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The Ferry Cover Problem on Regular Graphs and Small-Degree Graphs*

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Abstract The ferry problem may be viewed as generalizations of the classical wolf-goat-cabbage puzzle. The ferry cover problem is to determine the minimum required boat capacity to safely transport n items represented by a conflict graph. The Alcuin number of a conflict graph is the smallest capacity of a boat for which the graph possesses a feasible ferry schedule. In this paper the authors determine the Alcuin number of regular graphs and graphs with maximum degree at most five.

Keywords Graph, Alcuin number, Ferry problem, Vertex cover, Regular graph
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1 Introduction

A reachability question in discrete applied mathematics is to decide whether a given goal state is reachable from a given initial state (see [1–2]). An example of the oldest reachability problem is the well-known Alcuin's river crossing problem in the book *Propositiones ad acuendos iuvenes*, which was proposed by Alcuin of York more than 1000 years ago. The river crossing problem is described as follows.

A man has to take a wolf, a goat and a bunch of cabbages across a river. The only boat he could find was one which would carry only himself and one of them. How can he safely transport everything to the other side, without the wolf eating the goat or the goat eating the cabbages?

In Alcuin's river crossing problem, a safe transportation plan must satisfy that neither wolf and goat nor goat and cabbage can be left alone together.

Prisner [3] extend Alcuin's river crossing problem to an arbitrary conflict graph $G = (V, E)$, where V is a set of items and two items are connected by an edge in E if they are conflicting and thus cannot be left together without human supervision. Now the man has to ferry a set V of items across a river, while making sure that items that remain unattended on the same bank are safe from each other. The available boat has capacity $b \geq 1$, and thus can carry the

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man together with any subset of at most b items. In above graph-theoretic model, Alcuin's river crossing problem corresponds to the path P_3 with three vertices.

The Alcuin number of a (conflict) graph G is denoted by Alcuin(G) and is defined as the smallest capacity of a boat, for which G possesses a feasible ferry schedule. The ferry cover problem is to determine the Alcuin number of a conflict graph.

There is a close relationship between the Alcuin number and the (vertex) cover number of a graph (see Lemma 2.1). This naturally divides graphs into so-called small-boat and large-boat graphs: A graph is small-boat if its Alcuin number and vertex cover number are equal, and otherwise it is large-boat.

The problem of determining the Alcuin number of an arbitrary graph is NP-hard (see [4]). Lampis and Mitsou [4] further showed that the Alcuin number is hard to approximate. A complete analysis of the Alcuin number of trees can be found in [4].

Recently, Csorba et al. [1] found a quite surprising structural characterization of the Alcuin number. This characterization yields an NP-certificate for the Alcuin number and tells us that every feasible schedule (possibly of exponential length) can be transformed into a feasible schedule of (linear) length at most $2|V|+1$, and that this bound $2|V|+1$ is the strongest possible bound. Furthermore, they established that approximating the Alcuin number is exactly as hard as approximating the vertex cover number. On the positive side, it was proved that the Alcuin number of a bipartite graph can be determined in polynomial time. By the structural characterization theorem, the authors presented several combinatorial lemmas so as to determine whether a given graph is a small-boat graph. For chordal graphs, they gave a concise description of the division line between small-boat and large-boat graphs, and they also proved that every planar graph G with the vertex cover number at least five is a small-boat graph. In [5], Shan et al. gave the Alcuin number of a graph with maximum degree four and cover number at most two or maximum degree five and cover number at most four. Shan and Kong [6] further studied the Alcuin number of a hypergraph and its connections to the transversal number of the hypergraph. More information on this subject can be found in [7–11].

In this paper we study the Alcuin's numbers of graphs with maximum degree at most five and regular graphs. For these graphs, we present concise descriptions of the division line between small-boat and large-boat graphs. In Section 2, we first introduce some basic definitions and preliminaries. In Section 3, we give the sufficient conditions for a connected graph with maximum degree at most five to be a small-boat graph. In Section 4, we show that all k-regular graphs are small-boat graphs for $1 \leq k \leq 7$, this answers an open question posed by Seify and Shahmohamad [7]. Finally, conclusions and directions of further work are given in Section 5.

2 Definitions and Preliminaries

Let us first introduce some basic definitions. Two vertices u, v of $G = (V, E)$ are adjacent, or neighbours, if uv is an edge of G. The set of neighbours of a vertex v in G is denoted by $\Gamma_G(v)$, or briefly by $\Gamma(v)$. More generally for $U \subseteq V$, the neighbours in $V - U$ of vertices in U are called neighbours of U; their set is denoted by $\Gamma_G(U)$. For $X, Y \subseteq V$ and $X \cap Y = \emptyset$, we simply write $\Gamma_X(Y)$ for $\Gamma_G(Y) \cap X$. In particular, if $Y = \{v\}$, we write $\Gamma_X(v)$ for $\Gamma_G(v) \cap X$. The subgraph induced by X is denoted by $G[X]$. The degree of a vertex v in G is the number of edges of G incident with v, denoted by $\deg_G(v)$, or briefly by $\deg(v)$. A vertex of degree zero is called an isolated vertex. We denote by $\Delta(G)$ the maximum degree of the vertices of G. If $\deg_G(v) = k$ for all $v \in V$, then we call G k-regular. The complement $\overline{G}$ of G is the graph with the same vertex set but whose edge set consists of the edges not present in G. As usual, K_n, P_n and C_n denote the complete graph, path and cycle on n vertices, respectively.

A stable set in G is a set of vertices no two of which are adjacent. A stable set in G is maximum if the graph G contains no larger stable set. The cardinality of a maximum stable set in G is called the stability number of G, denoted $\alpha(G)$. A set $W \subseteq V$ is a vertex cover of G if every edge of G is incident with a vertex in W. The vertex cover number $\tau(G)$ of G is the size of a smallest vertex cover of G.

The Alcuin number of a graph is closely related to its vertex cover number. The following basic lemma follows almost immediately from the definitions.

Lemma 2.1 (see [3–4]) *For every graph G, $\tau(G) \le \text{Alcuin}(G) \le \tau(G) + 1$.*

Lemma 2.1 implies that the Alcuin number of a graph G is equal to either $\tau(G)$ or $\tau(G)+1$, which naturally divides graphs into so-called small-boat and large-boat graphs: A graph G is called a small-boat graph if $\text{Alcuin}(G) = \tau(G)$, and otherwise it is called a large-boat graph.

The problem of deciding whether or not a given graph is small-boat is clearly of great importance. A major step towards this goal is provided by the following structural characterization of the Alcuin number in graphs, due to Csorba, Hurkens and Woeginger [1].

Theorem 2.1 (see [1]) *Let $G = (V, E)$ be a graph. G possesses a feasible schedule for a boat of capacity $b \ge 1$ if and only if there exist five subsets X_1, X_2, X_3, Y_1, Y_2 of V that satisfy the following four conditions:*

(i) *$X_i \cap X_j = \emptyset$ for $1 \le i \ne j \le 3$ and $X = \bigcup_{i=1}^{3} X_i$ is a stable set of G.*

(ii) *The (not necessarily disjoint) sets $Y_1, Y_2 \subseteq Y$ are two nonempty subsets, where $Y = V - X$ and $|Y| \le b$.*

(iii) *$X_1 \cup Y_1$ and $X_2 \cup Y_2$ are stable sets of G.*

(iv) *$|Y_1| + |Y_2| \ge |X_3|$.*

If these four conditions are satisfied, then there exists a feasible schedule of length at most $2|V| + 1$.

Note that for a small-boat graph G with $b = \tau(G)$, the stable set X in Theorem 2.1 is a maximum-size stable set of G, and set Y is a minimum-size vertex cover of G.

The following lemmas provide tools for recognizing small-boat graphs.

Lemma 2.2 (see [1]) *Let $G = (V, E)$ be a graph and Y be a minimum vertex cover of G. If Y contains two (not necessarily distinct) vertices u and v that have at most two common neighbors in $V - Y$, then G is a small-boat graph.*

Lemma 2.3 (see [1]) *If $G = (V, E)$ is a graph that contains two distinct stable sets of* *maximum size (or, equivalently, two distinct vertex covers of minimum size), then G is a small-boat graph.*

3 Graphs with Maximum Degree Five

In this section we shall investigate the Alcuin's number for a graph $G=(V,E)$ with maximum degree at most five. For $S,T\subseteq V$, we denote $e(S,T)$ the set of edges between S and T.

Lemma 3.1 *Let $G=(V,E)$ be a non-trivial connected graph. If $\tau(G)\geq\lfloor\frac{\Delta(G)(\Delta(G)-1)}{3}\rfloor+1$, then G is a small-boat graph.*

Proof Suppose, by way of contradiction, that G is a large-boat graph with $\tau(G)\geq\lfloor\frac{\Delta(G)(\Delta(G)-1)}{3}\rfloor+1$. By Lemma 2.3, G has a unique maximum stable set X, and so $Y=V-X$ is the unique minimum vertex cover of G. Furthermore, by Lemma 2.2, we have $|\Gamma_X(u)\cap\Gamma_X(v)|\geq 3$ for any vertices u,v (not necessarily distinct) in Y. This implies that $\Delta(G)\geq 3$.

Let $\Delta(G)=k$. We claim that $|\Gamma_X(u)|=k$ for each $u\in Y$. If not, let $|\Gamma_X(u)|=k_1<k$. Since $|\Gamma_X(u)\cap\Gamma_X(v)|\geq 3$ for any $v\in Y-\{u\}$, we have

$$3(|Y-\{u\}|)\leq e(\Gamma_X(u),Y-\{u\})\leq\sum_{x\in\Gamma_X(u)}(\deg_G(x)-1)\leq k_1(k-1).$$

So

$$\tau(G)=|Y|\leq\Big\lfloor\frac{k_1(k-1)}{3}\Big\rfloor+1<\Big\lfloor\frac{k(k-1)}{3}\Big\rfloor+1,$$

a contradiction. Thus Y is a stable set of G and each vertex of Y has maximum degree k. This implies that there exits at least a vertex of X that has degree $<k$, for otherwise X and Y would be two distinct maximum stable sets of G. Without loss of generality, let $x_1\in\Gamma_X(u)$ and $\deg(x_1)\leq k-1$.

Next we show that $k\equiv 2\pmod 3$ and $\deg(x)\geq k-2$ for $x\in\Gamma_X(u)$. If $k\equiv 0,1\pmod 3$, then

$$\begin{aligned}3(|Y-\{u\}|)&\leq e(\Gamma_X(u),Y-\{u\})\\&\leq\sum_{x\in\Gamma_X(u)-\{x_1\}}(\deg_G(x)-1)+(\deg_G(x_1)-1)\\&\leq(k-1)^2+(k-2)\\&=k(k-1)-1.\end{aligned}$$

Hence

$$\tau(G)=|Y|\leq\Big\lfloor\frac{k(k-1)-1}{3}\Big\rfloor+1<\Big\lfloor\frac{k(k-1)}{3}\Big\rfloor+1,$$

a contradiction. If $k\equiv 2\pmod 3$ and $\deg(x)\leq k-3$ for some $x\in\Gamma_X(u)$, then

$$3(|Y-\{u\}|)\leq e(\Gamma_X(u),Y-\{u\})\leq\sum_{x\in\Gamma_X(u)}(\deg(x)-1)\leq k(k-1)-3.$$

Hence

$$\tau(G) = |Y| \le \left\lfloor \frac{k(k-1)-3}{3} \right\rfloor + 1 < \left\lfloor \frac{k(k-1)}{3} \right\rfloor + 1,$$

a contradiction. Thus

$$(k-2)\alpha(G) = (k-2)|X| \le k|Y| = k\tau(G),$$

or equivalently,

$$|X| = \alpha(G) \le \Big(\frac{k}{k-2}\Big)\tau(G) \le 2\tau(G) = 2|Y|.$$

We set $X_1 = X_2 = \emptyset$, $X_3 = X$, and $Y_1 = Y_2 = Y$. This clearly satisfies all conditions of the structure theorem, Theorem 2.1, and so G is a small-boat graph. This contradiction establishes the assertion.

Theorem 3.1 *Let $G = (V, E)$ be a nontrivial connected graph with $\Delta(G) \le 4$. If $\tau(G) \ge \Delta(G) - 1$, then G is a small-boat graph.*

Proof If $\Delta(G) \le 2$, the assertion follows directly from Lemma 2.2. If $\Delta(G) = 3$, then, by Lemma 3.1, G is a small-boat graph in the case when $\tau(G) \ge 3$. If $\tau(G) = 2$, then either $G = K_{2,3}$ or there exist two vertices $u, v \in Y$ such that $|\Gamma_X(u) \cap \Gamma_X(v)| \le 2$. In both cases, by Lemma 2.2, we see that G is a small-boat graph. Thus we may assume that $\Delta(G) = 4$.

We can prove it by contradiction. Suppose that G is a large-boat graph. By Lemma 3.1, we have $\tau(G) \le \lfloor \frac{\Delta(G)(\Delta(G)-1)}{3} \rfloor$=4. By Lemma 2.3, G has a unique maximum stable set X, and so $Y = V - X$ is the unique minimum vertex cover of G. Furthermore, by Lemma 2.2, we have $|\Gamma_X(u) \cap \Gamma_X(v)| \ge 3$ for any $u, v \in Y$. This implies that $\deg(u) \ge 3$ for each $u \in Y$, and $\Delta(G[Y]) \le 1$ as $\Delta(G) = 4$.

If $\Delta(G[Y]) = 0$, i.e., Y is a stable set of G. Then, since $|Y| = \tau(G) \ge \Delta(G) - 1 = 3$, we have

$$\begin{aligned}
\alpha(G) &= |X| \\
&\le |\Gamma_X(u)| + \sum_{v \in Y - \{u\}} |\Gamma_X(v) - (\Gamma_X(v) \cap \Gamma_X(u))| \\
&\le 4 + |Y - \{u\}| \\
&\le 2|Y| = 2\tau(G).
\end{aligned}$$

We set $X_1 = X_2 = \emptyset$, $X_3 = X$, and $Y_1 = Y_2 = Y$. It can be verified easily that the sets satisfy all conditions in Theorem 2.1, so G is a small-boat graph, a contradiction.

If $\Delta(G[Y]) = 1$, let $u \in Y$ be a vertex of degree one in $G[Y]$ and $uv \in E(G[Y])$. Clearly, there exists a maximum stable set of $G[Y]$ containing u. Let Y^* be such a stable set of $G[Y]$. Then $|Y^*| \ge 2$ as $\tau(G) \ge \Delta(G) - 1 = 3$. Hence

$$\begin{aligned}
\alpha(G) &= |X| \\
&\le |\Gamma_X(u)| + \sum_{v \in Y^* - \{u\}} |\Gamma_X(v) - (\Gamma_X(v) \cap \Gamma_X(u))| \\
&\le 3 + |Y^* - \{u\}| \\
&\le 2|Y^*|.
\end{aligned}$$

We set $X_1 = X_2 = \emptyset$, $X_3 = X$, and $Y_1 = Y_2 = Y^*$. Clearly, $|Y_1| + |Y_2| = 2|Y^*| \geq |X| = |X_3|$ and all other conditions in Theorem 2.1 are satisfied. So G is a small-boat graph, and derive a contradiction.

Remark 3.1 The constraint $\tau(G) \geq \Delta(G) - 1$ in Theorem 3.1 cannot be dropped. Let $\mathcal{F} = \{K_{1,3}, K_{1,4}, K_2 \circ \overline{K}_3, K^*_{2,3}\}$, where $K^*_{2,3}$ is the bipartite graph obtained from $K_{2,3}$ by adding one pendant edge to each vertex in the small partition class of $K_{2,3}$. Clearly, every graph G in $\mathcal{F}$ satisfies $\tau(G) = \Delta(G) - 2$. It is easy to see that every graph G in $\mathcal{F}$ is a large-boat graph.

Theorem 3.2 *Let $G = (V, E)$ a connected graph with $\Delta(G) = 5$. If $\tau(G) \geq \Delta(G)$, then G is a small-boat graph.*

Proof By Lemmas 2.2–2.3, we may assume that G has a unique maximum stable set, say X, and $|\Gamma_X(u) \cap \Gamma_X(v)| \geq 3$ for any $u, v \in Y$, where $Y = V - X$. We proceed by contradiction. Let Y^* be a maximum stable set of $G[Y]$, and let Y^*_0 be the set of isolated vertices in $G[Y]$. Obviously, $Y^*_0 \subseteq Y^*$.

First, we have the following claim.

Claim 1 For each $u \in Y$, there exists a vertex $v \in Y$ such that $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$.

If not, let $|\Gamma_X(u) \cap \Gamma_X(v)| \geq 4$ for all $v \in Y - \{u\}$. Then $\deg(v) \geq 4$ for each $v \in Y - \{u\}$ and so $\Delta(G[Y]) \leq 1$. Hence $|Y^*| = |Y^*_0| + \frac{|Y - Y^*_0|}{2} = \frac{|Y^*_0| + |Y|}{2}$. Since $|Y| = \tau(G) \geq \Delta(G) = 5$, we have

$$\begin{aligned}\alpha(G) &= |X| \\ &\leq |\Gamma_X(u)| + \sum_{v \in Y - \{u\}} |\Gamma_X(v) - (\Gamma_X(u) \cap \Gamma_X(v))| \\ &= |\Gamma_X(u)| + \sum_{v \in Y^*_0 - \{u\}} |\Gamma_X(v) - (\Gamma_X(u) \cap \Gamma_X(v))| \\ &\leq |\Gamma_X(u)| + |Y^*_0 - \{u\}|.\end{aligned}$$

We show that $|X| \leq 2|Y^*|$ by the above inequality. If $u \in Y^*_0$, then

$$|X| \leq |\Gamma_X(u)| + |Y^*_0 - \{u\}| \leq 5 + |Y^*_0| - 1 < |Y| + |Y^*_0| = 2|Y^*|.$$

If $u \notin Y^*_0$, then

$$|X| \leq |\Gamma_X(u)| + |Y^*_0 - \{u\}| \leq 4 + |Y^*_0| < |Y| + |Y^*_0| = 2|Y^*|.$$

Now we set $X_1 = X_2 = \emptyset$, $X_3 = X$, and $Y_1 = Y_2 = Y^*$. It is easy to check that the sets satisfy all conditions in Theorem 2.1, so G is a small-boat graph, a contradiction.

Claim 2 Let $u, v, w \in Y$, and $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$, If $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \leq 2$, then $|\Gamma_X(y) \cap (\Gamma_X(\{u, v, w\})| \geq 4$ for all $y \in Y - \{u, v, w\}$.

If not, let $y \in Y$ and $|\Gamma_X(y) \cap \Gamma_X(\{u, v, w\})| < 4$. Then $|\Gamma_X(y) \cap \Gamma_X(\{u, v, w\})| = 3$. Since $|\Gamma_X(y) \cap \Gamma_X(w)| \geq 3$ and $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \leq 2$, we have $|\Gamma_X(y) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \leq 2$. This implies that $|\Gamma_X(y) \cap \Gamma_X(u)| \leq 2$ or $|\Gamma_X(y) \cap \Gamma_X(v)| \leq 2$, a contradiction.

By Lmma 3.1, we see that $5 \leq \tau(G) \leq \lfloor \frac{k(k-1)}{3} \rfloor = 6$. Next we consider two cases depending on the value of $\tau(G)$.

Case 1 $\tau(G) = 6$. In this case, we first claim that $|\Gamma_X(u)| \geq 4$ for each $u \in Y$. Indeed, if not, let $u \in Y$ and $|\Gamma_X(u)| \leq 3$, then $|\Gamma_X(u)| = 3$ (as $|\Gamma_X(u)| \geq 3$). Note that $|\Gamma_X(u) \cap \Gamma_X(v)| \geq 3$ for any $v \in Y - \{u\}$ and $\tau(G) = 6$. Hence $\deg(x) \geq 6$ for each $x \in \Gamma_X(u)$, contradicting our assumption $\Delta(G) = 5$. Thus $|\Gamma_X(u)| \geq 4$ for all $u \in Y$. This also implies that $\Delta(G[Y]) \leq 1$ as $\Delta(G) = 5$. Thus $|Y^*| = |Y_0^*| + \frac{|Y|-|Y_0^*|}{2} = \frac{|Y_0^*|+|Y|}{2}$.

Claim 3 Let $u, v \in Y$ and $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$. Then there exists a vertex $w \in Y - \{u, v\}$ such that $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \leq 2$.

Suppose not, let $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| = 3$ for all $w \in Y - \{u, v\}$. Then $\deg(x) \geq 6$ for any $x \in \Gamma_X(u) \cap \Gamma_X(v)$, a contradiction.

By Claim 1, there exist $u, v \in Y$ such that $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$. By Claim 3, let $w \in Y - \{u, v\}$ satisfying $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \leq 2$. Note that $|\Gamma_X(w) \cap \Gamma_X(u)| \geq 3$ and $|\Gamma_X(w) \cap \Gamma_X(v)| \geq 3$. Then

$$|\Gamma_X(w) \cap (\Gamma_X(\{u, v\}))| \geq 4.$$

Hence, by Claim 2, we obtain

$$\begin{aligned}
\alpha(G) &= |X| \\
&\leq |\Gamma_X(u, v, w)| + \sum_{y \in Y - \{u,v,w\}} |\Gamma_X(y) - \Gamma_X(\{u, v, w\})| \\
&= |\Gamma_X(u, v, w)| + \sum_{y \in Y_0^* - \{u,v,w\}} |\Gamma_X(y) - (\Gamma_X(y) \cap \Gamma_X(\{u, v, w\}))| \\
&\leq (|\Gamma_X(\{u, v\})| + |\Gamma_X(w)| - |\Gamma_X(w) \cap \Gamma_X(\{u, v\})|) + |Y_0^* - \{u, v, w\}| \\
&\leq |\Gamma_X(u)| + |\Gamma_X(v)| - |\Gamma_X(u) \cap \Gamma_X(v)| + |\Gamma_X(w)| - 4 + |Y_0^* - \{u, v, w\}| \\
&\leq |\Gamma_X(u)| + |\Gamma_X(v)| + |\Gamma_X(w)| - 7 + |Y_0^* - \{u, v, w\}|. \qquad (3.1)
\end{aligned}$$

We show that $|X| \leq 2|Y^*|$ by the above inequality. If $\{u, v, w\} \subseteq Y_0^*$, then, by (3.1),

$$|X| \leq 15 - 7 + |Y_0^*| - 3 = 5 + |Y_0^*| = |Y| + |Y_0^*| \leq 2|Y^*|.$$

If $u, v, w \notin Y_0^*$, then, by (3.1), we have

$$|X| \leq 12 - 7 + |Y_0^*| = 5 + |Y_0^*| \leq 2|Y^*|.$$

If exactly both of $\{u, v, w\}$ belong to Y_0^*, then

$$|X| \leq 14 - 7 + |Y_0^*| - 2 = 5 + |Y_0^*| \leq 2|Y^*|.$$

If exactly one of $\{u, v, w\}$ belongs to Y_0^*, then

$$|X| \leq 13 - 7 + |Y_0^*| - 1 = 5 + |Y_0^*| \leq 2|Y^*|.$$

Thus $|X| \leq 2|Y^*|$. Now let $X_1 = X_2 = \emptyset, X_3 = X$, and let $Y_1 = Y_2 = Y^*$. It is easy to see that all conditions in Theorem 2.1 are satisfied, which is a contradiction.

Case 2 $\tau(G) = 5$. In this case, since $|\Gamma_X(u)| \geq 3$ for each $u \in Y$, we have $\Delta(G[Y]) \leq 2$. Let $Y^1 = \{y \in Y \mid y \text{ has degree one in } G[Y]\}$ and $Y^2 = \{y \in Y \mid y \text{ has degree two in } G[Y]\}$.

By Claim 1, Y contains two vertices u, v such that $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$. Let $Y_2^* = \{y \in Y^* \mid y$ has degree two in $G[Y]\}$.

Case 2.1 $\Delta(G[Y]) = 2$. Then $G[Y]$ is isomorphic to one of $\{C_5, P_5, C_4 \cup P_1, C_3 \cup P_2, C_3 \cup \overline{P}_2, P_4 \cup P_1, P_3 \cup P_2, P_3 \cup \overline{P}_2\}$. In this subcase, we can take $u \in Y^2$ that satisfies $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$ for each $v \in Y - \{u\}$. Obviously, $|\Gamma_X(y)| = 3$ for any $y \in Y^2$ and $\Gamma_X(Y^2) = \Gamma_X(u) = \Gamma_X(u) \cap \Gamma_X(v)$.

If $G[Y] \cong C_5$, then $|X| = 3$, while if $G \cong P_5$ or $C_4 \cup P_1$, then $|X| \le 5$. In these cases, we let $X_1 = X_2 = \emptyset,\ X_3 = X$, and let $Y_1 = Y_2 = Y^*$. Then clearly $|X_3| = |X| \le |Y_1| + |Y_2|$ and all other conditions in Theorem 2.1 are satisfied, so G is a small-boat graph, a contradiction.

If $G[Y]$ is isomorphic to one of $\{C_3 \cup P_2, C_3 \cup \overline{P}_2, P_4 \cup P_1, P_3 \cup P_2, P_3 \cup \overline{P}_2\}$, then it is easy to see that $|Y^*| = |Y_0^*| + \frac{|Y^1|}{2} + 1$. Hence, we have

$$\begin{aligned} |X| &\le |\Gamma_X(\{u,v\})| + \sum_{w \in Y - \{u,v\}} |\Gamma_X(w) - \Gamma_X(\{u,v\})| \\ &\le |\Gamma_X(Y^2)| + \sum_{w \in Y_0^*} |\Gamma_X(w) - \Gamma_X(Y^2)| + \sum_{w \in Y^1} |\Gamma_X(w) - \Gamma_X(Y^2)| \\ &\le 3 + 2|Y_0^*| + |Y^1|. \end{aligned} \tag{3.2}$$

If $|X| \le 2 + 2|Y_0^*| + |Y^1|$, then $|X| \le 2|Y^*|$. As before, we can deduce that G is a small-boat graph, a contradiction. Thus $|X| = 3 + 2|Y_0^*| + |Y^1|$. From (3.2), it follows that each vertex of $\bigcup_{y \in Y_0^* \cup Y^1} \Gamma_X(y) - \Gamma_X(Y^2)$ has degree one in G. If $G[Y]$ contains paths as its components, then we choose one end, say y_1, in such a path. We set $X_1 = \Gamma_X(Y_0^* \cup (Y^1 - \{y_1\})) - \Gamma_X(Y^2)$, $X_2 = \Gamma_X(y_1) - \Gamma_X(Y^2)$, $X_3 = \Gamma_X(Y^2)$, and let $Y_1 = \{y_1\}, Y_2 = Y_0^* \cup (Y^1 - \{y_1\}) \cup \{y_2\}$ where y_2 lies in the component C_3 of $G[Y]$ (if C_3 is a component of $G[Y]$). If $G[Y]$ contains no paths as its components, then $G[Y] \cong C_3 \cup \overline{P}_2$ and thus $Y_0^* = V(\overline{P}_2)$. We set $Y_1 = \{y_1\}, Y_2 = \{y_2\}$, and $X_1 = \Gamma_X(y_2) - \Gamma_X(Y^2), X_2 = \Gamma_X(y_1) - \Gamma_X(Y^2), X_3 = \Gamma_X(Y^2)$, where $y_1, y_2 \in Y_0^*$ and $y_1 \ne y_2$. In both cases, clearly X_1, X_2, X_3 form a partition of X, and $Y_1 \ne \emptyset, Y_2 \ne \emptyset$. Obviously, $|X_3| = 3 \le |Y_1| + |Y_2|$ and all other conditions in Theorem 2.1 are satisfied. So G is a small-boat graph, a contradiction.

Case 2.2 $\Delta(G[Y]) \le 1$. In this subcase, $|Y| = 5$ implies that $Y_0^* \ne \emptyset$ and $G[Y]$ is isomorphic to one of $\{2P_2 \cup P_1, P_2 \cup \overline{C}_3, \overline{K}_5\}$. Clearly, $|Y^*| = |Y_0^*| + \frac{|Y - Y_0^*|}{2} = \frac{|Y| + |Y_0^*|}{2}$.

Suppose that there exists a vertex $w \in Y - \{u, v\}$ such that $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \le 2$. Note that $|\Gamma_X(u) \cap \Gamma_X(v)| = 3$, $|\Gamma_X(w) \cap \Gamma_X(u))| \ge 3$ and $|\Gamma_X(w) \cap \Gamma_X(v))| \ge 3$, so $|\Gamma_X(w) \cap \Gamma_X(\{u,v\})| \ge 4$. Hence, by Claim 2, we have

$$\begin{aligned} |X| &\le |\Gamma_X(\{u,v,w\})| + \sum_{y \in Y - \{u,v,w\}} |\Gamma_X(y) - (\Gamma_X(y) \cap \Gamma_X(\{u,v,w\}))| \\ &\le (|\Gamma_X(\{u,v\})| + |\Gamma_X(w)| - |\Gamma_X(w) \cap \Gamma_X(\{u,v\})|) + |Y_0^* - \{u,v,w\}| \\ &\le |\Gamma_X(u)| + |\Gamma_X(v)| - |\Gamma_X(u) \cap \Gamma_X(v)| + |\Gamma_X(w)| - 4 + |Y_0^* - \{u,v,w\}| \\ &\le |\Gamma_X(u)| + |\Gamma_X(v)| + |\Gamma_X(w)| - 7 + |Y_0^* - \{u,v,w\}|. \end{aligned} \tag{3.3}$$

To obtain a contradiction, it suffices to show that $|X| \le 2|Y^*|$. If $\{u, v, w\} \subseteq Y_0^*$, then, by (3.3),

$|X| \leq 8+|Y_0^*|-3 = 5+|Y_0^*| = |Y|+|Y_0^*| \leq 2|Y^*|$. If $u, v, w \notin Y_0^*$, then, by (3.3), $|X| \leq 5+|Y_0^*| \leq 2|Y^*|$. If exactly both of $\{u, v, w\}$ belong to Y_0^*, then $|X| \leq 7 + |Y_0^*| - 2 = 5 + |Y_0^*| \leq 2|Y^*|$. If exactly one of $\{u, v, w\}$ belongs to Y_0^*, then $|X| \leq 6 + |Y_0^*| - 1 = 5 + |Y_0^*| \leq 2|Y^*|$. Thus $|X| \leq 2|Y^*|$. We set $X_1 = X_2 = \emptyset$, $X_3 = X$, and $Y_1 = Y_2 = Y^*$. Clearly, the sets satisfy all conditions in Theorem 2.1, a contradiction.

Suppose that $|\Gamma_X(w) \cap (\Gamma_X(u) \cap \Gamma_X(v))| \geq 3$ for any $w \in Y - \{u, v\}$, i.e., $\Gamma_X(u) \cap \Gamma_X(v) \subseteq \Gamma_X(w)$. Then,

$$\begin{aligned} |X| &\leq |\Gamma_X(\{u, v\})| + \sum_{y \in Y - \{u,v\}} |\Gamma_X(y) - \Gamma_X(\{u, v\})| \\ &\leq |\Gamma_X(\{u, v\})| + 2|Y_0^* - \{u, v\}| + |Y^1 - \{u, v\}| \\ &\leq |\Gamma_X(\{u, v\})| + 2|Y_0^* - \{u, v\}| + |Y - (Y_0^* \cup \{u, v\})|. \end{aligned} \tag{3.4}$$

We first claim that $|X| \leq 3 + |Y_0^*| + |Y|$. Indeed, if $u, v \in Y_0^*$, then $|X| \leq 7 + 2|Y_0^* - \{u, v\}| + |Y - (Y_0^* \cup \{u, v\})| = 7 + 2(|Y_0^*| - 2) + |Y| - |Y_0^*| \leq 3 + |Y_0^*| + |Y|$; if $u, v \notin Y_0^*$, then $|X| \leq 5 + 2|Y_0^* - \{u, v\}| + |Y - (Y_0^* \cup \{u, v\})| = 5 + 2|Y_0^*| + |Y| - |Y^*| - 2 \leq 3 + |Y_0^*| + |Y|$; if exactly one of $\{u, v\}$ belongs to Y_0^*, then $|X| \leq 6 + 2|Y_0^* - \{u, v\}| + |Y - (Y_0^* \cup \{u, v\})| = 6 + 2(|Y_0^*| - 1) + |Y| - (|Y_0^*| + 1) \leq 3 + |Y_0^*| + |Y|$, as claimed.

If $|X| \leq |Y_0^*| + |Y|$, then $|X| \leq 2|Y^*|$. As before, we would deduce that G is a small-boat graph. Thus we may assume that $1 + |Y_0^*| + |Y| \leq |X| \leq 3 + |Y_0^*| + |Y|$. If $|X| = 3 + |Y_0^*| + |Y|$, then $\deg(x) = 1$ for each $x \in X - (\Gamma_X(u) \cap \Gamma_X(v))$ by (3.4). We take $y_0 \in Y_0^*$, and let

$$\begin{aligned} X_1 &= \Gamma_X(y_0) - (\Gamma_X(u) \cap \Gamma_X(v)), \\ X_2 &= X - (\Gamma_X(y_0) \cup (\Gamma_X(u) \cap \Gamma_X(v))), \\ X_3 &= \Gamma_X(u) \cap \Gamma_X(v), \end{aligned}$$

and $Y_2 = \{y_0\}, Y_1 = Y^* - \{y_0\}$. If $|X| = 2 + |Y_0^*| + |Y|$, then $X - (\Gamma_X(u) \cap \Gamma_X(v))$ contains at most one vertex, say x, of degree two in G, while all other vertices in $X - (\Gamma_X(u) \cap \Gamma_X(v))$ have degree one in G. If such a vertex x exists, we choose $y_1, y_2 \in Y$ such that $y_i x \in E(G)$ for $i = 1, 2$, otherwise, we can choose both vertices $y_1, y_2 \in Y$ such that $\Gamma_X(\{y_1, y_2\}) - (\Gamma_X(u) \cap \Gamma_X(v)) \neq \emptyset$. Set

$$\begin{aligned} X_1 &= \Gamma_X(\{y_1, y_2\}) - (\Gamma_X(u) \cap \Gamma_X(v)), \\ X_2 &= \bigcup_{y \in Y - \{y_1, y_2\}} \Gamma_X(y) - (\Gamma_X(u) \cap \Gamma_X(v)), \\ X_3 &= \Gamma_X(u) \cap \Gamma_X(v), \end{aligned}$$

and let Y_1 and Y_2 be maximum stable sets of $Y - \{y_1, y_2\}$ and $\{y_1, y_2\}$, respectively. If $|X| = 1 + |Y_0^*| + |Y|$, then $X - (\Gamma_X(u) \cap \Gamma_X(v))$ contains either at most two vertices of degree two in G and all other vertices in $X - (\Gamma_X(u) \cap \Gamma_X(v))$ have degree one in G or at most one of degree three in G and all other vertices in $X - (\Gamma_X(u) \cap \Gamma_X(v))$ have degree one in G. Denote by W the set of vertices of Y that are adjacent to the vertices of degree two or degree three of $X - (\Gamma_X(u) \cap \Gamma_X(v))$ in G. Then clearly $0 \leq |W| \leq 4$. If $W \neq \emptyset$, let $X_1 = \bigcup_{y \in W} \Gamma_X(y) -$

$(\Gamma_X(u) \cap \Gamma_X(v))$, $X_2 = \bigcup_{y \in Y-W} \Gamma_X(y) - (\Gamma_X(u) \cap \Gamma_X(v))$, $X_3 = \Gamma_X(u) \cap \Gamma_X(v)$, and let Y_1 and Y_2 be maximum stable sets of W and $Y - W$, respectively. Otherwise, we choose one vertex $y_1 \in Y$ such that $\Gamma_X(\{y_1\}) - (\Gamma_X(u) \cap \Gamma_X(v)) \neq \emptyset$ and set

$$X_1 = \Gamma_X(\{y_1\}) - (\Gamma_X(u) \cap \Gamma_X(v)),$$
$$X_2 = \bigcup_{y \in Y-\{y_1\}} \Gamma_X(y) - (\Gamma_X(u) \cap \Gamma_X(v)),$$
$$X_3 = \Gamma_X(u) \cap \Gamma_X(v),$$

and let $Y_2 = \{y_1\}$ and Y_1 be maximum stable set of $Y - \{y_1\}$. In either case, one see that X_1, X_2, X_3 form a partition of X and $Y_1 \neq \emptyset, Y_2 \neq \emptyset$. Clearly, $X_i \cup Y_i$ $(i = 1, 2)$ is stable and $|Y_1| + |Y_2| \geq 3 = |X_3|$. Thus all conditions in Theorem 2.1 are satisfied. So G is a small-boat graph, a contradiction.

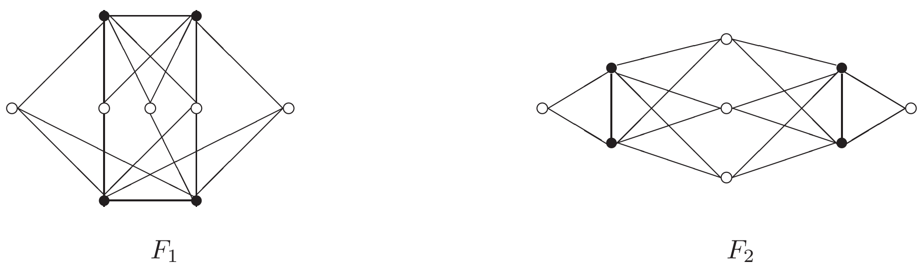

Figure 1 The graphs F_1 and F_2.

Remark 3.2 The constraint $\tau(G) \geq \Delta(G)$ in Theorem 3.2 cannot be dropped. Let F_1, F_2 be the graphs depicted in Figure 1. Clearly, $\Delta(F_i) = 5, \tau(F_i) = 4$ and the set of vertices indicated by solid dots is the unique minimum vertex cover of F_i for $i = 1, 2$. It is easy to verify that F_i is a large-boat graph.

4 Regular Graphs

In this section we turn our attention to regular graphs. In [7] Seify and Shahmohamad proved that every k-regular graph with $2 \leq r \leq 5$ is a small-boat graph, and asked the question: Whether all regular graphs are small-boat graphs? We shall give a concise description of the division line between small-boat and large-boat graphs for regular graphs.

To obtain our main result in this section, we need the following lemmas.

Lemma 4.1 (see [8]) *For a graph G on n vertices and average degree $d(G)$, $\alpha(G) \geq \lceil \frac{n}{d(G)+1} \rceil$.*

Lemma 4.2 *Let G be a large-boat graph and Y be a minimum vertex cover of G. Then $\tau(G) \leq \lfloor \frac{(\Delta(G)-\Delta(G[Y]))(\Delta(G)-1)}{3} \rfloor + 1$.*

Proof Since G is a large-boat graph, by Lemma 2.3, one see that Y is the unique vertex cover of minimum size. Furthermore, by Lemma 2.2, we have $|\Gamma_X(u) \cap \Gamma_X(v)| \geq 3$ for any $u, v \in Y$. Let $u \in Y$ and $\deg_{G[Y]}(u) = \Delta(G[Y])$. Then

$$\begin{aligned}3(|Y - \{u\}|) &\leq e(\Gamma_X(u), Y - \{u\}) \\ &\leq \sum_{x \in \Gamma_X(u)} (\deg_G(x) - 1) \\ &\leq |\Gamma_X(u)|(\Delta(G) - 1) \\ &\leq (\Delta(G) - \Delta(G[Y]))(\Delta(G) - 1).\end{aligned}$$

Hence, $\tau(G) = |Y| \leq \lfloor \frac{(\Delta(G) - \Delta(G[Y]))(\Delta(G)-1)}{3} \rfloor + 1$, as desired.

Theorem 4.1 *Let $G = (V, E)$ be a k-regular graph. If $1 \leq k \leq 7$, then G is a small-boat graph.*

Proof We may assume that G is connected as otherwise we look at each (connected) component separately. Let X be a maximum stable set of G and let $Y = V - X$. Then Y is a minimum vertex cover of G. Since G is regular, $|Y| \geq k$. If $1 \leq k \leq 2$, the assertion follows directly from Lemma 2.2. If $3 \leq k \leq 5$, then $\tau(G) = |Y| \geq k$. The assertion follows from Theorems 3.1–3.2. So we may assume that $6 \leq k \leq 7$.

We next proceed by contradiction. Suppose that G is a large-boat graph. Then, by Lemmas 2.2–2.3, X is the unique maximum stable set of G and $|\Gamma_X(u) \cap \Gamma_X(v)| \geq 3$ for any $u, v \in Y$. Hence Y is not a stable set of G and we have $|\Gamma_X(u)| \geq 3$ for any $u \in Y$. Thus $|X| \geq 3$ and $1 \leq \Delta(G[Y]) \leq k - 3$. This implies that $|Y| \geq \max\{k, |X| + 1\}$. By Lemma 4.2, we have

$$\max\{k, |X| + 1\} \leq |Y| \leq \Big\lfloor \frac{(k-1)^2}{3} \Big\rfloor + 1. \tag{4.1}$$

Let Y^* be a maximum stable set of $G[Y]$. If $|X| = 3$, then $|\Gamma_X(u)| = 3$ for each $u \in Y$, so $G[Y]$ is $(k-3)$-regular. By (4.1) and Lemma 4.2, we have $|Y| = k$. Furthermore, by Lemma 4.1, $\alpha(G[Y]) \geq \frac{k}{k-2}$, and so $\alpha(G[Y]) \geq 2$. Set $X_1 = X_2 = \emptyset$, $X_3 = X$, and $Y_1 = Y_2 = Y^*$. Clearly, $|Y_1| + |Y_2| = 2|Y^*| \geq |X_3|$ and all other conditions in Theorem 2.1 are satisfied, a contradiction. So we may assume that $|X| \geq 4$. We claim that $k|X| \geq 4|Y|$. Indeed, if $k|X| < 4|Y|$, then we have $3|Y| \leq \sum_{u \in Y} |\Gamma_X(u)| = k|X| < 4|Y|$. Thus there exists a vertex $v \in Y$ such that $|\Gamma_X(v)| \leq 3$, and so $|\Gamma_X(v)| = 3$. Hence $\Delta(G[Y]) \geq k - |\Gamma_X(v)| = k - 3$. By $|Y| \geq k$ and Lemma 4.2, we have $|Y| = k$. Then $3k \leq k|X| < 4k$, i.e., $|X| = 3$, this contradicting our assumption that $|X| \geq 4$. Consequently, we obtain

$$4k \leq 4|Y| \leq \sum_{u \in Y} |\Gamma_X(u)| = e(Y, X) = k|X| < k|Y|. \tag{4.2}$$

By (4.2), we obtain contradiction in the following three cases.

Case 1 $4|Y| \leq k|X| < 5|Y|$. Then $\sum_{u \in Y} |\Gamma_X(u)| = e(Y, X) = k|X| < 5|Y|$. This implies that Y contains a vertex v with $|\Gamma_X(v)| \leq 4$. Hence $\Delta(G[Y]) \geq k - 4$. By Lemma 4.2, we have

$k \le |Y| \le \lfloor \frac{4(k-1)}{3} \rfloor + 1$. Then

$$4k \le 4|Y| \le k|X| < 5|Y| \le 5\Big(\Big\lfloor \frac{4(k-1)}{3} \Big\rfloor + 1\Big), \tag{4.3}$$

or equivalently,

$$4 \le |X| < \frac{5}{k}\Big(\Big\lfloor \frac{4(k-1)}{3} \Big\rfloor + 1\Big).$$

By the above inequality, we obtain $4 \le |X| \le 5$ if $k = 6$, and $4 \le |X| \le 6$ if $k = 7$.

We first consider the case when $k = 6$. In this case, $|X| = 4$ or 5, we obtain $|Y| = 6$ or 7, respectively, by (4.3). Furthermore, note that $|Y| = |X| + 2$, the average degree of $G[Y]$ is $d(G[Y]) = \frac{6|Y|-6|X|}{|Y|} = \frac{12}{|Y|}$. Then $|Y^*| = \alpha(G[Y]) \ge \lceil \frac{|Y|^2}{12+|Y|} \rceil$ by Lemma 4.1. We set $X_1 = X_2 = \emptyset, X_3 = X$, and $Y_1 = Y_2 = Y^*$. Then

$$|Y_1| + |Y_2| = 2|Y^*| \ge 2\Big\lceil \frac{(|X|+2)^2}{14+|X|} \Big\rceil \ge \frac{2(|X|+2)^2}{14+|X|} \ge |X| = |X_3|,$$

and also all other conditions in Theorem 2.1 are satisfied, a contradiction.

Secondly, we consider the case when $k = 7$. In this case, $|X| = 4, 5$ or 6, and we obtain $|Y| = 7, 8$ or 9, respectively, by (4.3). Note that $|Y| = |X| + 3$. Then $2|E(G[Y])| = \sum_{y \in Y} \deg_{G[Y]}(y) = 7|Y| - 7|X| = 21$, this is a contradiction.

Case 2 $5|Y| \le k|X| < 6|Y|$. As in Case 1, one see that $\Delta(G[Y]) \ge k - 5$. By Lemma 4.2, we have $k \le |Y| \le \lfloor \frac{5(k-1)}{3} \rfloor + 1$. Then

$$5k \le 5|Y| \le k|X| < 6|Y| \le 6\Big(\Big\lfloor \frac{5(k-1)}{3} \Big\rfloor + 1\Big), \tag{4.4}$$

or equivalently,

$$5 \le |X| < \frac{6}{k}\Big(\Big\lfloor \frac{5(k-1)}{3} \Big\rfloor + 1\Big).$$

By the above inequality, we obtain $5 \le |X| \le 8$ if $k = 6$, and $5 \le |X| \le 9$ if $k = 7$.

We first consider the case when $k = 6$. In this case, $|X| = 5, 6, 7$ or 8, and we obtain $|Y| = 6, 7, 8$ or 9, respectively, by (4.4). This implies that $|Y| = |X| + 1$. Then the average degree of $G[Y]$ is $d(G[Y]) = \frac{6|Y|-6|X|}{|Y|} = \frac{6}{|Y|}$. Then $|Y^*| = \alpha(G[Y]) \ge \lceil \frac{|Y|^2}{6+|Y|} \rceil$ by Lemma 4.1. We set $X_1 = X_2 = \emptyset, X_3 = X$, and $Y_1 = Y_2 = Y^*$. Then

$$|Y_1| + |Y_2| \ge 2|Y^*| \ge 2\Big\lceil \frac{(|X|+1)^2}{7+|X|} \Big\rceil \ge \frac{2(|X|+1)^2}{7+|X|} \ge |X| = |X_3|,$$

and also all other conditions in Theorem 2.1 are satisfied, a contradiction.

Secondly, we consider the case when $k = 7$. In this case, $|X| = 5, 6, 7, 8$ or 9, and we obtain $|Y| = 7, 8, 9, 10$ or 11, respectively, by (4.4). This implies that $|Y| = |X| + 2$. Then the average degree of $G[Y]$ is $d(G[Y]) = \frac{7|Y|-7|X|}{|Y|} = \frac{14}{|Y|}$. Then

$$|Y^*| = \alpha(G[Y]) \ge \Big\lceil \frac{|Y|^2}{14+|Y|} \Big\rceil$$

by Lemma 4.1. We set $X_1 = X_2 = \emptyset, X_3 = X$, and $Y_1 = Y_2 = Y^*$. Then

$$|Y_1| + |Y_2| \geq 2|Y^*| \geq 2\left\lceil \frac{(|X|+2)^2}{16+|X|} \right\rceil \geq |X| = |X_3|,$$

and also all other conditions in Theorem 2.1 are satisfied, a contradiction.

Case 3 $6|Y| \leq k|X| < 7|Y|$. In this case, we have $k = 7$ by (4.2). From (4.1), it follows that $|Y| \leq 13$. If $|Y| \geq |X| + 2$, then $k|X| = 7|X| \geq 6|Y| = 6(|X| + 2)$, and so $|X| \geq 12$. This implies that $|Y| \geq 14$, a contradiction. So $|Y| = |X| + 1$. But then

$$2|E(G[Y])| = \sum_{y \in Y} \deg_{G[Y]}(y) = 7|Y| - 7|X| = 7,$$

a contradiction again.

In either case, we always arrive at a contradiction, the assertion follows.

Remark 4.1 In general, the assertion in Theorem 4.1 is not true for $k \geq 8$. For example, let $k \geq 8$ is not a prime, then k can be decompose into $k = k_1 k_2$, where k_1 $(\neq 1)$ is the smallest divisor of k. Let X be a stable set with $k_1 k_2 - (k_2 - 1)$ vertices and Y be the disjoin union of k_1 copies of complete graph K_{k_2}. Let H_k be the k-regular graph obtained from the disjoint union X and Y be by joining each vertex of X to each vertex of Y. By our construction, clearly X is the unique maximum stable set of H_k and Y is the unique minimum vertex cover of H_k. We show that G is a large-boat graph. If not, then, by Theorem 2.1, there exist the sets X_1, X_2, X_3, and Y_1, Y_2 satisfying conditions (i)–(iv) in the structure theorem. Since every vertex of X is adjacent to all the vertices of Y, we have $X_3 = X$. Then $X_1 = X_2 = \emptyset$. This implies that $Y_1 = Y_2 = Y^*$, where Y^* is a maximum stable set of $G[Y]$. Clearly, Y^* contains at most one vertex of each copy of K_{k_2}. So $|Y^*| = k_1$. But then

$$|Y_1| + |Y_2| = 2|Y^*| = 2k_1 < k_1 k_2 - (k_2 - 1) = |X_3|,$$

which is a contradiction.

5 Conclusions

In this paper we investigated the Alcuin's numbers of regular graphs and graphs with maximum degree at most five, we presented concise descriptions of the division line between small-boat and large-boat graphs. Finally, we propose the following open problem.

Problem 5.1 For every connected graph G with $\Delta(G) \geq 6$, determine the sharp lower bound $\varphi(\Delta(G))$ on $\tau(G)$ such that G is a small-boat graph when $\tau(G) \geq \varphi(\Delta(G))$.

For positive integers $k \geq 3$ and $1 \leq t \leq k - 2$, let $I_1 = \{x_1, x_2, \cdots, x_{2k+t}\}$ be a stable set on $2k + t$ vertices and let $I_2 = \{e_1, e_2, \cdots, e_k\}$ be a set of pairwise nonadjacent k edges. Now let $F_{k,t}$ be the graph obtained from the disjoin union of I_1 and I_2 by joining each vertex x_i $(1 \leq i \leq k)$ of I_1 to precisely two ends of the edge e_i of I_2 and joining each vertex x_j $(k + 1 \leq j \leq 2k + t)$ of I_1 to each vertex of I_2. By our construction, $F_{k,t}$ has maximum degree

$2k$. Clearly, I_1 is the unique maximum stable set of $F_{k,t}$ and the set of vertices in I_2 is the minimum vertex cover of $F_{k,t}$. It is easy to verify that $F_{k,t}$ is a large-boat graph. This implies that $\varphi(\Delta(G)) \geq \Delta(G) + 1$.

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