\mathbb{R}$ are solutions of (3.24) and we see from the last two equations of (3.21) that the maps given by these solutions are actually harmonic maps. This gives the case (B).

If otherwise, i.e., $m^2+n^2\neq 0$, then it follows from (3.24) that the map φ is biharmonic if and only if

$$\begin{cases} \cos(2\rho)=-\dfrac{4(m^2+n^2)(a^2+b^2)+4(am+bn)^2}{(m^2+n^2)^2}, \\ (am+bn)\cos(2\rho)=0. \end{cases} \tag{3.25}$$

If $am+bn\neq 0$, then we can easily check that Equation (3.25) has no solution. If $am+bn=0$, then the map φ is biharmonic if and only if

$$\begin{cases} \cos(2\rho)=-\dfrac{4(a^2+b^2)}{m^2+n^2}, \\ am+bn=0. \end{cases} \tag{3.26}$$

Since the first equation of (3.26) means that $\cos(2ar+2b\theta+2c)=-\frac{4(a^2+b^2)}{m^2+n^2}$ for any $r,\theta\in\mathbb{R}$, we conclude that $a=b=0$ and hence $\cos(2c)=0$. Recalling that $2\rho=2c\in(0,2\pi)$, we obtain solutions $c=\frac{\pi}{4},\frac{3\pi}{4}$. In these cases, $\cos\rho=\cos c\neq 0$. It follows from the third equation of (3.21) that the first component of the tension field $\tau^1=-\frac{1}{2}(m^2+n^2)\sin(2\rho)=\pm\frac{1}{2}(m^2+n^2)\neq 0$. Thus, the biharmonic maps $\varphi(r,\theta)=\big(\frac{\pi}{4},mr+n\theta+l\big)$, $\varphi(r,\theta)=\big(\frac{3\pi}{4},mr+n\theta+l\big)$ are proper biharmonic maps. From this we obtain the case (C).

Summarizing the above results we obtain the theorem.

As immediate consequences of Theorem 3.1, we have the following corollaries.

Corollary 3.1 *For $m^2+n^2\neq 0$, the map $(T^2,\mathrm{d}r^2+\mathrm{d}\theta^2)\longrightarrow(S^2,\mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2)$ with $\varphi(r,\theta)=\big(\pm\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\mathrm{e}^{(mr+n\theta+l)\mathrm{i}}\big)$ is a proper biharmonic map. In particular, the compositions of the family of harmonic embeddings $F_s:(T^2,\mathrm{d}r^2+\mathrm{d}\theta^2)\longrightarrow S^3$, $F_s(r,\theta)=(\mathrm{e}^{\mathrm{i}r}\cos s,\mathrm{e}^{\mathrm{i}\theta}\sin s)$ followed by the Hopf fibration $H:S^3\longrightarrow S^2$,*

$$\varphi_s=H\circ F_s:(T^2,\mathrm{d}r^2+\mathrm{d}\theta^2)\longrightarrow(S^2,\mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2),$$
$$\varphi_s(r,\theta)=(\rho(r,\theta),\phi(r,\theta)$$

with

$$\begin{cases} \rho(r,\theta)=2s, \quad s=\text{constant}, \\ \phi(r,\theta)=r-\theta, \end{cases}$$

are proper biharmonic maps if and only if $s=\frac{\pi}{8}$ or $s=\frac{3\pi}{8}$.

Remark 3.1 Note that Corollary 3.1 provides many new examples of proper biharmonic maps from a flat torus into a sphere, including special cases $\varphi : (S^1\times S^1, \mathrm{d}r^2+\mathrm{d}\theta^2) \longrightarrow (S^2, g^{S^2})$ with $\varphi(r,\theta) = (\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\sin\theta)$. These special cases, up to an isometry of the target sphere, are the same maps obtained in [39] by construction of orthogonal multiplication of complex numbers (see [39, Theorem 2.2] for details). These special cases of proper biharmonic map were also known as special solutions to the biharmonic equation for rotationally symmetric maps from a flat torus into a 2-sphere (see [33, 50]).

Corollary 3.2 *The map*

$$\begin{aligned}&\varphi = H\circ f : (T^2, \mathrm{d}r^2+\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2),\\ &\varphi(r,\theta) = (\rho(r,\theta), \phi(r,\theta))\end{aligned}$$

with

$$\begin{cases}\rho(r,\theta) = -r-\theta,\\ \phi(r,\theta)=\theta\end{cases}$$

is neither a harmonic map nor a biharmonic map.

Corollary 3.3 *The Gauss map of the torus* $X : T^2 \longrightarrow \mathbb{R}^3$, $X(r,\theta) = (a\sin r,\ (b+a\cos r)\cos\theta,\ (b+a\cos r)\sin\theta)$, *viewed as a map* $(T^2, \mathrm{d}r^2+\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2)$ *from a flat torus into a sphere, is neither harmonic nor biharmonic.*

Corollary 3.4 *Let* $f : (T^2, \mathrm{d}r^2+\mathrm{d}\theta^2) \longrightarrow S^3$ *with* $f(r,\theta) = (\cos(kr)\mathrm{e}^{\mathrm{i}m\theta}, \sin(kr)\mathrm{e}^{\mathrm{i}n\theta})$ *be the family of immersions studied in* [27]. *Then, the family of maps from a torus into a 2-sphere defined by the construction via Hopf fibration*

$$\begin{aligned}&\varphi = H\circ f : (T^2, \mathrm{d}r^2+\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2+\sin^2\rho\mathrm{d}\phi^2),\\ &\phi(r,\theta) = (\rho(r,\theta), \phi(r,\theta)),\end{aligned}$$

with

$$\begin{cases}\rho(r,\theta) = 2kr,\\ \phi(r,\theta)=(m-n)\theta\end{cases}$$

contains no proper biharmonic map.

4 Biharmonic Maps from a Non-Flat Torus into a 2-Sphere

In this section, we will study biharmonic maps from a non-flat torus into a 2-sphere. The non-flat metric on the torus we consider is $\mathrm{d}r^2 + (k+\cos r)^2\mathrm{d}\theta^2$ which is homothetic to the induced metric $g_T = a^2\mathrm{d}r^2 + (b+a\cos r)^2\mathrm{d}\theta^2$ (see Example 2.4 for details) from the standard embedding $X : T^2 \longrightarrow \mathbb{R}^3$, $X(r,\theta) = (a\sin r,\ (b+a\cos r)\cos\theta,\ (b+a\cos r)\sin\theta)$.

Theorem 4.1 *The map from a non-flat torus into a 2-sphere, $\varphi : (T^2, \mathrm{d}r^2+(k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2+\cos^2\rho\mathrm{d}\phi^2)$ $(k>1)$ with $\varphi(r,\theta)=(ar+b\theta+c, mr+n\theta+l)$ is biharmonic if and only if it is harmonic.*

Proof Let $\varphi : (T^2, \mathrm{d}r^2+(k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2+\cos^2\rho\mathrm{d}\phi^2)$ $(k>1)$ with $\varphi(r,\theta)=(ar+b\theta+c, mr+n\theta+l)$. Using Lemma 3.2 with $\sigma(r)=k+\cos r$, $\lambda(\rho)=\cos\rho$, we see that φ is biharmonic if and only if it solves the system

$$\begin{cases} \dfrac{x_{\theta\theta}}{(k+\cos r)^2}+x_{rr}-\dfrac{\sin r}{k+\cos r}x_r+\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\cos 2\rho\, x \\ +\Big(2my_r+\dfrac{2n}{(k+\cos r)^2}y_\theta+y^2\Big)\sin\rho\cos\rho=0, \\ \dfrac{y_{\theta\theta}}{(k+\cos r)^2}+y_{rr}-\dfrac{\sin r}{k+\cos r}y_r-2\Big(ay_r+mx_r+\dfrac{by_\theta+nx_\theta}{(k+\cos r)^2}\Big)\dfrac{\sin\rho}{\cos\rho} \\ +2\Big(\dfrac{m\sin r\sin\rho}{(k+\cos r)\cos\rho}-\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)\dfrac{\cos 2\rho}{\cos^2\rho}\Big)x=0, \\ x=\tau^1=-\dfrac{a\sin r}{k+\cos r}+\dfrac{1}{2}\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\sin 2\rho, \\ y=\tau^2=-\dfrac{m\sin r}{k+\cos r}-2\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)\tan\rho. \end{cases} \tag{4.1}$$

A straightforward computation using the last two equations of (4.1) yields

$$\begin{cases} x_r=-\dfrac{ak\cos r+a}{(k+\cos r)^2}+\dfrac{n^2\sin r\sin 2\rho}{(k+\cos r)^3}+a\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\cos 2\rho, \\ x_{rr}=-\dfrac{a(2-k^2)\sin r+ak\sin r\cos r-4an^2\sin r\cos 2\rho}{(k+\cos r)^3} \\ \qquad -\dfrac{2a^2m^2(k+\cos r)^4+2a^2n^2(k+\cos r)^2-n^2(k\cos r+\cos^2 r+3\sin^2 r)}{(k+\cos r)^4}\sin 2\rho, \\ x_\theta=b\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\cos 2\rho, \\ x_{\theta\theta}=-2b^2\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\sin 2\rho, \\ y_r=-\dfrac{mk\cos r+m}{(k+\cos r)^2}-\dfrac{4bn\sin r\tan\rho}{(k+\cos r)^3}-\dfrac{2a\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)}{\cos^2\rho}, \\ y_{rr}=-\dfrac{m(2-k^2)\sin r+mk\sin r\cos r}{(k+\cos r)^3}-\dfrac{4bn(k\cos r+\cos^2 r+3\sin^2 r)\tan\rho}{(k+\cos r)^4} \\ \qquad -\dfrac{8abn\sin r}{(k+\cos r)^3\cos^2\rho}-\dfrac{4a^2\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)\sin\rho}{\cos^3\rho}, \\ y_\theta=-\dfrac{2b\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)}{\cos^2\rho}, \\ y_{\theta\theta}=\dfrac{4b^2\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)\sin\rho}{\cos^3\rho}. \end{cases} \tag{4.2}$$

Substituting (4.2) into the first equation of (4.1), we have

$$\begin{cases} \dfrac{(a(k^2-1)-2bmn)\sin r}{(k+\cos r)^3}+\dfrac{2am^2\sin r}{k+\cos r}+\Big[\dfrac{n^2(k\cos r+\sin^2 r+1)}{(k+\cos r)^4} \\ +\dfrac{\frac{m^2}{2}\sin^2 r-m^2k\cos r-m^2}{(k+\cos r)^2}-2\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)^2 \\ -2\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\Big(a^2+\dfrac{b^2}{(k+\cos r)^2}\Big)\Big]\sin 2\rho \\ +\Big[\dfrac{2an^2\sin r}{(k+\cos r)^3}-\dfrac{4am^2\sin r}{k+\cos r}+\dfrac{2bmn\sin r}{(k+\cos r)^3}\Big]\cos 2\rho \\ +\dfrac{1}{4}\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)^2\sin 4\rho=0, \\ \rho=ar+b\theta+c. \end{cases} \tag{4.3}$$

We will solve Equation (4.3) by the following cases.

Case I $b\neq 0$.

In this case, using the assumption that $b\neq 0$ and the fact that the functions $1,\ \sin 2\rho,\ \cos 2\rho$ and $\sin 4\rho$ are linearly independent as functions of variable θ, we conclude from (4.3) that

$$\begin{cases} \dfrac{(a(k^2-1)-2bmn)\sin r}{(k+\cos r)^3}+\dfrac{2am^2\sin r}{k+\cos r}=0, \\ \dfrac{n^2(k\cos r+\sin^2 r+1)}{(k+\cos r)^4}+\dfrac{\frac{m^2}{2}\sin^2 r-m^2k\cos r-m^2}{(k+\cos r)^2} \\ -2\Big(am+\dfrac{bn}{(k+\cos r)^2}\Big)^2-2\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)\Big(a^2+\dfrac{b^2}{(k+\cos r)^2}\Big)=0, \\ \dfrac{2an^2\sin r}{(k+\cos r)^3}-\dfrac{4am^2\sin r}{k+\cos r}+\dfrac{2bmn\sin r}{(k+\cos r)^3}=0, \\ \dfrac{1}{4}\Big(m^2+\dfrac{n^2}{(k+\cos r)^2}\Big)^2=0. \end{cases} \tag{4.4}$$

From the fourth and the first equation of (4.4), we have $a=m=n=0$. In this case, we use the last two equations in (4.1) to conclude that the components of the tension field $x=\tau^1=0,\ y=\tau^2=0$. This implies that the map φ is actually harmonic.

Case II $b=0$ and $a=0$.

In this case, substituting $a=b=0$ and (4.2) into the second equation of (4.1), we obtain

$$\begin{cases} \dfrac{m(k^2-1)\sin r}{(k+\cos r)^3}-\dfrac{2mn^2\sin r}{(k+\cos r)^3}\sin^2\rho+\dfrac{2m^3\sin r}{k+\cos r}\sin^2\rho=0, \\ \rho=c. \end{cases} \tag{4.5}$$

If $c=0$, then $\sin\rho=0$ and (4.5) reduces to $m(k^2-1)=0$ which implies $m=0$ since $k>1$ by assumption. In this case, the last two equations in (4.1) imply that $\tau(\varphi)=0$ and hence φ is harmonic.

If $c\neq 0$, then (4.5) is equivalent to

$$\begin{cases} m[(k^2-1)+2(m^2k^2-n^2)\sin^2\rho+4m^2k\sin^2\rho\cos r+2m^2\sin^2\rho\cos^2 r]=0, \\ \rho=c\neq 0. \end{cases} \tag{4.6}$$

If $m \neq 0$, we have

$$\begin{cases} k^2 - 1 + 2(m^2k^2 - n^2)\sin^2\rho + 4m^2k\sin^2\rho\cos r + 2m^2\sin^2\rho\cos^2 r = 0, \\ \rho = c \neq 0. \end{cases} \tag{4.7}$$

Note that $k > 1$ and the functions $1,\ \cos r,\ \cos^2 r$ are linearly independent, then Equation (4.7) implies that $\sin^2\rho = 0$ and reduces to $k^2 - 1 = 0$, a contradiction.

If otherwise, i.e., $m = 0$, substituting $a = b = m = 0$ and (4.2) into the first equation of (4.1), we have

$$\begin{cases} \dfrac{n^2\sin 2\rho}{2(k+\cos r)^4}(2k\cos r - 2\cos^2 r + 4 + n^2\cos 2\rho) = 0, \\ \rho = c \neq 0. \end{cases} \tag{4.8}$$

The above equation implies that $n^2\sin 2\rho = 0$, i.e., $n = 0$. It follows that $x = \tau^1 = 0,\ y = \tau^2 = 0$, meaning that the map φ is harmonic in this case.

Case III $b = 0,\ a \neq 0$. We will show that Equation (4.3) has no solution in this case.

Multiplying $(k+\cos r)^4$ to both sides of Equation (4.3) and simplifying the resulting equation by using the product-to-sum formulas, we have

$$a_0\sin(2ar+2c) + \sum_{i=1}^{4}\frac{a_i \pm b_i}{2}\sin[(2a \pm i)r + 2c]$$
$$+ c_0\sin(4ar+4c) + \sum_{i=1}^{4}\frac{c_i}{2}\sin[(4a \pm i)r + 4c] + d_1\sin r + d_2\sin 2r + d_3\sin 3r + d_4\sin 4r = 0,$$

or, equivalently

$$\begin{aligned} &a_0\cos(2c)\sin(2ar) + a_0\sin(2c)\cos(2ar) \\ &+ \cos(2c)\sum_{i=1}^{4}\frac{a_i \pm b_i}{2}\sin(2a \pm i)r + \sin(2c)\sum_{i=1}^{4}\frac{a_i \pm b_i}{2}\cos(2a \pm i)r \\ &+ c_0\cos(4c)\sin(4ar) + c_0\sin(4c)\cos(4ar) \\ &+ \cos(4c)\sum_{i=1}^{4}\frac{c_i}{2}\sin(4a \pm i)r + \sin(4c)\sum_{i=1}^{4}\frac{c_i}{2}\cos(4a \pm i)r \\ &+ d_1\sin r + d_2\sin 2r + d_3\sin 3r + d_4\sin 4r = 0, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} a_0 = &-\frac{7m^2k^2}{4} - 2a^2n^2k^2 - a^2n^2 - \frac{7m^2}{16} - 4a^2m^2k^4 \\ &- 12a^2m^2k^2 - \frac{3a^2m^2}{2} + \frac{3n^2}{2}, \end{aligned} \tag{4.10}$$

$$a_1 = -\frac{5m^2k}{2} - m^2k^3 - 4a^2n^2k - 16a^2m^2k^3 - 12a^2m^2k + n^2k, \tag{4.11}$$

$$a_2 = -\frac{5m^2k^2}{4} - a^2n^2 - \frac{m^2}{2} - 12a^2m^2k^2 - 2a^2m^2 - \frac{n^2}{2}, \tag{4.12}$$

$$a_3 = -\frac{m^2k}{2} - 4a^2m^2k, \quad a_4 = -\frac{m^2}{16} - \frac{a^2m^2}{2}, \tag{4.13}$$

$$b_1 = -4am^2k^3 - 3am^2k + 2an^2k, \quad b_2 = -6am^2k - am^2 + an^2, \tag{4.14}$$

$$b_3 = -3am^2k, \quad b_4 = -\frac{am^2}{2}, \tag{4.15}$$

$$c_0 = \frac{m^4k^4}{4} + \frac{3m^4k^2}{4} + \frac{3m^4}{32} + \frac{m^2n^2k^2}{2} + \frac{m^2n^2}{4} + \frac{n^4}{4}, \tag{4.16}$$

$$c_1 = \frac{m^4k^3 + 3m^4k}{4} + m^2n^2k, \quad c_2 = \frac{3m^4k^2}{4} + \frac{m^2n^2}{4} + \frac{m^4}{8}, \tag{4.17}$$

$$c_3 = \frac{m^4k}{4}, \quad c_4 = \frac{m^4}{32}, \tag{4.18}$$

$$d_1 = a(k^2-1)k + am^2\Big(2k^3 + \frac{3k}{2}\Big), \tag{4.19}$$

$$d_2 = \frac{a(k^2-1)}{2} + am^2\Big(3k^2 + \frac{1}{2}\Big), \tag{4.20}$$

$$d_3 = \frac{3am^2k}{2}, \quad d_4 = \frac{am^2}{4}. \tag{4.21}$$

We observe that the 40 trigonometric functions appearing in the linear combination on the left hand side of Equation (4.9) are linearly independent for the values of a that produce no like terms among them. Note also that even in the case the values of a produce like terms, we can collect the like terms so that the resulting set of functions are linearly independent.

Case A For those values of a that do not turn any of $\sin(2ar),\ \sin(2a\pm i)r, \sin(4ar), \sin(4a\pm i)r$ $(i = 1, 2, 3, 4)$ into a like term of $\sin r, \sin 2r, \sin 3r, \sin 4r$. In this case, we have all coefficients vanish, including $d_1 = d_4 = 0$, which imply $a = 0$, a contradiction.

Case B For those values of a that turn one of $\sin(2ar), \sin(2a \pm i)r, \sin(4ar), \sin(4a \pm i)r$ $(i = 1, 2, 3, 4)$ into a like term to one of $\sin r, \sin 2r, \sin 3r, \sin 4r$. We can check that the only values of a that turn one of $\sin(2ar), \sin(2a \pm i)r, \sin(4ar), \sin(4a \pm i)r$ $(i = 1, 2, 3, 4)$ into a like term to one of $\sin r, \sin 2r, \sin 3r, \sin 4r$ are the following:

$$a = \pm j,\ \pm\frac{2j-1}{2},\ \pm\frac{2j-1}{4}, \quad j = 1, 2, 3, 4. \tag{4.22}$$

It is not difficult to check that none of positive values of a given in (4.22) can produce like term of $\sin(4a+4)r$ and none of the negative values of a given in (4.22) can produce like term of $\sin(4a-4)r$. It follows that for each value of a given in (4.22), we can, after a possible collecting of like terms in (4.9), use the linear independence of the resulting set of functions to conclude that $c_4 = \frac{m^4}{32} = 0$. This implies that $m = 0$ for any values of a given in (4.22). Substituting $m = 0$ into Equation (4.9), we have

$$\begin{aligned}
&\sin(2c)\Big\{a_0\cos(2ar) + \frac{a_1+b_1}{2}\cos[(2a+1)r] + \frac{a_1-b_1}{2}\cos[(2a-1)r] \\
&+ \frac{a_2+b_2}{2}\cos[(2a+2)r] + \frac{a_2-b_2}{2}\cos[(2a-2)r]\Big\} + c_0\sin(4c)\cos(4ar) \\
&+ \cos(2c)\Big\{a_0\sin(2ar) + \frac{a_1+b_1}{2}\sin[(2a+1)r] + \frac{a_1-b_1}{2}\sin[(2a-1)r] \\
&+ \frac{a_2+b_2}{2}\sin[(2a+2)r] + \frac{a_2-b_2}{2}\sin[(2a-2)r]\Big\} + c_0\cos(4c)\sin(4ar)
\end{aligned}$$

$$+ d_1 \sin r + d_2 \sin 2r = 0, \tag{4.23}$$

where

$$a_0 = -2a^2n^2k^2 - a^2n^2 + \frac{3n^2}{2}, \quad a_1 = -4a^2n^2k + n^2k, \tag{4.24}$$

$$a_2 = -a^2n^2 - \frac{n^2}{2}, \quad b_1 = 2an^2k, \quad b_2 = an^2, \quad c_0 = \frac{n^4}{4}, \tag{4.25}$$

$$d_1 = a(k^2-1)k, \quad d_2 = \frac{a(k^2-1)}{2}. \tag{4.26}$$

We can check that the only values of a that turn $\sin(2ar)$ into a like term to one of $\sin(2a \pm i)r\ (i=1,2), \sin(4ar), \sin r, \sin 2r$ are $a = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$.

For the case $a \neq \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$, $\sin(2ar)$ is not a like term to any other term and hence we can use linear independence of functions in Equation (4.23) to conclude that $a_0 = 0$. Using (4.24) we have either $\frac{3}{2} - a^2(2k^2+1) = 0$ or $n = 0$. The first case gives the values of a contradicting those given by (4.22). The latter case, i.e., $n = 0$, implies that $a_0 = a_1 = a_2 = b_1 = b_2 = c_0 = 0$. Substituting these into (4.23), we have $d_1 = d_2 = 0$ which implies $a = 0$, a contradiction.

For the case $a = \pm 1, \pm\frac{1}{2}$, or $\pm\frac{1}{4}$, we first check that for $a = \pm\frac{1}{4}$, we have $d_2 = 0$, a contradiction. Secondly, we check that for $a = \pm 1, \pm\frac{1}{2}$, we have $a_1 \pm b_1 = 0, a_2 \pm b_2 = 0$ respectively. A further checking shows that in each of these four cases, we have $n = 0$, which, as we have seen in the above argument, will lead to a contradiction. This ends the proof that the biharmonic map equation has no solution in Case B.

Summarizing the results in Cases A and B we conclude that for the case $b = 0,\ a \neq 0$, the biharmonic map equations have no solution.

Combining the results proved in Cases I–III, we obtain the theorem.

From the proof of Theorem 4.1 we have seen the following corollary.

Corollary 4.1 *For $a \neq 0$, there exists no biharmonic map in the family of the maps from a non-flat torus into a 2-sphere, $\varphi : (T^2, \mathrm{d}r^2 + (k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \cos^2\rho\mathrm{d}\phi^2)$ $(k > 1)$ with $\varphi(r,\theta) = (ar + b\theta + c, mr + n\theta + l)$.*

Noting that the models $(S^2, \mathrm{d}\rho^2 + \sin^2\rho\mathrm{d}\phi^2)$ and $(S^2, \mathrm{d}\rho^2 + \cos^2\rho\mathrm{d}\phi^2)$ of a 2-sphere are isometric to each other, we can use Theorem 4.1 to have the following classification results.

Corollary 4.2 *The composition of the family of immersions*

$$f_s : (T^2, \mathrm{d}r^2 + (k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow S^3, \quad f_s(r,\theta) = (\mathrm{e}^{\mathrm{i}r}\cos s, \mathrm{e}^{\mathrm{i}\theta}\sin s)$$

followed by the Hopf fiberation $H : S^3 \longrightarrow S^2$,

$$\varphi_s = H \circ f_s : (T^2, \mathrm{d}r^2 + (k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho\mathrm{d}\phi^2),$$
$$\varphi_s(r,\theta) = (\rho(r,\theta), \phi(r,\theta))$$

with

$$\begin{cases} \rho(r,\theta) = 2s,\ s = \text{constant}, \\ \phi(r,\theta) = r - \theta, \end{cases}$$

is neither a harmonic map nor a biharmonic map.

Corollary 4.3 *The composition of the map* $f : (T^2, \mathrm{d}r^2 + (k + \cos r)^2 \mathrm{d}\theta^2) \longrightarrow S^3$ *with*

$$f(r,\theta) = \Big(\cos\frac{-(r+\theta)}{2}\mathrm{e}^{\mathrm{i}\frac{x+y}{2}},\ \sin\frac{-(r+\theta)}{2}\,\mathrm{e}^{\mathrm{i}\frac{r-\theta}{2}}\Big)$$

given in [39] *followed by the construction via Hopf fibration*

$$\begin{aligned}&\varphi = H \circ f : (T^2, \mathrm{d}r^2 + (k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho \mathrm{d}\phi^2),\\ &\varphi(r,\theta) = (\rho(r,\theta), \phi(r,\theta))\end{aligned}$$

with

$$\begin{cases}\rho(r,\theta) = -r-\theta,\\ \phi(r,\theta) = \theta,\end{cases}$$

is neither a harmonic map nor a biharmonic map.

Corollary 4.4 *Let* $f : (T^2, \mathrm{d}r^2 + (k + \cos r)^2 \mathrm{d}\theta^2) \longrightarrow S^3$ *with*

$$f(r,\theta) = (\cos(cr)\mathrm{e}^{\mathrm{i}m\theta}, \sin(cr)\mathrm{e}^{\mathrm{i}n\theta})$$

be the family of immersions studied in [27]*. Then, the family of maps from a torus into a 2-sphere defined by the construction via Hopf fibration*

$$\begin{aligned}&\varphi = H \circ f : (T^2, \mathrm{d}r^2 + (k+\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, \mathrm{d}\rho^2 + \sin^2\rho \mathrm{d}\phi^2),\\ &\phi(r,\theta) = (\rho(r,\theta), \phi(r,\theta))\end{aligned}$$

with

$$\begin{cases}\rho(r,\theta) = 2cr,\\ \phi(r,\theta) = (m-n)\theta\end{cases}$$

is neither a harmonic map nor a biharmonic map.

Corollary 4.5 *Let* $X : T^2 \longrightarrow \mathbb{R}^3$ *be an embedding with*

$$X(r,\theta) = (a\sin r,\ (b + a\cos r)\cos\theta,\ (b+a\cos r)\sin\theta)$$

with $b > a > 0$*. Then, the Gauss map*

$$\varphi : (T^2, g_T = a^2\mathrm{d}r^2 + (b + a\cos r)^2\mathrm{d}\theta^2) \longrightarrow (S^2, h = \mathrm{d}r^2 + \cos^2 r\mathrm{d}\theta^2)$$

with $\varphi(r,\theta) = (r,\theta)$ *is neither harmonic nor biharmonic.*

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