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Variation Inequalities Related to Schrödinger Operators on Morrey Spaces*

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Abstract The authors establish the boundedness of the variation operators associated with the heat semigroup, Riesz transforms and commutators generated by the Riesz transforms and BMO-type functions in the Schrödinger setting on the Morrey spaces.

Keywords Schrödinger operators, Variation operators, Heat semigroups, Riesz transforms, Commutators, Morrey spaces
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1 Introduction

Let $\mathcal{T} = \{T_t\}_{t>0}$ be a family of operators such that the limit $\lim\limits_{t\to 0} T_t f(x) = Tf(x)$ exists in some sense. A classical method of measuring the speed of convergence of the family $\{T_t\}_{t>0}$ is to consider "square function" of the type $\Big(\sum\limits_{i=1}^{\infty} |T_{t_i} f - T_{t_{i+1}} f|^2\Big)^{\frac{1}{2}}$, where $t_i \searrow 0$, or more generally the ρ -variation operator defined by

$$\mathcal{V}_\rho(\mathcal{T} f)(x) := \sup_{t_i \searrow 0} \Big(\sum_{i=1}^{\infty} |T_{t_{i+1}} f(x) - T_{t_i} f(x)|^\rho\Big)^{\frac{1}{\rho}}, \tag{1.1}$$

where $\rho > 1$ and the supremum is taken over all sequence $\{t_i\}$ decreasing to zero. We denote F_ρ the space that includes all the functions $\varphi : (0,\infty) \to \mathbb{R}$, such that

$$\|\varphi\|_{F_\rho} = \sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}} \Big(\sum_{j=0}^{\infty} |\varphi(\varepsilon_j) - \varphi(\varepsilon_{j+1})|^\rho\Big)^{\frac{1}{\rho}} < \infty. \tag{1.2}$$

Then $\|\cdot\|_{F_\rho}$ is a seminorm on F_ρ. It can be written as

$$\mathcal{V}_\rho(T_t)(f) = \|T_t f\|_{F_\rho}. \tag{1.3}$$

The variation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis. We refer the readers to [2, 7–8, 11, 16, 18–19] and the references therein for more background information.

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Recently, Betancor et al. [3] studied the bounded behaviors of variation operators for some Schrödinger type operators in Lebesgue spaces. Precisely, let $n \geq 3$ and $\mathcal{L} = -\Delta + V$ be the Schrödinger operator defined on $\mathbb{R}^n$ associated with a fixed non-negative potential $V \in \mathrm{RH}_q$ (the reverse Hölder class) for $q \geq \frac{n}{2}$, that is, there exists $C > 0$, such that

$$\Big(\frac{1}{|B|}\int_B V(x)^q \mathrm{d}x\Big)^{\frac{1}{q}} \leq \frac{C}{|B|}\int_B V(x)\mathrm{d}x \tag{1.4}$$

for every ball B in $\mathbb{R}^n$. Consider the heat semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ generated by the operator $-\mathcal{L}$, which can be written as

$$W_t^{\mathcal{L}}(f)(x) = \mathrm{e}^{-t\mathcal{L}}f(x) := \int_{\mathbb{R}^n} \mathcal{W}_t^{\mathcal{L}}(x,y)f(y)\mathrm{d}y \quad \text{for } f \in L^2(\mathbb{R}^n),\ t > 0.$$

It follows from [14] that $\mathcal{W}_t^{\mathcal{L}}(x,y)$ satisfies the estimates:

$$0 \leq \mathcal{W}_t^{\mathcal{L}}(x,y) \leq (4\pi)^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big),$$

and the semigroup $\{W_t^{\mathcal{L}}\}$ is C_0 in $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. In [3], Betancor et al. showed that the variation operator $\mathcal{V}_\rho(W_t^{\mathcal{L}})$ associated with $\{W_t^{\mathcal{L}}\}_{t>0}$ is bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$, and is of weak type $(1,1)$.

Moreover, for $\ell = 1, \cdots, n$, we consider the ℓ-th Riesz transform in the $\mathcal{L}$-context defined by

$$R_\ell^{\mathcal{L}}(f)(x) = \lim_{\varepsilon\to 0^+} R_{l,\varepsilon}^{\mathcal{L}}(f)(x) := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} \mathfrak{R}_\ell^{\mathcal{L}}(x,y)f(y)\mathrm{d}y, \quad \text{a.e. } x \in \mathbb{R},$$

and its adjoint operator defined by

$$R_\ell^{\mathcal{L},*}(f)(x) = \lim_{\varepsilon\to 0^+} R_{\ell,\varepsilon}^{\mathcal{L},*}(f)(x) := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} \mathfrak{R}_\ell^{\mathcal{L}}(y,x)f(y)\mathrm{d}y, \ \text{ a.e. } x \in \mathbb{R}.$$

Here, for every x, $y \in \mathbb{R}^n$, $x \neq y$,

$$\mathfrak{R}_\ell^{\mathcal{L}}(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} (-\mathrm{i}\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_\ell}\Gamma(x,y,\tau)\mathrm{d}\tau,$$

where $\Gamma(x,y,\tau)$ represents the fundamental solution to the operator $\mathcal{L} + \mathrm{i}\tau$ (see [3, 29, 34]).

Betancor et al. [3] proved that when $\frac{n}{2} \leq q < n$, for $\ell = 1, 2, \cdots, n$, the variation operators $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})$ (resp., $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L},*})$) associated with the family of truncations $\{R_{l,\varepsilon}^{\mathcal{L}}\}_{\varepsilon>0}$ (resp., $\{R_{\ell,\varepsilon}^{\mathcal{L},*}\}_{\varepsilon>0}$) are bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < p_0$ (resp., $p_0' < p < \infty$) with $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, and $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})$ are of weak type (1, 1). Moreover, when $q \geq n$, both $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})$ and $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L},*})$ are bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$ and of weak type (1, 1).

In addition, for every $V \in \mathrm{RH}_{\frac{n}{2}}$, Shen [28] introduced the function γ, which is called as the critical radius and defined as

$$\gamma(x) = \sup\Big\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\mathrm{d}y \leq 1\Big\}, \quad x \in \mathbb{R}^n, \tag{1.5}$$

and plays key roles in the theory of Harmonic analysis operators associated with $\mathcal{L}$. In [4], Bongioann et al. defined the space $\mathrm{BMO}_\theta(\gamma)$, $\theta \geq 0$, as follows.

Definition 1.1 *A locally integrable function b in $\mathbb{R}^n$ is* $\mathrm{BMO}_\theta(\gamma)$ *provided that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|b(y)-b_{B(x,r)}|\mathrm{d}y \le C\Big(1+\frac{r}{\gamma(x)}\Big)^{\theta} \tag{1.6}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = |B|^{-1}\int_B |b(x) - b_B|\mathrm{d}x$. We denote

$$\|b\|_{\mathrm{BMO}_\theta(\gamma)} = \sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|b(y)-b_{B(x,r)}|\mathrm{d}y\Big(1+\frac{r}{\gamma(x)}\Big)^{-\theta}.$$

It is easy to check that $\mathrm{BMO}(\mathbb{R}^n) = \mathrm{BMO}_0(\gamma) \subset \mathrm{BMO}_\theta(\gamma) \subset \mathrm{BMO}_{\theta'}(\gamma)$ for $0 \le \theta \le \theta'$. Set $\mathrm{BMO}_\infty(\gamma) = \bigcup\limits_{\theta>0} \mathrm{BMO}_\theta(\gamma)$. Then $\mathrm{BMO}_\infty(\gamma)$ is larger than $\mathrm{BMO}(\mathbb{R}^n)$ in general (see [4]).

For $b \in \mathrm{BMO}_\infty(\gamma)$ and $\ell = 1, \cdots, n$, the commutators $R^{\mathcal{L}}_{b,\ell}$ and $R^{\mathcal{L},*}_{b,\ell}$ are respectively defined by

$$R^{\mathcal{L}}_{b,\ell}(f) = bR^{\mathcal{L}}_\ell - R^{\mathcal{L}}_\ell(bf) \quad \text{and} \quad R^{\mathcal{L},*}_{b,\ell}(f) = bR^{\mathcal{L},*}_\ell - R^{\mathcal{L},^*}_\ell(bf) \quad \text{for } f \in C^\infty_c(\mathbb{R}^n).$$

It was shown in [4] that, for every $b \in \mathrm{BMO}_\infty(\gamma)$ and $\ell = 1, \cdots, n$, the operators $R^{\mathcal{L}}_{b,\ell}$ (resp., $R^{\mathcal{L},*}_{b,\ell}$) are bounded on $L^p(\mathbb{R}^n)$, provided that $1 < p < p_0$ (resp., $p_0' < p < \infty$) with $\frac{1}{p_0} = \big(\frac{1}{q} - \frac{1}{n}\big)_+$ for $V \in \mathrm{RH}_q$, $q \ge \frac{n}{2}$. In [3], Betancor et al. obtained the following point-wise representations of the commutator operators by a principal value integral:

$$R^{\mathcal{L}}_{b,\ell}(f)(x) = \lim_{\varepsilon\to 0} R^{\mathcal{L}}_{b,\ell,\varepsilon}(f)(x) := \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}(b(x)-b(y))\mathfrak{R}^{\mathcal{L}}_\ell(x,y)f(y)\mathrm{d}y, \quad \text{a.e. } x \in \mathbb{R}^n$$

and

$$R^{\mathcal{L},*}_{b,\ell}(f)(x) = \lim_{\varepsilon\to 0} R^{\mathcal{L},*}_{b,\ell,\varepsilon}(f)(x) := \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}(b(x)-b(y))\mathfrak{R}^{\mathcal{L}}_\ell(y,x)f(y)\mathrm{d}y, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Moreover, the authors in [3] proved that for every $b \in \mathrm{BMO}_\infty(\gamma)$ and $\ell = 1, \cdots, n$, the variation operators $\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})$ (resp., $\mathcal{V}_\rho(R^{\mathcal{L},*}_{b,\ell,\varepsilon})$) associated with the family of truncations $\{R^{\mathcal{L}}_{b,\ell,\varepsilon}\}_{\varepsilon>0}$ (resp., $\{R^{\mathcal{L},*}_{b,\ell,\varepsilon}\}_{\varepsilon>0}$) are bounded from $L^p(\mathbb{R}^n)$ into itself, provided that $1 < p < p_0$ (resp., $p_0' < p < \infty$) with $\frac{1}{p_0} = \big(\frac{1}{q} - \frac{1}{n}\big)_+$ for $V \in \mathrm{RH}_q$, $q \ge \frac{n}{2}$.

On the other hand, in order to extend the boundedness of Schrödinger type operators in Lebesgue spaces, Tang and Dong [32] introduced the following Morrey spaces related to the non-negative potential V, denoted by $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$.

Definition 1.2 *Let $p \in [1, \infty)$, $\alpha \in (-\infty, +\infty)$ and $\lambda \in [0, n)$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \in \mathrm{RH}_q$ $(q > 1)$, we say $f \in L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ provided that*

$$\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} = \sup_{B(x_0,r)\subset\mathbb{R}^n}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{\alpha} r^{-\lambda}\int_{B(x_0,r)}|f(x)|^p\mathrm{d}x < \infty, \tag{1.7}$$

where $B = B(x_0, r)$ denotes a ball centered at x_0 and with radius r, $\gamma(x_0)$ is the critical radius at x_0 defined as in (1.5).

Clearly, when $\alpha = 0$ or $V = 0$ and $0 < \lambda < n$, the spaces $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ are the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, which were introduced by Morrey [23] in 1938 and were subsequently found to have many important applications to the elliptic equations (see [6, 9, 13–14, 17, 24]), the Navier-Stokes equations (see [21–22, 33]) and the Schrödinger equations (see [26–27, 30–31]) etc. It is easy to see that $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n) \subset L^{p,\lambda}(\mathbb{R}^n)$ for $\alpha > 0$ and $L^{p,\lambda}(\mathbb{R}^n) \subset L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ for $\alpha < 0$. In [32], the authors established the $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$-boundedness of the Riesz transforms $R^{\mathcal{L}}_\ell$, $R^{\mathcal{L},*}_\ell$, and the corresponding commutators with $V \in B_n$ (see also [25, 32] for more results related to Schrödinger operators in $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$).

Based on the above results, it is a natural and interesting question that whether we can establish the $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$-boundedness of the variation operators aforementioned in Schrödinger setting. The main purpose of this paper is to answer this question. Our results can be formulated as follows.

Theorem 1.1 *Let $V \in \mathrm{RH}_q$ for $q \geq \frac{n}{2}$ and $\rho > 2$. Assume $\alpha \in (-\infty, +\infty)$ and $\lambda \in (0, n)$.*

(i) *If $1 < p < \infty$, then $\|\mathcal{V}_\rho(W^{\mathcal{L}}_t f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}$, where C is independent of f.*

(ii) *If $p = 1$, then for any $\eta > 0$,*

$$\eta\Big(1 + \frac{r}{\gamma(x)}\Big)^{\alpha} |\{y \in B(x, r) : |\mathcal{V}_\rho(W^{\mathcal{L}}_t f)(y)| > \eta\}| \leq Cr^{\lambda}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}$$

holds for all balls $B(x, r)$, where C is independent of x, r, η and f.

Theorem 1.2 *Let $\ell = 1, \cdots, n$, $\rho > 2$ and $V \in \mathrm{RH}_q$ with $q \geq \frac{n}{2}$ and assume $\alpha \in (-\infty, +\infty)$ and $\lambda \in \big(0, \big(1 - \frac{p}{p_0}\big)n\big)$. Then*

(i) *for $1 < p < p_0$, where $\frac{1}{p_0} = \big(\frac{1}{q} - \frac{1}{n}\big)_+$,*

$$\|\mathcal{V}_\rho(R^{\mathcal{L}}_{\ell,\varepsilon} f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C is independent of f;

(ii) *for $p = 1$ and any $\eta > 0$,*

$$\eta\Big(1 + \frac{r}{\gamma(x)}\Big)^{\alpha} |\{y \in B(x, r) : |\mathcal{V}_\rho(R^{\mathcal{L}}_{\ell,\varepsilon} f)(y)| > \eta\}| \leq Cr^{\lambda}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}$$

holds for all balls B, where C is independent of x, r, η and f;

(iii) *for $p'_0 < p < \infty$,*

$$\|\mathcal{V}_\rho(R^{\mathcal{L},*}_{\ell,\varepsilon} f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C is independent of f.

Theorem 1.3 *Let $\ell = 1, \cdots, n$, $\rho > 2$, $b \in \mathrm{BMO}_\infty(\gamma)$ and $V \in \mathrm{RH}_q$ with $q \geq \frac{n}{2}$ and assume $\alpha \in (-\infty, +\infty)$ and $\lambda \in \big(0, \big(1 - \frac{p}{p_0}\big)n\big)$. Then for $1 < p < p_0$, where $\frac{1}{p_0} = \big(\frac{1}{q} - \frac{1}{n}\big)_+$,*

$$\|\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon} f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C\|b\|_\theta\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)};$$

and for $p'_0 < p < \infty$,

$$\|\mathcal{V}_\rho(R^{\mathcal{L},*}_{b,\ell,\varepsilon} f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C\|b\|_\theta\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C is independent of f.

Remark 1.1 In [29], it was proved that if V is a nonnegative polynomial, then $V \in \mathrm{RH}_q$ for any $1 < q < \infty$. Therefore, as special cases of our results, the corresponding ones to the Hermite operator: $H = -\Delta + |x|^2$ hold. This can be regarded as the generalization of the corresponding results in [10–11].

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1.1, and the proofs of Theorems 1.2–1.3 will be given in Section 3. Throughout this paper, the letter C always denotes a positive constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $\frac{1}{p} + \frac{1}{p'} = 1$.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We first recall some properties of the auxiliary function $\gamma(x)$, which will be used below.

Lemma 2.1 (see [29]) *If $V \in B_{\frac{n}{2}}$, then there exist c_0 and $k_0 \geq 1$, such that for all $x, y \in \mathbb{R}^n$,*

$$\frac{1}{c_0}\gamma(x)\Big(1 + \frac{|x-y|}{\gamma(x)}\Big)^{-k_0} \leq \gamma(y) \leq c_0\gamma(x)\Big(1 + \frac{|x-y|}{\gamma(x)}\Big)^{\frac{k_0}{k_0+1}}. \tag{2.1}$$

In particular, $\gamma(x) \sim \gamma(y)$ if $|x-y| < C\gamma(x)$, and the ball $B(x, \gamma(x))$ is called critical.

Proof of Theorem 1.1 Without loss of generally, we may assume that $\alpha < 0$. Picking any $x_0 \in \mathbb{R}^n$ and $r > 0$, we write

$$f(x) = f_0(x) + \sum_{i=1}^{\infty} f_i(x), \tag{2.2}$$

where $f_0 = f\chi_{B(x_0,2r)}$, $f_i = f\chi_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}$ for $i \geq 1$. Then

$$\begin{aligned}\Big(\int_{B(x_0,r)} |\mathcal{V}_\rho(W_t^{\mathcal{L}})(f)(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} &\leq C\Big(\int_{B(x_0,r)} |\mathcal{V}_\rho(W_t^{\mathcal{L}})(f_0)(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \\ &\quad + C\sum_{i=1}^{\infty}\Big(\int_{B(x_0,r)} |\mathcal{V}_\rho(W_t^{\mathcal{L}})(f_i)(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}}.\end{aligned}$$

By the L^p-boundedness of $\mathcal{V}_\rho(W_t^{\mathcal{L}})$, we have

$$\int_{B(x_0,r)} |\mathcal{V}_\rho(W_t^{\mathcal{L}})(f_0)(x)|^p \mathrm{d}x \leq \int_{B(x_0,2r)} |f(x)|^p \mathrm{d}x \leq \Big(1 + \frac{r}{\gamma(x_0)}\Big)^{-\alpha} r^\lambda \|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}. \tag{2.3}$$

Assume that $\{t_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero. For every $i \geq 1$, we can write

$$\begin{aligned}&\Big(\sum_{j=0}^{\infty}\Big|\int_{\mathbb{R}^n}(\mathcal{W}_{t_j}^{\mathcal{L}}(x,y) - \mathcal{W}_{t_{j+1}}^{\mathcal{L}}(x,y))f_i(y)\mathrm{d}y\Big|^\rho\Big)^{\frac{1}{\rho}} \\ &\leq C\sum_{j=0}^{\infty}\int_{\mathbb{R}^n}|f_i(y)|\int_{t_{j+1}}^{t_j}\Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t\mathrm{d}y \\ &\leq C\int_{\mathbb{R}^n}|f_i(y)|\int_0^{\infty}\Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t\mathrm{d}y, \qquad x \in B(x_0, r).\end{aligned} \tag{2.4}$$

Note that for every $k \in \mathbb{N}$, there exist c, $C > 0$ such that (see [14, (2.7)])

$$\Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big| \leq \frac{C}{t^{\frac{n}{2}+1}}\Big(1 + \frac{t}{\gamma(x)^2} + \frac{t}{\gamma(y)^2}\Big)^{-k}\mathrm{e}^{-\frac{c|x-y|^2}{t}} \quad \text{for } x,\ y \in \mathbb{R}^n,\ t > 0. \tag{2.5}$$

Then, for $x \in B(x_0, r)$ and $y \in \mathbb{R}^n \setminus B(x_0, 2r)$, we claim that for any $N \in \mathbb{N}$, there exists $C > 0$ such that

$$\int_0^\infty \Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t \le \frac{C}{(1+\frac{|x_0-y|}{\gamma(x_0)})^N}\frac{1}{|x_0-y|^n}. \tag{2.6}$$

Indeed, let $r_0 = \gamma(x_0)$ and $r_1 = |x_0 - y|$. Without loss of generality, we may assume $r_0 \le r_1$, otherwise (2.6) holds obviously. Then

$$\int_0^\infty \Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t = \int_0^{|x_0-y|^2} \Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t + \int_{|x_0-y|^2}^\infty \Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t =: \mathrm{I}_1 + \mathrm{I}_2.$$

For term I_2, applying (2.5) and Lemma 2.1, we have

$$\begin{aligned}
\mathrm{I}_2 &\le C\int_{|x_0-y|^2}^\infty t^{-\frac{n}{2}-1}\Big(1+\frac{t}{\gamma(y)^2}\Big)^{-k}\mathrm{e}^{-\frac{c|x-y|^2}{t}}\mathrm{d}t \\
&\le C\Big(1+\frac{|x_0-y|^2}{\gamma(y)^2}\Big)^{-k}\int_{|x_0-y|^2}^\infty t^{-\frac{n}{2}-1}\mathrm{d}t \\
&\le C\Big(1+\frac{|x_0-y|^2}{\gamma(y)^2}\Big)^{-k}|x_0-y|^{-n} \\
&\le C|x_0-y|^{-n}\Bigg(1+\frac{\big(\frac{|x_0-y|}{\gamma(x_0)}\big)^2}{c_0\big(1+\frac{|x_0-y|}{\gamma(x_0)}\big)^{\frac{2k_0}{k_0+1}}}\Bigg)^{-k} \\
&\le C|x_0-y|^{-n}\Big(1+\frac{|x_0-y|}{\gamma(x_0)}\Big)^{\frac{(k_0-1)k}{k_0+1}} \\
&\le C|x_0-y|^{-n}\Big(1+\frac{|x_0-y|}{\gamma(x_0)}\Big)^{-N}
\end{aligned}$$

by taking $N = \big[\frac{k(k_0-1)}{k_0+1}\big]$ for any $k \in \mathbb{N}$.

For term I_1, using (2.5) and Lemma 2.1 again, we have

$$\begin{aligned}
\mathrm{I}_1 &\le \int_0^{r_0^2}\Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t + \int_{r_0^2}^{r_1^2}\Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t \\
&\le C\int_{\frac{r_1^2}{r_0^2}}^\infty \Big(\frac{|x_0-y|^2}{u}\Big)^{-\frac{n}{2}-1}\frac{|x_0-y|^2}{u^2}\mathrm{e}^{-cu}\mathrm{d}u + C\int_{r_0^2}^{r_1^2} t^{-\frac{n}{2}-1}\mathrm{e}^{-\frac{c|x-y|^2}{t}}\mathrm{d}t \\
&\le Cr_1^{-n}\int_{\frac{r_1^2}{r_0^2}}^\infty u^{\frac{n}{2}-1}\mathrm{e}^{-cu}\mathrm{d}u + Cr_0^{-n-2}\mathrm{e}^{-\frac{cr_1^2}{r_0^2}}r_1^2 \\
&\le Cr_1^{-n}\mathrm{e}^{-\frac{cr_1^2}{r_0^2}} + Cr_0^{-n-2}\mathrm{e}^{-\frac{cr_1^2}{r_0^2}}r_1^2 \\
&\le C|x_0-y|^{-n}\Big(1+\frac{|x_0-y|}{\gamma(x_0)}\Big)^{-N}.
\end{aligned}$$

This together with the estimate of I_2 implies that our claim holds.

Now, by (2.4) and (2.6), we get

$$\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}|f(y)|\int_0^\infty\Big|\frac{\partial}{\partial t}\mathcal{W}_t^{\mathcal{L}}(x,y)\Big|\mathrm{d}t\mathrm{d}y$$

$$\le C\int_{B(x_0,2^{i+1}r)\backslash B(x_0,2^ir)}|x_0-y|^{-n}\Big(1+\frac{|x_0-y|}{\gamma(x_0)}\Big)^{-N}|f(y)|\mathrm{d}y$$
$$\le C(2^ir)^{-\frac{n}{p}}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-N}\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|^p\mathrm{d}y\Big)^{\frac{1}{p}}$$

and

$$\begin{aligned}\int_{B(x_0,r)}|\mathcal{V}_\rho(W_t^{\mathcal{L}})(f_i)(x)|^p\mathrm{d}x &\le C(2^i)^{-n}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np}\int_{B(x_0,2^{i+1}r)}|f(y)|^p\mathrm{d}y\\ &\le C(2^i)^{\lambda-n}r^\lambda\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np-\alpha}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\\ &\le C(2^i)^{\lambda-n}r^\lambda\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},\end{aligned}$$

by taking $N=[-\alpha]+1$. Note that $\lambda<n$, we get

$$\|\mathcal{V}_\rho(W_t^{\mathcal{L}})(f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}.$$

As for the case $p=1$, by replacing (2.3) with the corresponding weak estimate, we have

$$\begin{aligned}\Big|\Big\{y\in B(x,r):|\mathcal{V}_\rho(W_t^{\mathcal{L}})(f_0)(y)|>\frac{\eta}{2}\Big\}\Big| &\le \frac{C}{\eta}\int_{B(x_0,2r)}|f(y)|\mathrm{d}y\\ &\le \frac{C}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}r^\lambda. \qquad (2.7)\end{aligned}$$

Using (2.4) and (2.6), we get

$$\begin{aligned}&\Big|\Big\{y\in B(x,r):\Big|\mathcal{V}_\rho(W_t^{\mathcal{L}})\Big(\sum_{i=1}^\infty f_i\Big)(y)\Big|>\frac{\eta}{2}\Big\}\Big|\\ &\le \frac{C}{\eta}\sum_{i=1}^\infty\int_{B(x,r)}|\mathcal{V}_\rho(W_t^{\mathcal{L}})(f_i)(y)|\mathrm{d}y\\ &\le \frac{C}{\eta}\sum_{i=1}^\infty\int_{B(x,r)}\Big(\int_{B(x_0,2^{i+1}r)\backslash B(x_0,2^ir)}|x_0-z|^{-n}\Big(1+\frac{|x_0-z|}{\gamma(x_0)}\Big)^{-N}|f(z)|\mathrm{d}z\Big)\mathrm{d}y\\ &\le \frac{C}{\eta}\sum_{i=1}^\infty(2^i)^{-n}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-N}\int_{B(x_0,2^{i+1}r)}|f(z)|\mathrm{d}z\\ &\le \frac{C}{\eta}\sum_{i=1}^\infty(2^i)^{\lambda-n}r^\lambda\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-N-\alpha}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}\\ &\le \frac{C}{\eta}\sum_{i=1}^\infty(2^i)^{\lambda-n}r^\lambda\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)} \qquad (2.8)\end{aligned}$$

by taking $N=[-\alpha]+1$. Noting $\lambda<n$ and combing the estimates (2.7) and (2.8), we get for any $\eta>0$,

$$\eta\Big(1+\frac{r}{\gamma(x)}\Big)^{\alpha}|\{y\in B(x,r):|\mathcal{V}_\rho(W_t^{\mathcal{L}}f)(y)|>\eta\}|\le Cr^\lambda\|f\|_{L^{1,\lambda}_{\alpha,V}}. \qquad (2.9)$$

Theorem 1.1 is proved.

3 Proofs of Theorems 1.2–1.3

Let $\mathfrak{R}_\ell^{\mathcal{L}}(x,y)$ and $\mathfrak{R}_\ell^{\mathcal{L},*}(x,y)$ be the kernel function of $R_\ell^{\mathcal{L}}$ and $R_\ell^{\mathcal{L},*}$, repectively.

Lemma 3.1 (see [3]) *Let $\ell = 1,\cdots,n$ and $V \in \mathrm{RH}_q$ with $q > \frac{n}{2}$. Then*
(i) *for every $k \in \mathbb{N}$, there exists $C > 0$ such that*

$$|\mathfrak{R}_\ell^{\mathcal{L}}(x,y)| \le C\frac{1}{\big(1+\frac{|x-y|}{\gamma(y)}\big)^k}\frac{1}{|x-y|^{n-1}}\Big(\int_{B\left(x,\frac{|x-y|}{4}\right)}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}+\frac{1}{|x-y|}\Big); \tag{3.1}$$

(ii) *when $q > n$, the term involving V can be dropped from inequalities* (3.1).

Proof of Theorem 1.2 It is enough to prove the result when $\frac{n}{2} < q < n$. Indeed, according to [15], if $V \in \mathrm{RH}_{\frac{n}{2}}$, then $V \in \mathrm{RH}_{\varepsilon+\frac{n}{2}}$ for some $\varepsilon > 0$. Moreover, $\mathrm{RH}_{q_1} \subset \mathrm{RH}_{q_2}$, when $q_1 > q_2$. On the other hand, noting that $R_{\ell,\varepsilon}^{\mathcal{L},*}$ is the adjoint of $R_{\ell,\varepsilon}^{\mathcal{L}}$, for $V \in \mathrm{RH}_q$ with $q \ge \frac{n}{2}$, we can write

$$\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L},*}f)(x) = \Big\|\int_{\varepsilon_{i+1}<|x-y|<\varepsilon_i}\mathfrak{R}_\ell^{\mathcal{L},*}(x,y)f(y)\mathrm{d}y\Big\|_{F_\rho} = \Big\|\int_{\varepsilon_{i+1}<|x-y|<\varepsilon_i}\mathfrak{R}_\ell^{\mathcal{L}}(y,x)f(y)\mathrm{d}y\Big\|_{F_\rho}.$$

Therefore, we need only to prove the results of $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})$, and the proof of $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L},*})$ is similar. Without loss of generality, we may assume that $\alpha < 0$. Pick any $x_0 \in \mathbb{R}^n$ and $r > 0$. As in the proof of Theorem 1.1, we write

$$f(x) = f_0(x) + \sum_{i=1}^{\infty} f_i(x),$$

where $f_0 = f\chi_{B(x_0,2r)}$, $f_i = f\chi_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}$ for $i \ge 1$. By the L^p-boundedness of $\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})$, we get

$$\int_{B(x_0,r)}|\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})(f_0)(x)|^p\mathrm{d}x \le \int_{B(x_0,2r)}|f(x)|^p\mathrm{d}x \le \Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}r^\lambda\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}. \tag{3.2}$$

From (1.3), we have

$$\begin{aligned}\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})(f_i)(x) &= \Big\|\int_{\varepsilon_{i+1}<|x-y|<\varepsilon_i}\mathfrak{R}_\ell^{\mathcal{L}}(x,y)f_i(y)\mathrm{d}y\Big\|_{F_\rho}\\ &\le \int_{\mathbb{R}^n}\|\chi_{\{\varepsilon_{i+1}<|x-y|<\varepsilon_i\}}(y)\|_{F_\rho}|\mathfrak{R}_\ell^{\mathcal{L}}(x,y)||f_i(y)|\mathrm{d}y\\ &\le \int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}|\mathfrak{R}_\ell^{\mathcal{L}}(x,y)||f(y)|\mathrm{d}y.\end{aligned} \tag{3.3}$$

In the term last but one, we have used $\|\chi_{\varepsilon_{i+1}<|x-y|<\varepsilon_i}(y)\|_{F_\rho} \le 1$.

Now it follows from Lemma 3.1 that

$$\begin{aligned}&\int_{B(x_0,r)}|\mathcal{V}_\rho(R_{\ell,\varepsilon}^{\mathcal{L}})(f_i)(x)|^p\mathrm{d}x\\ &\le C\int_{B(x_0,r)}\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}|\mathfrak{R}_\ell^{\mathcal{L}}(x,y)||f(y)|\mathrm{d}y\Big)^p\mathrm{d}x\\ &\le C\int_{B(x_0,r)}\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-k}|x-y|^{-n}|f(y)|\mathrm{d}y\Big)^p\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+C\int_{B(x_0,r)}\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-k}|x-y|^{1-n}|f(y)|\\
&\times\Big(\int_{B\left(x,\frac{|x-y|}{4}\right)}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)\mathrm{d}y\Big)^p\mathrm{d}x\\
=:{}&\mathrm{A}_1+\mathrm{A}_2.
\end{aligned}
$$

For term A_1, using Lemma 2.1, we have

$$
\begin{aligned}
\mathrm{A}_1&\le C\int_{B(x_0,r)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}(2^ir)^{-np}\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\mathrm{d}y\Big)^p\mathrm{d}x\\
&\le C\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}(2^ir)^{-np}r^n\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\mathrm{d}y\Big)^p\\
&\le C\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}(2^ir)^{-np+\frac{np}{p'}}r^n\int_{B(x_0,2^{i+1}r)}|f(y)|^p\mathrm{d}y\\
&\le C\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}-\alpha}r^\lambda(2^i)^{\lambda-n}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\\
&\le C(2^i)^{\lambda-n}r^\lambda\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}-\alpha}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}.
\end{aligned}\tag{3.4}
$$

Now, we estimate the term A_2. Using (2.1) and Hölder's inequality, we can write

$$
\begin{aligned}
\mathrm{A}_2\le{}&C\int_{B(x_0,r)}(2^ir)^{(1-n)p}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}\\
&\times\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}|f(y)|\Big(\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)\mathrm{d}y\Big)^p\mathrm{d}x\\
\le{}&C\int_{B(x_0,r)}(2^ir)^{(1-n)p}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}\\
&\times\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\Big(\int_{B(x_0,2^{i+2}r)}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)\mathrm{d}y\Big)^p\mathrm{d}x\\
\le{}&C\int_{B(x_0,r)}(2^ir)^{(1-n)p}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}\\
&\times\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\mathrm{d}y\Big)^p\mathrm{d}x\\
\le{}&C\int_{B(x_0,r)}(2^ir)^{(1-n)p}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)|^p\\
&\times\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\mathrm{d}y\Big)^p\mathrm{d}x\\
\le{}&C(2^ir)^{(1-n)p}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}(2^{i+1}r)^{\frac{pn}{p'}}\int_{B(x_0,2^{i+1}r)}|f(y)|^p\mathrm{d}y\\
&\times\int_{B(x_0,r)}|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)|^p\mathrm{d}x.
\end{aligned}
$$

Using Hölder's inequality again and the boundedness of the 1-th Euclidean fractional integral $\mathcal{I}_1 : L^q \to L^{p_0}$ with $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$ (see [1]), we obtain that

$$\begin{aligned}\int_{B(x_0,r)} |\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)|^p \mathrm{d}x &\le C\|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})\|_{L^{p_0}}^p |B(x_0,r)|^{1-\frac{p}{p_0}} \\ &\le Cr^{n-\frac{pn}{p_0}} \|V\chi_{B(x_0,2^{i+2}r)}\|_{L^q}^p.\end{aligned}$$

Moreover, $V \in B_q$ for some $q > 1$ implies that V satisfies the doubling condition, i.e., there exist constants $\mu \ge 1$ and C such that

$$\int_{tB} V(x)\mathrm{d}x \le Ct^{n\mu} \int_B V(x)\mathrm{d}x$$

holds for every ball B and $t > 1$. Therefore,

$$\|V\chi_{B(x_0,2^{i+2}r)}\|_{L^q} \le C(2^i r)^{\frac{n}{q}-n} \int_{B(x_0,2^{i+1}r)} V(x)\mathrm{d}x \le C(2^i r)^{\frac{n}{q}-n} \Big(\frac{2^i r}{\gamma(x_0)}\Big)^{n\mu} \gamma(x_0)^{n-2}.$$

Then we have

$$\begin{aligned}\mathrm{A}_2 &\le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}^p (2^i r)^{2p-np} \Big(1 + \frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}} (2^i r)^\lambda (2^i)^{\frac{pn}{p_0}-n} \Big(\frac{2^i r}{\gamma(x_0)}\Big)^{pn\mu} \gamma(x_0)^{p(n-2)} \\ &\le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}^p (2^i)^{\lambda+\frac{pn}{p_0}-n} r^\lambda \Big(1 + \frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}} \Big(\frac{2^i r}{\gamma(x_0)}\Big)^{2p-pn+pn\mu} \\ &\le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}^p (2^i)^{\lambda+\frac{pn}{p_0}-n} r^\lambda \Big(1 + \frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+2p-pn+pn\mu} \\ &\le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}^p (2^i)^{\lambda+\frac{pn}{p_0}-n} r^\lambda \Big(1 + \frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+2p-pn+pn\mu} \Big(1 + \frac{r}{\gamma(x_0)}\Big)^{-\alpha}.\end{aligned}$$

This together with (3.4) by taking $k = ([-\alpha + 2 + n(\mu - 1)] + 1)(k_0 + 1)$ implies that

$$\|\mathcal{V}_\rho(R^{\mathcal{L}}_{\ell,\varepsilon})(f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}.$$

As for the case $p = 1$, by replacing (3.2) with the corresponding weak estimate, we have

$$\begin{aligned}\Big|\Big\{y \in B(x,r) : |\mathcal{V}_\rho(R^{\mathcal{L}}_{\ell,\varepsilon})(f_0)(y)| > \frac{\eta}{2}\Big\}\Big| &\le \frac{C}{\eta} \int_{B(x_0,2r)} |f(y)|\mathrm{d}y \\ &\le \frac{C}{\eta} \|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)} \Big(1 + \frac{r}{\gamma(x_0)}\Big)^{-\alpha} r^\lambda. \qquad (3.5)\end{aligned}$$

According to (3.3) and (3.1), we get

$$\begin{aligned}&\Big|\Big\{y \in B(x,r) : \Big|\mathcal{V}_\rho(R^{\mathcal{L}}_{\ell,\varepsilon})\Big(\sum_{i=1}^\infty f_i\Big)(y)\Big| > \frac{\eta}{2}\Big\}\Big| \\ &\le \frac{C}{\eta} \sum_{i=1}^\infty \int_{B(x,r)} |\mathcal{V}_\rho(R^{\mathcal{L}}_{\ell,\varepsilon})(f_i)(y)|\mathrm{d}y \\ &\le \frac{C}{\eta} \sum_{i=1}^\infty \int_{B(x,r)} \Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)} |\mathfrak{R}^{\mathcal{L}}_\ell(x,z)||f(z)|\mathrm{d}z\Big)\mathrm{d}y\end{aligned}$$

$$\begin{aligned}&\leq \frac{C}{\eta}\sum_{i=1}^{\infty}\int_{B(x,r)}\Big(\int_{B(x_0,2^{i+1}r)\backslash B(x_0,2^ir)}\frac{|f(z)|}{(1+\frac{|x-z|}{\gamma(z)})^k|x-z|^n}\mathrm{d}z\Big)\mathrm{d}y\\&\quad+\frac{C}{\eta}\sum_{i=1}^{\infty}\int_{B(x,r)}\Big(\int_{B(x_0,2^{i+1}r)\backslash B(x_0,2^ir)}\frac{|f(z)|\mathrm{d}z}{(1+\frac{|x-z|}{\gamma(z)})^k|x-z|^{n-1}}\\&\quad\times\Big(\int_{B(x,\frac{|x-z|}{4})}\frac{V(u)\mathrm{d}u}{|u-x|^{n-1}}\Big)\Big)\mathrm{d}y\\&=:\mathrm{D}_1+\mathrm{D}_2.\end{aligned}$$

In a similar way to the estimates for terms A_1 and A_2 with $p=1$, we can take $k=([-\alpha+2+n(\mu-1)]+1)(k_0+1)$ and note that $0<\lambda<\big(1-\frac{1}{p_0}\big)n$ to obtain

$$\mathrm{D}_1\leq \frac{C}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}r^{\lambda}$$

and

$$\mathrm{D}_2\leq \frac{C}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}r^{\lambda}.$$

These together with (3.5) imply that

$$\eta\Big(1+\frac{r}{\gamma(x)}\Big)^{\alpha}|\{y\in B(x,r):|\mathcal{V}_\rho(R^{\mathcal{L},*}_{\ell,\varepsilon}f)(y)|>\eta\}|\leq Cr^{\lambda}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.2.

In what follows, we will prove Theorem 1.3. The following property of $\mathrm{BMO}_\infty(\gamma)$ functions will be useful.

Lemma 3.2 (see [4]) *Let $\theta>0$ and $1\leq s<\infty$. If $b\in\mathrm{BMO}_\theta(\gamma)$, then*

$$\Big(\frac{1}{|B|}\int_B|b-b_B|^s\Big)^{\frac{1}{s}}\leq C\|b\|_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{\theta'}\tag{3.6}$$

for all $B=B(x_0,r)$, with $x_0\in\mathbb{R}^n$ and $r>0$, where $\theta'=(k_0+1)\theta$ and k_0 is the constant appearing in (2.1).

Proof of Theorem 1.3 We need only to prove the results of $\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})(f)$ in part (i). Without loss of generality, we may assume that $\alpha<0$. Pick any $x_0\in\mathbb{R}^n$ and $r>0$, and write

$$f(x)=f_0(x)+\sum_{i=1}^{\infty}f_i(x),$$

where $f_0=f\chi_{B(x_0,2r)}$, $f_i=f\chi_{B(x_0,2^{i+1}r)\backslash B(x_0,2^ir)}$ for $i\geq 1$. By the L^p-boundedness of $\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})$, we have

$$\begin{aligned}\int_{B(x_0,r)}|\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})(f_0)(x)|^p\mathrm{d}x&\leq C\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\int_{B(x_0,2r)}|f(x)|^p\mathrm{d}x\\&\leq C\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}r^{\lambda}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}\tag{3.7}$$

Set

$$b_r = |B(x_0,r)|^{-1}\int_{B(x_0,r)} b(x)\mathrm{d}x.$$

For $i \geq 1$, according to (1.3), we have

$$\begin{aligned}
\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})(f_i)(x) &= \|R^{\mathcal{L}}_{b,\ell,\varepsilon}(f_i)(x))\|_{F_\rho} \\
&= \Big\|\int_{\varepsilon_{i+1}<|x-y|<\varepsilon_i}(b(x)-b(y))\mathfrak{R}^{\mathcal{L}}_\ell(x,y)f_i(y)\mathrm{d}y\Big\|_{F_\rho} \\
&\leq \int_{\mathbb{R}^n}\|\chi_{\{\varepsilon_{i+1}<|x-y|<\varepsilon_i\}(y)}\|_{F_\rho}|b(x)-b(y)||\mathfrak{R}^{\mathcal{L}}_\ell(x,y)||f_i(y)|\mathrm{d}y \\
&\leq \int_{\mathbb{R}^n}|b(x)-b(y)||\mathfrak{R}^{\mathcal{L}}_\ell(x,y)||f_i(y)|\mathrm{d}y \\
&\leq |b(x)-b_r|\int_{B(x_0,2^{i+1}r\setminus B(x_0,2^ir))}|\mathfrak{R}^{\mathcal{L}}_\ell(x,y)||f(y)|\mathrm{d}y \\
&\quad + \int_{B(x_0,2^{i+1}r\setminus B(x_0,2^ir))}|b(y)-b_r||\mathfrak{R}^{\mathcal{L}}_\ell(x,y)||f(y)|\mathrm{d}y.
\end{aligned}$$

Applying Lemma 3.1, we can write

$$\begin{aligned}
&\int_{B(x_0,r)}|\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})(f_i)(x)|^p\mathrm{d}x \\
&\leq C\int_{B(x_0,r)}|b(x)-b_r|^p\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-k}|x-y|^{-n}|f(y)|\mathrm{d}y\Big)^p\mathrm{d}x \\
&\quad + C\int_{B(x_0,r)}|b(x)-b_r|^p\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-k}|x-y|^{1-n}|f(y)| \\
&\quad \times\Big(\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)\mathrm{d}y\Big)^p\mathrm{d}x \\
&\quad + C\int_{B(x_0,r)}\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}|b(y)-b_r|\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-k}|x-y|^{-n}|f(y)|\mathrm{d}y\Big)^p\mathrm{d}x \\
&\quad + C\int_{B(x_0,r)}\Big(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}|b(y)-b_r|\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-k}|x-y|^{1-n}|f(y)| \\
&\quad \times\Big(\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)\mathrm{d}y\Big)^p\mathrm{d}x \\
&=: \mathrm{B}_1+\mathrm{B}_2+\mathrm{B}_3+\mathrm{B}_4.
\end{aligned}$$

Using Lemma 2.1, Hölder's inequality and Lemma 3.2, we have

$$\begin{aligned}
\mathrm{B}_1 &\leq \int_{B(x_0,r)}|b(x)-b_r|^p\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}(2^ir)^{-np}\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\mathrm{d}y\Big)^p\mathrm{d}x \\
&\leq C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}}(2^ir)^{\lambda-np}(2^ir)^{\frac{pn}{p'}}\int_{B(x_0,r)}|b(x)-b_r|^p\mathrm{d}x \\
&\leq C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}(2^ir)^{\lambda-n}r^n\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}
\end{aligned}$$

$$\le C\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}(2^i)^{\lambda-n}r^\lambda$$

$$\le C\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}(2^i)^{\lambda-n}r^\lambda. \tag{3.8}$$

Similarly, we can get

$$\begin{aligned}
\mathrm{B}_3 &\le C\int_{B(x_0,r)}(2^ir)^{-np}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}\Big(\int_{B(x_0,2^{i+1}r)}|b(y)-b_r||f(y)|\mathrm{d}y\Big)^p\mathrm{d}x\\
&\le C(2^ir)^{-np}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}r^n\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|^p\mathrm{d}y\Big)\Big(\int_{B(x_0,2^{i+1}r)}|b(y)-b_r|^{p'}\mathrm{d}y\Big)^{\frac{p}{p'}}\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}}(2^ir)^{\lambda-np+\frac{pn}{p'}}r^n\\
&\quad\times\Big(\frac{1}{|B(x_0,2^{i+1}r)|}\int_{B(x_0,2^{i+1}r)}|b(y)-b_{2^{i+1}r}|^{p'}\mathrm{d}y+|b_{2^{i+1}r}-b_r|^{p'}\Big)^{\frac{p}{p'}}\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}(\gamma)}(i+1)^p\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}(2^ir)^{\lambda-n}r^n\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}(\gamma)}(i+1)^p\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}\\
&\quad\times\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}(2^i)^{\lambda-n}r^\lambda.
\end{aligned}\tag{3.9}$$

Imitating the estimation of term A_2, using Lemma 2.1, Hölder's inequality and Lemma 3.2 again, we obtain

$$\begin{aligned}
\mathrm{B}_2 &\le C\int_{B(x_0,r)}|b(x)-b_r|^p(2^ir)^{(1-n)p}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}\\
&\quad\times\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|\Big(\int_{B(x_0,2^{i+2}r)}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)\mathrm{d}y\Big)^p\mathrm{d}x\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}}(2^ir)^{\lambda-n+p}\\
&\quad\times\int_{B(x_0,r)}|b(x)-b_r|^p\Big(\int_{B(x_0,2^{i+2}r)}\frac{V(z)\mathrm{d}z}{|z-x|^{n-1}}\Big)^p\mathrm{d}x\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}}(2^ir)^{\lambda-n+p}\|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\|^p_{L^{p_0}}\\
&\quad\times\Big(\int_{B(x_0,r)}|b(x)-b_r|^{\frac{pp_0}{p_0-p}}\mathrm{d}x\Big)^{\frac{p_0-p}{p_0}}\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}(2^ir)^{\lambda+p-n}r^{\frac{(p_0-p)n}{p_0}}\\
&\quad\times\|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\|^p_{L^{p_0}}\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta'p}(2^ir)^{\lambda+p-\frac{pn}{p_0}}(2^i)^{\frac{(p-p_0)n}{p_0}}
\end{aligned}$$

$$\times \|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\|^p_{L^{p_0}}.$$

According to the boundedness of the 1-th Euclidean fractional integral $\mathcal{I} : L^q \to L^{p_0}$ with $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, we have

$$\|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\|^p_{L^{p_0}} \le C(2^i r)^{\frac{pn}{q}-np}\Big(\frac{2^i r}{\gamma(x_0)}\Big)^{\mu np}\gamma(x_0)^{(n-2)p}.$$

Therefore, we obtain

$$\begin{aligned}
\mathrm{B}_2 &\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p}(2^i r)^{\lambda+2p-np}\\
&\quad\times (2^i)^{\frac{(p-p_0)n}{p_0}}\Big(\frac{2^i r}{\gamma(x_0)}\Big)^{\mu np}\gamma(x_0)^{(n-2)p}\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p}(2^i r)^{\lambda+2p-np}\\
&\quad\times (2^i)^{\frac{(p-p_0)n}{p_0}}\Big(\frac{2^i r}{\gamma(x_0)}\Big)^{\mu np}\gamma(x_0)^{(n-2)p}\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p}\Big(\frac{2^i r}{\gamma(x_0)}\Big)^{2p+(\mu-1)np}\\
&\quad\times (2^i)^{\lambda+\frac{(p-p_0)n}{p_0}}r^\lambda\\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}_\theta(\gamma)}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p+2p+(\mu-1)np}\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}\\
&\quad\times (2^i)^{\lambda+\frac{pn}{p_0}-n}r^\lambda.
\end{aligned}\tag{3.10}$$

Similarly, we can estimate B_4 as follows:

$$\begin{aligned}
\mathrm{B}_4 &\le C\int_{B(x_0,r)}(2^i r)^{(1-n)p}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)|^p\\
&\qquad\times\Big(\int_{B(x_0,2^{i+1}r)}|b(y)-b_r||f(y)|\mathrm{d}y\Big)^p\mathrm{d}x\\
&\quad\le C(2^i r)^{(1-n)p}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\frac{kp}{k_0+1}}\|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\|^p_{L^{p_0}}r^{(1-\frac{p}{p_0})n}\\
&\qquad\times\Big(\int_{B(x_0,2^{i+1}r)}|f(y)|^p\mathrm{d}y\Big)\Big(\int_{B(x_0,2^{i+1}r)}|b(y)-b_r|^{p'}\mathrm{d}y\Big)^{\frac{p}{p'}}\\
&\quad\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}(\gamma)}(i+1)^p\Big(1+\frac{2^i r}{\gamma(x)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p}(2^i r)^{p-pn+\frac{pn}{p'}+\lambda}\\
&\qquad\times r^{(1-\frac{p}{p_0})n}\|\mathcal{I}_1(V\chi_{B(x_0,2^{i+2}r)})(x)\|^p_{L^{p_0}}\\
&\quad\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}(\gamma)}(i+1)^p\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p}(2^i)^{\frac{pn}{p_0}-n+\lambda}\\
&\qquad\times r^\lambda\Big(\frac{2^i r}{\gamma(x_0)}\Big)^{2p+(\mu-1)np}
\end{aligned}$$

$$\leq C\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\|b\|^p_{\mathrm{BMO}(\gamma)}(i+1)^p\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{kp}{k_0+1}+\theta' p+2p+(\mu-1)np}$$
$$\times\Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha}(2^i)^{\frac{pn}{p_0}-n+\lambda}r^\lambda. \tag{3.11}$$

Then (3.8)–(3.11) by taking $k=([-\alpha+\theta'+2+(\mu-1)n]+1)(k_0+1)$ imply that

$$\|\mathcal{V}_\rho(R^{\mathcal{L}}_{b,\ell,\varepsilon})(f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\leq C\|b\|_{\mathrm{BMO}_\theta(\gamma)}\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.3.

Remark 3.1 The fluctuations of a family $\mathcal{T}=\{T_t\}_{t>0}$ of operators when $t\to 0^+$ can also be analyzed by using oscillation operators (see, for instance, [5, 20] etc.). If $\{t_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero, the oscillation operator $\mathcal{O}(\mathcal{T})$ is defined by

$$\mathcal{O}(\mathcal{T}f)(x):=\Big(\sum_{i=1}^{\infty}\sup_{t_{i+1}\leq\varepsilon_{i+1}<\varepsilon_i\leq t_i}|T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)|^2\Big)^{\frac{1}{2}}.$$

Then, by using the procedures developed in this paper, we can establish the corresponding conclusions in Theorem 1.1 for $\mathcal{O}(W^{\mathcal{L}}_t)$, Theorem 1.2 for $\mathcal{O}(R^{\mathcal{L}}_{\ell,\varepsilon})$ and $\mathcal{O}(R^{\mathcal{L},*}_{\ell,\varepsilon})$, Theorem 1.3 for $\mathcal{O}(R^{\mathcal{L}}_{b,\ell,\varepsilon})$ and $\mathcal{O}(R^{\mathcal{L},*}_{b,\ell,\varepsilon})$. The details are omitted.

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