Chin. Ann. Math. Ser. B
39(6), 2018, 963–972
DOI: 10.1007/s11401-018-0107-3

Chinese Annals of Mathematics, Series B

Besov Functions and Tangent Space to the Integrable Teichmüller Space*

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Abstract The authors identify the function space which is the tangent space to the integrable Teichmüller space. By means of quasiconformal deformation and an operator induced by a Zygmund function, several characterizations of this function space are obtained.

Keywords Universal Teichmüller space, Integrable Teichmüller space, Zygmund function, Quasiconformal deformation, Besov function
2010 MR Subject Classification 30C62, 30F60, 32G15, 30H25

1 Introduction and Statement of Results

We begin with some basic definitions and notations. Let $\Delta = \{z : |z| < 1\}$ denote the unit disk in the extended complex plane $\widehat{\mathbb{C}}$. $\Delta^* = \widehat{\mathbb{C}} - \overline{\Delta}$ is the exterior of Δ, and $S^1 = \partial\Delta = \partial\Delta^*$ is the unit circle. For any function $f = f(\zeta)$ defined on the unit circle S^1, we always denote by $\widehat{f}$ the function defined by $\widehat{f}(\theta) = f(\mathrm{e}^{\mathrm{i}\theta})$. The letter C denotes a positive constant that may change at different occurrences. The notation $A \asymp B$ means that there is a positive constant C independent of A and B such that $A/C \le B \le CA$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a positive constant C independent of A and B such that $A \le CB$ ($A \ge CB$).

One of its models of the universal Teichmüller space T can be defined as the right coset space $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$, where $\mathrm{QS}(S^1)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle, and $\mathrm{Möb}(S^1)$ the subgroup of Möbius transformations of the unit disk. Recall that a sense preserving self-homeomorphism h of the unit circle S^1 is quasisymmetric, if there exists a constant $M > 0$, such that

$$\frac{1}{M} \le \Big|\frac{\widehat{h}(\theta+t) - \widehat{h}(\theta)}{\widehat{h}(\theta) - \widehat{h}(\theta-t)}\Big| \le M \tag{1.1}$$

for all real numbers θ and $t > 0$. Beurling-Ahlfors [4] proved that a sense preserving self-

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*This work was supported by the National Natural Science Foundation of China (Nos. 11371268, 11171080, 11601100, 11701459), the Jiangsu Provincial Natural Science Foundation of China (No. BK201 41189) and the Ph.D Research Startup Foundation of Guizhou Normal University (No. 11904-05032130006).

homeomorphism h is quasisymmetric if and only if there exists some quasiconformal homeomorphism of Δ onto itself which has boundary values h.

Let $M(\Delta)$ denote the open unit ball of the Banach space $L^\infty(\Delta)$ of essentially bounded measurable functions on Δ. For $\mu \in M(\Delta)$, let f_μ be the quasiconformal mapping of Δ onto itself with complex dilatation equal to μ and keeping the points 1, -1 and i fixed. We say two elements μ and ν in $M(\Delta)$ are equivalent, denoted by $\mu \sim \nu$, if $f_\mu|_{S^1} = f_\nu|_{S^1}$. Then $M(\Delta)/_\sim$ is the Bers model of the universal Teichmüller space T. There exists the one to one map Ψ which maps $M(\Delta)/_\sim$ onto $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ by sending an equivalence class $[\mu]$ to $f_\mu|_{S^1}$. It is known that $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) = M(\Delta)/_\sim$ carries a natural complex structure so that the natural projection Φ from $M(\Delta)$ onto T is a holomorphic split submersion (see [12–13]).

Let Λ denote the Zygmund space in the usual sense (see [27]), which consists of all continuous functions H on the real line $\mathbb{R}$ satisfying the condition

$$|H(x+t) - 2H(x) + H(x-t)| = O(t) \tag{1.2}$$

for all real number x and $t > 0$. Then Reimann [16] (see also [8]) identified the tangent space to T at the identity map as the set of all functions H on the unit circle which satisfy the condition $\widehat{H} \in \Lambda$ and the normalized conditions

$$\mathrm{Re}\, \overline{\zeta} H(\zeta) = 0 \tag{1.3}$$

and

$$H(1) = H(-1) = H(\mathrm{i}) = 0. \tag{1.4}$$

In this paper, we will identify the function space which is the tangent space to the integrable Teichmüller space, a subspace of the universal Teichmüller space which we define below. Let $p \geq 2$ be a fixed number throughout the paper. Given an open subset Ω in the extended complex plane, we denote by $\mathcal{L}^p(\Omega)$ the Banach space of all essentially bounded measurable functions μ on Ω with norm

$$\|\mu\|_{\mathcal{L}^p} = \|\mu\|_\infty + \Big(\frac{1}{\pi}\iint_\Omega \frac{|\mu(z)|^p}{(1-|z|^2)^2}\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}}. \tag{1.5}$$

Set $\mathcal{M}^p(\Delta) = M(\Delta) \cap \mathcal{L}^p(\Delta)$. Then $T_p = \mathcal{M}^p(\Delta)/_\sim$ is one of the models of the p-integrable Teichmüller space. T_2 was first introduced by Cui [5] and was much investigated in recent years (see [19–20]), and nowadays T_2 is usually called the Weil-Petersson Teichmüller space. For a general p, T_p was first introduced and investigated by Guo [10] (see also [21–22, 25]). We say a quasisymmetric homeomorphism h is a p-integrable asymptotic affine homeomorphism if it represents a point in T_p. Let $\mathrm{QS}_p(S^1)$ denote the set of p-integrable asymptotic affine homeomorphisms of S^1. Then the right coset space $\mathrm{QS}_p(S^1)/\mathrm{Möb}(S^1)$ is another model of the p-integrable Teichmüller space T_p. It is known that $T_p = \mathrm{QS}_p(S^1)/\mathrm{Möb}(S^1) = \mathcal{M}^p(\Delta)/_\sim$ carries a natural complex structure so that the natural projection Φ from $\mathcal{M}^p(\Delta)$ onto T_p is a holomorphic split submersion (see [19–20, 22]). To guess what the tangent space to T_p should be, we recall the following result, which characterizes intrinsically the elements in $\mathrm{QS}_p(S^1)$ without using quasiconformal extensions.

Theorem 1.1 *A sense-preserving homeomorphism h on the unit circle belongs to the class $\mathrm{QS}_p(S^1)$ if and only if h is absolutely continuous* (*with respect to the arc-length measure*) *such that* $\log \widehat{h'}$ *belongs to the Besov class $B_p(S^1)$.*

Theorem 1.1 was first proved in [19] for $p = 2$ and then was extended to a general p in [22]. Recall that the Besov space $B_p(S^1)$ is the collection of measurable functions u on the unit circle S^1 with the semi-norm

$$\|u\|_{B_p} = \Big(\int_0^{2\pi}\int_0^{2\pi}\frac{|u(\mathrm{e}^{\mathrm{i}t}) - u(\mathrm{e}^{\mathrm{i}\theta})|^p}{|t-\theta|^2}\mathrm{d}t\mathrm{d}\theta\Big)^{\frac{1}{p}}. \tag{1.6}$$

To identify the tangent space to T_p at the identity map, we denote by Λ_p the set of all functions H on the unit circle S^1 such that H is absolutely continuous with $\widehat{H}' \in B_p(S^1)$. We will prove the following theorem.

Theorem 1.2 *The tangent space at the identity to the manifold T_p is the function space consisting of all functions $H \in \Lambda_p$ with the normalized conditions* (1.3)–(1.4).

We will also give several characterizations of the function space Λ_p. In our previous paper [11], we associate a continuous function H on the unit circle with a holomorphic function ϕ_H by

$$\phi_H(\zeta, z) = \frac{1}{2\pi\mathrm{i}}\int_{S^1}\frac{H(w)}{(1-\zeta w)^2(1-zw)^2}\mathrm{d}w, \quad (\zeta, z) \in (\Delta\times\Delta)\cup(\Delta^*\times\Delta^*), \tag{1.7}$$

and consequently a kernel function by

$$K_H(\zeta, z) = (\chi_\Delta(\zeta)\chi_\Delta(z) - \chi_{\Delta^*}(\zeta)\chi_{\Delta^*}(z))\phi_H(\zeta, z), \quad (\zeta, z) \in (\Delta\cup\Delta^*)\times(\Delta\cup\Delta^*), \tag{1.8}$$

where χ is the characteristic function of a set. Consider the standard Bergman space $\mathcal{A}^2$ which is the complex Hilbert space of all holomorphic functions ψ on $\Delta\cup\Delta^*$ with inner product and norm

$$\langle\phi,\psi\rangle = \frac{1}{\pi}\iint_{\Delta\cup\Delta^*}\phi(w)\overline{\psi(w)}\mathrm{d}u\mathrm{d}v, \quad \|\phi\|_{\mathcal{A}^2} = \langle\phi,\phi\rangle^{\frac{1}{2}}. \tag{1.9}$$

Then K_H (formally) induces an integral operator T_H by the formula

$$T_H\psi(\zeta) = \iint_{\Delta\cup\Delta^*}K_H(\zeta, z)\psi(\overline{z})\mathrm{d}x\mathrm{d}y, \tag{1.10}$$

or more precisely, for $\psi \in \mathcal{A}^2$,

$$T_H\psi(\zeta) = \begin{cases}\dfrac{1}{\pi}\displaystyle\iint_\Delta \phi_H(\zeta,\overline{z})\psi(z)\mathrm{d}x\mathrm{d}y, & \text{if } \zeta\in\Delta,\\ -\dfrac{1}{\pi}\displaystyle\iint_{\Delta^*}\phi_H(\zeta,\overline{z})\psi(z)\mathrm{d}x\mathrm{d}y, & \text{if } \zeta\in\Delta^*.\end{cases} \tag{1.11}$$

We proved in [11] that T_H is a bounded operator from $\mathcal{A}^2$ into itself if and only if $\widehat{H} \in \Lambda$, furthermore, T_H is a Hilbert-Schmidt operator if and only if $\widehat{H}$ belongs to the Sobolev space $H^{\frac{3}{2}}(S^1)$, or equivalently, $H \in \Lambda_2$. We will extend the latter result for a general p. Recall that a linear operator T from a Hilbert space E into itself is a p-Schatten class operator if and only if $\sum|\langle T(e_n), e_n\rangle|^p < \infty$ for any orthonormal basis e_n of E (see [26]). A 2-Schatten class operator is also called a Hilbert-Schmidt operator.

Theorem 1.3 *Let H be continuous on the unit circle S^1. Then $T_H : \mathcal{A}^2 \to \mathcal{A}^2$ is a p-Schatten class operator if and only if $H \in \Lambda_p$.*

In the proof of Theorems 1.2–1.3, we will give some more characterizations of the function space Λ_p. Our proof will be based on the theory of quasiconformal deformations, especially on the discussion from our previous paper [24] (see also [11, 18]). For completeness and for the paper to be self-contained, we will repeat some discussion from the papers [11, 18, 24].

2 Quasiconformal Deformation Extensions for Functions in Λ_p

According to Ahlfors [3], a complex-valued function F defined in a domain Ω is called a quasiconformal deformation (abbreviated to q.d.) if it has the generalized derivative $\overline{\partial}F$ such that $\overline{\partial}F \in L^\infty(\Omega)$. There are several reasons for being interested in quasiconformal deformations because of their close relation with quasiconformal mappings and Teichmüller spaces (see [1–2, 7–8, 12–13, 24]) and also of their own interests (see [3, 11, 15, 18]). In particular, the notion of quasiconformal deformations is closely related to that of Zygmund functions. Reich and Chen [15] proved that any function H on S^1 with $\widehat{H} \in \Lambda$ has a q.d. extension to the unit disk and conversely, any continuous function H on the unit circle which has a q.d. extension to the unit disk must satisfy $\widehat{H} \in \Lambda$, if H also satisfies the normalized condition (1.3). Later, we showed in [18] that for a continuous function H on the unit circle, $\widehat{H} \in \Lambda$, if and only if H can be extended to a quasiconformal deformation $\widetilde{H}$ of the whole plane $\mathbb{C}$ so that $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$. Furthermore, it was proved that

$$E(H)(z) = \frac{|1-|z|^2|^3}{2\pi \mathrm{i}} \int_{S^1} \frac{H(\zeta)}{(1-\overline{z}\zeta)^3(\zeta - z)} \mathrm{d}\zeta, \quad z \in \mathbb{C} \setminus S^1 \tag{2.1}$$

is a desired extension of H when $\widehat{H} \in \Lambda$. For details, see [11, 15].

In this section, we are concerned with the q.d. extensions for functions in Λ_p based on the discussion from our previous papers [18, 24]. We first recall some classical analytic function spaces. We denote by $\mathrm{H}^1(\Delta)$ the Hardy space in the usual sense, namely, $\phi \in \mathrm{H}^1(\Delta)$ if ϕ is holomorphic in Δ with

$$\|\phi\|_{\mathrm{H}^1} = \sup_{0<r<1} \int_0^{2\pi} |\phi(r\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}\theta < \infty. \tag{2.2}$$

We also denote by $\mathrm{H}^\infty(\Delta)$ the Banach space of all bounded holomorphic functions in Δ. We denote by $\mathrm{BMOA}(\Delta)$ the subspace of $\mathrm{H}^1(\Delta)$ which consists of those holomorphic functions ϕ in Δ with $\phi|_{S^1} \in \mathrm{BMO}(S^1)$, the space of all integrable functions u on S^1 of bounded mean oscillation

$$\|u\|_{\mathrm{BMO}} = \sup_{I \subset S^1} \frac{1}{|I|} \int_I \Big| u(z) - \frac{1}{|I|} \int_I u(z) |\mathrm{d}z| \Big| |\mathrm{d}z| < \infty. \tag{2.3}$$

We also denote by $B_p(\Delta)$ the space of functions ϕ holomorphic in Δ with semi-norm

$$\|\phi\|_{B_p} = \Big(\frac{1}{\pi} \iint_\Delta |\phi'(z)|^p (1-|z|^2)^{p-2} \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{p}}. \tag{2.4}$$

It is well known that for a holomorphic function $\phi \in \mathrm{H}^1(\Delta)$, $\phi \in B_p(\Delta)$ if and only if $\phi|_{S^1} \in B_p(S^1)$. We say $\phi \in \mathrm{BMOA}(\Delta^*)$ if $\phi(z^{-1}) \in \mathrm{BMOA}(\Delta)$, and $\phi \in B_p(\Delta^*)$ if $\phi(z^{-1}) \in B_p(\Delta)$. For more information on these function spaces, we refer to the books [9, 23, 26].

Lemma 2.1 *Let f be analytic in the unit disk Δ. Then $f(z) \in B_p(\Delta)$ if and only if $zf(z) \in B_p(\Delta)$.*

Proof We first assume that $f(z) \in B_p(\Delta)$. Noting that (see [26])

$$\iint_\Delta |f(z)|^p (1-|z|^2)^{p-2} \mathrm{d}x\mathrm{d}y \asymp \iint_\Delta |f'(z)|^p (1-|z|^2)^{2p-2} \mathrm{d}x\mathrm{d}y + |f(0)|^p, \tag{2.5}$$

we have

$$\begin{aligned} \iint_\Delta |(zf(z))'|^p (1-|z|^2)^{p-2} \mathrm{d}x\mathrm{d}y &\lesssim \iint_\Delta (|f(z)|^p + |f'(z)|^p)(1-|z|^2)^{p-2} \mathrm{d}x\mathrm{d}y \\ &\lesssim \iint_\Delta |f'(z)|^p (1-|z|^2)^{p-2} \mathrm{d}x\mathrm{d}y + |f(0)|^p. \end{aligned}$$

This implies that $zf(z) \in B_p(\Delta)$.

Conversely, suppose that $g(z) := zf(z) \in B_p(\Delta)$. Recall that given a function $v \in L^\infty(S^1)$, the associated Toeplitz operator T_v is defined for $f \in \mathrm{H}^1(\Delta)$ by

$$(T_v f)(z) = \frac{1}{2\pi \mathrm{i}} \int_{S^1} \frac{v(\zeta) f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \quad z \in \Delta.$$

It is known that $T_{\overline{v}} : B_p(\Delta) \to B_p(\Delta)$ is a bounded operator for any $v \in \mathrm{H}^\infty(\Delta)$ (see [17]). Noting that $B_p(\Delta) \subset \mathrm{BMOA}(\Delta)$, we conclude by [24, Lemma 3.2] that $f(z) \in \mathrm{BMOA}(\Delta) \subset \mathrm{H}^1(\Delta)$. Then the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{S^1} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{S^1} \frac{\overline{\zeta} g(\zeta)}{\zeta - z} \mathrm{d}\zeta = (T_{\overline{\zeta}} g)(z).$$

We obtain $f(z) \in B_p(\Delta)$ by the above mentioned result of Shamoyan [17].

We now prove the following proposition.

Proposition 2.1 *Let f be analytic in Δ. Then the following statements are equivalent*:

(1) *f is continuous in $\Delta \cup S^1$ with $f|_{S^1} \in \Lambda_p$;*

(2) $f' \in B_p(\Delta)$;

(3) $\iint_\Delta |f'''(z)|^p (1 - |z|^2)^{2p-2} \mathrm{d}x\mathrm{d}y < \infty$;

(4) *f can be extended to a quasiconformal deformation F to the whole plane so that $\overline{\partial} F \in \mathcal{L}^p(\Delta^*)$, and $F(z) = O(z^2)$ as $z \to \infty$.*

Proof It is well known that (2) $\Leftrightarrow$ (3) (see (2.5)). It is also well known (see [6]) that for an analytic function f on the unit disk Δ, f is continuous in $\Delta \cup S^1$ such that $f|_{S^1}$ is absolutely continuous in S^1 if and only if $f' \in \mathrm{H}^1(\Delta)$, and in this case

$$\widehat{f}'(\theta) = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} f'(\mathrm{e}^{\mathrm{i}\theta}). \tag{2.6}$$

Now suppose that (1) holds. Then $f' \in \mathrm{H}^1(\Delta)$ and therefore $zf'(z) \in \mathrm{H}^1(\Delta)$. This yields $zf'(z) \in B_p(\Delta)$, which implies by Lemma 2.1 that $f' \in B_p(\Delta)$. Conversely, we assume that (2) holds, then f is continuous in $\Delta \cup S^1$ and $f|_{S^1}$ is absolutely continuous in S^1. By Lemma 2.1, we have $zf'(z) \in B_p(\Delta)$, which implies that $\widehat{f}' \in B_p(S^1)$. This shows that (1) $\Leftrightarrow$ (2).

We now show that (3) $\Rightarrow$ (4). Suppose that (3) holds so that (see [26])

$$\sup_{z \in \Delta} |f'''(z)| (1 - |z|^2)^2 \lesssim \iint_\Delta |f'''(z)|^p (1 - |z|^2)^{2p-2} \mathrm{d}x\mathrm{d}y < \infty. \tag{2.7}$$

Consider the function F defined as $F(z) = f\big(\frac{1}{\overline{z}}\big) - \big(\frac{1}{\overline{z}} - z\big) f'\big(\frac{1}{\overline{z}}\big) + \frac{1}{2}\big(\frac{1}{\overline{z}} - z\big)^2 f''\big(\frac{1}{\overline{z}}\big)$, $z \in \Delta^* \backslash \{\infty\}$. Then $F(z) = O(z^2)$ as $z \to \infty$ and a direct computation gives $|\overline{\partial} F(z)| = \frac{1}{2} |f'''\big(\frac{1}{\overline{z}}\big)| \big(1 - \frac{1}{|z|^2}\big)^2$. It follows from (2.7) that $\overline{\partial} F \in L^\infty(\Delta^*)$, and

$$\iint_{\Delta^*} \frac{|\overline{\partial} F(z)|^p}{(|z|^2 - 1)^2} \mathrm{d}x\mathrm{d}y \lesssim \iint_\Delta |f'''(z)|^p (1 - |z|^2)^{2p-2} \mathrm{d}x\mathrm{d}y < \infty.$$

Consequently, $\overline{\partial} F \in \mathcal{L}^p(\Delta^*)$.

Finally, we show that (4) $\Rightarrow$ (3). Suppose that (4) holds and set $\overline{\partial} F = \mu$. By Cauchy integral formula and Green formula, we conclude that

$$f'''(z) = \frac{3}{\pi \mathrm{i}} \int_{S^1} \frac{F(\zeta)}{(\zeta - z)^4} \mathrm{d}\zeta = -\frac{6}{\pi} \iint_{\Delta^*} \frac{\mu(\zeta)}{(\zeta - z)^4} \mathrm{d}\xi \mathrm{d}\eta.$$

Consequently, by Hölder inequality, we deduce that

$$\iint_{\Delta} |f'''(z)|^p (1-|z|^2)^{2p-2} \mathrm{d}x\mathrm{d}y = \frac{36}{\pi^2} \iint_{\Delta} \Big| \iint_{\Delta^*} \frac{\mu(\zeta)}{(\zeta-z)^4} \mathrm{d}\xi \mathrm{d}\eta \Big|^p (1-|z|^2)^{2p-2} \mathrm{d}x\mathrm{d}y$$
$$\lesssim \iint_{\Delta} \iint_{\Delta^*} \frac{|\mu(\zeta)|^p}{|\zeta-z|^4} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}x\mathrm{d}y = \iint_{\Delta^*} \frac{|\mu(\zeta)|^p}{(|\zeta|^2-1)^2} \mathrm{d}\xi \mathrm{d}\eta.$$

This completes the proof Proposition 2.1.

Repeating the reasoning in the proof of Proposition 2.1, we are able to obtain the following proposition.

Proposition 2.2 *Let g be analytic in Δ^*. Then the following statements are all equivalent*:

(1) *g is continuous in $\Delta^* \cup S^1$ with $g|_{S^1} \in \Lambda_p$*;

(2) $g' \in B_p(\Delta^*)$;

(3) $\iint_{\Delta^*} |g'''(z)|^p (|z|^2-1)^{2p-2} \mathrm{d}x\mathrm{d}y < \infty$;

(4) *g can be extended to a quasiconformal deformation G to the whole plane so that $\overline{\partial} G \in \mathcal{L}^p(\Delta)$.*

We proceed to discuss the q.d. extensions for functions in Λ_p. For a continuous function H on the unit circle, we consider the Cauchy integral

$$C(H)(z) = \frac{1}{2\pi \mathrm{i}} \int_{S^1} \frac{H(\zeta)}{\zeta - z} \mathrm{d}\zeta, \quad z \in \Delta \cup \Delta^*. \tag{2.8}$$

More precisely, we always set $f(z) = C(H)(z)$ for $z \in \Delta$, $g(z) = C(H)(z)$ for $z \in \Delta^*$ in the rest of this section. Then f and g are holomorphic in Δ and Δ^*, respectively. We also let J denote the harmonic conjugation operator in the usual sense (see [6, 9]), namely, $J(H)$ is the following Cauchy principle value integral

$$J(H)(z) = -\frac{1}{\pi} \int_{S^1} \frac{H(\zeta)}{\zeta - z} \mathrm{d}\zeta, \quad z \in S^1. \tag{2.9}$$

It is well known that J preserves the Zygmund space Λ and the Besov space $B_p(S^1)$ as well (see [9, 23]). We now show that J also preserves the space Λ_p.

Proposition 2.3 *The harmonic conjugation operator J keeps the space Λ_p invariant.*

Proof Suppose $H \in \Lambda_p$ so that H is absolutely continuous on S^1 with $\widehat{H}' \in B_p(S^1)$. As done in [24], for $f(z) = C(H)(z)$, we find out that $zf'(z) \in \mathrm{H}^1(\Delta)$ with boundary values $\frac{\widehat{H}' + \mathrm{i}J(\widehat{H}')}{2}$ up to a constant. Since $\widehat{H}' \in B_p(S^1)$, and J preserves $B_p(S^1)$, we conclude that $zf'(z) \in B_p(\Delta)$, which implies $f' \in B_p(\Delta)$ by Lemma 2.1. Thus, f is continuous in $\Delta \cup S^1$ and absolutely continuous in S^1. Noting that $f = \frac{H + \mathrm{i}J(H)}{2}$ on S^1, we conclude that $J(H)$ is absolutely continuous in S^1, and $\widehat{H}'(\theta) + \mathrm{i}\widehat{J(H)}'(\theta) = 2\widehat{f}'(\theta) = 2\mathrm{i}e^{\mathrm{i}\theta} f'(e^{\mathrm{i}\theta}) \in B_p(S^1)$, which implies that $\widehat{J(H)}' \in B_p(S^1)$. Thus, $J(H) \in \Lambda_p$ as required.

Now we can prove the main results in this section.

Theorem 2.1 *Let H be continuous on the unit circle. Then the following statements are equivalent*:

(1) $H \in \Lambda_p$;

(2) *$f' \in B_p(\Delta)$, and $g' \in B_p(\Delta^*)$*;

(3) *f and g have q.d. extensions F and G respectively to the whole plane so that $\overline{\partial}F \in \mathcal{L}^p(\Delta^*)$, $\overline{\partial}G \in \mathcal{L}^p(\Delta)$, and $F(z) = O(z^2)$ as $z \to \infty$;*

(4) *H can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\overline{\partial}\widetilde{H} \in \mathcal{L}^p(\mathbb{C})$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$.*

Proof Noting that $f = \frac{H+\mathrm{i}J(H)}{2}$, $g = \frac{-H+\mathrm{i}J(H)}{2}$ on S^1, we conclude that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by means of Propositions 2.1–2.3.

(3) $\Rightarrow$ (4) Suppose that f and g have q.d. extensions F and G respectively to the whole plane so that $\overline{\partial}F \in \mathcal{L}^p(\Delta^*)$, $\overline{\partial}G \in \mathcal{L}^p(\Delta)$, and $F(z) = O(z^2)$ as $z \to \infty$. Define $\widetilde{H}$ by $\widetilde{H}(z) = F(z) - \widetilde{G}(z)$ on $\Delta \cup S^1$, and $\widetilde{H}(z) = \widetilde{F}(z) - G(z)$ on $\Delta^* \cup S^1 \setminus \{\infty\}$. Then $\widetilde{H}$ is the desired q.d. extension of H to the whole plane.

(4) $\Rightarrow$ (3) Suppose that H can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\overline{\partial}\widetilde{H} \in \mathcal{L}^p(\mathbb{C})$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$. Denote $\overline{\partial}\widetilde{H} = \mu$. Set $G(z) = \frac{1}{\pi}\iint_\Delta \frac{\mu(\zeta)}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta$ and $F(z) = \widetilde{H}(z) + G(z)$. We proved in [24] that F and G are q.d. extensions of f and g, respectively. It is clear that $\overline{\partial}F|_{\Delta^*} \in \mathcal{L}^p(\Delta^*)$, $\overline{\partial}G|_\Delta \in \mathcal{L}^p(\Delta)$, and $F(z) = O(z^2)$ as $z \to \infty$.

When H satisfies the normalized condition (1.3), we can obtain some stronger results, which will be used to prove Theorem 1.2 in the next section.

Theorem 2.2 *Let H be continuous on the unit circle which satisfies the normalized condition* (1.3). *Then the following statements are equivalent*:

(1) *$H \in \Lambda_p$;*

(2) *H can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\overline{\partial}\widetilde{H} \in \mathcal{L}^p(\mathbb{C})$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$;*

(3) *H can be extended to a quasiconformal deformation H_1 to Δ so that $\overline{\partial}H_1 \in \mathcal{L}^p(\Delta)$;*

(4) *H can be extended to a quasiconformal deformation H_2 to $\Delta^* \setminus \{\infty\}$ so that $\overline{\partial}H_2 \in \mathcal{L}^p(\Delta^*)$ and $H_2(z) = O(z^2)$ as $z \to \infty$.*

Proof Repeat the proof of Theorem 3.4 in [24].

3 Proof of Theorem 1.2

We reproduce the proof of Theorem 1.1 in [24]. Suppose that we are given a curve of strongly quasisymmetric mappings $h^t(\zeta)$ ($t > 0$ is small) normalized to fix ± 1 and i, which is the identity for $t = 0$ and differentiable with respect to t for the manifold structure on T_p. Denote

$$h^t(\zeta) = \zeta + tH(\zeta) + o(t), \quad t \to 0.$$

Since the natural projection $\Phi : \mathcal{M}^p(\Delta) \to T_p$ is a holomorphic split submersion, we conclude that there is a differentiable curve of Beltrami coefficients $\nu_t \in \mathcal{M}(\Delta)$ such that h^t is the restriction to the unit circle of the normalized quasiconformal mapping f_{ν_t}. Now there exists some $\mu \in \mathcal{L}^p(\Delta)$ such that $\nu_t = t\mu + o(t)$. Consequently,

$$f_{\nu_t}(z) = z + t\dot{f}[\mu](z) + o(t), \quad t \to 0.$$

Here $\dot{f}[\mu]$ satisfies the normalized conditions (1.3) and (1.4) and is uniquely determined by the condition $\overline{\partial}\dot{f}[\mu] = \mu$ (see [1–2, 13–14]). Noting that $H = \dot{f}[\mu]|_{S^1}$, we conclude by Theorem 2.2 that $H \in \Lambda_p$ and satisfies the normalized conditions (1.3)–(1.4).

Conversely, suppose that we are given a function $H \in \Lambda_p$ satisfying the normalized conditions (1.3)–(1.4). By Theorem 2.2, we deduce that H can be extended to the unit disk to a quasiconformal deformation $\widetilde{H}$ with $\overline{\partial}-$derivative $\mu = \overline{\partial}\widetilde{H} \in \mathcal{L}^p(\Delta)$. Set $\mu_t = t\mu$ for small $t > 0$. Then

$$f_{\mu_t}(z) = z + t\dot{f}[\mu](z) + o(t), \quad t \to 0.$$

Noting that both $\dot{f}[\mu]$ and $\widetilde{H}$ satisfy the normalized conditions (1.3)–(1.4) and have the same $\overline{\partial}$-derivative μ, we conclude that $\dot{f}[\mu] = \widetilde{H}$. Then

$$f_{\mu_t}(z) = z + t\widetilde{H}(z) + o(t), \quad t \to 0.$$

Set $h^t = f_{\mu_t}|_{S^1}$. Then it holds that

$$h^t(\zeta) = \zeta + tH(\zeta) + o(t), \quad t \to 0,$$

which implies that h^t is a differentiable curve in T_p with the tangent vector H.

4 Proof of Theorem 1.3

We continue to use the notations in previous sections. Recall that

$$E(H)(z) = \frac{|1-|z|^2|^3}{2\pi \mathrm{i}} \int_{S^1} \frac{H(\zeta)}{(1-\overline{z}\zeta)^3(\zeta - z)} \mathrm{d}\zeta, \quad z \in \mathbb{C} \setminus S^1 \tag{4.1}$$

defines a q.d. extension of a continuous function H on S^1 with $\widehat{H} \in \Lambda$. By differentiating (4.1) with respect to $\overline{z}$, we obtain

$$\overline{\partial}E(H)(z) = \frac{3}{2\pi \mathrm{i}}(\chi_\Delta(z) - \chi_{\Delta^*}(z))(1-|z|^2)^2 \int_{S^1} \frac{H(\zeta)}{(1-\overline{z}\zeta)^4} \mathrm{d}\zeta. \tag{4.2}$$

If we set

$$\phi_H(z) = \phi_H(z,z) = \frac{1}{2\pi \mathrm{i}} \int_{S^1} \frac{H(w)}{(1-zw)^4} \mathrm{d}w, \quad z \in \Delta \cup \Delta^*, \tag{4.3}$$

then we have

$$\overline{\partial}E(H)(z) = 3(\chi_\Delta(z) - \chi_{\Delta^*}(z))(1-|z|^2)^2 \phi_H(\overline{z}), \quad z \in \Delta \cup \Delta^*. \tag{4.4}$$

Theorem 1.3 is contained in the following result.

Theorem 4.1 *Let H be a continuous function on the unit circle. Then the following statements are all equivalent*:

(1) $H \in \Lambda_p$;

(2) *H can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\overline{\partial}\widetilde{H} \in \mathcal{L}^p(\mathbb{C})$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$*;

(3) *T_H is a p-Schatten class operator from $\mathcal{A}^2$ into itself*;

(4) $\iint_{\Delta \cup \Delta^*} |\phi_H(z)|^p |1-|z|^2|^{2p-2} \mathrm{d}x\mathrm{d}y < \infty$;

(5) $\overline{\partial}E(H) \in \mathcal{L}^p(\mathbb{C})$.

Proof We consider a special subset of $\mathcal{A}^2$. For fixed $z \in \Delta \cup \Delta^*$, we set $\psi_z \in \mathcal{A}^2$ by

$$\psi_z(\zeta) = \frac{1-|z|^2}{(1-z\zeta)^2}\chi_\Delta(\zeta), \tag{4.5}$$

when $z \in \Delta$, while when $z \in \Delta^*$,

$$\psi_z(\zeta) = \frac{1-|z|^2}{(1-z\zeta)^2}\chi_{\Delta^*}(\zeta). \tag{4.6}$$

It is easy to see that $\|\psi_z\|_{\mathcal{A}^2} = 1$. It was proved in [11] that for any continuous function H on the unit circle, it holds that

$$T_H\psi_z(\zeta) = (1-|z|^2)K_H(\zeta, z), \quad (\zeta, z) \in (\Delta \cup \Delta^*) \times (\Delta \cup \Delta^*). \tag{4.7}$$

Moreover, if for each fixed $z \in \Delta \cup \Delta^*$, $K_H(\cdot, z) \in \mathcal{A}^2$, then for all $z \in \Delta \cup \Delta^*$, it holds that

$$\langle K_H(\cdot, z), \psi_{\overline{z}}\rangle = |1-|z|^2|\phi_H(z). \tag{4.8}$$

(1) $\Leftrightarrow$ (2) follows from Theorem 2.1, while (5) $\Rightarrow$ (2) is obvious. To prove (2) $\Rightarrow$ (3), let $\widetilde{H}$ be a q.d. extension of H to the whole plane so that $\overline{\partial}\widetilde{H} \in \mathcal{L}^p(\mathbb{C})$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$. Let ϕ_n be any orthonormal basis of $\mathcal{A}^2$. Then, $\|\phi_n\|_{\mathcal{A}^2} = 1$, and (see [26, Theorem 4.19])

$$\sum |\phi_n(z)|^2 = \frac{1}{(1-|z|^2)^2}, \quad z \in \Delta \cup \Delta^*. \tag{4.9}$$

On the other hand, it follows from the inequality (3.8) in [11] that

$$\|T_H(\phi_n)\|^2 \lesssim \iint_{\Delta\cup\Delta^*} |\phi_n(z)\overline{\partial}\widetilde{H}(z)|^2 \mathrm{d}x\mathrm{d}y.$$

The Hölder inequality yields

$$\begin{aligned}\|T_H(\phi_n)\|^p &\lesssim \Big(\iint_{\Delta\cup\Delta^*} |\phi_n(z)\overline{\partial}\widetilde{H}(z)|^2 \mathrm{d}x\mathrm{d}y\Big)^{\frac{p}{2}} \\ &\lesssim \iint_{\Delta\cup\Delta^*} |\phi_n(z)|^2|\overline{\partial}\widetilde{H}(z)|^p \mathrm{d}x\mathrm{d}y \Big(\iint_{\Delta\cup\Delta^*} |\phi_n(z)|^2 \mathrm{d}x\mathrm{d}y\Big)^{\frac{p}{2}-1} \\ &= \iint_{\Delta\cup\Delta^*} |\phi_n(z)|^2|\overline{\partial}\widetilde{H}(z)|^p \mathrm{d}x\mathrm{d}y.\end{aligned}$$

By (4.9) we have

$$\sum \|T_H(\phi_n)\|^p \lesssim \sum \iint_{\Delta\cup\Delta^*} |\phi_n(z)|^2|\overline{\partial}\widetilde{H}(z)|^p \mathrm{d}x\mathrm{d}y = \iint_{\Delta\cup\Delta^*} \frac{\overline{\partial}\widetilde{H}(z)|^p}{(1-|z|^2)^2} \mathrm{d}x\mathrm{d}y < \infty.$$

Thus, T_H is a p-Schatten class operator from $\mathcal{A}^2$ into itself.

To prove (3) $\Rightarrow$ (4), suppose that $T_H : \mathcal{A}^2 \to \mathcal{A}^2$ is a p-Schatten class operator. Consider the map J defined by $J\phi(z) = \overline{\phi(\overline{z})}$, which is an isometric isomorphism of $\mathcal{A}^2$ onto itself. Then $JT_H : \mathcal{A}^2 \to \mathcal{A}^2$ is also a p-Schatten class operator. Now it is known (see [26, Corollary 6.7])

$$\iint_{\Delta\cup\Delta^*} |\langle JT_H(\psi_z), \psi_z\rangle|^p (1-|z|^2)^{-2} \mathrm{d}x\mathrm{d}y < \infty. \tag{4.10}$$

Noting that $J\psi_z = \psi_{\overline{z}}$ for each $z \in \Delta \cup \Delta^*$, we obtain that

$$\langle JT_H(\psi_z), \psi_z\rangle = \langle T_H(\psi_z), J\psi_z\rangle = \langle T_H(\psi_z), \psi_{\overline{z}}\rangle,$$

which implies by (4.7)–(4.8) that (4.10) is equivalent to

$$\iint_{\Delta\cup\Delta^*} |\phi_H(z)|^p |1-|z|^2|^{2p-2} \mathrm{d}x\mathrm{d}y < \infty$$

as desired.

(4) $\Rightarrow$ (5) follows directly from (4.4). This completes the proof of Theorem 4.1.

Acknowledgement The authors express their gratitude to the referee for useful advice.

References

[1] Ahlfors, L. V., Remarks on Teichmüller's space of Riemann surfaces, *Ann. Math.,* **74**, 1961, 171–191.

[2] Ahlfors, L. V., Lectures on Quasiconformal Mapping, D. Van Nostrand, Princeton, New York, 1966.

[3] Ahlfors, L. V., Quasiconformal deformations and mappings in $\mathbb{R}^n$, *J. Anal. Math.,* **30**, 1976, 74–97.

[4] Beurling, A. and Ahlfors, L. V., The boundary correspondence under quasiconformal mappings, *Acta Math.,* **96**, 1956, 125–142.

[5] Cui, G., Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, *Sci. China Ser. A,* **43**, 2000, 267–279.

[6] Duren, L., Theory of H^p Spaces, Academic Press, New York, London, 1970.

[7] Gardiner, F. P. and Lakic, N., Quasiconformal Teichmüller Theory, Mathematical Surveys and Monographs, **76**, American Mathematical Society, Providence, RI, 2000.

[8] Gardiner, F. P. and Sullivan, D., Symmetric structure on a closed curve, *Amer. J. Math.,* **114**, 1992, 683–736.

[9] Garnett, J. B., Bounded Analytic Functions, Academic Press, New York, 1981.

[10] Guo, H., Integrable Teichmüller spaces, *Sci. China Ser. A,* **43**, 2000, 47–58.

[11] Hu, Y., Song, J., Wei, H. and Shen, Y., An integral operator induced by a Zygmund function, *J. Math. Anal. Appl.,* **401**, 2013, 560–567.

[12] Lehto, O., Univalent Functions and Teichmüller Spaces, John Wiley & Sons Inc., Springer-Verlag, New York, 1986.

[13] Nag, S., The Complex Analytic Theory of Teichmüller Spaces, Wiley-Interscience, 1988.

[14] Nag, S. and Verjovsky, A., Diff(S^1) and the Teichmüller spaces, *Comm. Math. Phys.,* **130**, 1990, 123–138.

[15] Reich, E. and Chen, J., Extensions with bounded $\overline{\partial}-$derivative, *Ann. Acad. Sci. Fenn.,* **16**, 1991, 377–389.

[16] Reimann, H., Ordinary differential equations and quasiconformal mappings, *Invent. Math.,* **33**, 1976, 247–270.

[17] Shamoyan, F. A., Toeplitz operators and division by an inner function in some spaces of analytic functions, *Dokl. Akad. Nauk Armenii,* **76**, 1983, 215–219.

[18] Shen, Y., Fourier coefficients of Zygmund functions and analytic functions with quasiconformal deformation extensions, *Sci. China Math.,* **55**, 2012, 607–624.

[19] Shen, Y., Weil-Peterssen Teichmüller space, *Amer. J. Math.*, arXiv: 1304.3197.

[20] Takhtajan, L. and Teo, L., Weil-Petersson metric on the universal Teichmüller space, *Mem. Amer. Math. Soc.,* **183**(861), 2006, 119pp.

[21] Tang, S., Some characterizations of the integrable Teichmüller space, *Sci. China Math.,* **56**, 2013, 541–551.

[22] Tang, S. and Shen, Y., Integrable Teichmüller space, preprint.

[23] Triebel, H., Theory of Function Spaces, Monographs in Mathematics, **78**, Birkhäuser, Basel, 1983.

[24] Wei, H. and Shen, Y., On the tangent space to the BMO-Teichmüller space, *J. Math. Anal. Appl.,* **419**, 2014, 715–726.

[25] Yanagishita, M., Introduction of a complex structure on the p-integrable Teichmüller space, *Ann. Acad. Sci. Fenn. Math.,* **39**, 2014, 947–971.

[26] Zhu, K., Operator Theory in Function Spaces, Math. Surveys Monogr., **138**, American Mathematical Society, Providence, RI, 2007.

[27] Zygmund, A., Smooth functions, *Duke Math. J.,* **12**, 1945, 47–76.