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One-Sided Exact Boundary Null Controllability of Entropy Solutions to a Class of Hyperbolic Systems of Conservation Laws with Characteristics with Constant Multiplicity*

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Abstract This paper proves the local exact one-sided boundary null controllability of entropy solutions to a class of hyperbolic systems of conservation laws with characteristics with constant multiplicity. This generalizes the results in [Li, T. and Yu, L., One-sided exact boundary null controllability of entropy solutions to a class of hyperbolic systems of conservation laws, To appear in Journal de Mathématiques Pures et Appliquées, 2016.] for a class of strictly hyperbolic systems of conservation laws.

Keywords Characteristics with constant multiplicity, One-sided boundary null controllability, Semi-global entropy solution, ε-Approximate front tracking solution

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1 Introduction

In this paper, we study the local one-sided exact boundary null controllability for $n \times n$ hyperbolic system of conservation laws in one space dimension:

$$\partial_t H(u) + \partial_x G(u) = 0, \quad t > 0,\ 0 < x < L, \tag{1.1}$$

where u is an n-vector valued unknown function of (t, x), G and H are smooth n-vector valued functions of u, defined on a ball $B_r(0)$ centered at the origin in $\mathbb{R}^n$ with suitable small radius r.

For system (1.1), we require the following assumptions:

(H1) System (1.1) is hyperbolic, that is, for any given $u \in B_r(0)$, the matrix $DH(u)$ is non-singular and the matrix $(DH(u))^{-1}DG(u)$ has n real eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ and a complete set of left (resp. right) eigenvectors $\{l_1(u), \cdots, l_n(u)\}$ (resp. $\{r_1(u), \cdots, r_n(u)\}$).

(H2) For any given $u \in B_r(0)$, each eigenvalue of $(DH(u))^{-1}DG(u)$ has a constant multiplicity. To fix the idea and without loss of generality, we suppose that

$$\lambda_1(u) < \cdots < \lambda_k(u) < \lambda_{k+1}(u) \equiv \cdots \equiv \lambda_{k+p}(u) < \lambda_{k+p+1}(u) < \cdots < \lambda_n(u), \tag{1.2}$$

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where $\lambda(u) := \lambda_{k+1}(u) \equiv \cdots \equiv \lambda_{k+p}(u)$ is an eigenvalue with constant multiplicity $p \geq 1$. When $p = 1$, system (1.1) is strictly hyperbolic.

(H3) There are no zero eigenvalues, that is, there exists an $m \in \{1, \cdots, n\}$ and a constant $c > 0$, such that

$$\lambda_m(u) < -c < 0 < c < \lambda_{m+1}(u), \quad \forall u \in B_r(0). \tag{1.3}$$

Under this assumption, $DG(u)$ is also a non-singular matrix. Without loss of generality, we assume that $1 \leq k < \cdots < k+p \leq m$, i.e., the eigenvalue $\lambda(u)$ is negative. The other situation is similar.

(H4) All negative characteristics are linear degenerate and all positive characteristics are either genuinely nonlinear or linear degenerate in the sense of Lax (see [7, 10]). Recall that the i-th characteristic is linearly degenerate if

$$D\lambda_i(u) \cdot r_i(u) \equiv 0, \quad \forall u \in B_r(0), \tag{1.4}$$

while, the i-th characteristic is genuinely nonlinear if

$$D\lambda_i(u) \cdot r_i(u) \neq 0, \quad \forall u \in B_r(0). \tag{1.5}$$

In fact, the characteristic $\lambda(u)$ with constant multiplicity $p \geq 2$ must be linearly degenerate (see Lemma 2.1).

By (H3), the boundary $x = 0$ and $x = L$ are non-characteristic. We prescribe the following general nonlinear boundary conditions:

$$\begin{aligned} x = 0: &\ b_1(u) = g_1(t), \\ x = L: &\ b_2(u) = g_2(t), \end{aligned}$$

where $g_1 : \mathbb{R}^+ \to \mathbb{R}^{n-m}$, $g_2 : \mathbb{R}^+ \to \mathbb{R}^m$ are given boundary functions and $b_1 \in \mathbf{C^1}(B_r(0); \mathbb{R}^{n-m})$, $b_2 \in \mathbf{C^1}(B_r(0); \mathbb{R}^m)$. In order to guarantee the well-posedness for the forward mixed initial-boundary value problem of system (1.1), we assume the following we assume the following (H5).

(H5) b_1 and b_2 satisfy the following conditions, respectively (see [10]):

$$\left.\begin{aligned} &\det[Db_1(u) \cdot r_{m+1}(u) \mid \cdots \mid Db_1(u) \cdot r_n(u)] \neq 0, \\ &\det[Db_2(u) \cdot r_1(u) \mid \cdots \mid Db_2(u) \cdot r_m(u)] \neq 0, \end{aligned}\right. \quad \forall u \in B_r(0). \tag{1.6}$$

Without loss of generality, we may assume that $b_i(0) = 0$ $(i = 1, 2)$. Here the value of $u(t, 0)$ and $u(t, L)$ should be understood as the inner trace of the function $u(t, x)$ on the boundary $x = 0$ and $x = L$, respectively.

Thus, the mixed initial-boundary value problem can be written as

$$\begin{cases} \partial_t H(u) + \partial_x G(u) = 0, & t > 0,\ 0 < x < L, \\ t = 0: \ u = \overline{u}(x), & 0 < x < L, \\ x = 0: \ b_1(u) = g_1(t), & t > 0, \\ x = L: \ b_2(u) = g_2(t), & t > 0. \end{cases} \tag{1.7}$$

The entropy condition can be defined the same as in [9]. We recall that a continuously differentiable convex function $\eta(u) : \mathbb{R}^n \to \mathbb{R}$ is called a convex entropy of system (1.1), with an entropy flux $\zeta(u) : \mathbb{R}^n \to \mathbb{R}$, if we have

$$D\eta(u)(DH(u))^{-1}DG(u) = D\zeta(u). \tag{1.8}$$

Definition 1.1 *For any given $T > 0$, $u = u(t,x) \in \mathbb{L}^1((0,T)\times(0,L))$ is an entropy solution to system* (1.1) *on the domain $\mathbb{D} := \{\ 0 < t < T,\ 0 < x < L\}$ if*

(1) *u is a weak solution to* (1.1) *on the domain $\mathbb{D}$ in the sense of distributions, that is, for every $\phi \in C_c^1(\mathbb{D})$, we have*

$$\int_0^T \int_0^L [\partial_t\phi(t,x)H(u(t,x)) + \partial_x\phi(t,x)G(u(t,x))]\mathrm{d}x\mathrm{d}t = 0. \tag{1.9}$$

(2) *u is entropy admissible in the sense that there exists a convex entropy $\eta(u)$ with entropy flux $\zeta(u)$ for system* (1.1), *such that for every non-negative function $\phi \in C_c^1(\mathbb{D})$, we have*

$$\int_0^T \int_0^L [\partial_t\phi(t,x)\eta(u(t,x)) + \partial_x\phi(t,x)q(u(t,x))]\mathrm{d}x\mathrm{d}t \geq 0. \tag{1.10}$$

Moreover, if u also satisfies the following initial-boundary conditions:

(3) *for a.e. $x \in (0,L)$, $\lim\limits_{t\to 0+} u(t,x) = \overline{u}(x)$ and*

$$\lim_{x\to 0+} b_1(u(t,x)) = g_1(t), \quad \lim_{x\to L-} b_2(u(t,x)) = g_2(t), \quad a.e.\ t \in (0,T),$$

then we say that u is an entropy solution to the mixed initial-boundary value problem (1.7) *on the domain $\mathbb{D}$.*

For a class of strictly hyperbolic systems of conservation laws which satisfy assumption (H3)–(H5), we obtained in [9] the local one-sided exact boundary null controllability of entropy solutions, by means of a similar constructive method (with essential modifications) proposed in [11] in the framework of classical solutions. Since there are also physical models of conservation laws which are not strictly hyperbolic but with characteristics with constant multiplicity (see [5]), it is worthwhile to study the boundary controllability problem for this kind of systems. Following a similar strategy, we can generalize the corresponding controllability results to a class of non-strictly hyperbolic systems of conservation laws with characteristics with constant multiplicity. More precisely, we have the following theorem.

Theorem 1.1 *Let system* (1.1) *and $b_i(u)$ $(i = 1,2)$ satisfy assumptions* (H1)–(H5). *Assume that system* (1.1) *possesses a convex entropy $\eta(u)$ together with an entropy flux $\zeta(u)$. Let*

$$T > L\Big\{\frac{1}{|\lambda_m(0)|} + \frac{1}{\lambda_{m+1}(0)}\Big\}. \tag{1.11}$$

Then, for any given initial data $\overline{u} \in BV(0,L)$ with $\underset{0<x<L}{\mathrm{Tot.Var.}}(\overline{u}) + |\overline{u}(0+)|$ sufficiently small, there exists a boundary control $g_2 \in BV(0,T)$ with $\underset{0<t<T}{\mathrm{Tot.Var.}}(g_2) + |g_2(0+)|$ sufficiently small, acting on the boundary $x = L$, such that system (1.1) *together with the initial condition*

$$t = 0:\ u = \overline{u}, \quad x \in (0,L) \tag{1.12}$$

and the boundary conditions

$$\begin{cases} x = 0: \ b_1(u) = 0, \\ x = L: \ b_2(u) = g_2(t), \end{cases} \quad t \in (0, T) \tag{1.13}$$

admits an entropy solution $u = u(t, x)$ *on the domain* $\{0 < t < T,\ 0 < x < L\}$*, that satisfies*

$$t = T: \quad u \equiv 0, \quad \forall x \in (0, L). \tag{1.14}$$

As in [9], throughout this paper, the solution to the mixed initial-boundary value problem (1.7) means the limit of a convergent sequence of corresponding ε-approximate front tracking solutions. This kind of solution is actually an entropy solution, provided that the system possesses a convex entropy.

The proof of Theorem 1.1 is mainly based on the following three basic ingredients: The well-posedness of semi-global solutions as the limits of ε-approximate solutions to the mixed initial-boundary value problem; the solution to the forward mixed problem of system (1.1) is also a solution to the corresponding rightward mixed problem of the system; the determinate domain of solutions to the one-sided rightward mixed initial-boundary value problem of the system. We will show that all these facts are also valid for a class of non-strictly hyperbolic systems considered in this paper.

The paper is organized as follows. In Section 2, we give all the results about semi-global solutions to the mixed problem (1.7), which are needed for proving Theorem 1.1. Since these results were proved for the strictly hyperbolic case in [9], we only need to add a supplementary discussion associated with the characteristic with constant multiplicity $p \geq 2$. In Section 3, we give the proof of Theorem 1.1, following the main strategy in [9].

2 Semi-global Solutions

Consider the general hyperbolic system of conservation laws

$$\partial_t H(u) + \partial_x G(u) = 0, \quad t > 0,\ 0 < x < L \tag{2.1}$$

with the following initial and boundary conditions:

$$\begin{cases} t = 0: \ u = \overline{u}(x), & 0 < x < L, \\ x = 0: \ b_1(u) = g_1(t), & t > 0, \\ x = L: \ b_2(u) = g_2(t), & t > 0. \end{cases} \tag{2.2}$$

Throughout this paper, in order to avoid abusively using constants, we denote by the notation C a positive constant which depends only on system (2.1), constant L and functions b_1, b_2, but is independent of the special choice of initial data $\overline{u}$, boundary data g_1, g_2 and time T. Moreover, we denote by $C(T)$ a positive constant which depends also on time T.

All results in this section (except those in Subsection 2.3) hold for more general systems whose characteristic families are either genuinely nonlinear or linearly degenerate.

2.1 Riemann problem

The basic building block for constructing solutions to system (2.1) is the solution to Riemann problems, i.e., the initial value problem and the one-sided mixed problem with piecewise constant data with a single jump. First we consider the Riemann initial value problem at a point (τ, ξ):

$$\begin{cases} \partial_t H(u) + \partial_x G(u) = 0, \\ t = \tau: \ u = \begin{cases} u^{\mathrm{L}}, & \text{if } x < \xi, \\ u^{\mathrm{R}}, & \text{if } x > \xi, \end{cases} \end{cases} \tag{2.3}$$

where $u^{\mathrm{L}}, u^{\mathrm{R}} \in B_r(0)$.

We normalize the left and right eigenvectors $l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ of $(DH)^{-1}DG(u)$, so that

$$l_i(u) \cdot r_j(u) \equiv \delta_{ij}, \quad i, j = 1, \cdots, n,$$

where δ_{ij} is the Kronecker symbol.

For two states $\omega, \omega' \in \mathbb{R}^n$, let

$$A(\omega, \omega') = \int_0^1 [DH](\theta\omega + (1-\theta)\omega')\mathrm{d}\theta \tag{2.4}$$

and

$$B(\omega, \omega') = \int_0^1 [DG](\theta\omega + (1-\theta)\omega')\mathrm{d}\theta. \tag{2.5}$$

We have

$$H(\omega') - H(\omega) = A(\omega, \omega')(\omega' - \omega), \quad G(\omega') - G(\omega) = B(\omega, \omega')(\omega' - \omega).$$

By hyperbolicity, for two ω and ω' sufficiently close to the origin, we denote by $\lambda_i(\omega, \omega')$ the i-th real eigenvalue of the matrix $A^{-1}(\omega, \omega')B(\omega, \omega')$ (see [10]).

For any simple eigenvalue $\lambda_i(u)$ and for any given $u \in B_r(0)$, let $\sigma \mapsto R_i(\sigma)[u]$ denote the i-rarefaction curve passing through u for $\sigma \in [-\sigma_0, \sigma_0]$ with σ_0 sufficiently small and let $\sigma \mapsto S_i(\sigma)[u]$ denote the i-shock curve passing through u for $\sigma \in [-\sigma_0, \sigma_0]$ with σ_0 sufficiently small (see [9]).

If the i-th characteristic is linearly degenerate, we choose $r_i(u)$ to have the unit length, and we let the coinciding i-rarefaction curve and i-shock curve be parameterized by the arc-length. If the i-th characteristic is genuinely nonlinear, we choose $r_i(u)$ such that $\nabla\lambda_i(u) \cdot r_i(u) \equiv 1$, and we let the i-rarefaction curve and the i-shock curve be parameterized in such a way that

$$\lambda_i(R_i(\sigma)[u]) - \lambda_i(u) = \sigma \quad \text{and} \quad \lambda_i(S_i(\sigma)[u]) - \lambda_i(u) = \sigma,$$

respectively. This parametrization leads to a useful property:

$$u = S_i(-\sigma)S_i(\sigma)[u] \quad \text{for all } \sigma \in [-\sigma_0, \sigma_0], \ u \in B_r(0), \quad i \in \{1, \cdots, n\}. \tag{2.6}$$

With this parametrization, a straightforward computation shows that the composite function

$$\Psi_i(\sigma)[u] = \begin{cases} R_i(\sigma)[u], & \text{if } \sigma > 0, \\ S_i(\sigma)[u], & \text{if } \sigma < 0, \end{cases}$$

which is smooth for $\sigma \neq 0$ and of class C^2 at $\sigma = 0$, is called the i-th elementary wave curve.

Suppose that $u^{\mathrm{R}} = \Psi_i(\sigma)[u^{\mathrm{L}}]$ for some $\sigma \in [-\sigma_0, \sigma]$ with $i \in \{1, \cdots, k, k+p+1, \cdots, n\}$, i.e. the i-th eigenvalue is simple. In this case, when the i-th characteristic is genuinely nonlinear with $\sigma > 0$, the solution u_i is an i-rarefaction wave; when the i-th characteristic is genuinely nonlinear with $\sigma < 0$ or the i-th characteristic is linearly degenerate, the solution u_i is an i-shock wave or an i-contact discontinuity, respectively.

For the characteristic with constant multiplicity $p \geq 2$, one has the following lemma.

Lemma 2.1 (see [5]) *The characteristic with constant multiplicity $p \geq 2$ must be linearly degenerate, that is,*

$$\nabla\lambda(u) \cdot r_j(u) \equiv 0, \quad j = k+1, \cdots, k+p, \ \forall u \in B_r(0).$$

Moreover, for any $u^- \in B_r(0)$, there exists a p-dimensional connected smooth manifold $\Sigma(u^-)$ in a neighborhood of u^- with $u^- \in \Sigma(u^-)$, where $\Sigma(u^-)$ can be expressed by the following smooth parametric representation:

$$u = \Psi_{k+1}(\sigma_{k+p}, \cdots, \sigma_{k+1})[u^-], \quad \sigma_j \in [-\sigma_0, \sigma_0], \quad j = k+1, \cdots, k+p$$

for some small σ_0, such that

$$\frac{\partial}{\partial \sigma_j} u(0, \cdots, 0)[u^-] = r_j, \quad j = k+1, \cdots, k+p.$$

In other words, for any $u^+ \in \Sigma(u^-)$, there exist uniquely small numbers $\sigma_{k+1}, \cdots, \sigma_{k+p}$ such that $u^+ = \Psi_{k+1}(\sigma_{k+p}, \cdots, \sigma_{k+1})[u^-]$, and the entropy solution u_{k+1} to the Riemann problem (2.3) *with initial data $[u^{\mathrm{L}} = u^-, u^{\mathrm{R}} = u^+]$ is always a contact discontinuity, that is,*

$$u_{k+1} = \begin{cases} u^+, & x > st, \\ u^-, & x < st, \end{cases}$$

where $s = \lambda(u^-) = \lambda(u^+)$.

The following lemma gives the existence of entropy solutions to Riemann initial value problem (see [8]).

Lemma 2.2 *For the Riemann initial value problem* (2.3) *with general initial data $u^{\mathrm{L}}, u^{\mathrm{R}} \in B_r(0)$, under assumptions* (H1)–(H3)*, there exist n uniquely determined small numbers $\sigma_1, \cdots, \sigma_n$ such that*

$$\begin{aligned} u^{\mathrm{R}} = {} & \Psi_n(\sigma_n) \circ \cdots \circ \Psi_{k+p+1}(\sigma_{k+p+1}) \\ & \circ \Psi_{k+1}(\sigma_{k+p}, \cdots, \sigma_{k+1}) \circ \Psi_k(\sigma_k) \circ \cdots \circ \Psi_1(\sigma_1)[u^{\mathrm{L}}]. \end{aligned}$$

Let

$$\omega_0 = u^{\mathrm{L}}, \quad \omega_{k+1} = \Psi_{k+1}[\omega_k],$$
$$\omega_i = \Psi_i(\sigma_i)[\omega_{i-1}], \quad i \in \{1, \cdots, k, k+p+1, \cdots, n\}.$$

One can choose intermediate speeds

$$-\infty = \widehat{\lambda}_0 < \widehat{\lambda}_1 < \cdots < \widehat{\lambda}_k < \widehat{\lambda}_{k+1} = \cdots = \widehat{\lambda}_{k+p} < \widehat{\lambda}_{k+p+1} < \cdots < \widehat{\lambda}_{n-1} < \widehat{\lambda}_n = \infty,$$

such that for each $i \in \{1, \cdots, k+1, k+p+1, \cdots, n\}$, *the speed of waves for the solution* u_i *to the elementary Riemann initial value problem with the initial data*

$$u(\tau, x) = \begin{cases} \omega_{i-1}, & \text{if } x < \xi, \\ \omega_i, & \text{if } x > \xi \end{cases}$$

is contained in the interval $(\widehat{\lambda}_{i-1}, \widehat{\lambda}_i)$. *Then the solution* $u = u(t,x)$ *to the Riemann initial value problem* (2.3) *can be constructed by piecing together these solutions* $\{u_i\}$, *that is,*

$$u(t,x) = u_i(t,x) \quad \text{for} \quad \widehat{\lambda}_{i-1} < \frac{x}{t} < \widehat{\lambda}_i, \quad i = 1, \cdots, k+1, k+p+1, \cdots, n.$$

Consider now the left-sided Riemann mixed problem at a point $(\tau, 0)$, viz.,

$$\begin{cases} \partial_t H(u) + \partial_x G(u) = 0, & t > \tau, \ x > 0, \\ t = \tau: \ u = \overline{u}, & x > 0, \\ x = 0: \ b_1(u) = \overline{g}_1, & t > \tau \end{cases} \tag{2.7}$$

and the right-sided Riemann mixed problem at a point (τ, L), viz.,

$$\begin{cases} \partial_t H(u) + \partial_x G(u) = 0, & t > \tau, \ x < L, \\ t = \tau: \ u = \overline{u}, & x < L, \\ x = L: \ b_2(u) = \overline{g}_2, & t > \tau, \end{cases} \tag{2.8}$$

respectively, where $\overline{u}$ and $\overline{g}_i$ $(i = 1, 2)$ are constants with $|\overline{u}|$ and $|\overline{g_i}|$ $(i = 1, 2)$ sufficiently small.

In a similar way to the strictly hyperbolic case (see [1, 2]), one can prove the following lemma.

Lemma 2.3 *Let* $b_1 : \Omega \to \mathbb{R}^{n-m}$ *and* $b_2 : \Omega \to \mathbb{R}^m$ *be* $\mathbf{C^1}$*-maps satisfying assumption* (H5), *and let* $\overline{g}_1 \in \mathbb{R}^{n-m}$, $\overline{g}_2 \in \mathbb{R}^m$. *Then, for any given* $\overline{u} \in B_r(0)$, *there exists a positive constant* δ^* *with the following property: If* $|\overline{g}_1 - b_1(\overline{u})| \le \delta^*$, *then there is a unique choice of* $(n-m)$ *small numbers* $\sigma_{m+1}, \cdots, \sigma_n$ *such that* $b_1(\Psi_{m+1}(\sigma_{m+1}) \circ \cdots \circ \Psi_n(\sigma_n)[\overline{u}]) = \overline{g}_1$; *Similarly, if* $|\overline{g}_2 - b_2(\overline{u})| \le \delta^*$, *then there is a unique choice of* m *small numbers* $\sigma_1, \cdots, \sigma_m$ *such that* $b_2(\Psi_m(\sigma_m) \circ \cdots \circ \Psi_{q+1}(\sigma_{k+p+1}) \circ \Psi_{k+1}(\sigma_{k+p}, \cdots, \sigma_{k+1}) \circ \Psi_k(\sigma_k) \circ \cdots \circ \Psi_1(\sigma_1)[\overline{u}]) = \overline{g}_2$.

Therefore, by property (2.6), the solution to problem (2.7) on the domain $\{t > \tau, \ x > 0\}$ coincides with the solution to the Riemann initial value problem at $(\tau, 0)$ with initial states $u^{\mathrm{L}} = \Psi_{m+1}(\sigma_{m+1}) \circ \cdots \circ \Psi_n(\sigma_n)[\overline{u}]$, $u^{\mathrm{R}} = \overline{u}$. While, on the domain $\{t > \tau, \ x < L\}$, the solution to problem (2.8) coincides with the solution to the Riemann initial value problem at

(τ, L) with the initial data $u^{\mathrm{L}} = \overline{u},\ u^{\mathrm{R}} = \Psi_m(\sigma_m)\circ\cdots\circ\Psi_{k+p+1}(\sigma_{k+p+1})\circ\Psi_{k+1}(\sigma_{k+p}, \cdots, \sigma_{k+1})\circ \Psi_k(\sigma_k)\circ\cdots\circ\Psi_1(\sigma_1)[\overline{u}]$.

Following the fact that Ψ_i is smooth with respect to σ_i for $i \in \{1, \cdots, k, k+p+1, \cdots, n\}$ and Ψ_{k+1} is smooth with respect to $\sigma_{k+1}, \cdots, \sigma_{k+p}$, the next two lemmas on estimates for boundary interaction of fronts can be obtained by similar proofs as in [4] for the strictly hyperbolic case.

Lemma 2.4 *Suppose that b_1 and b_2 satisfy assumption* (H5). *If*

$$\begin{aligned} u^- &= \Psi_m(\sigma_m)\circ\cdots\circ\Psi_{k+p+1}(\sigma_{k+p+1}) \\ &\quad \circ\Psi_{k+1}(\sigma_{k+p}, \cdots, \sigma_{k+1})\circ\Psi_k(\sigma_k)\circ\cdots\circ\Psi_1(\sigma_1)[\overline{u}], \\ \overline{u} &= \Psi_n(\widetilde{\sigma}_n)\circ\cdots\circ\Psi_{m+1}(\widetilde{\sigma}_{m+1})[u^+] \end{aligned}$$

with $\overline{u}, u^-, u^+ \in B_r(0)$, then

$$\sum_{i=m+1}^{n} |\widetilde{\sigma}_i| \le C\Big(\sum_{i=1}^{m} |\sigma_i| + |b_1(u^+) - b_1(u^-)|\Big).$$

Similarly, if

$$\begin{aligned} \overline{u} &= \Psi_n(\sigma_n)\circ\cdots\circ\Psi_{m+1}(\sigma_{m+1})[u^-], \\ u^+ &= \Psi_m(\widetilde{\sigma}_m)\circ\cdots\circ\Psi_{k+p+1}(\widetilde{\sigma}_{k+p+1}) \\ &\quad \circ\Psi_{k+1}(\widetilde{\sigma}_{k+p}, \cdots, \widetilde{\sigma}_{k+1})\circ\Psi_k(\widetilde{\sigma}_k)\circ\cdots\circ\Psi_1(\widetilde{\sigma}_1)[\overline{u}], \end{aligned}$$

then

$$\sum_{i=1}^{m} |\widetilde{\sigma}_i| \le C\Big(\sum_{i=m+1}^{n} |\sigma_i| + |b_2(u^+) - b_2(u^-)|\Big).$$

Lemma 2.5 *Suppose that b_1 and b_2 satisfy assumption* (H5). *If $u^*, v^* \in B_r(0)$ and there exist small numbers $q_1^*, \cdots, q_n^*$ such that*

$$\begin{aligned} v^* &= S_n(q_n^*)\circ\cdots\circ S_{k+p+1}(q_{k+p+1}^*) \\ &\quad \circ\Psi_{k+1}(q_{k+p}^*, \cdots, q_{k+1}^*)\circ S_k(q_k^*)\circ\cdots\circ S_1(q_1^*)[u^*], \end{aligned}$$

then we have

$$\sum_{j=m+1}^{n} |q_j^*| \le C\Big(\sum_{i=1}^{m} ||q_i^*| + |b_1(u^*) - b_1(v^*)|\Big)$$

and

$$\sum_{i=1}^{m} |q_i^*| \le C\Big(\sum_{j=m+1}^{n} |q_j^*| + |b_2(u^*) - b_2(v^*)|\Big).$$

2.2 Solution as the limit of ε-approximate front tracking solutions

We first give the definition of ε-approximate front tracking solutions. The only difference with the one given in [9] is that now there are fronts corresponding to the characteristics with constant multiplicity $p \ge 2$.

Definition 2.1 *For any given time $T>0$ and any fixed $\varepsilon>0$, we say that a continuous map*

$$t \mapsto u^\varepsilon(t,\cdot) \in \mathbb{L}^1(0,L), \quad \forall t \in (0,T)$$

is an ε-approximate front tracking solution to system (2.1) *if*

(1) *$u^\varepsilon = u^\varepsilon(t,x) \in B_r(0)$ for all $(t,x) \in \overline{\mathbb{D}} := \{0 \le t \le T,\ 0 \le x \le L\}$, and, as a function of two variables, it is piecewise constant with discontinuities occurring along finitely many straight lines with non-zero slope in the domain $\overline{\mathbb{D}}$. Jumps can be of two types: physical fronts* (*shocks, contact discontinuities or rarefaction fronts*) *and non-physical fronts, denoted by $\mathcal{P}$ and $\mathcal{NP}$, respectively.*

(2) *Along each physical front $x = x_\alpha(t)$ $(\alpha \in \mathcal{P})$, the left and right limits of $u^\varepsilon(t,\cdot)$ on it are selected by*

$$\begin{aligned} &u^{\mathrm{R}} = \Psi_{k_\alpha}(\sigma_\alpha)[u^{\mathrm{L}}], \quad \text{if } k_\alpha \in \{1,\cdots,k,k+p+1,\cdots,n\},\\ &u^{\mathrm{R}} = \Psi_{k_\alpha}(\sigma_{\alpha,p},\cdots,\sigma_{\alpha,1})[u^{\mathrm{L}}], \quad \text{if } k_\alpha = k+1, \end{aligned}$$

where $u^{\mathrm{L}} := u^\varepsilon(t,x_\alpha(t)-)$, $u^{\mathrm{R}} := u^\varepsilon(t,x_\alpha(t)+)$, and σ_α or $(\sigma_{\alpha,p},\cdots,\sigma_{\alpha,1})$ denotes the corresponding wave amplitude. Moreover, if the k_α-th characteristic is simple and genuinely nonlinear with $\sigma_\alpha < 0$, i.e., the front is a shock, then

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^{\mathrm{L}},u^{\mathrm{R}})| \le C\varepsilon.$$

If the k_α-th characteristic is linear degenerate (*no matter it is simple or not*), *i.e., the front is a contact discontinuity, then*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^{\mathrm{L}})| \le C\varepsilon.$$

If the k_α-th characteristic family is genuinely nonlinear with $0 < \sigma_\alpha \le \varepsilon$, i.e., the front is a rarefaction front, then

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^{\mathrm{R}})| \le C\varepsilon.$$

(3) *All non-physical fronts $x = x_\alpha(t)$ $(\alpha \in \mathcal{NP})$ have the constant speed $\dot{x}_\alpha \equiv \widehat{\lambda}$ with either $\widehat{\lambda} > \max\limits_{1\le i\le n} \sup\limits_{u\in B_r(0)} |\lambda_i(u)|$ or $0 < \widehat{\lambda} < c$, where c is given by* (1.3)*. Moreover, the total amplitude of all non-physical waves in $u^\varepsilon(t,\cdot)$ is uniformly bounded by ε, i.e.,*

$$\sum_{\alpha\in\mathcal{NP}} |u^\varepsilon(t,x_\alpha+) - u^\varepsilon(t,x_\alpha-)| \le \varepsilon, \quad \forall t \in (0,T).$$

In addition, if the initial-boundary values of u^ε satisfy approximatively the initial-boundary conditions (2.2)*, namely, if*

$$\begin{aligned} &\|u^\varepsilon(0,\cdot) - \overline{u}\|_{\mathbb{L}^1(0,L)} \le \varepsilon,\\ &\|b_1(u^\varepsilon(\cdot,0+)) - g_1\|_{\mathbb{L}^1(0,T)} \le \varepsilon, \quad \|b_2(u^\varepsilon(\cdot,L-)) - g_2\|_{\mathbb{L}^1(0,T)} \le \varepsilon, \end{aligned}$$

then $u^\varepsilon = u^\varepsilon(t,x)$ is called the ε-approximate front tracking solution to the initial-boundary value problem (2.1)–(2.2)*. For brevity, the ε-approximate front tracking solution will be called the ε-solution in what follows.*

The construction of ε-solutions was given in [9] in the strictly hyperbolic case (i.e. $p = 1$). The algorithm can be roughly described as follows: We first choose a piecewise constant vector function $(\overline{u}^\varepsilon, g_1^\varepsilon, g_2^\varepsilon)$ which is a good approximation to the given initial-boundary data $(\overline{u}, g_1, g_2)$. At the time $t = 0$, we approximately solve the Riemann initial value problem or the Riemann initial-boundary value problem at each jump point, such that each rarefaction wave of the solution to the corresponding Riemann problem is replaced by an approximate wave consisting of several fronts with wave amplitude smaller than ε, while the shock or the contact discontinuity is not modified at all. When these fronts interact at some points, we approximately solve again the Riemann problem at these interaction points. In order to avoid the infinitely increment of fronts number, we apply three different approximate Riemann solvers (accurate Riemann solver, simplified Riemann solver and crude Riemann solver) according to specific rules. We repeat this process and the algorithm can be extended up to a given time T, provided that the total amplitude of waves is sufficiently small and the total number of fronts is finite.

For system (1.1) with $\lambda(u)$ with constant multiplicity p (≥ 2), by Lemma 2.1, the wave corresponding to $\lambda(u)$ is always a contact discontinuity, then we treat it as the same as the contact discontinuities or shocks corresponding to simple characteristics. The only difference is that its wave amplitude is now a vector with p components.

In order to prove that the total amplitude of waves is sufficiently small, we need to define the linear Glimm functional V^ε and nonlinear Glimm functional Q^ε similar to those in [9]. For notational convenience, for a front α corresponding to a simple characterisitc with amplitude σ_α or a front corresponding to $\lambda(u)$ with amplitude $(\sigma_{\alpha,k+1}, \cdots, \sigma_{\alpha,k+p})$, we write

$$\widehat{\sigma}_\alpha = \begin{cases} \sigma_\alpha, & \text{if } \alpha \text{ is a front corresponding to a simple characterisitic,} \\ (\sigma_{\alpha,k+1}, \cdots, \sigma_{\alpha,k+p}), & \text{if } \alpha \text{ is a front corresponding to } \lambda(u). \end{cases}$$

The absolute value $|\widehat{\sigma}_\alpha|$ of wave corresponding to $\lambda(u)$ is

$$|\widehat{\sigma}_\alpha| = \sum_{j=1}^{p} |\sigma_{\alpha,k+j}|.$$

With this notation, we define Glimm-type functionals $V^\varepsilon(t)$ and $Q^\varepsilon(t)$ as follows:

$$V^\varepsilon(t) := \sum_\alpha K_\alpha |\widehat{\sigma}_\alpha(t)| + C_1 \sum_{i=1,2} \underset{t<s<T}{\text{Tot. Var.}} (g_i^\varepsilon(s)),$$

where the first sum takes over all fronts α across the segment $\{t\} \times (0, L)$ and C_1 is a positive constant which can be specified as in [9].

$$Q^\varepsilon(t) := \sum_{(\alpha,\beta)\in\mathcal{A}} |\widehat{\sigma}_\alpha(t)||\widehat{\sigma}_\beta(t)|$$

measures the wave interaction potential, where $\mathcal{A}$ is the set of all approaching waves (see [3, 9]).

It is easy to see that all of the interaction estimates (see [9, Lemmas 5.6–5.8]) in [9]) are valid in our case by replacing $|\sigma_\alpha|$ with $|\widehat{\sigma}_\alpha|$. Then we can similarly use the argument in [9] to get the same results on semi-global solutions to the mixed initial-boundary value problem (1.7) satisfying assumptions (H1)–(H5), by means of the new Glimm-type functionals V^ε and Q^ε. In fact, we have the following existence results of ε-solutions: For any given $T>0$, any given initial-boundary data $(\overline{u}, g_1, g_2)$ and any given $\varepsilon>0$ small enough, if $\Lambda(\overline{u}, g_1, g_2)$ is sufficiently small, where

$$\begin{aligned}\Lambda(\overline{u}, g_1, g_2) := \underset{0<x<L}{\mathrm{Tot.Var.}}\,(\overline{u}) + |\overline{u}(0+)| + \sum_{i=1,2} \underset{0<t<T}{\mathrm{Tot.Var.}}\,(g_i) \\ + |b_1(\overline{u}(0+)) - g_1(0+)| + |b_2(\overline{u}(L-)) - g_2(0+)|,\end{aligned}$$

we can construct an ε-solution to problem (2.1)–(2.2) on the domain $\mathbb{D}$ via an algorithm given in [9], such that for all $\varepsilon>0$ small, the maps $t \mapsto u^\varepsilon(t,\cdot)$ are uniformly Lipschitz continuous in $\mathbb{L}^1$ norm with respect to t and $\underset{0<x<L}{\mathrm{Tot.Var.}}\,(u^\varepsilon(t,\cdot))$ remains sufficiently small uniformly for all $t \in (0,T)$.

Moreover, as in [9], we can prove that the approximate stability holds for ε-solutions u^ε and v^ε on the triangle domains

$$\mathfrak{L}(x_1) := \Big\{(t,x) \ \Big| \ 0<t<\widehat{\tau}_1(x_1),\ 0<x<\frac{x_1(\widehat{\tau}_1(x_1)-t)}{\widehat{\tau}_1(x_1)}\Big\}$$

and

$$\mathfrak{R}(x_0) := \Big\{(t,x) \ \Big| \ 0<t<\widehat{\tau}_2(x_0),\ \frac{(L-x_0)t}{\widehat{\tau}_2(x_0)} + x_0 < x < L\Big\}$$

for any given $x_1 \in (0,L]$ and $x_0 \in [0,L)$, where

$$\widehat{\tau}_1(x_1) = x_1 \min_{u\in B_r(0)} \{|\lambda_1(u)|^{-1}\} \quad \text{and} \quad \widehat{\tau}_2(x_0) = (L-x_0) \min_{u \in B_r(0)} \{\lambda_n(u)^{-1}\}.$$

In fact, at each point $(t,x) \in \mathfrak{L}(x_1) \cup \mathfrak{R}(x_0)$, we define the vector function $\widehat{q} = (q_1, .., q_n)$ implicitly by

$$\begin{aligned}v^\varepsilon(t,x) = S_n(q_n) \circ \cdots \circ S_{k+p+1}(q_{k+p+1}) \\ \circ\, \Psi_{k+1}(q_{k+p}, \cdots, q_{k+1}) \circ S_k(q_k) \circ \cdots \circ S_1(q_1)[u^\varepsilon(t,x)].\end{aligned}$$

Then we can define functional $\Gamma(u^\varepsilon, v^\varepsilon)(t)$ measuring the distance between $u^\varepsilon(t,\cdot)$ and $v^\varepsilon(t,\cdot)$ in the same way as in [9]. Combining the standard argument in [4] and the stability results in [6] for the Cauchy problem, we can get the same results of the approximate stability on $\mathfrak{L}(x_1)$ and $\mathfrak{R}(x_0)$ in our case with characteristics with constant multiplicity $p \geq 2$. By induction, we can obtain the approximate stability of ε-solutions on the domain $\mathbb{D}$.

Now, fix a sequence $\varepsilon^\nu \searrow 0$ as $\nu \to +\infty$. By Helly's theorem (see [3, Theorem 2.3]), we can extract a subsequence of $\{u^\nu\}$, which converges to a limit function $u = u(t,x)$ in $\mathbb{L}^1((0,T)\times(0,L))$. In fact, we have the following proposition.

Proposition 2.1 *For any fixed $T>0$, there exist positive constants δ and $C(T)$ such that for every initial-boundary data $(\overline{u}, g_1^u, g_2^u)$ with*

$$\Lambda(\overline{u}, g_1^u, g_2^u) \leq \delta,$$

problem (2.1)–(2.2) *associated with the initial-boundary data* $(\overline{u}, g_1^u, g_2^u)$ *admits a solution* $u = u(t,x)$ *on the domain* $\mathbb{D} = \{0 < t < T,\ 0 < x < L\}$ *as the limit of a sequence of* ε*-solutions, satisfying*

$$\begin{aligned}&\operatorname*{Tot.Var.}_{0<x<L}(u(t,\cdot)) \le C(T)\Lambda(\overline{u}, g_1, g_2), \quad \forall t \in (0,T),\\ &\|u(t,\cdot) - u(s,\cdot)\|_{\mathbb{L}^1(0,L)} \le C(T)|t-s|, \quad \forall t,s \in (0,T)\end{aligned}$$

and $u(t,x) \in B_r(0)$ *for a.e.* $(t,x) \in \mathbb{D}$.

Moreover, if $v = v(t,x)$ *is a solution as the limit of a sequence of* ε^ν*-solutions of system* (2.1), *associated with the initial-boundary data* $(\overline{v}, g_1^v, g_2^v)$ *with* $\Lambda(\overline{v}, g_1^v, g_2^v) \le \delta$, *then for any given* $x_0 \in [0,L)$ *and* $x_1 \in (0,L]$, *there exists a positive constant* C *independent of* x_0 *and* x_1, *such that*

$$\begin{aligned}&\|u(t,\cdot) - v(t,\cdot)\|_{\mathbb{L}^1(\mathfrak{L}_t(x_1))}\\ &\le C\Big(\|\overline{u} - \overline{v}\|_{\mathbb{L}^1(0,x_1)} + \int_0^t |g_1^u(s) - g_1^v(s)\Big)|\mathrm{d}s\Big), \quad \forall t \in [0, \widehat{\tau}_1(x_1)],\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\|u(t,\cdot) - v(t,\cdot)\|_{\mathbb{L}^1(\mathfrak{R}_t(x_0))}\\ &\le C\Big(\|\overline{u} - \overline{v}\|_{\mathbb{L}^1(x_0,L)} + \int_0^t |g_2^u(s) - g_2^v(s)|\mathrm{d}s\Big), \quad \forall t \in [0, \widehat{\tau}_2(x_0)],\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}\mathfrak{L}_t(x_1) &:= \Big\{x \;\Big|\; 0 < x < \frac{x_1(\widehat{\tau}_1(x_1) - t)}{\widehat{\tau}_1(x_1)}\Big\},\\ \mathfrak{R}_t(x_0) &:= \Big\{x \;\Big|\; \frac{(L - x_0)t}{\widehat{\tau}_2(x_0)} + x_0 < x < L\Big\},\end{aligned}$$

and there exists a positive constant $C(T)$ *depending on time* T, *such that*

$$\begin{aligned}&\|u(t,\cdot) - v(t,\cdot)\|_{\mathbb{L}^1(0,L)}\\ &\le C(T)\Big(\|\overline{u} - \overline{v}\|_{\mathbb{L}^1(0,L)} + \sum_{i=1,2}\int_0^t |g_i^u(s) - g_i^v(s)|\mathrm{d}s\Big), \quad \forall t \in (0,T).\end{aligned} \tag{2.11}$$

In particular, (2.11) implies that the solution provided by Proposition 2.1 is independent of different choices of the convergent sequence of ε-solutions.

Remark 2.1 Under the assumption that system (2.1) possesses a convex entropy $\eta(u)$, the solution $u = u(t,x)$ given by Proposition 2.1 is actually an entropy solution to the problem (2.1)–(2.2) on the domain $\mathbb{D}$ (see [3, Section 7.4]).

Remark 2.2 According to (2.9) (resp. (2.10)), the triangle domain $\mathfrak{L}(x_1)$ (resp. $\mathfrak{R}(x_0)$) is the determinate domain of the solution to one-sided initial-boundary value problem (2.1) with the initial data on the interval $(0, x_1)$ (resp. (x_0, L)) and the boundary condition on $x = 0$ (resp. $x = L$).

In particular, let $u = u(t,x)$ be the solution to problem (2.1)–(2.2) on the domain $\mathbb{D}$ given by Proposition 2.1, with $\Lambda(\overline{u}, g_1, g_2)$ sufficiently small. For any given $x_0 \in (0,L)$, if $\overline{u} \equiv 0$ on (x_0, L) and $g_2 \equiv 0$ on the interval $(0, \widehat{\tau}_2(x_0))$, then $u \equiv 0$ on the domain $\mathfrak{R}(x_0) \cap \mathbb{D}$.

2.3 Some further properties of ε-approximate front tracking solutions and solutions

Using the same arguments in the proof of Lemma 2.11 in [9], we can prove the following lemma, which asserts that the trace of ε-solutions on the boundary converges to the corresponding trace of the limit solution.

Lemma 2.6 *Suppose that* $\{u^\nu\}$ *is a sequence of* ε^ν*-solutions to the mixed initial-boundary value problem* (2.1)–(2.2)*. Then, up to a subsequence, as* $\nu \to \infty$*, we have*

$$\|u^\nu(\cdot, 0+) - u(\cdot, 0+)\|_{\mathbb{L}^\infty} \to 0,$$
$$\|u^\nu(\cdot, L-) - u(\cdot, L-)\|_{\mathbb{L}^\infty} \to 0.$$

Using the fact that all negative eigenvalues are linearly degenerate, we can prove, in the same way as in [9], that the equivalence between the solution to the forward problem and the solution to the corresponding rightward problem. In fact, by checking the ε-solution to the forward problem satisfying Definition 2.1 in the rightward sense (see Appendix in [9] for the proof), we can obtain the following lemma.

Lemma 2.7 *Let system* (2.1) *satisfy assumptions* (H1)–(H4)*. Suppose that* $u^\varepsilon(t,x)$ *is an* ε*-solution in the forward sense of problem* (2.1)–(2.2) *on the domain* $\mathbb{D} = \{0 < t < T,\ 0 < x < L\}$*. Then, if we exchange the role of* t *and* x*, namely, regard* x *as the "time" variable and* t *as the "space" variable,* u^ε *is also an* ε*-solution in the rightward sense of the system*

$$\partial_x G(u) + \partial_t H(u) = 0 \tag{2.12}$$

on the domain $\mathbb{D}$*.*

Since we can apply Helly's theorem to the ε-solutions in the rightward sense, by passing to the limit, we can prove the corresponding results (see Appendix in [9] for the proofs).

Proposition 2.2 *Under the same assumptions of Lemma* 2.7*, if* u *is a solution to system* (2.1) *in the forward sense, given by Proposition* 2.1*, then* u *is also a solution to system* (2.12) *in the rightward sense. Similar results hold from the solution in the rightward sense to that in the forward sense.*

Proposition 2.3 *Under the same assumptions of Lemma* 2.7*, assume that* $u = u(t,x)$ *is a forward solution to problem* (2.1)–(2.2) *on the domain* $\{0 < t < T_1,\ 0 < x < L\}$ *with* $T_1 \geq L \max\limits_{u \in B_r(0)} \frac{1}{|\lambda_m(u)|}$*, given by Proposition* 2.1*. Then on the triangular domain* $\big\{0 < t < T_1,\ 0 < x < \frac{L(T_1 - t)}{T_1}\big\}$*,* u *coincides with the rightward solution* $\widetilde{u}$ *to system* (2.12)*, given by Proposition* 2.1*, associated with the initial condition*

$$x = 0: \ \widetilde{u} = u(\cdot, 0+)$$

and the following boundary condition corresponding to the original initial data $\overline{u}$*:*

$$t = 0: \ \widetilde{b}_1(\widetilde{u}) = \widetilde{b}_1(\overline{u}),$$

where $\widetilde{b}_1 \in \mathbf{C^1}(B_r(0); \mathbb{R}^{n-m})$ *is an arbitrarily given function that satisfies the same assumption* (1.6) *as* b_1*.*

3 Proof of Theorem 1.1

In the same spirit of [9], we repeatedly apply the well-posedness of semi-global solutions to prove Theorem 1.1, namely, to realize the one-sided local exact boundary null controllability for a class of general hyperbolic systems of conservation laws with characteristics with constant multiplicity.

$$\partial_t H(u) + \partial_x G(u) = 0, \quad t > 0,\ 0 < x < L \tag{3.1}$$

with additional assumption that all negative characteristics are linearly degenerate. The proof here is quite similar to that in [9], so we write it completely here for readers' convenience.

In order to get Theorem 1.1, it suffices to establish the following lemma.

Lemma 3.1 *Under the same assumptions of Theorem* 1.1*, let* $T > 0$ *satisfy* (1.11)*. For any given initial data* $\overline{u}$ *and boundary data* g_1 *with* $\underset{0<x<L}{\mathrm{Tot.Var.}}(\overline{u}) + |\overline{u}(0+)|$ *and* $\underset{0<t<T}{\mathrm{Tot.Var.}}(g_1)+|g(0+)|$ *sufficiently small, system* (1.1) *together with the boundary condition*

$$x = 0:\ b_1(u) = 0, \quad t \in (0, T) \tag{3.2}$$

admits a solution $u = u(t,x)$ *on the domain* $\{0 < t < T,\ 0 < x < L\}$ *with small* $\underset{0<t<T}{\mathrm{Tot.Var.}}(u(\cdot, L-)) + u(0, L-)$*, satisfying simultaneously the initial condition* (1.12) *and the final condition* (1.14)*.*

In fact, let $u = u(t,x)$ be a solution given by Lemma 3.1. Taking the boundary control as

$$g_2(t) := b_2(u(t, L-)), \quad \forall t \in (0, T),$$

which has small amplitude and total variation, we obtain the local exact boundary null controllability desired by Theorem 1.1.

Proof of Lemma 3.1 If (1.11) holds, then for $r > 0$ sufficiently small, we have

$$T > L \max_{u \in B_r(0)} \Big\{ \frac{1}{|\lambda_m(u)|} + \frac{1}{\lambda_{m+1}(u)} \Big\}. \tag{3.3}$$

Step 1 Let

$$T_1 := L \max_{u \in B_r(0)} \frac{1}{|\lambda_m(u)|}. \tag{3.4}$$

Choosing an artificial function g_f with $\underset{0<t<T_1}{\mathrm{Tot.Var.}}(g_f)+|g_f(0+)|$ sufficiently small, we consider the forward problem of system (3.1) with the following initial condition and boundary conditions:

$$\begin{aligned} t = 0: &\quad u = \overline{u}, \\ x = 0: &\quad b_1(u) = 0, \\ x = L: &\quad b_2(u) = g_f. \end{aligned}$$

By Proposition 2.1, there exists a unique solution $u_f = u_f(t,x)$ obtained as the limit of a sequence of ε^ν-solutions $u_f^\nu = u_f^\nu(t,x)$ on the domain $\{0 < t < T_1,\ 0 < x < L\}$ with

$\underset{0<x<L}{\text{Tot. Var.}}\,(u_f(t,\cdot)) + \underset{0<t<T_1}{\text{Tot. Var.}}\,(u_f(\cdot,0+)) + \underset{0<t<T_1}{\text{Tot. Var.}}\,(u_f(\cdot,L-))$ sufficiently small and $u_f(t,x) \in B_r(0)$.

Step 2 Let

$$a(t) = \begin{cases} u_f(t,0+), & 0<t<T_1, \\ 0, & T_1<t<T. \end{cases}$$

Obviously, $a(t) \in B_r(0)$ with sufficiently small total variation, and $u = a(t)$ satisfies the boundary condition (3.2) at $x = 0$ on the whole time interval $(0,T)$.

Now we exchange the role of variables t and x and consider the rightward problem for the system

$$\partial_x G(u) + \partial_t H(u) = 0, \quad 0<x<L,\ 0<t<T$$

with the initial condition

$$x = 0: \ u = a(t), \quad 0<t<T$$

and the following boundary conditions corresponding to the initial state $u = \overline{u}$ and the final state $u = 0$:

$$\begin{aligned} &t = 0: \quad l_s(u)u = l_s(\overline{u})\overline{u}, \quad s = m+1,\cdots,n, \\ &t = T: \quad l_r(u)u = 0, \quad r = 1,\cdots,m, \end{aligned} \tag{3.5}$$

where $l_i(u)$ $(i = 1,\cdots,n)$ are the left eigenvectors of $(DH(u)^{-1}DG(u)$, or equivalently, the left eigenvectors of $(DG(u))^{-1}DH(u)$. A direct computation shows that this boundary condition satisfies assumption (1.6).

By Proposition 2.1, the rightward problem admits a solution $u = u(t,x)$ on the domain $\{0<t<T,\ 0<x<L\}$ as the limit of a sequence of ε^ν-solutions u^ν. By Proposition 2.2, the function u is also a solution of the system (1.1) in the forward sense on $\{0<t<T,\ 0<x<L\}$. Since $u(t,0) = a(t)$ for a.e. $t \in (0,T)$, we have

$$b_1(u(t,0+)) = 0, \quad \text{a.e. } t \in (0,T).$$

Step 3 It now remains to show that u verifies the initial condition (1.12) and the final condition (1.14).

By Proposition 2.2, both u_f and u are solutions in the rightward sense. Then by Proposition 2.3 and Remark 2.2 for the rightward problem, and by (3.4), u_f coincides with u on the triangular domain $\big\{0 \le t \le T_1,\ 0 \le x \le \frac{L(T_1-t)}{T_1}\big\}$. This implies (1.12).

Since $u(t,0) \equiv 0$ for $T_1 \le t \le T$ and u satisfies (3.5) for $0 \le x \le L$, by (3.3)–(3.4) and by Remark 2.2 for the rightward problem, we have $u(t,x) \equiv 0$ on the triangular domain $\big\{T_1 \le t \le T,\ 0 \le x \le \frac{L(t-T_1)}{T-T_1}\big\}$. In particular, we get (1.14).

Thus $u = u(t,x)$ is a desired solution and the proof of Lemma 3.1 is complete.

Remark 3.1 By Remark 2.1, under the assumption that system (3.1) possesses a convex entropy $\eta(u)$, the solution $u = u(t,x)$ is actually an entropy solution.

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