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Measure Estimates of Nodal Sets of Polyharmonic Functions*

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Abstract This paper deals with the function u which satisfies $\triangle^k u = 0$, where $k \geq 2$ is an integer. Such a function u is called a polyharmonic function. The author gives an upper bound of the measure of the nodal set of u, and shows some growth property of u.

Keywords Polyharmonic function, Nodal set, Frequency, Measure estimate, Growth property
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1 Introduction

The nodal sets are zero level sets. We want to study the measure estimates of nodal sets of polyharmonic functions in this paper. In 1979, Almgren [1] introduced the frequency concept of harmonic functions. Then in 1986 and 1987, Garofalo and Lin [4–5] established the monotonicity formula of the frequency and the doubling conditions for solutions of the uniformly second order elliptic equations, and showed the unique continuation of such solutions by using the doubling conditions. In 2000, Han [6] studied the structure of the nodal sets of solutions of a class of uniformly high order elliptic equations. In 2003, Han, Hardt and Lin in [7] investigated structures and measure estimates of singular sets of solutions of high order uniformly elliptic equations. In 2014, the author and Yang in [13] gave the measure estimates of nodal sets for bi-harmonic functions.

The classical frequency of a harmonic function is defined as follows.

Definition 1.1 *If u is a harmonic function in B_1, then for any $r \leq 1$, one can define the frequency function of u centered at the origin with radius r as follows:*

$$N(r) = r\frac{D(r)}{H(r)} = r\frac{\int_{B_r} |\nabla u|^2 \mathrm{d}x}{\int_{\partial B_r} u^2 \mathrm{d}\sigma}, \tag{1.1}$$

where $\mathrm{d}\sigma$ means the $(n-1)$-Hausdorff measure on the sphere ∂B_r. Similarly, one can define the frequency centered at other point.

Based on this idea, we define the frequency of a polyharmonic function as follows. We first show some notations in this paper as follows:

$$u_1 = u, \quad u_2 = \triangle u, \quad \cdots, \quad u_k = \triangle^{k-1} u, \quad u_{k+1} = \triangle^k u = 0.$$

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Definition 1.2 *Suppose that u satisfies that $\triangle^k u = 0$, where k is a positive integer more than or equal to 2. Such a function u is called a k-polyharmonic function in the rest of this paper. Then we define*

$$N(r) = r\frac{D(r) + E(r)}{H(r)}, \tag{1.2}$$

where

$$D(r) = \sum_{i=1}^{k} D_i(r), \quad E(r) = \sum_{i=1}^{k} E_i(r), \quad H(r) = \sum_{i=1}^{k} H_i(r),$$
$$D_i(r) = \int_{B_r} |\nabla u_i|^2 \mathrm{d}x, \quad E_i(r) = \int_{B_r} u_i u_{i+1} \mathrm{d}x, \quad H_i(r) = \int_{\partial B_r} u_i^2 \mathrm{d}\sigma.$$

The function $N(r)$ is called the frequency of u centered at the origin with radius r. Similarly, we can define the frequency centered at other point.

Remark 1.1 Noting that for any $j = 1, 2, \cdots, k$, u_j is a $(k-j+1)$-polyharmonic function, and u_k is a harmonic function. Thus one can also define the frequency for u_j as above. We denote such frequency as $N_j(r)$. It is easy to see that $N_1(r) = N(r)$, and $N_k(r)$ is just the classical frequency of a harmonic function as in Definition 1.1.

Remark 1.2 This frequency is in fact the following form

$$N(r) = r\frac{\sum\limits_{i=1}^{k} \int_{\partial B_r} u_i u_{i\nu} \mathrm{d}\sigma}{\sum\limits_{i=1}^{k} \int_{\partial B_r} u_i^2 \mathrm{d}\sigma}. \tag{1.3}$$

Here $u_{i\nu}$ is $\nabla u \cdot \nu$ and ν is the outer unit normal on ∂B_r.

Now we state the main results of this paper.

Theorem 1.1 *Let u be a polyharmonic function in $B_1 \subseteq \mathbb{R}^n$, i.e., $\triangle^k u = 0$ in B_1. Then*

$$\mathcal{H}^{n-1}(\{x \in B_{\frac{1}{16}} : u(x) = 0\}) \leq C\sum_{i=1}^{k} N_i(1) + C, \tag{1.4}$$

where C is a positive constant depending only on n and k.

Theorem 1.2 *Let u be a k-polyharmonic function in the whole space $\mathbb{R}^n$.*

(1) *If the frequency of u centered at the origin is bounded in $\mathbb{R}^n$, then u is a polynomial. Moreover, if $N(r) < N_0$ for any $r > 0$, then it holds that*

$$\deg(u) \leq CN_0 + C, \tag{1.5}$$

where $\deg(u)$ *means the order of degree of u and C is a positive constant depending only on n and k. In this case, for any $i = 2, \cdot, k$, the functions u_i are also polynomials.*

(2) *If u is a polynomial, then the frequency of u is bounded by the order of degree of u in the whole space $\mathbb{R}^n$.*

The rest of this paper is organized as follows. In the second section we introduce some interesting properties of the frequency and prove the monotonicity formula of the frequency. In the third section, the doubling conditions of the polyharmonic functions are proved. The forth section gives the measure estimates of nodal sets of polyharmonic functions, i.e., the proof of Theorem 1.1. The last section shows the growth property of polyharmonic functions.

2 Monotonicity Formula of Frequency

In this section, we will give some interesting properties for the frequency of polyharmonic functions, and then prove the monotonicity formula for this frequency function.

Lemma 2.1 *If u satisfies $\triangle^k u = 0$, where $k \in \mathbb{N}$ and the vanishing order of u at the origin is $l \geq 2(k-1)$, then*

$$\lim_{r\to 0} N(r) \geq l - 2(k-1). \tag{2.1}$$

Proof Note that u is k-polyharmonic. So each u_i is analytic near the origin, thus we may assume that for each $i = 1, 2, \cdots, k$, $u_i(x) = P_i(x) + R_i(x)$, where $P_i(x)$ is a homogeneous polynomial. Assume that the order of degree of $P_i(x)$ is l_i, and then $R_i(x) = o(|x|^{l_i})$ as $|x| \to 0$. Because the vanishing order of u at the origin is l, it is known that $l_1 = l$, and for each $i = 2, 3, \cdots, k$, $l_i \geq l - 2(i-1)$. Let $l_0 = \inf\{l_1, l_2, \cdots, l_k\}$. Because each $P_i(x)$ is a homogeneous polynomial of degree l_i, $P_i(x)$ can be written as $P_i(x) = r^{l_i}\phi_i(\theta)$, where (r, θ) is the polar coordinate system. Then

$$\begin{aligned}
N(r) &= r\frac{\sum\limits_{i=1}^{k}\int_{B_r}|\nabla u_i|^2\mathrm{d}x + \sum\limits_{i=1}^{k}\int_{B_r}u_iu_{i+1}\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_i^2\mathrm{d}\sigma} = r\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_i^2\mathrm{d}\sigma}\\
&= \frac{\sum\limits_{i=1}^{k}l_ir^{2l_i}\int_{\partial B_r}\phi_i^2(\theta)\mathrm{d}\sigma + \sum\limits_{i=1}^{k}o(r^{2l_i})}{\sum\limits_{i=1}^{k}r^{2l_i}\int_{\partial B_r}\phi_i^2(\theta)\mathrm{d}\sigma + \sum\limits_{i=1}^{k}o(r^{2l_i})}\\
&= \frac{l_0\sum\limits_{i=1}^{k}r^{2l_0}\int_{\partial B_r}\phi_i^2(\theta)\mathrm{d}\sigma + o(r^{2l_0})}{\sum\limits_{i=1}^{k}r^{2l_0}\sum\limits_{\partial B_r}\phi_i^2(\theta)\mathrm{d}\sigma + o(r^{2l_0})},
\end{aligned}$$

where $\mathrm{d}\sigma = r^{n-1}\mathrm{d}\omega$, $\mathrm{d}\omega$ is the $(n-1)$-Hausdorff measure on the unit sphere $\mathcal{S}^{n-1}$. Let $r \to 0$ in the above form, one can get $\lim\limits_{r\to 0} N(r) = l_0 \geq l - 2(k-1)$. That is the desired result.

In order to prove some properties of the proposed frequency, we need the following two lemmas which can be seen in [9, 13].

Lemma 2.2 *If u is a harmonic function in B_r, then*

$$\int_{B_r} u^2\mathrm{d}x \leq \frac{r}{n}\int_{\partial B_r} u^2\mathrm{d}\sigma. \tag{2.2}$$

Lemma 2.3 *For any $u \in W_0^{1,2}(B_r)$, it holds that*

$$\int_{B_r} u^2\mathrm{d}x \leq \frac{4r^2}{n^2}\int_{B_r}|\nabla u|^2\mathrm{d}x. \tag{2.3}$$

Now we show some properties of such frequency.

Lemma 2.4 *If $n \geq 2$, $r \leq 1$, and u is a k-polyharmonic function as above, then the frequency of u satisfies that*

$$N(r) \geq -Cr,$$

where C is a positive constant depending only on n.

Proof For any fixed i and r, define the function u_{i1}^r and u_{i2}^r as follows:

$$\begin{aligned} &\triangle u_{i1}^r = u_{i+1} \quad \text{in } B_r, \quad u_{i1}^r = 0 \quad \text{on } \partial B_r, \\ &\triangle u_{i2}^r = 0 \quad \text{in } B_r, \quad u_{i2}^r = u_i \quad \text{on } \partial B_r. \end{aligned}$$

So $u_i = u_{i1}^r + u_{i2}^r$. Note that for any $i = 1, 2, \cdots, k$, u_{i2}^r are harmonic functions, we have

$$\int_{B_r} |u_{i2}^r|^2 \mathrm{d}\sigma \leq \frac{r}{n} \int_{\partial B_r} |u_{i2}^r|^2 \mathrm{d}\sigma = \frac{r}{n} \int_{\partial B_r} |u_i|^2 \mathrm{d}\sigma, \tag{2.4}$$

which is presented in [8, Chapter 2]. On the other hand, the functions u_{i1}^r are all in $W_0^{1,p}(B_r)$, so from the Poincaré's inequality, we have

$$\int_{B_r} |u_{i1}^r|^2 \mathrm{d}\sigma \leq \frac{4r^2}{n^2} \int_{B_r} |\nabla u_{i1}^r|^2 \mathrm{d}\sigma.$$

Because

$$\int_{B_r} |\nabla u_i|^2 \mathrm{d}\sigma = \int_{B_r} |\nabla u_{i1}^r|^2 \mathrm{d}\sigma + |\nabla u_{i2}^r|^2 + 2\nabla u_{i1}^r u_{i2}^r$$

and

$$\int_{B_r} \nabla u_{i1}^r \nabla u_{i2}^2 \mathrm{d}\sigma = 0,$$

we have

$$\int_{B_r} |u_{i1}^r|^2 \mathrm{d}\sigma \leq \frac{4r^2}{n^2} \int_{B_r} |\nabla u_{i1}^r|^2 \mathrm{d}\sigma \leq \int_{B_r} |\nabla u_i|^2 \mathrm{d}\sigma. \tag{2.5}$$

We write the term $\int_{B_r} u_i u_{i+1} \mathrm{d}\sigma$ as

$$\begin{aligned} \int_{B_r} u_i u_{i+1} \mathrm{d}\sigma &= \int_{B_r} u_{i1}^r u_{i+1,1}^r \mathrm{d}\sigma + \int_{B_r} u_{i1}^r u_{i+1,2}^r \mathrm{d}\sigma + \int_{B_r} u_{i2}^r u_{i+1,1}^r \mathrm{d}\sigma + \int_{B_r} u_{i2}^r u_{i+1,2}^r \mathrm{d}\sigma \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

Now we will give the estimates of $|\mathrm{I}|$, $|\mathrm{II}|$, $|\mathrm{III}|$ and $|\mathrm{IV}|$ separately. First consider the term $|\mathrm{I}|$. By using the form (2.5), we have

$$|\mathrm{I}| \leq \frac{1}{2}\Big(\int_{B_r} |u_{i1}^r|^2 \mathrm{d}\sigma + \int_{B_r} |u_{i+1,1}^r|^2 \mathrm{d}\sigma\Big) \leq \frac{4r^2}{n^2}\Big(\int_{B_r} |\nabla u_i|^2 \mathrm{d}\sigma + \int_{B_r} |\nabla u_{i+1}|^2 \mathrm{d}\sigma\Big).$$

For $|\mathrm{IV}|$, by using (2.4), we have

$$|\mathrm{IV}| \leq \frac{r}{2n}\Big(\int_{\partial B_r} u_i^2 \mathrm{d}\sigma + \int_{\partial B_r} u_{i+1}^2 \mathrm{d}\sigma\Big).$$

Now we focus on $|\mathrm{II}|$. Also from the forms (2.4)–(2.5), we have for any $\epsilon > 0$, it holds that

$$|\mathrm{II}| \leq \frac{\epsilon}{2} \int_{B_r} |u_{i1}^r|^2 \mathrm{d}\sigma + \frac{1}{2\epsilon} \int_{B_r} |u_{i+1,2}^r|^2 \mathrm{d}\sigma \leq \frac{2\epsilon r^2}{n^2} \int_{B_r} |\nabla u_i|^2 \mathrm{d}\sigma + \frac{r}{2\epsilon n} \int_{\partial B_r} u_{i+1}^2 \mathrm{d}\sigma.$$

Similarly, for |III|, we have

$$|\mathrm{III}| \leq \frac{C\epsilon r^2}{2}\int_{B_r}|\nabla u_{i+1}|^2\mathrm{d}\sigma + \frac{r}{2\epsilon n}\int_{\partial B_r}u_i^2\mathrm{d}\sigma.$$

So

$$|E(r)| \leq \Big(\frac{4r^2\epsilon}{n^2}+\frac{2r^2}{n^2}\Big)D(r) + \Big(\frac{r}{n}+\frac{2r}{\epsilon n}\Big)H(r).$$

Choose $\epsilon = \frac{1}{4}$. Then from Lemmas 2.2–2.3 and the fact that $n \geq 2$, $r \leq 1$, we have

$$\frac{4r^2\epsilon}{n^2}+\frac{2r^2}{n^2} = \frac{3r^2}{n^2} < 1.$$

So

$$|E(r)| \leq CrH(r) + D(r).$$

Thus

$$N(r) = r\frac{D(r)+E(r)}{H(r)} \geq r\frac{D(r)-|E(r)|}{H(r)} \geq -Cr,$$

which is the desired result.

Remark 2.1 It is obvious that the result of the above lemmas also hold for the frequency centered at other points.

Remark 2.2 The frequency of a harmonic function is obviously nonnegative. For a polyharmonic function, the frequency may not be nonnegative, but from Lemma 2.4, one knows that it also has a lower bound.

Next we will show the monotonicity formula for this frequency.

Theorem 2.1 *Let u be a k-polyharmonic function. Then there exists two positive constants C_0 and C depending only on n and k such that if $N(r) \geq C_0$, then it holds that*

$$\frac{N'(r)}{N(r)} \geq -C. \tag{2.6}$$

Proof It is easy to check that

$$\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{D'(r)+E'(r)}{D(r)+E(r)} - \frac{H'(r)}{H(r)}. \tag{2.7}$$

We calculate $D'(r)$, $E'(r)$ and $H'(r)$ separately. We write $H(r)$ as follows:

$$H(r) = \int_{|x=r|} u^2(x)\mathrm{d}\sigma_x = r^{n-1}\int_{|y|=1} u^2(ry)\mathrm{d}\sigma_y,$$

where $\mathrm{d}\sigma_x$ and $\mathrm{d}\sigma_y$ are the $(n-1)$-Hausdorff measures on the corresponding spheres. This implies that

$$H_i'(r) = (n-1)r^{n-2}\int_{|y|=1} u_i^2(ry)\mathrm{d}\sigma_y + 2r^{n-1}\int_{|y|=1} u_i(ry)u_{i\nu}(ry)\mathrm{d}\sigma_y$$

$$= \frac{n-1}{r} H_i(r) + 2\int_{\partial B_r} u_i u_{i\nu} \mathrm{d}\sigma.$$

So

$$H'(r) = 2\sum_{i=1}^{k} \int_{\partial B_r} u_i u_{i\nu} \mathrm{d}\sigma + \frac{n-1}{r} H(r). \tag{2.8}$$

Now consider $D'(r)$ and $E'(r)$. First note that

$$D'(r) = \sum_{i=1}^{k} D_i'(r),$$

$$E'(r) = \sum_{i=1}^{k} E_i'(r),$$

where

$$D_i'(r) = \int_{\partial B_r} |\nabla u_i|^2 \mathrm{d}\sigma$$

and

$$E_i'(r) = \int_{\partial B_r} u_i u_{i+1} \mathrm{d}\sigma.$$

For $D_i'(r)$, it holds that

$$\begin{aligned} D_i'(r) &= \int_{\partial B_r} |\nabla u_i|^2 \mathrm{d}x = \frac{1}{r}\int_{\partial B_r} |\nabla u_i|^2 \cdot x \cdot \frac{x}{r} \mathrm{d}x = \frac{1}{r}\int_{B_r} \mathrm{div}(|\nabla u_i|^2 \cdot x) \mathrm{d}x \\ &= \frac{n}{r}\int_{B_r} |\nabla u_i|^2 \mathrm{d}x + \frac{2}{r}\int_{B_r} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial^2 u_i}{\partial x_j \partial x_l} \cdot x_l \mathrm{d}x \\ &= \frac{n}{r} D_i(r) + \frac{2}{r}\int_{B_r} \frac{\partial}{\partial x_j}\Big(\frac{\partial u_i}{\partial x_l}\Big)\Big(\frac{\partial u_i}{\partial x_j} \cdot x_l\Big) \mathrm{d}x \\ &= \frac{n}{r} D_i(r) + \frac{2}{r}\int_{\partial B_r} \frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}\frac{x_j}{r} \cdot x_l \mathrm{d}\sigma - \frac{2}{r}\int_{B_r} \frac{\partial u_i}{\partial x_l} \cdot \frac{\partial}{\partial x_j}\Big(\frac{\partial u_i}{\partial x_j} \cdot x_l\Big) \mathrm{d}x \\ &= \frac{n-2}{r} D_i(r) + 2\int_{\partial B_r} u_{i\nu}^2 \mathrm{d}\sigma - \frac{2}{r}\int_{B_r} \nabla u_i \cdot x \cdot u_{i+1} \mathrm{d}x. \end{aligned}$$

For $E_i'(r)$, we have

$$\begin{aligned} E_i'(r) &= \int_{\partial B_r} u_i u_{i+1} \mathrm{d}\sigma = \frac{1}{r}\int_{\partial B_r} u_i u_{i+1} x \cdot \frac{x}{r} \mathrm{d}x = \frac{1}{r}\int_{B_r} \mathrm{div}(u_i u_{i+1} x) \mathrm{d}x \\ &= \frac{n}{r}\int_{B_r} u_i u_{i+1} \mathrm{d}x + \frac{1}{r}\int_{B_r} u_{i+1} \nabla u_i \cdot x \mathrm{d}x + \frac{1}{r}\int_{B_r} u_i \nabla u_{i+1} \cdot x \mathrm{d}x \\ &= \frac{n}{r} E_i(r) + \frac{1}{r}\int_{B_r} u_{i+1} \nabla u_i \cdot x \mathrm{d}x + \frac{1}{r}\int_{B_r} u_i \nabla u_{i+1} \cdot x \mathrm{d}x. \end{aligned}$$

Thus

$$\begin{aligned} D_i'(r) + E_i'(r) = \frac{n-2}{r}(D_i(r) + E_i(r)) + 2\int_{\partial B_r} u_{i\nu}^2 \mathrm{d}\sigma \\ + \frac{1}{r}\int_{B_r} \nabla u_{i+1} \cdot x \cdot u_i \mathrm{d}x - \frac{1}{r}\int_{B_r} u_i \cdot x \cdot u_{i+1} \mathrm{d}x \end{aligned}$$

$$+\frac{2}{r}\int_{B_r} u_iu_{i+1}\mathrm{d}x.$$

So

$$\begin{aligned}\frac{D'(r)+E'(r)}{D(r)+E(r)} &= \frac{n-2}{r}+2\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_{i\nu}^2\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}\\ &\quad+\frac{1}{r}\frac{\sum\limits_{i=1}^{k}(\int_{B_r}\nabla u_{i+1}\cdot x\cdot u_i\mathrm{d}x-\int_{B_r}\nabla u_i\cdot x\cdot u_{i+1}\mathrm{d}x+2\int_{B_r}u_iu_{i+1}\mathrm{d}x)}{\sum\limits_{i=1}^{k}(\int_{B_r}|\nabla u_i|^2\mathrm{d}x+\int_{B_r}u_iu_{i+1}\mathrm{d}x)}\\ &=\frac{n-2}{r}+2\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_{i\nu}^2\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}+\frac{1}{r}\frac{\mathrm{R}_1-\mathrm{R}_2+2\mathrm{R}_3}{D(r)+E(r)}.\end{aligned}$$

Then we will estimate $|\mathrm{R}_1|$, $|\mathrm{R}_2|$ and $|\mathrm{R}_3|$ separately.

$$\begin{aligned}|\mathrm{R}_1| &\le \sum_{i=1}^{k}\Big|\int_{B_r}\nabla u_{i+1}\cdot x\cdot u_i\mathrm{d}x\Big| \le \frac{r}{2}\Big(\sum_{i=1}^{k}\int_{B_r}|\nabla u_{i+1}|^2\mathrm{d}x+\sum_{i=1}^{k}\int_{B_r}u_i^2\mathrm{d}x\Big)\\ &\le Cr\sum_{i=1}^{k}\Big(\int_{B_r}|\nabla u_i|^2\mathrm{d}x+\int_{\partial B_r}u_i^2\mathrm{d}\sigma\Big).\end{aligned}$$

From the similar arguments, we have

$$\begin{aligned}|\mathrm{R}_2| &\le \sum_{i=1}^{k}\Big|\int_{B_r}\nabla u_i\cdot x\cdot u_{i+1}\mathrm{d}x\Big| \le \frac{r}{2}\sum_{i=1}^{k}\Big(\int_{B_r}|\nabla u_i|^2\mathrm{d}x+\int_{B_r}|u_{i+1}|^2\mathrm{d}x\Big)\\ &\le Cr\sum_{i=1}^{k}\Big(\int_{B_r}|\nabla u_i|^2\mathrm{d}x+\int_{\partial B_r}u_i^2\mathrm{d}\sigma\Big)\end{aligned}$$

and

$$\begin{aligned}|\mathrm{R}_3| &\le \sum_{i=1}^{k}\Big|\int_{B_r}u_iu_{i+1}\mathrm{d}x\Big| \le \frac{1}{2}\sum_{i=1}^{k}\Big(\int_{B_r}u_i^2\mathrm{d}x+\int_{B_r}u_{i+1}^2\mathrm{d}x\Big)\\ &\le Cr\Big(\int_{B_r}|\nabla u_i|^2\mathrm{d}x+\int_{\partial B_r}u_i^2\mathrm{d}\sigma\Big).\end{aligned}$$

From the assumption that $N(r)\ge C_0$ and the proof of Lemma 2.4, we have

$$|E(r)|\le CH(r)+\frac{3}{4}D(r)\le\frac{C}{C_0}(D(r)+|E(r)|)+\frac{3}{4}D(r),$$

where C is the constant in Lemma 2.4. Choose C_0 large enough such that

$$\Big(\frac{C}{C_0+\frac{3}{4}}\Big)\frac{C_0}{C_0-C}=\frac{7}{8}.$$

Then

$$D(r)+E(r)\geq\frac{1}{8}D(r).$$

So

$$\frac{|\mathrm{R}_1-\mathrm{R}_2+2\mathrm{R}_3|}{D(r)+E(r)}\leq\frac{CD(r)+CH(r)}{D(r)+E(r)}\leq C.$$

Thus

$$\frac{D'(r)+E'(r)}{D(r)+E(r)}\geq -C+\frac{n-2}{r}+2\frac{\sum\limits_{i=1}^{n}\int_{\partial B_r}u_{i\nu}^2\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}.$$

From (2.8), we have

$$\frac{H'(r)}{H(r)}=\frac{n-1}{r}+2\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_i^2\mathrm{d}\sigma}.$$

So we finally get

$$\frac{N'(r)}{N(r)}\geq 2\Big(\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_{i\nu}^2\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}-\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_iu_{i\nu}\mathrm{d}\sigma}{\sum\limits_{i=1}^{k}\int_{\partial B_r}u_i^2\mathrm{d}\sigma}\Big)-C\geq -C.$$

This ends the proof.

Remark 2.3 The above theorem also holds for the frequency centered at other point, i.e., if u is a polyharmonic function and $N(p,r)$ is the frequency of u centered at the point p with radius r, then it holds that

$$\frac{\mathrm{d}N(p,r)}{\mathrm{d}r}\cdot\frac{1}{N(p,r)}\geq -C, \tag{2.9}$$

if $N(p,r)\geq C_0$, where C_0 and C are two positive constants depending only on n and k.

Lemma 2.5 *For any $p\in\overline{B_{\frac14}}$, we have*

$$N\Big(p,\frac12(1-|p|)\Big)\leq C_1N(1)+C_2, \tag{2.10}$$

where C_1 and C_2 are positive constants depending only on n and k.

Proof We only prove the case that $|p|=\frac14$. Other cases are similar. Note that $B_{\frac34}(p)\subseteq B_1$ and $B_{\frac14}\subseteq B_{\frac12}(p)$. From Theorem 2.1, we have

$$\sum_{i=1}^{k}\int_{B_{\frac34}(p)}u_i^2\mathrm{d}x\leq 4^{CN(1)+C}\sum_{i=1}^{k}\int_{B_{\frac12}(p)}u_i^2\mathrm{d}\sigma. \tag{2.11}$$

Now we claim that

$$\sum_{i=1}^{k}\frac{1}{|\partial B_{\frac{5}{8}}(p)|}\int_{\partial B_{\frac{5}{8}}(p)}u_i^2\mathrm{d}\sigma\le 4^{CN(1)+C}\sum_{i=1}^{k}\frac{1}{|\partial B_{\frac{1}{2}}(p)|}\int_{\partial B_{\frac{1}{2}}(p)}u_i^2\mathrm{d}\sigma. \tag{2.12}$$

In fact, from (1.3), (2.8), Lemma 2.4 and some direct calculation, we know that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\log\Big(\sum_{i=1}^{k}\frac{1}{|\partial B_r(p)|}\int_{\partial B_r(p)}u_i^2\mathrm{d}\sigma\Big)&=\frac{1-n}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\log(H(p,r))\\&=\frac{1-n}{r}+\frac{H'(p,r)}{H(p,r)}\\&=\frac{n-1}{r}+\frac{n-1}{r}+2\frac{\sum\limits_{i=1}^{k}\int_{\partial B_r(p)}u_iu_{i\nu}\mathrm{d}\sigma}{H(p,r)}\\&=\frac{2}{r}N(p,r)\ge -C.\end{aligned}\tag{2.13}$$

Thus

$$\begin{aligned}\sum_{i=1}^{k}\int_{B_{\frac{3}{4}}(p)}u_i^2\mathrm{d}x&\ge\sum_{i=1}^{k}\int_{B_{\frac{3}{4}}(p)-B_{\frac{5}{8}}(p)}u_i^2\mathrm{d}\sigma\\&=\sum_{i=1}^{k}\int_{\frac{5}{8}}^{\frac{3}{4}}r^{n-1}\frac{1}{|\partial B_r(p)|}\int_{\partial B_r(p)}u_i^2\mathrm{d}\sigma\\&\ge C\sum_{i=1}^{k}\frac{1}{|\partial B_{\frac{5}{8}}(p)|}\int_{\partial B_{\frac{5}{8}}(p)}u_i^2\mathrm{d}\sigma,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{k}\int_{B_{\frac{1}{2}}(p)}u_i^2\mathrm{d}x&=\sum_{i=1}^{k}\int_{0}^{\frac{1}{2}}r^{n-1}\frac{1}{|\partial B_r(p)|}\int_{\partial B_r(p)}u_i^2\mathrm{d}\sigma\\&\le C\sum_{i=1}^{k}\frac{1}{|\partial B_{\frac{1}{2}}(p)|}\int_{\partial B_{\frac{1}{2}}(p)}u_i^2\mathrm{d}\sigma.\end{aligned}$$

So the claim (2.12) holds. Integrating (2.13) from $\frac{1}{2}$ to $\frac{5}{8}$, we obtain

$$\log\Big(\sum_{i=1}^{k}\frac{1}{|\partial B_{\frac{5}{8}}(p)|}\int_{\partial B_{\frac{5}{8}}(p)}u_i^2\mathrm{d}\sigma\Big)-\log\Big(\sum_{i=1}^{k}\frac{1}{|\partial B_{\frac{1}{2}}(p)|}\int_{\partial B_{\frac{1}{2}}(p)}u_i^2\mathrm{d}\sigma\Big)=\int_{\frac{1}{2}}^{\frac{5}{8}}\frac{2N(p,r)}{r}\mathrm{d}r.$$

From Theorem 2.1, we know that

$$\int_{\frac{1}{2}}^{\frac{5}{8}}\frac{2N(p,r)}{r}\mathrm{d}r\ge CN\Big(p,\frac{1}{2}\Big)-C.$$

So

$$N\Big(p,\frac{1}{2}\Big)\le CN(1)+C.$$

Then from Theorem 2.1 again, we get

$$N(p,r)\le CN(1)+C$$

for any $r\le\frac{1}{2}$, and that is the desired result.

3 Doubling Conditions

In this section, we will show the doubling condition of a polyharmonic function u. In fact, from the proof of Lemma 2.5, it is easy to see that the following doubling condition holds.

Lemma 3.1 *Let u be a k-polyharmonic function, and assume that $2r < 1$. Then it holds that*

$$\sum_{i=1}^{k} \frac{1}{|\partial B_{2r}|} \int_{\partial B_{2r}} u_i^2 \mathrm{d}\sigma \leq 2^{CN(1)+C} \sum_{i=1}^{k} \frac{1}{|\partial B_r|} \int_{\partial B_r} u_i^2 \mathrm{d}\sigma \tag{3.1}$$

and

$$\sum_{i=1}^{k} \frac{1}{|B_{2r}|} \int_{B_{2r}} u_i^2 \mathrm{d}x \leq 2^{C'N(1)+C'} \sum_{i=1}^{k} \frac{1}{|B_r|} \int_{B_r} u_i^2 \mathrm{d}x, \tag{3.2}$$

where C and C' are two positive constants depending only on n and k.

Proof We only need to prove the form (3.1). Because one can simply integrate (3.1) from 0 to r to get (3.2).

Integrating (2.13) from r to $2r$, we know that

$$\log\Big(\frac{H(2r)}{(2r)^{n-1}}\Big) - \log\Big(\frac{H(r)}{r^{n-1}}\Big) = 2\int_r^{2r} \frac{N(t)}{t} \mathrm{d}t,$$

and thus

$$H(2r) \leq 2^n H(r) \mathrm{e}^{2\int_r^{2r} \frac{N(t)}{t} \mathrm{d}t}.$$

From Theorem 2.1, we know that $N(t) \leq \max\{CN(2r), C_0\} \leq CN(2r) + C$ for any $t \in (r, 2r)$. Here C is some positive constant depending only on n and k, and C_0 is the same constant as in Theorem 2.1. So

$$H(2r) \leq 2^n H(r) \mathrm{e}^{CN(2r)+C} = 2^{CN(2r)+C} H(r),$$

which is the desired result.

It is known that the doubling condition for harmonic functions and bi-harmonic functions as follows.

Lemma 3.2 *Let u be a harmonic function and $2r < 1$. Then*

$$\frac{1}{|B_{2r}|} \int_{B_{2r}} u^2 \mathrm{d}x \leq 2^{CN(1)+C} \frac{1}{|B_r|} \int_{B_r} u^2 \mathrm{d}x, \tag{3.3}$$

where $N(r)$ is the frequency of u and C is a positive constant depending only on n.

Lemma 3.3 *Let u be a bi-harmonic function and $2r < 1$. Then*

$$\frac{1}{|B_{2r}|} \int_{B_{2r}} u^2 \mathrm{d}x \leq \frac{1}{r^4} 2^{C(N_1(1)+N_2(1))+C} \frac{1}{|B_r|} \int_{B_r} u^2 \mathrm{d}x, \tag{3.4}$$

where $N_1(r)$ is the frequency of u, $N_2(r)$ is the frequency of $\triangle u$, and C is a positive constant depending only on n.

Lemmas 3.2–3.3 can be seen in [9] and [13], respectively.

Now we will show the doubling condition for a polyharmonic function.

Theorem 3.1 *Let u be a k-polyharmonic function, i.e., u satisfies that $\triangle^k u = 0$ in $B_1 \subseteq \mathbb{R}^n$ and assume that $2r < 1$, $n \geq 2$. Then it holds that*

$$\frac{1}{|B_{2r}|}\int_{B_{2r}} u^2 \mathrm{d}x \leq \frac{1}{r^C} 2^{C\left(\sum\limits_{i=1}^{k} N_i(1)\right)+C} \frac{1}{|B_r|}\int_{B_r} u^2 \mathrm{d}x, \tag{3.5}$$

where C is a positive constant depending only on n and k.

Proof We prove this lemma by the inductions.

Assume that we have already known that for any j satisfies $k \geq j \geq l$, form (3.5) and the following inequality

$$\int_{B_r} u_{j+1}^2 \mathrm{d}x \leq \frac{1}{r^C} 2^{C\sum\limits_{i=j+1}^{k} N_i(1)+C} \int_{B_r} u_j^2 \mathrm{d}x \tag{3.6}$$

holds for u_j. From the above two lemmas, we know that for $j = k$ and $j = k-1$, these two inequalities hold. Now we will prove that the inequalities (3.5) and (3.6) hold for u replaced by u_{l-1} and thus the theorem is proved.

Noting that

$$\triangle^2 u_{l-1} = u_{l+1},$$

it holds that for any text function $\psi \in C_0^\infty(B_1)$,

$$\int_{B_1} \triangle u_{l-1} \triangle \psi \mathrm{d}x = \int_{B_1} u_{l+1} \psi \mathrm{d}x. \tag{3.7}$$

Choose $\psi = u_{l-1}\phi^2$, where ϕ satisfies

$$\phi = 1 \quad \text{in } B_r, \quad \phi = 0 \quad \text{outside } B_{2r},$$

and

$$|\nabla \phi| < \frac{C}{r}, \quad |\nabla^2 \phi| < \frac{C}{r^2}.$$

Put this Ψ into (3.7), we have

$$\begin{aligned}
\int_{B_1} u_{l+1} u_{l-1} \phi^2 \mathrm{d}x &= \int_{B_1} \triangle u_{l-1} \triangle (u_{l-1}\phi^2) \mathrm{d}x \\
&= \int_{B_1} u_l^2 \phi^2 \mathrm{d}x + 4\int_{B_1} u_l \phi \nabla u_{l-1} \nabla \phi \mathrm{d}x + 2\int_{B_1} u_l u_{l-1} (|\nabla \phi|^2 + \phi \triangle \phi) \mathrm{d}x \\
&= \int_{B_1} u_l^2 \phi^2 \mathrm{d}x - 4\int_{B_1} u_{l-1} \phi \nabla u_l \nabla \phi \mathrm{d}x - 2\int_{B_1} u_l u_{l-1} (|\nabla \phi|^2 + \phi \triangle \phi) \mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{B_1} u_l^2 \phi^2 \mathrm{d}x &= \int_{B_1} u_{l-1} u_{l+1} \phi^2 \mathrm{d}x + 4\int_{B_1} u_{l-1} \phi \nabla u_l \nabla \phi \mathrm{d}x + 2\int_{B_1} u_l u_{l-1} (|\nabla \phi|^2 + \phi \triangle \phi) \mathrm{d}x \\
&\leq \Big(\int_{B_1} u_{l+1}^2 \phi^2 \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{B_1} u_{l-1}^2 \phi^2 \mathrm{d}x\Big)^{\frac{1}{2}} + 4\Big(\int_{B_1} u_{l-1}^2 |\nabla \phi|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{B_1} |\nabla u_l|^2 \phi^2 \mathrm{d}x\Big)^{\frac{1}{2}}
\end{aligned}$$

$$+2\Big(\int_{B_1} u_l^2(|\nabla\phi|^2+|\phi\triangle\phi|)\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{B_1} u_{l-1}^2(|\nabla\phi|^2+|\phi\triangle\phi|)\mathrm{d}x\Big)^{\frac{1}{2}}.$$

Now we consider the estimate of the term $\int_{B_1}|\nabla u_l|^2\phi^2\mathrm{d}x$.

$$\begin{aligned}&\int_{B_1}|\nabla u_l|^2\phi^2\mathrm{d}x\\ =&-\int_{B_1}u_lu_{l+1}\phi^2\mathrm{d}x-2\int_{B_1}u_l\phi\nabla u_l\nabla\phi\mathrm{d}x\\ \le&\Big(\int_{B_1}u_l^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{B_1}u_{l+1}^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}+\Big(\int_{B_1}u_l^2|\nabla\phi|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{B_1}|\nabla u_l|^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}\\ \le&\Big(\int_{B_1}u_l^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{B_1}u_{l+1}^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}+2\int_{B_1}u_l^2|\nabla\phi|^2\mathrm{d}x+\frac{1}{2}\int_{B_1}|\nabla u_l|^2\phi^2\mathrm{d}x.\end{aligned}$$

Thus we have

$$\int_{B_1}|\nabla u_l|^2\phi^2\mathrm{d}x\le 2\Big(\int_{B_1}u_l^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{B_1}u_{l+1}^2\phi^2\mathrm{d}x\Big)^{\frac{1}{2}}+4\int_{B_1}u_l^2|\nabla\phi|^2\mathrm{d}x.$$

From

$$\int_{B_r}u_{l+1}^2\mathrm{d}x\le\frac{1}{r^C}2^{C\sum\limits_{i=l+1}^{k}N_i(1)+C}\int_{B_r}u_l^2\mathrm{d}x$$

and the doubling condition for u_l, we have

$$\int_{B_r}u_l^2\mathrm{d}x\le\frac{1}{r^C}2^{C\sum\limits_{i=l}^{k}N_i(1)+C}\int_{B_{2r}}u_{l-1}^2\mathrm{d}x. \tag{3.8}$$

This shows that (3.6) holds for $j=l-1$. Then from Lemma 3.1 and the induction assumptions, we have

$$\int_{B_r}u_{l-1}^2\mathrm{d}x\le\frac{1}{r^C}2^{C\sum\limits_{i=l-1}^{k}N_i(1)+C}\int_{B_{2r}}u_{l-1}^2\mathrm{d}x, \tag{3.9}$$

and thus the desired result holds by inductions.

4 Measure Estimates of Nodal Sets

In this section, we will show the upper bound of the measure of the nodal set for a polyharmonic function u, i.e., we will give the proof of Theorem 1.1.

To estimate the measure of the nodal set, we need an estimate for the number of zero points of analytic functions which was first proved in [2].

Lemma 4.1 *Suppose that f: $B_1\subseteq\mathbb{C}\to\mathbb{C}$ is analytic with*

$$|f(0)|=1\quad and\quad \sup_{B_1}|f|\le 2^N$$

for some positive constant N. Then for any $r\in(0,1)$, there holds

$$\mathcal{H}^0(\{z\in B_r: f(z)=0\})\le CN, \tag{4.1}$$

where C is a positive constant depending only on r.

We also need the following priori estimate.

Lemma 4.2 *Let u be a polyharmonic function. Then if $2r < 1$, we have*

$$|u|_{L^\infty(B_r)} \le \frac{1}{r^C} 2^{C\sum\limits_{i=1}^{k} N_i(1)+C} \sum_{i=1}^{k} \Big(\frac{1}{|B_R|}\int_{B_r} u_i^2 \mathrm{d}x\Big)^{\frac{1}{2}}, \tag{4.2}$$

where C is a positive constant depending only on n and k.

Proof Let

$$w_{i,r}(x) = \int_{B_r} \Gamma(x-y) u_{i+1}(y)\mathrm{d}y, \quad i = 1, 2, \cdots, k-1,$$

where $\Gamma(x-y) = c|x-y|^{2-n}$ is the fundamental solution of the Laplace operator. Then

$$|w_{i,r}(x)| = \Big|\int_{B_r} \Gamma(x-y) u_{i+1}(y)\mathrm{d}y\Big| \le Cr^2 \sup_{B_r} |u_{i+1}(y)|.$$

It also holds that

$$\triangle w_{i,r} = u_{i+1} \quad \text{in } B_r.$$

So

$$\triangle(u_i - w_{i,r}) = 0 \quad \text{in } B_r.$$

Because u_k is a harmonic function, it is known that for any $y \in B_r$,

$$\begin{aligned}|u_k(y)| &= \Big|\frac{1}{|B_r(y)|}\int_{B_r(y)} u_k(z)\mathrm{d}z\Big| \le \Big(\frac{1}{|B_r(y)|}\int_{B_r(y)} u_k^2(z)\mathrm{d}z\Big)^{\frac{1}{2}} \\ &\le C\Big(\frac{1}{|B_{2r}|}\int_{B_{2r}} u_k^2(z)\mathrm{d}z\Big)^{\frac{1}{2}} \le 2^{CN_k(1)+C}\Big(\frac{1}{|B_r|}\int_{B_r} u_k^2(z)\mathrm{d}z\Big)^{\frac{1}{2}}.\end{aligned}$$

Thus for any $x \in B_r$,

$$|w_{k-1,r}(x)| \le 2^{CN_k(1)+C} r^2 \Big(\frac{1}{|B_r|}\int_{B_r} u_k^2(z)\mathrm{d}z\Big)^{\frac{1}{2}}.$$

On the other hand, from the fact that $u_{k-1} - w_{k-1,2r}$ is harmonic in B_{2r}, we know that for any $x \in B_r$,

$$\begin{aligned}&|u_{k-1}(x) - w_{k-1,2r}(x)| \\ &= \Big|\frac{1}{|B_r(x)|}\int_{B_r(x)} (u_{k-1}(z) - w_{k-1,2r}(z))\mathrm{d}z\Big| \\ &\le \Big(\frac{1}{|B_r(x)|}\int_{B_r(x)} u_{k-1}^2(z)\mathrm{d}z\Big)^{\frac{1}{2}} + \Big(\frac{1}{|B_r(x)|}\int_{B_r(x)} |w_{k-1,2r}(z)|\mathrm{d}z\Big) \\ &\le C\Big(\frac{1}{|B_{2r}|}\int_{B_{2r}} u_{k-1}^2(z)\mathrm{d}z\Big)^{\frac{1}{2}} + Cr^2 \sup_{y\in B_{2r}} |u_k(y)| \\ &\le \frac{1}{r^C} 2^{C(N_k(1)+N_{k-1}(1))+C}\Big(\frac{1}{|B_r|}\int_{B_r} u_{k-1}^2(z)\mathrm{d}z\Big)^{\frac{1}{2}} + 2^{CN_k(1)+C}\Big(\frac{1}{|B_r|}\int_{B_r} u_k^2(z)\mathrm{d}z\Big)^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{1}{r^C} 2^{C(N_k(1)+N_{k-1}(1))+C} \Big(\Big(\frac{1}{|B_r|} \int_{B_r} u_{k-1}^2(z) \mathrm{d}z \Big)^{\frac{1}{2}} + \Big(\frac{1}{|B_r|} \int_{B_r} u_k^2(z) \mathrm{d}z \Big) \Big).$$

Then for $x \in B_r$,

$$\begin{aligned} |u_{k-1}(x)| &\leq |u_{k-1}(x) - w_{k-1,2r}(x)| + |w_{k-1,2r}(x)| \\ &\leq \frac{1}{r^C} 2^{C(N_k(1)+N_{k-1}(1))+C} \Big(\Big(\frac{1}{|B_r|} \int_{B_r} u_{k-1}^2(z) \mathrm{d}z \Big)^{\frac{1}{2}} + \Big(\frac{1}{|B_r|} \int_{B_r} u_k^2(z) \mathrm{d}z \Big)^{\frac{1}{2}} \Big). \end{aligned}$$

That is the desired result for u_{k-1}. Repeat this argument k times, the desired result can be proved.

Now we show the measure estimate of the nodal set $\{x : u(x) = 0\}$.

Proof of Theorem 1.1 Without loss of generality, we may assume

$$\frac{1}{|B_1|} \int_{B_1} u^2 \mathrm{d}x = 1.$$

Then from Theorem 3.1 and Lemma 2.5, it holds that

$$\frac{1}{|B_{\frac{1}{16}}(p)|} \int_{B_{\frac{1}{16}}(p)} u^2 \mathrm{d}x \geq 4^{-C \sum\limits_{i=1}^{k} N_i(1) - C}$$

for any $p \in \partial B_{\frac{1}{4}}$. Then there exists a point $x_p \in B_{\frac{1}{16}}(p)$ such that

$$|u(x_p)| \geq 2^{-C \sum\limits_{i=1}^{k} N_i(1) - C}.$$

On the other hand, from Lemma 4.2 and (3.6), one knows that for any $x \in B_{\frac{1}{4}}$,

$$|u(x)| \leq 2^{C \sum\limits_{i=1}^{k} N_i(1) + C}.$$

Choose $p_j \in \partial B_{\frac{1}{4}}$ to be the point on the j-axis and take $f_j(\omega; t) = u(x_{p_j} + t\omega)$ for $t \in \big(-\frac{5}{8}, \frac{5}{8}\big)$, where $\omega \in \mathcal{S}^{n-1}$. Then f_j is an analytic function with respect to t. Extend it to a complex analytic function $f_j(\omega; z)$, and keep the upper bound. Then we have

$$|f_j(\omega; 0)| \geq 2^{-C \sum\limits_{i=1}^{k} N_i(1) - C} \tag{4.3}$$

and

$$|f_j(\omega; z)| \leq 2^{C \sum\limits_{i=1}^{k} N_i(1) + C}. \tag{4.4}$$

Using Lemma 4.1, we have

$$\mathcal{H}^0 \Big(\Big\{ |t| < \frac{5}{8} : u(x_{p_j} + t\omega) = 0 \Big\} \Big) \leq C \sum_{i=1}^{k} N_i(1) + C.$$

That means

$$\mathcal{H}^0(\{t : u(x_{p_j} + t\omega) = 0, x_{p_j} + t\omega \in B_{\frac{1}{16}}\}) \leq C \sum_{i=1}^{k} N_i(1) + C.$$

Then from the integral geometric formula, which can be seen in [3, 10], we have

$$\mathcal{H}^{n-1}(\{x \in B_{\frac{1}{16}} : u(x) = 0\}) \leq C \sum_{i=1}^{k} N_i(1) + C, \tag{4.5}$$

and this is the desired result.

5 Growth Property of Polyharmonic Functions

In this section, we will derive a growth behavior of the polyharmonic functions in the whole space $\mathbb{R}^n$. The result is written in Theorem 1.2.

Proof of Theorem 1.2 First assume that $N(r)$ is bounded, i.e., $N(r) \leq N_0$ on $\mathbb{R}^n$. Then we need to show that u is a polynomial.

Without loss of generality, assume

$$\sum_{i=1}^{k} \frac{1}{|\partial B_1|} \int_{\partial B_1} u_i^2 \mathrm{d}\sigma = 1. \tag{5.1}$$

From the mean value formula and the fact that u_k is a harmonic function, we have that

$$\sup_{B_r} |u_k| \leq Cr\Big(\frac{1}{|\partial B_{2r}|} \int_{\partial B_{2r}} u_k^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}$$

holds for any $r > 1$. For each $i = 1, 2, \cdots, k-1$, write u_i as $u_i = u_{i1}^{2r} + u_{i2}^{2r}$ as in the proof of Lemma 2.4, i.e.,

$$\begin{aligned} \triangle u_{i1}^{2r} &= u_{i+1} \quad \text{in } B_{2r}, \\ u_{i1}^{2r} &= 0 \qquad\ \ \text{on } \partial B_{2r} \end{aligned}$$

and

$$\begin{aligned} \triangle u_{i2}^{2r} &= 0 \quad \text{in } B_{2r}, \\ u_{i2}^{2r} &= u_i \quad \text{on } \partial B_{2r}. \end{aligned}$$

Then from the priori estimate of u_{i1}^{2r} and the mean value property of u_{i2}^{2r}, we have

$$\begin{aligned} \sup_{B_r} |u_i| &\leq \sup_{B_r} |u_{i1}^{2r}| + \sup_{B_r} |u_{i2}^{2r}| \\ &\leq Cr^2 \sup_{B_{2r}} |u_{i+1}| + Cr\Big(\frac{1}{|\partial B_{2r}|} \int_{\partial B_{2r}} u_i^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}. \end{aligned}$$

Thus for u_{k-1}, it holds that

$$\begin{aligned} \sup_{B_r} |u_{k-1}| &\leq Cr^2 \sup_{B_{2r}} |u_k| + Cr\Big(\frac{1}{|\partial B_{2r}|} \int_{\partial B_{2r}} u_i^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}} \\ &\leq Cr^2 \Big(\Big(\frac{1}{|\partial B_{4r}|} \int_{\partial B_{4r}} u_k^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}} + \Big(\frac{1}{|\partial B_{4r}|} \int_{\partial B_{4r}} u_{k-1}^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big). \end{aligned}$$

Continue these arguments for k times, we get

$$\sup_{B_r} |u| \leq Cr^{2k-2} \sum_{i=1}^{k} \Big(\frac{1}{|\partial B_{2^k r}|} \int_{\partial B_{2^k r}} u_i^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}. \tag{5.2}$$

Thus from Lemma 3.1 and the assumption (5.1), we have that

$$\sup_{B_r} |u| \leq Cr^{CN_0+C} \tag{5.3}$$

holds for any $r > 1$. Thus u must be a polynomial and the order of degree of u is less than or equal to $CN_0 + C$, where C is a positive constant depending only on n and k.

If a k-polyharmonic function u is a polynomial, then from the fact that

$$N(r) = r\frac{\sum\limits_{i=1}^{k} \int_{\partial B_r} u_i u_{i\nu} \mathrm{d}\sigma}{\sum\limits_{\partial B_r} u_i^2},$$

it is easy to check that $N(r)$ is bounded by the order of degree of u. Of course, for any $i = 2, \cdot, k$, the functions u_i are all polynomials.

References

[1] Almgren, F. J., Dirichlet's problem for muptiple valued functions and the regularity of Mass minimizing integral currents, Minimal Submanifolds and Geodesics, Obata, M. (ed.), North Holland, Amsterdam, 1979, 1–6.

[2] Donnelly, H. and Fefferman, C., Nodal sets of eigenfunctions on Riemannian manifolds, *Invent. Math.*, **93**, 1988, 161–183.

[3] Federer, H., Geometric Measure Theory, Springer-Verlag, New York, 1969.

[4] Garofalo, N. and Lin, F. H., Monotonicity properties of variational integrals, A_p-weights and unique continuation, *Indiana Univ. Math. J.*, **35**, 1986, 245–267.

[5] Garofalo, N. and Lin, F. H., Unique continuation for elliptic operators: a geometric-variational approach, *Comm. Pure. Appl. Math.*, **40**, 1987, 347–366.

[6] Han, Q., Schäuder estimates for elliptic operators with applications to nodal sets, *The Journal of Geometric Analysis*, **10**(3), 2000, 455–480.

[7] Han, Q., Hardt, R. and Lin, F. H., Singular sets of higher order elliptic equations, *Communications in Partial Differential Equations*, **28**, 2003, 2045–2063.

[8] Han, Q. and Lin, F. H., Nodal sets of solutions of elliptic differential equations, http://nd.edu/qhan/nodal.pdf.

[9] Lin, F. H., Nodal sets of solutions of elliptic and parabolic equations, *Comm. Pure. Appl. Math.*, **44**, 1991, 287–308.

[10] Lin, F. H. and Yang, X. P., Geometric Measure Theory: An Introduction, Advanced Mathematics, 1, Science Press, Beijing; International Press, Boston, 2002.

[11] Liu, H. R., Tian, L. and Yang, X. P., The growth of H-harmonic functions on the Heisenberg group, *Science China Mathematics*, **54**(4), 2014, 795–806.

[12] Schoen, R. M. and Yau, S. T., Lectures on Differential Geometry, International Press, Cambridge, MA, 1994.

[13] Tian, L. and Yang, X. P., Measure estimates of nodal sets of bi-harmonic functions, *Journal of Differential Equations*, **256**(2), 2014, 558–576.