Chin. Ann. Math. Ser. B
39(6), 2018, 989–1000
DOI: 10.1007/s11401-018-0109-1

Chinese Annals of Mathematics, Series B

Strongly Gauduchon Spaces

Wei XIA[1] Lingxu MENG[2]

Abstract The authors define strongly Gauduchon spaces and the class $\mathscr{SG}$, which are generalization of strongly Gauduchon manifolds in complex spaces. Comparing with the case of Kählerian, the strongly Gauduchon space and the class $\mathscr{SG}$ are similar to the Kähler space and the Fujiki class $\mathscr{C}$ respectively. Some properties about these complex spaces are obtained. Moreover, the relations between the strongly Gauduchon spaces and the class $\mathscr{SG}$ are studied.

Keywords Strongly Gauduchon metric, Strongly Gauduchon space, Class $\mathscr{SG}$, Topologically essential map
2000 MR Subject Classification 32C15, 32C10, 53C55

1 Introduction

The complex manifold with a strongly Gauduchon metric is an important object in non-Kähler geometry. In [13, 15], Popovici first defined the strongly Gauduchon metric in the study of limits of projective manifolds under deformations. A strongly Gauduchon metric on a complex n-dimensional manifold is a hermitian metric ω such that $\partial\omega^{n-1}$ is $\overline{\partial}$-exact. A compact complex manifold is called a strongly Gauduchon manifold, if there exists a strongly Gauduchon metric on it.

Proposition 1.1 *Let M be a compact complex manifold of dimension n. Then the following is equivalent:*

(1) *M is a strongly Gauduchon manifold.*

(2) *There exists a strictly positive $(n-1, n-1)$-form Ω, such that $\partial\Omega$ is $\overline{\partial}$-exact.*

(3) *There exists a real closed $(2n-2)$-form Ω whose $(n-1, n-1)$-component $\Omega^{n-1,n-1}$ is strictly positive.*

In [13], Popovici observed that (1) and (3) are equivalent. "(1) $\Rightarrow$ (2)" is obvious by the definition of strongly Gauduchon manifolds. Conversely, for any strictly positive $(n-1, n-1)$-form Ω, there exists a unique strictly positive $(1, 1)$-form ω such that $\omega^{n-1} = \Omega$ (see [12, p. 280]). So we have "(2) $\Rightarrow$ (1)".

Popovici proved the following two important theorems.

Manuscript received November 27, 2015. Revised June 14, 2016.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: 11110180005@fudan.edu.cn
[2]Department of Mathematics, North University of China, Taiyuan 030051, China.
E-mail: 20160012@nuc.edu.cn

Theorem 1.1 (see [13, Proposition 3.3]) *Let M be a compact complex manifold. Then M is a strongly Gauduchon manifold if and only if there is no nonzero positive current T of bidegree $(1,1)$ on M which is d-exact on M.*

Theorem 1.2 (see [14, Theorem 1.3]) *Let $f : M \to N$ be a modification of compact complex manifolds. Then M is a strongly Gauduchon manifold if and only if N is a strongly Gauduchon manifold.*

On the other hand, in [8], Fujiki generalized the concept "Kähler" to general complex spaces. A kind of generalization is the Kähler space which is a complex space admitting a strictly positive closed $(1,1)$-form, and the other kind is the Fujiki class $\mathscr{C}$ consisting of the reduced compact complex spaces which are the meoromorphic images of compact Kähler spaces. In [16–17], Varouchas proved that any reduced complex space in the Fujiki class $\mathscr{C}$ has a proper modification which is a compact Kähler manifold. Now, many authors use it as the definition of the Fujiki class $\mathscr{C}$. Inspired by the method of Fujiki and the theorem of Varouchas, we give two kinds of generalizations of strongly Gauduchon manifolds to complex spaces — the strongly Gauduchon spaces and the class $\mathscr{SG}$. In view of definitions of them, the strongly Gauduchon spaces (see Definition 2.1) is similar to the Kähler spaces, and the class $\mathscr{SG}$ (see Definition 3.1) is similar to the Fujiki class $\mathscr{C}$.

In Section 2, we study the properties of strongly Gauduchon spaces and give a method of constructing examples which are singular strongly Gauduchon spaces, but not in $\mathscr{B}$, where $\mathscr{B}$ is the set of reduced compact complex spaces which are bimeromorphic to compact balanced manifolds.

In Section 3, we study the class $\mathscr{SG}$ and propose a conjecture on the relation between strongly Gauduchon spaces and the class $\mathscr{SG}$ as follows.

Conjecture 1.1 *Any strongly Gauduchon space belongs to class $\mathscr{SG}$.*

We prove it in some special cases (see Theorems 3.2–3.4).

In Section 4, we study a family of reduced complex spaces over a nonsingular curve and give a theorem on the total space being in $\mathscr{SG}$.

2 Strongly Gauduchon Spaces

First, we give a proposition about strongly Gauduchon manifolds which is similar to the case of balanced manifolds.

Proposition 2.1 *Let M and N be compact complex manifolds of pure dimension.*

(1) *If $f : M \to N$ is a holomorphic submersion and M is a strongly Gauduchon manifold, then N is a strongly Gauduchon manifold.*

(2) *$M \times N$ is a strongly Gauduchon manifold if and only if M and N are both strongly Gauduchon manifolds.*

Proof Set $\dim M = m$, $\dim N = n$.

(1) Let Ω_M be a strictly positive $(m-1, m-1)$-form, such that $\partial\Omega_M = \overline{\partial}\alpha$, where α is a

$(2m-2)$-form on M. Define

$$\Omega_N := f_*\Omega_M.$$

By the proof of Proposition 1.9(ii) in [12], we know that Ω_N is a strictly positive $(n-1, n-1)$-form. Obviously, $\partial\Omega_N = \overline{\partial}(f_*\alpha)$ is $\overline{\partial}$-exact. So N is a strongly Gauduchon manifold.

(2) If $M \times N$ is a strongly Gauduchon manifold, then M and N are both strongly Gauduchon manifolds by (1).

Conversely, let M and N be both strongly Gauduchon manifolds. Suppose that ω_M and ω_N are strongly Gauduchon metrics on M and N, respectively, such that $\partial\omega_M^{m-1} = \overline{\partial}\alpha$ and $\partial\omega_N^{n-1} = \overline{\partial}\beta$, where α and β are $(2m-2)$- and $(2n-2)$-form on M and N, respectively. We define a metric on $M \times N$

$$\omega := \omega_M + \omega_N.$$

Then

$$\omega^{m+n-1} := C_1\omega_M^{m-1} \wedge \omega_N^n + C_2\omega_M^m \wedge \omega_N^{n-1},$$

where C_1, C_2 are constants. So

$$\begin{aligned}\partial\omega^{m+n-1} : &= C_1\partial\omega_M^{m-1} \wedge \omega_N^n + C_2\omega_M^m \wedge \partial\omega_N^{n-1} \\ &= \overline{\partial}(C_1\alpha \wedge \omega_N^n + C_2\omega_M^m \wedge \beta)\end{aligned}$$

is $\overline{\partial}$-exact on $M \times N$. Hence ω is a strongly Gauduchon metric on $M \times N$.

We recall the definitions of forms and currents on complex spaces (see [11]).

Let X be a reduced complex space and X_{reg} be the set of nonsingular points on X. Obviously, X_{reg} is a complex manifold.

Suppose that X is an analytic subset of a complex manifold M. Set $I_X^{p,q}(M) = \{\alpha \in A^{p,q}(M) \mid i^*\alpha = 0\}$, where $i : X_{\mathrm{reg}} \to M$ is the inclusion. Define $A^{p,q}(X) := A^{p,q}(M)/I_X^{p,q}(M)$. It can be easily shown that $A^{p,q}(X)$ does not depend on the embedding of X into M. Hence, for any complex space X, we can define $A^{p,q}(X)$ through the local embeddings in $\mathbb{C}^N$. More precisely, we define a sheaf of germs $\mathcal{A}_X^{p,q}$ of (p,q)-forms on X and $A^{p,q}(X)$ as the group of its global sections. Similarly, we can also define $A_c^{p,q}(X)$ (the space of (p,q)-forms with compact supports), $A^k(X)$ and $A_c^k(X)$.

We can naturally define $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$, $\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$ and $d : A^k(X) \to A^{k+1}(X)$.

If $f : X \to Y$ is a holomorphic map between reduced complex spaces, then we can naturally define $f^* : A^{p,q}(Y) \to A^{p,q}(X)$, such that f^* commutes with ∂, $\overline{\partial}$, d.

When X is a subvariety of a complex manifold M, we define the space of currents on X

$$\mathcal{D}'^r(X) := \{T \in \mathcal{D}'^r(M) \mid T(u) = 0,\ \forall u \in I_{X,c}^{2n-r}(M)\},$$

where $\mathcal{D}'^r(M)$ is the space of currents on M and $I_{X,c}^{2n-r}(M) = \{\alpha \in A_c^{2n-r}(M) \mid i^*\alpha = 0\}$. We can define a space $\mathcal{D}'^r(X)$ of the currents on any reduced complex space X as the case of $A^r(X)$. Define

$$\mathcal{D}'^{p,q}(X) := \{T \in \mathcal{D}'^{p+q}(X) \mid T(u) = 0,\ \forall u \in A_c^{r,s}(M),\ (r,s) \neq (n-p, n-q)\}.$$

A current T is called a (p,q)-current on X, if $T \in \mathcal{D}'^{p,q}(X)$. If $T \in \mathcal{D}'^{r}(X)$, we call r the degree. If $T \in \mathcal{D}'^{p,q}(X)$, we call (p,q) the bidegree. We also denote $\mathcal{D}'_r(X) = \mathcal{D}'^{2n-r}(X)$ and $\mathcal{D}'_{p,q}(X) = \mathcal{D}'^{n-p,n-q}(X)$. A current $T \in \mathcal{D}'^{p,p}(X)$ is called real if for every $\alpha \in A_c^{2n-2p}(X)$, $T(\overline{\alpha}) = \overline{T(\alpha)}$.

If $f : X \to Y$ is a holomorphic map of reduced compact complex spaces, we define $f_* : \mathcal{D}'_r(X) \to \mathcal{D}'_r(Y)$ as $f_*T(u) := T(f^*u)$ for any $u \in A_c^r(Y)$.

A real (p,p)-form ω on X is called strictly positive, if there exists an open covering $\mathcal{U} = \{U_\alpha\}$ of X with an embedding $i_\alpha : U_\alpha \to V_\alpha$ of U_α into a domain V_α in $\mathbb{C}^{n_\alpha}$ and a strictly positive (p,p)-form ω_α on V_α, such that $\omega\,|_{U_\alpha} = i_\alpha^*\omega_\alpha$, for each α.

Now, following [2], we give a kind of generalization of strongly Gauduchon manifolds.

Definition 2.1 *A purely n-dimensional reduced compact complex space X is called a strongly Gauduchon space, if there exists a strictly positive $(n-1,n-1)$-form Ω, such that $\partial\Omega$ is $\overline{\partial}$-exact.*

By its definition, it is easy to see that X is a strongly Gauduchon space if and only if there exists a real closed $(2n-2)$-form Ω' on X whose $(n-1,n-1)$-component $\Omega'^{n-1,n-1}$ is strictly positive. Indeed, if Ω is a strictly positive $(n-1,n-1)$-form, such that $\partial\Omega = \overline{\partial}\alpha$, where α is a $(n,n-2)$-form, then

$$\Omega' := \Omega - \alpha - \overline{\alpha}$$

is the desired form. Conversely, since Ω' is real and d-closed, $\partial\Omega'^{n-1,n-1} = -\overline{\partial}\Omega'^{n,n-2}$. Hence, $\Omega := \Omega'^{n-1,n-1}$ is the desired form.

Obviously, strongly Gauduchon manifolds and compact balanced spaces are strongly Gauduchon spaces.

Proposition 2.2 *Let X be a reduced compact complex space of pure dimension and M be a compact complex manifold of pure dimension. If $X \times M$ is a strongly Gauduchon space, then M is a strongly Gauduchon manifold.*

Proof Let X_{reg} be the set of nonsingular points on X and Ω be a strictly positive $(n+m-1,n+m-1)$-form on $X \times M$, such that $\partial\Omega$ is $\overline{\partial}$-exact, where $n = \dim X$ and $m = \dim M$. Suppose that $\pi : X_{\mathrm{reg}} \times M \to M$ is the second projection. By the proof of Proposition 1.9(ii) in [12], we know that $\pi_*(\Omega\,|_{X_{\mathrm{reg}}\times M})$ is a strictly positive $(m-1,m-1)$-form on M. Obviously, $\partial\pi_*(\Omega\,|_{X_{\mathrm{reg}}\times M})$ is $\overline{\partial}$-exact. So M is a strongly Gauduchon manifold.

We know that, on a compact balanced manifold M, the fundamental class $[V]$ of any hypersurface V is not zero in $H^2(M,\mathbb{R})$ (see [12, Corollary 1.7]). It is equivalent to that, the current $[V]$ on M defined by any hypersurface V is not d-exact. For strongly Gauduchon spaces, we have the following proposition.

Proposition 2.3 *If X is a strongly Gauduchon space, then the current $[V]$ defined by any hypersurface V of X is not $\partial\overline{\partial}$-exact.*

Proof Suppose $\dim X = n$. Let Ω be a strictly positive $(n-1,n-1)$-form on X such that $\partial\Omega = \overline{\partial}\alpha$, where α is a $(2n-2)$-form on X. If $[V] = \partial\overline{\partial}Q$ for some current Q on X, then

$$[V](\Omega) = \int_V \Omega > 0.$$

On the other hand,

$$[V](\Omega) = (\partial\overline{\partial}Q)(\Omega) = -Q(\overline{\partial}\partial\Omega) = -Q(\overline{\partial}\overline{\partial}\alpha) = 0.$$

It is a contradiction.

Proposition 2.4 *If $f : X \to Y$ is a finite holomorphic unramified covering map of reduced compact complex spaces of pure dimension, then X is a strongly Gauduchon space if and only if Y is a strongly Gauduchon space.*

Proof Set $n = \dim X = \dim Y$ and $d = \deg f$.

Let X be a strongly Gauduchon space and Ω_X be a strictly positive $(n-1, n-1)$-form on X, such that $\partial\Omega_X = \overline{\partial}\alpha_X$, where α_X is a $2(n-1)$-form on X. For every $y \in Y$, we set $f^{-1}(y) = \{x_1, \cdots, x_d\}$. Then there exists an open neighbourhood $V \subseteq Y$ of y, and open neighbourhoods $U_1, \cdots, U_d$ of $x_1, \cdots, x_d$ in X, respectively, which do not intersect with each other, such that $f^{-1}(V) = \bigcup\limits_{i=1}^{d} U_i$ and the restriction $f|_{U_i} : U_i \to V$ is an isomorphism for $i = 1, \cdots, d$. We define two forms on V as

$$\Omega_V := \sum_{i=1}^{d} (f|_{U_i}^{-1})^* (\Omega_X|_{U_i}),$$
$$\alpha_V := \sum_{i=1}^{d} (f|_{U_i}^{-1})^* (\alpha_X|_{U_i}).$$

If V and V' are two open subsets in Y as above (possible for different points in Y) and $V \cap V' \neq \emptyset$, we can easily check $\Omega_V = \Omega_{V'}$ on $V \cap V'$. Hence, we can construct a global $(n-1, n-1)$-form Ω_Y on Y such that $\Omega_Y|_V = \Omega_V$. By the same reason, we can define a global $2(n-1)$-form α_Y on Y such that $\alpha_Y|_V = \alpha_V$. Obviously, Ω_Y is strictly positive and $\partial\Omega_Y = \overline{\partial}\alpha_Y$. Therefore, Y is a strongly Gauduchon space.

Conversely, suppose that Ω_Y is a strictly positive $(n-1, n-1)$-form on Y, such that $\partial\Omega_Y$ is $\overline{\partial}$-exact on Y. For all $x \in X$, there is an open neighbourhood U of x in X, an open neighbourhood V of $f(x)$ in Y, such that $f\ |_U \colon U \to V$ is an isomorphism. $(f^*\Omega_Y)|U = (f|_U)^*(\Omega_Y|_V)$ is obviously strictly positive on U, so is $f^*\Omega_Y$ on X. Obviously, $f^*\Omega_Y$ is $\overline{\partial}$-exact on X. Therefore, X is a strongly Gauduchon space.

3 The Class $\mathscr{SG}$

Now, we give the other generalization of strongly Gauduchon manifolds.

Definition 3.1 *A reduced compact complex space X of pure dimension is called in class $\mathscr{SG}$, if it has a desingularization $\widetilde{X}$ which is a strongly Gauduchon manifold.*

If one desingularization of X is a strongly Gauduchon manifold, then every desingularization of X is a strongly Gauduchon manifold. Indeed, if $X_1 \to X$ and $X_2 \to X$ are two desingularizations of X, then there exists a bimeromorphic map $f : X_1 \dashrightarrow X_2$. Let $\Gamma \subseteq X_1 \times X_2$ be the graph of f, and $p_1 : \Gamma \to X_1$, $p_2 : \Gamma \to X_2$ be the two projections on X_1, X_2, respectively. Then p_1, p_2 are modifications. If $\widetilde{\Gamma}$ is a desingularization of Γ, then $\widetilde{\Gamma} \to X_1$ and $\widetilde{\Gamma} \to X_2$ are modifications of compact complex manifolds. By Theorem 1.2, we know that X_1 is a strongly Gauduchon manifold if and only if $\widetilde{\Gamma}$ is a strongly Gauduchon manifold, and then if and only if X_2 is a strongly Gauduchon manifold. Hence Definition 2.1 is not dependent on the choice of the desingularization of X. So, if $X \in \mathscr{SG}$ is nonsingular, then X is a strongly Gauduchon manifold.

Using the same method as above, we can prove the following proposition.

Proposition 3.1 *The class $\mathscr{SG}$ is invariant under bimeromorphic maps.*

Obviously, strongly Gauduchon manifolds and the normalizations of complex spaces in class $\mathscr{SG}$ are in class $\mathscr{SG}$. Recall that a reduced compact complex space X is called in class $\mathscr{B}$, if it has a desingulariztion $\widetilde{X}$ which is a balanced manifold (see [7]). Then complex spaces in class $\mathscr{B}$ are in class $\mathscr{SG}$.

Proposition 3.2 *If X and Y are reduced compact complex spaces, then $X \times Y$ is in the class $\mathscr{SG}$ if and only if X and Y are both in the class $\mathscr{SG}$.*

Proof If $f : \widetilde{X} \to X$ and $g : \widetilde{Y} \to Y$ are desingulariztions, then $f \times g : \widetilde{X} \times \widetilde{Y} \to X \times Y$ is a desingulariztion of $X \times Y$. By Proposition 2.1(2), we know that $\widetilde{X} \times \widetilde{Y}$ is a strongly Gauduchon manifold if and only if $\widetilde{X}$ and $\widetilde{Y}$ are both strongly Gauduchon manifolds. So we get this proposition easily.

Using this proposition, we can construct some examples of complex spaces in $\mathscr{SG}$ which are neither strongly Gauduchon manifolds nor in class $\mathscr{B}$. If Y is a singular reduced compact complex space in class $\mathscr{B}$ and Z is a compact strongly Gauduchon manifold but not a balanced manifold, then $Y \times Z$ is in $\mathscr{SG}$, but it is neither a strongly Gauduchon manifold nor in $\mathscr{B}$. Indeed, $Y \times Z$ is singular, so it is not a strongly Gauduchon manifold. By Proposition 3.2, $Y \times Z \in \mathscr{SG}$. Assume $Y \times Z \in \mathscr{B}$. By [7, Proposition 2.3], we know $Z \in \mathscr{B}$. Since Z is nonsingular, Z is balanced, which contradicts the choice of Z. Hence we get the following relations:

$$\mathscr{C} \subsetneqq \mathscr{B} \subsetneqq \mathscr{SG},$$

where $\mathscr{C}$ is the Fujiki class, and the first "$\subsetneqq$" was proved in [7, Section 2].

If X is a reduced compact complex space of pure dimension, then $X \in \mathscr{SG}$ if and only if every irreducible component of X is in $\mathscr{SG}$. Indeed, if let $\widetilde{X_1}, \cdots, \widetilde{X_r}$ be the desingulariztions of $X_1, \cdots, X_r$, all the irreducible components of X, then the disjoint union $\widetilde{X} := \widetilde{X_1} \amalg \cdots \amalg \widetilde{X_r}$ is a desingulariztion of X. Hence the conclusion follows, since $\widetilde{X}$ is a strongly Gauduchon manifold if and only if $\widetilde{X_1}, \cdots, \widetilde{X_r}$ are all strongly Gauduchon manifolds.

In the following, we need the definition of a smooth morphism (see [4, (0.4)]). A surjective holomorphic map $f : X \to Y$ between reduced complex spaces is called a smooth morphism, if for all $x \in X$, there is an open neighbourhood W of x in X, an open neighbourhood U of $f(x)$ in Y, such that $f(W) = U$, and there is a commutative diagram

$$\begin{tikzcd} W \arrow[r, "f|_W"] \arrow[d, "g"'] & U \\ \Delta^r \times U \arrow[ur, "pr_2"'] & \end{tikzcd}$$

where $r = \dim X - \dim Y$, g is an isomorphism (i.e., biholomorphic map), pr_2 is the second projection, and Δ^r is a small polydisc. Moreover, if $\dim X = \dim Y$, a smooth morphism is exactly a surjective local isomorphism.

Obviously, if $f : X \to Y$ is a smooth morphism, and Y is a complex manifold, then X must also be a complex manifold, and f is a submersion between complex manifolds.

Proposition 3.3 *Let $f : X \to Y$ be a smooth morphism of reduced compact complex spaces. If $X \in \mathscr{SG}$, then $Y \in \mathscr{SG}$.*

Proof Suppose that $p : \widetilde{Y} \to Y$ is a desingulariztion. Consider the following Cartesian diagram:

$$\begin{array}{ccc} \widetilde{X} := X \times_Y \widetilde{Y} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{3.1}$$

where $X \times_Y \widetilde{Y} = \{(x, \widetilde{y}) \in X \times \widetilde{Y} \mid f(x) = p(\widetilde{y})\}$, q is the projection to X, and $\widetilde{f}$ is the projection to $\widetilde{Y}$. We can prove that $\widetilde{f}$ is a submersion of complex manifolds and q is a modification (see [7, Claims 1–2 in the proof of Proposition 2.4]). Since $X \in \mathscr{SG}$, $\widetilde{X}$ is a strongly Gauduchon manifold, so is $\widetilde{Y}$ by Proposition 2.1(1), hence $Y \in \mathscr{SG}$.

Proposition 3.4 *If $f : X \to Y$ is a finite unramified covering map of reduced compact complex spaces, then $X \in \mathscr{SG}$ if and only if $Y \in \mathscr{SG}$.*

Proof Suppose that $p : \widetilde{Y} \to Y$ is a desingulariztion. Consider the Cartesian diagram (3.1). We know that $\widetilde{f}$ is a surjective local isomorphism, and q is a modification. Since $\widetilde{Y}$ is locally compact, by [10, Lemma 2], $\widetilde{f}$ is a finite covering map in topological sense. Moreover, since $\widetilde{f}$ is a local isomorphism (in analytic sense), $\widetilde{f}$ is a finite unramifield covering map (in analytic sense). By Proposition 2.4, we know that $\widetilde{X}$ is a strongly Gauduchon manifold, if and only if $\widetilde{Y}$ is a strongly Gauduchon manifold. Hence $X \in \mathscr{SG}$ if and only if $Y \in \mathscr{SG}$.

We generalize Theorem 3.5(2) and Theorem 3.9(2) in [1] as follows.

Proposition 3.5 *Let $f : X \to Y$ be a smooth morphism of reduced compact complex spaces, and $n = \dim X > m = \dim Y \geq 2$. If $Y \in \mathscr{B}$ and there exists a point y_0 in Y such that the current $[f^{-1}(y_0)]$ is not d-exact on X, then $X \in \mathscr{SG}$.*

Proof Choose a desingulariztion $p : \widetilde{Y} \to Y$, such that $\widetilde{Y}$ is a compact balanced manifold. Considering the Catesian diagram (3.1), we know that $\widetilde{f}$ is a submersion of complex manifolds and q is a modification.

For every $\widetilde{y} \in p^{-1}(y_0)$, the current $[\widetilde{f}^{-1}(\widetilde{y})]$ can not be written as dQ for any current Q of degree $2m-1$ on $\widetilde{X}$. Otherwise, since $\widetilde{f}^{-1}(\widetilde{y}) = f^{-1}(y_0) \times \{\widetilde{y}\}$, we have

$$[f^{-1}(y_0)] = q_*[\widetilde{f}^{-1}(\widetilde{y})] = q_*(dQ) = dq_*Q,$$

which contradicts the assumption.

Now suppose that $\widetilde{y}'$ is any point in $\widetilde{Y}$. Then the fundamental classes $[\widetilde{y}] = [\widetilde{y}']$ in $H^{2m}(\widetilde{Y}, \mathbb{R})$. Since $\widetilde{f}$ is smooth,

$$[\widetilde{f}^{-1}(\widetilde{y}')] = \widetilde{f}^*[\widetilde{y}'] = \widetilde{f}^*[\widetilde{y}] = [\widetilde{f}^{-1}(\widetilde{y})]$$

in $H^{2m}(\widetilde{X},\mathbb{R})$, where $\widetilde{y} \in p^{-1}(y_0)$ and $\widetilde{f}^*: H^{2m}(\widetilde{Y},\mathbb{R}) \to H^{2m}(\widetilde{X},\mathbb{R})$ is the pull back of $\widetilde{f}$. Hence for every $\widetilde{y}' \in \widetilde{Y}$, the current $[\widetilde{f}^{-1}(\widetilde{y}')]$ is not d-exact on $\widetilde{X}$. By [1, Theorem 3.5(2) and Theorem 3.9(2)], $\widetilde{X}$ is a strongly Gauduchon manifold, hence, $X \in \mathscr{SG}$.

Next, we consider the relation between strongly Gauduchon spaces and class $\mathscr{SG}$. From definitions of them, the relation between strongly Gauduchon spaces and the class $\mathscr{SG}$ is similar to that of Kähler spaces and the Fujiki class $\mathscr{C}$. Moreover, in the nonsingular case, we know that a modification of a strongly Gauduchon manifold is also a strongly Gauduchon manifold, by Theorem 1.2. So we think that the following also holds.

Conjecture 3.1 *Any strongly Gauduchon space belongs to class $\mathscr{SG}$.*

We can prove it in some extra conditions. First, we recall a theorem and several notations.

Theorem 3.1 (see [3, Theorem 1.5]) *Let M be a complex manifold of dimension n, E be a compact analytic subset and $\{E_i\}_{i=1,\cdots,s}$ be all the p-dimensional irreducible components of E. If T is a $\partial\overline{\partial}$-closed positive $(n-p,n-p)$-current on M such that* $\operatorname{supp} T \subseteq E$*, then there exist constants $c_i \geq 0$, such that $T - \sum_{i=1}^{s} c_i[E_i]$ is supported on the union of the irreducible components of E of dimension greater than p.*

For a compact complex manifold M, the Bott-Chern cohomology group of degree (p,q) is defined as

$$H^{p,q}_{BC}(M) := \frac{\operatorname{Ker}(d: A^{p,q}(M) \to A^{p+q+1}(M))}{\partial\overline{\partial}A^{p-1,q-1}(M)},$$

and the Aeplli cohomology group of degree (p,q) is defined as

$$H^{p,q}_{A}(M) := \frac{\operatorname{Ker}(\partial\overline{\partial}: A^{p,q}(M) \to A^{p+1,q+1}(M))}{\partial A^{p-1,q}(M) + \overline{\partial}A^{p,q-1}(M)}.$$

It is well-known that all these groups can also be defined by means of currents of corresponding degree. For every $(p,q) \in \mathbb{N}^2$, the identity induces a natural map

$$i: H^{p,q}_{BC}(M) \to H^{p,q}_{A}(M).$$

In general, the map i is neither injective nor surjective. If M satisfies $\partial\overline{\partial}$-lemma, then for every $(p,q) \in \mathbb{N}^2$, i is an isomorphism (see [5, Lemma 5.15, Remarks 5.16, 5.21]).

Theorem 3.2 *Let X be a strongly Gauduchon space. If it has a desingularization $\widetilde{X}$ such that $i: H^{1,1}_{BC}(\widetilde{X}) \to H^{1,1}_{A}(\widetilde{X})$ is injective, then $X \in \mathscr{SG}$.*

Proof Set $\dim X = n$. Suppose that $\pi: \widetilde{X} \to X$ is the desingularization. We need to prove that $\widetilde{X}$ is a strongly Gauduchon manifold. By Theorem 1.1, it suffices to prove that if T is a positive $(1,1)$-current on $\widetilde{X}$ which is d-exact, then $T = 0$.

Let $E \subseteq \widetilde{X}$ be the exceptional set of π, Ω be the real closed $(2n-2)$-form on X whose $(n-1,n-1)$-part $\Omega^{n-1,n-1}$ is strictly positive. Since T is d-exact, we have $T(\pi^*\Omega) = 0$. On the other hand, since T is a $(1,1)$-current, we have

$$T(\pi^*\Omega) = T(\pi^*\Omega^{n-1,n-1}) = \int_{\widetilde{X}} T \wedge \pi^*\Omega^{n-1,n-1}$$

and $\pi^*\Omega^{n-1,n-1}$ is strictly positive on $\widetilde{X} - E$, so we obtain $\operatorname{supp} T \subseteq E$.

By Theorem 3.1 for $p = n-1$, we obtain

$$T = \sum_i c_i[E_i],$$

where $c_i \geq 0$ and E_i are the $(n-1)$-dimensional irreducible components of E. Since T is real and d-exact, $i([T]_{BC}) = 0$ in $H_A^{1,1}(\widetilde{X})$. Because i is injective, we know $[T]_{BC} = 0$ in $H_{BC}^{1,1}(\widetilde{X})$. So, there is a real 0-current Q on $\widetilde{X}$, such that $T = i\partial\overline{\partial}Q$. Since $T \geq 0$, Q is plurisubhamonic. By maximum principle, Q is a constant. Hence $T = 0$.

Lemma 3.1 (see [7, Lemma 3.6]) *Let $f : X \to Y$ be a modification between reduced compact complex spaces of dimension n. If Y is normal and the Betti number satisfies $b_{2n-1}(Y) = 0$, then there is an exact sequence*

$$0 \longrightarrow H_{2n-2}(E,\mathbb{R}) \xrightarrow{i_*} H_{2n-2}(X,\mathbb{R}) \xrightarrow{f_*} H_{2n-2}(Y,\mathbb{R}) \ ,$$

where E is the exceptional set of f, $i : E \to X$ is the inclusion. Moreover, $H_{2n-2}(E,\mathbb{R}) = \bigoplus_j \mathbb{R}[E_j]$, where $\{E_j\}_j$ are all the $(n-1)$-dimensional irreducible components of E (possibly there exist some other components of dimension $< n-1$ in E).

Theorem 3.3 *If X is a normal strongly Gauduchon space of dimension n with the Betti number $b_{2n-1}(X) = 0$, then $X \in \mathscr{SG}$.*

Proof Suppose that T is a positive $(1,1)$-current on $\widetilde{X}$ which is d-exact. As the proof in Theorem 3.2, we obtain

$$T = \sum_i c_i[E_i],$$

where $c_i \geq 0$, E_i are the $(n-1)$-dimensional irreducible components of E. Since T is d-exact, $\sum\limits_i c_i[E_i] = [T]_{\widetilde{X}} = 0$ in $H_{2n-2}(\widetilde{X},\mathbb{R})$. By Lemma 3.1, we get $c_i = 0$ for all i.

Theorem 3.4 *Let X be a compact strongly Gauduchon space. If it has a desingularization $\widetilde{X}$ whose exceptional set has codimension ≥ 2, then $X \in \mathscr{SG}$.*

Proof Suppose that $\dim X = n$, and that T is a positive $(1,1)$-current on $\widetilde{X}$ which is d-exact. As the proof in Theorem 3.2, we obtain $\operatorname{supp} T \subseteq E$. By Theorem 3.1, for $p = n-1$, we get $T = 0$ immediately.

4 Families of Complex Spaces over a Nonsingular Curve

In this section, we study families of complex spaces over a curve. It should be useful in the study of deformations and moduli spaces of complex spaces. The following definition is a generalization of the corresponding notion defined in [12].

Definition 4.1 *Let X be a reduced compact complex space of pure dimension n, and $f : X \to C$ be a holomorphic map onto a nonsingular compact complex curve C. f is called topologically essential, if for every $p \in C$, no linear combination $\sum\limits_j c_j[F_j]$ is zero in $H_{2n-2}(X,\mathbb{R})$, where the F_j's are all the irreducible components of the fibres $f^{-1}(p)$, $c_j \geq 0$, and at least one of the c_j's is positive.*

Note that, for any reduced compact complex space X of pure dimension n and the holomorphic map $f : X \to C$ onto a nonsingular compact complex curve C, f is an open map by the open mapping theorem (see [9, p. 109]). Hence for every $p \in C$, every irreducible component of $f^{-1}(p)$ has dimension $n-1$ (see [6, Subsection 3.10]).

Now, we can generalize [18, Theorem 4.1] as follows.

Theorem 4.1 *Suppose that X is a purely n-dimensional compact normal complex space which admits a topologically essential holomorphic map $f : X \to C$ onto a nonsingular compact complex curve C, and that X has a desingularization $\pi : \widetilde{X} \to X$, such that no nonzero nonnegative linear combination of hypersurfaces contained in the exceptional set of π is zero in $H_{2n-2}(\widetilde{X}, \mathbb{R})$. If every nonsingular fiber of f is a strongly Gauduchon manifold, then $X \in \mathscr{SG}$.*

Proof Set $\widetilde{f} := f \circ \pi$. For every $p \in C$, set $f^{-1}(p) = \bigcup\limits_i V_i$, where V_i are all the irreducible components of $f^{-1}(p)$ which have dimension $n-1$. Since X is normal, $\operatorname{codim} X_s \geq 2$, where X_s is the set of singular points of X. So

$$\pi^{-1}(V_i) = \widetilde{V}_i \bigcup \bigcup_j E_{ij},$$

where $\widetilde{V}_i = \overline{\pi^{-1}(V_i - X_s)}$ is the strict transform of V_i, and E_{ij} are all irreducible components of $\pi^{-1}(V_i)$ contained in the exceptional set of π. It is possible that some E_{ij} are contained in other E_{kl} or $\widetilde{V}_k$. We denote any E_{ij}, which is not properly contained in either E_{kl} or $\widetilde{V}_k$, by $E_{ij'}$, and we denote any E_{ij}, which is properly contained in either E_{kl} or $\widetilde{V}_k$, by $E_{ij''}$ (i.e., there exists either E_{kl} or $\widetilde{V}_k$, such that $E_{ij''} \subsetneqq E_{kl}$ or $\widetilde{V}_k$). Then

$$\widetilde{f}^{-1}(p) = \bigcup_i \Big(\widetilde{V}_i \bigcup \bigcup_{j'} E_{ij'} \Big)$$

is the irreducible decomposition of $\widetilde{f}^{-1}(p)$. Hence $\operatorname{codim} E_{ij'} = 1$.

We need the following two claims.

Claim 4.1 $\widetilde{f}$ is topologically essential.

Proof Otherwise, we have

$$\sum_i a_i[\widetilde{V}_i] + \sum_{ij'} b_{ij'}[E_{ij'}] = 0$$

in $H_{2n-2}(\widetilde{X}, \mathbb{R})$, for some $a_i, b_{ij'} \geq 0$, and at least one of the a_i's, $b_{ij'}$'s is positive. Since $\pi(E_{ij'}) \subseteq X_s$ has codimension ≥ 2, $\pi_*[E_{ij'}] = 0$ in $H_{2n-2}(X, \mathbb{R})$. In $H_{2n-2}(X, \mathbb{R})$, $\pi_*[\widetilde{V}_i] = [V_i]$. Hence

$$\sum_i a_i[V_i] = 0$$

through π_*. Since f is topologically essential, $a_i = 0$ for all i. So

$$\sum_{ij'} b_{ij'}[E_{ij'}] = 0$$

in $H_{2n-2}(\widetilde{X}, \mathbb{R})$, where $b_{ij'} \geq 0$ and at least one of the $b_{ij'}$'s is positive. It contradicts the assumption on $\widetilde{X}$.

Claim 4.2 For every $p \in C$, if $\widetilde{f}^{-1}(p)$ is nonsingular, then it is a strongly Gauduchon manifold.

Proof Since $\widetilde{f}^{-1}(p) = \bigcup_i \big(\widetilde{V_i} \bigcup \bigcup_{j'} E_{ij'}\big)$ is nonsingular, we have

$$\begin{aligned} \widetilde{V_i} \cap \widetilde{V_k} &= \emptyset, \quad \forall\, i \neq k; \\ \widetilde{V_i} \cap E_{kl'} &= \emptyset, \quad \forall\, i, k, l'. \end{aligned}$$

Since for any i, j, E_{ij} is contained in some $E_{kl'}$ or $\widetilde{V_k}$, we have $\widetilde{V_i} \cap E_{ij} = \emptyset$. On the other hand, if $V_i \cap X_s \neq \emptyset$, then the intersection of $\widetilde{V_i}$ and $\bigcup_j E_{ij}$ is not empty, which contradicts $\widetilde{V_i} \cap E_{ij} = \emptyset$. So for all i, $V_i \cap X_s = \emptyset$. Hence, the map

$$\pi \mid_{\widetilde{f}^{-1}(p)} : \widetilde{f}^{-1}(p) \to f^{-1}(p)$$

is an isomorphism. Since every nonsingular fiber of f is a strongly Gauduchon manifold and $\widetilde{f}^{-1}(p)$ is nonsingular, $\widetilde{f}^{-1}(p)$ is a strongly Gauduchon manifold.

Now, by Claims 4.1–4.2, $\widetilde{X}$ is a strongly Gauduchon manifold according to [18, Theorem 4.1]. Hence, $X \in \mathscr{SG}$.

By the above theorem, we have the following corollary immediately.

Corollary 4.1 *Suppose that X is a purely dimensional compact normal complex space which admits a topologically essential holomorphic map $f : X \to C$ onto a nonsingular compact complex curve C, and that X has a desingularization $\widetilde{X}$ whose exceptional set has codimension ≥ 2. If every nonsingular fiber of f is a strongly Gauduchon manifold, then $X \in \mathscr{SG}$.*

Corollary 4.2 *Let X be a purely n-dimensional normal compact complex space which admits a topologically essential holomorphic map onto a nonsingular compact complex curve. If the Betti number satisfies $b_{2n-1}(X) = 0$, then $X \in \mathscr{SG}$.*

Proof *By Lemma* 3.1*, we know that, for any desingularization $\pi : \widetilde{X} \to X$, $\{[E_j]\}_j$ are linearly independent in $H_{2n-2}(\widetilde{X}, \mathbb{R})$, where $\{E_j\}_j$ are all the $(n-1)$-dimensional irreducible components of the exceptional set of π. Using Theorem* 4.1*, we get this corollary immediately.*

Acknowledgement The authors would like to thank their supervisor Prof. Jixiang Fu for his constant encouragement and many helpful discussions.

References

[1] Alessandrini, L., Holomorphic submersions onto Kähler or balanced manifolds, *Tohoku Math. J.* (2), **68**(4), 2014, 607–619.

[2] Alessandrini, L. and Andreatta, M., Closed transverse (p, p)-forms on compact complex manifolds, *Compositio Math.*, **61**, 1987, 181–200; Erratum ibid., **61**, 1987, 143.

[3] Alessandrini, L. and Bassanelli, G., Positive $\partial\overline{\partial}$-closed currents and non-Kähler geometry, *J. Geom. Anal.*, **2**, 1992, 291–316.

[4] Deligne, P., Équations Différentielles à Points Singuliers Réguliers, Lecture Notes in Math., **163**, Springer-Verlag, Berlin, New York, 1970.

[5] Deligne, P., Griffiths, P., Morgan, J., et al., Real homotopy theory of Kähler manifolds. *Invent. Math.*, **29**(3), 1975, 245–274.

[6] Fischer, G., Complex Analytic Geometry, Lecture Notes in Math., **538**, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[7] Fu, J., Meng, L. and Xia, W., Complex balanced spaces, *Internat. J. Math.*, **26**(12), 2015, 1550105.

[8] Fujiki, A., Closedness of the Douady spaces of compact Kähler spaces, *Publ. RIMS, Kyoto Univ.*, **14**, 1978, 1–52.

[9] Grauert, H. and Remmert, R., Coherent Analytic Sheaves, Grundlehren der Math. Wiss., **265**, Springer-Verlag, Berlin, 1984.

[10] Ho, C.-W., A note on proper maps, *Proc. Amer. Math. Soc.*, **51**, 1975, 237–241.

[11] King, J., The currents defined by analytic varieties, *Acta Math.*, **127**, 1971, 185–220.

[12] Michelsohn, M. L., On the existence of special metrics in complex geometry, *Acta Math.*, **149**, 1982, 261–295.

[13] Popovici, D., Limits of projective manifolds under holomorphic deformations. arXiv: 0910.2032v1

[14] Popovici, D., Stability of strongly Gauduchon manifolds under modifications, *J. Geom. Anal.*, **23**, 2013, 653–659.

[15] Popovici, D., Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics, *Invent. Math.*, **194**, 2013, 515–534.

[16] Varouchas, J., Stabilité de la classe des variétés Kählériennes par certains morphismes propres, *Invent. Math.*, **77**, 1984, 117–127.

[17] Varouchas, J., Kähler spaces and proper open morphisms, *Math. Ann.*, **283**, 1989, 13–52.

[18] Xiao, J., On strongly Gauduchon metrics of compact complex manifolds, *J. Geom. Anal.*, **25**, 2015, 2011–2027.