Chin. Ann. Math. Ser. B
39(6), 2018, 1001–1016
DOI: 10.1007/s11401-018-0110-8

Chinese Annals of Mathematics, Series B

Some Weighted Norm Inequalities on Manifolds*

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Abstract Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property and the Gaussian upper bounds. Denote by Δ the Laplace-Beltrami operator and by ∇ the Riemannian gradient. In this paper, the author proves the weighted reverse inequality $\|\Delta^{\frac{1}{2}} f\|_{L^p(w)} \leq C\||\nabla f|\|_{L^p(w)}$, for some range of p determined by M and w. Moreover, a weak type estimate is proved when $p = 1$. Some weighted vector-valued inequalities are also established.

Keywords Weighted norm inequality, Poincaré inequality, Riesz transform.
2010 MR Subject Classification 42B20, 58J35

1 Introduction

Let M be a complete Riemannian manifold, Δ be the Laplace-Beltrami operator, and ∇ be the Riemannian gradient. Denote by $\nabla\Delta^{-\frac{1}{2}}$ the Riesz transform on M. To start with, we recall some facts about Riesz transform on the Euclidean space $\mathbb{R}^n$. Laplace-Beltrami operator on $\mathbb{R}^n$ is defined by $\Delta = -\sum\limits_{j=1}^{n} \partial^2 x_j$. The Riesz transform can be written as $\nabla\Delta^{-\frac{1}{2}} = (R_1, R_2, \cdots, R_n)$, where $R_j = \partial x_j \Delta^{-\frac{1}{2}}$ for $1 \leq j \leq n$. By the classical Calderón-Zygmund theory, there exists constant $C_p > 0$, such that for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\||\nabla\Delta^{-\frac{1}{2}} f|\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.1}$$

where $1 < p < \infty$ (see for example [13, 20]). In fact, by duality, one can get the reverse inequality that there exists constant $c_p > 0$, such that for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\|\Delta^{\frac{1}{2}} f\|_{L^p(\mathbb{R}^n)} \leq c_p \||\nabla f|\|_{L^p(\mathbb{R}^n)}$$

(see for example [10, Subsection 2.1]).

As for manifolds, Strichartz [22] proved that (1.1) holds for $1 < p < \infty$ on rank-one symmetric spaces and asked the sufficient conditions for (1.1) to hold on general manifolds. Since then, lots of partial answers have been given. In fact, the range of p such that (1.1) holds depends on the geometry of manifolds. It turns out that the range may not be $(1, \infty)$ (see for example [2–3, 12, 20]).

On the other hand, the weighted norm inequalities for singular integral operators have a long history and are of great interest in harmonic analysis (see [20, Chapter 5] for more details).

Manuscript received October 16, 2015. Revised May 3, 2016.
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*This work was supported by the China Scholarship Council (No. 201406100171).

In the series [4–7], Auscher and Martell established some criteria to prove the weighted norm inequalities. In particular, they considered the Riesz transform, the reverse inequalities, and some quadratic operators of Littlewood-Paley-Stein type associated with elliptic operators in [5]. In [6], they showed the weighted estimates of Riesz transform on manifolds (see [9] for the corresponding results on graphs). In [8–9], these criteria were used to prove the L^p boundedness of Riesz transform associated with Schrödinger operators.

In this paper, let M be a complete non-compact Riemannian manifold with d its geodesic distance, and μ be its Riemannian measure. Moreover, assume that it has the volume doubling property, i.e., there exists a constant $C > 0$ such that

$$V(B(x, 2r)) \le CV(B(x, r))$$

for all $x \in M$ and $r > 0$, where $B(x, r)$ is the geodesic ball centered at x with radius r, and $V(B)$ is the volume of B with respect to the Riemannian measure μ. For any $\lambda > 0$, denote $\lambda B = B(x, \lambda r)$. The volume doubling property implies that there exist $C, \nu \ge 1$ such that, for every ball B and $\lambda > 1$,

$$V(\lambda B) \le C\lambda^\nu V(B). \tag{1.2}$$

Let Δ be the Laplace-Beltrami operator on M. Denote by $p_t(x, y)$ the heat kernel of the semigroup $\mathrm{e}^{-t\Delta}$, where $t > 0$ and $x, y \in M$. It is said to have Gaussian upper bounds if there exist some constants $C, c > 0$ such that for all $t > 0$, $x, y \in M$,

$$p_t(x, y) \le \frac{C}{V(B(x, \sqrt{t}))} \mathrm{e}^{-c\frac{d^2(x,y)}{t}}.$$

Note also that M is stochastic complete when M is geodesic complete and has volume doubling property (see [16, p. 303]). That is

$$\mathrm{e}^{-t\Delta}1(x) = 1, \quad \mu\text{-a.e. } x \in M \qquad \text{for every } t > 0.$$

Now we recall the definition of Muckenhoupt weights and reverse Hölder classes in [3].

Definition 1.1 *We say that a nonnegative locally integrable function w belongs to the Muckenhoupt A_p-weight classes $A_p(\mu)$ on M for $1 < p < \infty$ if for some $A < \infty$ and all balls $B \subset M$,*

$$\Big(\frac{1}{V(B)} \int_B w \mathrm{d}\mu\Big)\Big(\frac{1}{V(B)} \int_B w^{\frac{1}{1-p}} \mathrm{d}\mu\Big)^{p-1} \le A,$$

and for $p = 1$ if

$$\frac{1}{V(B)} \int_B w \mathrm{d}\mu \le Aw(x), \quad \text{a.e. } x \in M.$$

We say that a nonnegative locally integrable function w belongs to the reverse Hölder classes $\mathrm{RH}_s(\mu)$ with exponent $s > 1$ if there exists a constant C such that, for every ball B,

$$\Big(\frac{1}{V(B)} \int_B w^s \mathrm{d}\mu\Big)^{\frac{1}{s}} \le \frac{C}{V(B)} \int_B w \mathrm{d}\mu.$$

The endpoint $s = \infty$ is defined as follows. For every ball B,

$$w(x) \le \frac{C}{V(B)} \int_B w \mathrm{d}\mu, \quad \text{a.e. } x \in M.$$

For properties of Muckenhoupt A_p-weights $A_p(\mu)$ and reverse Hölder classes $\mathrm{RH}_s(\mu)$, we refer the reader to [3, 20]. To proceed, we need the following definitions in [3].

Definition 1.2 *Given $w \in A_p(\mu)$, define*

$$r_w = \inf\{p > 1 : w \in A_p(\mu)\}, \quad s_w = \sup\{s > 1 : w \in \mathrm{RH}_s(\mu)\}.$$

Then for $1 \le p_0 < q_0 \le \infty$, define

$$W_w(p_0, q_0) = \Big(p_0 r_w, \frac{q_0}{(s_w)'}\Big) = \{p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}}(\mu) \cap \mathrm{RH}_{(\frac{q_0}{p})'}(\mu)\},$$

where $q' = \frac{q}{q-1}$ is the conjugate exponent to q. Set

$$q_+ = \sup\{p \in (1, \infty) : \textit{Riesz transform is bounded on } L^p(\mu)\}.$$

In [6], the weighted estimates of Riesz transform on manifolds were proved.

Theorem A *Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property and Gaussian upper bounds. Let $w \in A_\infty(\mu)$.*

(i) *For $p \in W_w(1, q_+)$, the Riesz transform is of strong-type (p,p) with respect to $w\mathrm{d}\mu$, that is,*

$$\||\nabla \Delta^{-\frac{1}{2}} f|\|_{L^p(M,w)} \le C_{p,w} \|f\|_{L^p(M,w)}$$

for all f bounded with compact support.

(ii) *If $w \in A_1(\mu) \cap \mathrm{RH}_{(q_+)'}(\mu)$, then the Riesz transform is of weak-type $(1,1)$ with respect to $w\mathrm{d}\mu$, that is,*

$$\||\nabla \Delta^{-\frac{1}{2}} f|\|_{L^{1,\infty}(M,w)} \le C_{1,w} \|f\|_{L^1(M,w)}$$

for all f bounded with compact support.

We will prove the weighted estimates for the reverse inequality. Before we state the main result of this paper, recall that M supports a p-Poincaré inequality for $1 \le p < \infty$ if there exists $C > 0$ such that, for every ball B and every locally Lipschitz function f,

$$\Big(\frac{1}{V(B)} \int_B |f - f_B|^p \mathrm{d}\mu\Big)^{\frac{1}{p}} \le Cr\Big(\frac{1}{V(B)} \int_B |\nabla f|^p \mathrm{d}\mu\Big)^{\frac{1}{p}},$$

where r is the radius of B and $f_B = (V(B))^{-1} \int_B f \mathrm{d}\mu$. We define

$$r_- = \inf\{p \ge 1 : p\text{-Poincaré inequality holds}\}.$$

Note that the unweighted estimates were proved in [1, Theorem 0.7].

The main result of this paper is as follows.

Theorem 1.1 *Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property and Poincaré inequality with $1 \le r_- < 2$. Let $w \in A_\infty(\mu)$.*

(i) *For $p \in W_w(r_-, \infty)$, there exists $C_{p,w} > 0$ such that*

$$\|\Delta^{\frac{1}{2}} f\|_{L^p(w)} \le C_{p,w} \||\nabla f|\|_{L^p(w)} \tag{1.3}$$

for all f smooth with compact support.

(ii) *If* 1*-Poincaré inequality holds, then for every* $w \in A_1(\mu)$ *there exists* $C_{1,w} > 0$ *such that*

$$\|\Delta^{\frac{1}{2}} f\|_{L^{1,\infty}(w)} \le C_{1,w} \||\nabla f|\|_{L^1(w)} \tag{1.4}$$

for all f *smooth with compact support.*

To prove Theorem 1.1, we need to estimate the operator S defined by setting for any $h : M \times (0, \infty) \to \mathbb{R}$,

$$Sh(x) = \int_0^\infty \Delta e^{-t\Delta} h(x,t) dt.$$

Moreover, for any $p \in [1, \infty]$, let

$$L^p_{\mathbb{H}}(\mu) = \{h : M \times (0,\infty) \to \mathbb{R} \mid \||h|_{\mathbb{H}}\|_{L^p(\mu)} < \infty\},$$

where $|h|_{\mathbb{H}}(x) = (\int_0^\infty |h(x,t)|^2 \frac{dt}{t})^{\frac{1}{2}}$ for any $x \in M$. Denote by $L^\infty_c(\mu)$ the function space consisting of bounded functions with compact support with respect to μ, and set

$$L^\infty_{c,\mathbb{H}}(\mu) = \{h : M \times (0,\infty) \to \mathbb{R} \mid |h|_{\mathbb{H}}(x) \in L^\infty_c(\mu)\}.$$

Then we have the following theorem.

Theorem 1.2 *Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property and Gaussian upper bounds. For any* $h \in L^\infty_{c,\mathbb{H}}(\mu)$, *we have*

(i) *If* $1 < p < \infty$ *and* $w \in A_p(\mu)$, *there exists* $C_{p,w} > 0$ *such that*

$$\|Sh\|_{L^p(w)} \le C_{p,w} \|h\|_{L^p_{\mathbb{H}}(w)}. \tag{1.5}$$

(ii) *If* $p = 1$ *and* $w \in A_1(\mu)$, *there exists* $C_{1,w} > 0$ *such that*

$$\|Sh\|_{L^{1,\infty}(w)} \le C_{1,w} \|h\|_{L^1_{\mathbb{H}}(w)}. \tag{1.6}$$

Now we introduce some notations. Given any ball B, denote $C_1(B) = 4B$, $C_j(B) = 2^{j+1}B \setminus 2^j B$ for $j \ge 2$. Denote by χ_E the indicator function of subset E of M, and set $f_j = f\chi_{C_j(B)}$ for any given function f. The constants C, c and c' may change from line to line.

2 Proof of Theorem 1.2

To begin with, we recall some properties of the operator S in the unweighted cases. Note that the Littlewood-Paley-Stein square operator for $f \in L^p(\mu)$ is defined by

$$Gf(x) = \Big(\int_0^\infty |t\Delta e^{-t\Delta} f|^2 \frac{dt}{t}\Big)^{\frac{1}{2}}.$$

Since $e^{-t\Delta}$ is a symmetric diffusion semigroup in our settings, G is bounded on $L^p(\mu)$ for $1 < p < \infty$ (see [21]). By duality, we have

$$\|Sh\|_{L^p(\mu)} \le \|h\|_{L^p_{\mathbb{H}}(\mu)}, \quad 1 < p < \infty. \tag{2.1}$$

In fact, for any $\varphi \in C_0^\infty(M)$, we have

$$\begin{aligned}\langle Sh, \varphi\rangle_{L^2(\mu)} &= \int_0^\infty \int_M h(x,t)\Delta \mathrm{e}^{-t\Delta}\varphi(x)\mathrm{d}\mu\mathrm{d}t\\ &\le \int_M \Big(\int_0^\infty |h(x,t)|^2\frac{\mathrm{d}t}{t}\Big)^{\frac12} G\varphi(x)\mathrm{d}\mu\\ &\le \|h\|_{L^p_{\mathbb{H}}(\mu)}\|G\varphi\|_{L^{p'}(\mu)},\end{aligned}$$

where $1 < p < \infty$ and $p' = \frac{p}{p-1}$.

Before proving Theorem 1.2, we need to prove some lemmas. Denote by $L^1_{\mathrm{loc}}(\mu)$ the function space consisting of locally integral functions with respect to μ.

Lemma 2.1 *Given any fixed $l \ge 1$, there exist $C, c > 0$, such that the following holds for every ball $B = B(x_0, r)$, $h \in L^1_{\mathrm{loc}}(\mu)$ and $j \ge 2$:*

$$\sup_{x\in B}(|\mathrm{e}^{-lr^2\Delta}h_j(x)| + lr^2|\Delta\mathrm{e}^{-lr^2\Delta}h_j(x)|) \le C\mathrm{e}^{-c\frac{4^j}{l}}\frac{1}{V(2^{j+1}B)}\int_{C_j(B)}|h(y)|\mathrm{d}\mu.$$

Proof We first deal with $|\mathrm{e}^{-lr^2\Delta}h_j(x)|$. According to the Gaussian upper bounds of the heat kernel, one gets

$$\begin{aligned}\sup_{x\in B}|\mathrm{e}^{-lr^2\Delta}h_j(x)| &\le \sup_{x\in B}\int_{C_j(B)} P_{lr^2}(x,y)|h(y)|\mathrm{d}\mu\\ &\le C\sup_{x\in B}\int_{C_j(B)}\frac{1}{V(x,\sqrt{l}r)}\mathrm{e}^{-c\frac{4^jr^2}{lr^2}}|h(y)|\mathrm{d}\mu\\ &\le C\sup_{x\in B}\int_{C_j(B)}\frac{1}{V(x,r)}\mathrm{e}^{-c\frac{4^j}{l}}|h(y)|\mathrm{d}\mu\\ &\le C\int_{C_j(B)}\frac{1}{V(x_0,2r)}\mathrm{e}^{-c\frac{4^j}{l}}|h(y)|\mathrm{d}\mu\\ &\le C\int_{C_j(B)}\frac{2^{j\nu}}{V(2^{j+1}B)}\mathrm{e}^{-c\frac{4^j}{l}}|h(y)|\mathrm{d}\mu\\ &\le C\mathrm{e}^{-c\frac{4^j}{l}}\frac{1}{V(2^{j+1}B)}\int_{C_j(B)}|h(y)|\mathrm{d}\mu,\end{aligned}$$

where the third and fifth inequalities follow from (1.2). Here and in what follows, we denote $V(B(x,r))$ by $V(x,r)$. Note that $\Delta\mathrm{e}^{-t\Delta}f = -\frac{\partial}{\partial t}\mathrm{e}^{-t\Delta}f$. By the Gaussian upper bounds of the time derivative of the heat kernel (see for example [15–16, 19]),

$$\Big|\frac{\partial^k}{\partial t^k}p_t(x,y)\Big| \le C_k\frac{1}{V(x,\sqrt{t})}t^{-k}\exp\Big(-\frac{d^2(x,y)}{C_kt}\Big), \tag{2.2}$$

we can get the estimate of $lr^2|\Delta\mathrm{e}^{-lr^2\Delta}h_j(x)|$. Then the lemma has been proved.

Remark 2.1 Given any $l \ge 1$ fixed, there exist $C, c > 0$, such that the following holds for every ball $B = B(x_0, r)$, $h \in L^1_{\mathrm{loc}}(\mu)$ supported in B and $j \ge 2$:

$$\sup_{x\in C_j(B)}(|\mathrm{e}^{-lr^2\Delta}h(x)| + lr^2|\Delta\mathrm{e}^{-lr^2\Delta}h(x)|) \le C\mathrm{e}^{-c\frac{4^j}{l}}\frac{1}{V(B)}\int_B|h(y)|\mathrm{d}\mu.$$

Observing that $V(x_0,r) \le C(1+\frac{d(x,x_0)}{r})^\nu V(x,r)$, the remark follows according to the proof of the lemma.

Lemma 2.2 *There exist $C, c > 0$, such that the followings hold for every $f \in L^1(\mu)$ and $a > 0$:*

(i) $|\Delta \mathrm{e}^{-t\Delta} f(x)| \leq \dfrac{c}{tV(x, \sqrt{t})} \|f\|_{L^1(\mu)}$.

(ii) $\|(I - \mathrm{e}^{-t\Delta})\mathrm{e}^{-at\Delta} f\|_{L^1(\mu)} \leq \dfrac{C}{a} \|f\|_{L^1(\mu)}$.

Proof Since $\Delta \mathrm{e}^{-t\Delta} f = -\frac{\partial}{\partial t} \mathrm{e}^{-t\Delta} f$. By (2.2), (i) follows.

Note that

$$\mathrm{e}^{-at\Delta} f - \mathrm{e}^{-(a+1)t\Delta} f = \int_{at}^{(a+1)t} \Delta \mathrm{e}^{-\tau\Delta} f \mathrm{d}\tau.$$

According to (2.2), there exists $C > 0$ such that $\|\Delta \mathrm{e}^{-t\Delta} f\|_{L^1(\mu)} \leq \frac{C}{t} \|f\|_{L^1(\mu)}$ for $t > 0$ and $f \in L^1(\mu)$. As a result, one gets

$$\begin{aligned} \|(I - \mathrm{e}^{-t\Delta})\mathrm{e}^{-at\Delta} f\|_{L^1(\mu)} &\leq C \ln\Big(1 + \frac{1}{a}\Big) \|f\|_{L^1(\mu)} \\ &\leq \frac{C}{a} \|f\|_{L^1(\mu)}. \end{aligned}$$

In order to prove (1.6), we need the following vector-valued extension of Theorem 3.3 in [6].

Lemma 2.3 *Let $1 \leq p_0 < q_0 \leq \infty$ and $w \in A_\infty(\mu)$. Let T be a sublinear operator defined on $L^2_{\mathbb{H}}(\mu)$, and $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from $L^\infty_{c,\mathbb{H}}(\mu)$ into $L^2_{\mathbb{H}}(\mu)$. Assume the following conditions:*

(a) *There exists $q \in W_w(p_0, q_0)$, such that T is bounded from $L^q_{\mathbb{H}}(w)$ to $L^{q,\infty}(w)$.*

(b) *For all $j \geq 1$, there exists a constant α_j, such that for any ball B with $r(B)$ as its radius and for any $f \in L^\infty_{c,\mathbb{H}}(\mu)$ supported in B,*

$$\Big(\fint_{C_j(B)} |\mathcal{A}_{r(B)} f|^{q_0}_{\mathbb{H}} \ \mathrm{d}\mu\Big)^{\frac{1}{q_0}} \leq \alpha_j \Big(\fint_B |f|^{p_0}_{\mathbb{H}} \ \mathrm{d}\mu\Big)^{\frac{1}{p_0}}.$$

(c) *There exists $\beta > (s_w)'$, i.e., $w \in \mathrm{RH}_{\beta'}(\mu)$, with the following property: For all $j \geq 2$, there exists a constant α_j, such that for any ball B with $r(B)$ as its radius and for any $f \in L^\infty_{c,\mathbb{H}}(\mu)$ supported in B and for $j \geq 2$,*

$$\Big(\fint_{C_j(B)} |T(I - \mathcal{A}_{r(B)}) f|^{\beta} \ \mathrm{d}\mu\Big)^{\frac{1}{\beta}} \leq \alpha_j \Big(\fint_B |f|^{p_0}_{\mathbb{H}} \ \mathrm{d}\mu\Big)^{\frac{1}{p_0}}.$$

(d) $\sum_j \alpha_j 2^{\nu_w j} < \infty$ *for α_j in* (b) *and* (c), *where ν_w is the doubling constant of $w\mathrm{d}\mu$.*

If $w \in A_1(\mu) \cap \mathrm{RH}_{(\frac{q_0}{p_0})'}(\mu)$, then there exists a constant $C > 0$ such that for all $f \in L^\infty_{c,\mathbb{H}}(\mu)$,

$$\|Tf\|_{L^{p_0,\infty}(w)} \leq C \|f\|_{L^{p_0}_{\mathbb{H}}(w)}.$$

Proof The proof of this lemma is similar to that of [7, Theorem 3.3]. Since we have the vector-valued Calderón-Zygmund decomposition, the methods in proving [7, Theorem 3.3] apply well.

Proof of (1.5) in Theorem 1.2 Fix $1 < p < \infty$ and $w \in A_p(\mu)$. Thanks to the self-improve property of A_p weights, there exists $1 < p_0 < p$ such that $w \in A_{\frac{p}{p_0}}$. We will use [3, Theorem 3.7] to prove (1.5) (see Appendix). In order to use the theorem, take p_0 as above,

$q_0 = \infty$ and $\mathcal{A}_r = I - (I - \mathrm{e}^{-r^2\Delta})^n$ with r the radius of the ball B, n large enough to be determined later. In our situation, we can set $\mathcal{E} = L^{p_0}_{\mathbb{H}}(\mu)$ and $\mathcal{D} = L^{\infty}_{c,\mathbb{H}}(\mu)$. Then it is sufficient to prove the following inequalities for any $f \in L^{\infty}_{c,\mathbb{H}}(\mu)$ and $x \in B$:

(I) $\displaystyle \Big(\frac{1}{V(B)}\int_B |S(I - \mathrm{e}^{-r^2\Delta})^n f|^{p_0}\mathrm{d}\mu\Big)^{\frac{1}{p_0}} \le CM(|f|^{p_0}_{\mathbb{H}})^{\frac{1}{p_0}}(x)$,

(II) $\displaystyle \sup_{x\in B} S(I - (I - \mathrm{e}^{-r^2\Delta})^n)f \le CM(|Sf|^{p_0})^{\frac{1}{p_0}}(x)$.

We will prove (II) first. Note that $S\mathcal{A}_r = \mathcal{A}_r S$. Then expand $\mathcal{A}_r$ and let $h_j = (Sf)\chi_{C_j(B)}$. For $j \ge 2$, apply Lemma 2.1. Since h_1 is supported in $C_1(B)$, it is a direct result of the Gaussian upper bounds of the heat kernel and (1.2).

To prove (I), let

$$f(x,t) = \sum_{j\ge 1} f(x,t)\chi_{C_j(B)} = \sum_{j\ge 1} f_j.$$

We will treat f_1 and f_j $(j \ge 2)$ separately.

According to (2.1) with $p = p_0$, one obtains

$$\begin{aligned}\|S(I - \mathcal{A}_r)f_1\|_{L^{p_0}(B)} &\le C\|(I - \mathcal{A}_r)f_1\|_{L^{p_0}_{\mathbb{H}}(\mu)} \\ &\le C\|f_1\|_{L^{p_0}_{\mathbb{H}}(\mu)} + C\sum_{l=1}^{n} \|\mathrm{e}^{-lr^2\Delta} f_1\|_{L^{p_0}_{\mathbb{H}}(\mu)}.\end{aligned}$$

Since the variable x of f_1 is supported in $C_1(B)$, it follows that

$$\Big(\frac{1}{V(B)}\Big)^{\frac{1}{p_0}} \|f_1\|_{L^{p_0}_{\mathbb{H}}(\mu)} \le CM(|f_1|^{p_0}_{\mathbb{H}})^{\frac{1}{p_0}}.$$

For $1 \le l \le n$, one has

$$\begin{aligned}&\Big(\frac{1}{V(B)}\Big)^{\frac{1}{p_0}} \|\mathrm{e}^{-lr^2\Delta} f_1\|_{L^{p_0}_{\mathbb{H}}(\mu)} \\ &= \Big(\frac{1}{V(B)}\sum_{i\ge 1}\int_{C_i(4B)} |\mathrm{e}^{-lr^2\Delta} f_1|^{p_0}_{\mathbb{H}}\ \mathrm{d}\mu\Big)^{\frac{1}{p_0}} \\ &\le C\sum_{i\ge 1}\Big(\frac{V(2^{i+3}B)}{V(B)}\Big)^{\frac{1}{p_0}} \sup_{x\in C_i(4B)} |\mathrm{e}^{-lr^2\Delta} f_1|_{\mathbb{H}}.\end{aligned}$$

Apply Remark 2.1 when $i \ge 2$ and Gaussian upper bounds when $i = 1$. Then one gets

$$\begin{aligned}&\Big(\frac{1}{V(B)}\Big)^{\frac{1}{p_0}} \|\mathrm{e}^{-lr^2\Delta} f_1\|_{L^{p_0}_{\mathbb{H}}(\mu)} \\ &\le C\sum_{i\ge 1} 2^{\frac{i\nu}{p_0}} \exp(-c4^i) M(|f|_{\mathbb{H}}) \\ &\le CM(|f|^{p_0}_{\mathbb{H}})^{\frac{1}{p_0}}(x),\end{aligned}$$

where we have used (1.2). Thus we have proved (I) for f_1.

To treat f_j for $j \geq 2$, we use the Minkowski integral inequality and get

$$\begin{aligned}
&\Big(\frac{1}{V(B)}\int_B \Big|\int_0^\infty \Delta e^{-t\Delta}(I-e^{-r^2\Delta})^n f_j \, dt\Big|^{p_0} d\mu\Big)^{\frac{1}{p_0}} \\
&\leq \int_0^\infty \Big(\frac{1}{V(B)}\int_B |\Delta e^{-t\Delta}(I-e^{-r^2\Delta})^n f_j|^{p_0} d\mu\Big)^{\frac{1}{p_0}} dt \\
&= \int_0^{2^{j+1}r^2 n} \Big(\frac{1}{V(B)}\int_B |\Delta e^{-t\Delta}(I-e^{-r^2\Delta})^n f_j|^{p_0} d\mu\Big)^{\frac{1}{p_0}} dt \\
&\quad + \int_{2^{j+1}r^2 n}^\infty \Big(\frac{1}{V(B)}\int_B |\Delta e^{-t\Delta}(I-e^{-r^2\Delta})^n f_j|^{p_0} d\mu\Big)^{\frac{1}{p_0}} dt \\
&= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

For I_1, we expand $(I-e^{-r^2\Delta})^n$, and Lemma 2.1 gives

$$\begin{aligned}
\mathrm{I}_1 &\leq C\sum_{l=0}^n \int_0^{2^{j+1}r^2 n} \frac{1}{t+lr^2}\frac{e^{-c\frac{4^j r^2}{t+lr^2}}}{V(2^{j+1}B)}\int_{2^{j+1}B}|f(y,t)| d\mu dt \\
&= C\sum_{l=0}^n \frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}\int_0^{2^{j+1}r^2 n} |f(y,t)|\frac{1}{t+lr^2}e^{-c\frac{4^j r^2}{t+lr^2}} dt d\mu \\
&\leq C\sum_{l=0}^n \frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}\Big(\int_0^\infty |f|^2\frac{dt}{t}\Big)^{\frac12}\Big(\int_0^{2^{j+1}r^2 n}\frac{t}{(t+lr^2)^2}e^{-2c\frac{4^j r^2}{t+lr^2}} dt\Big)^{\frac12} d\mu.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^{2^{j+1}r^2 n} \frac{t}{(t+lr^2)^2}e^{-2c\frac{4^j r^2}{t+lr^2}} dt \\
&= \int_0^{2^{j+1}n} \frac{s}{(s+l)^2}e^{-2c\frac{4^j}{(s+l)}} ds \quad (\text{let } t = r^2 s) \\
&\leq Ce^{-\frac{c}{2n}2^j} = Ce^{-c'2^j},
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{I}_1 &\leq Ce^{-c'2^j}\frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}\Big(\int_0^\infty |f|^2\frac{dt}{t}\Big)^{\frac12} d\mu \\
&\leq Ce^{-c'2^j} M(|f|_{\mathbb{H}}^{p_0})^{\frac{1}{p_0}}.
\end{aligned}$$

For I_2, from Lemma 2.2, we have that for any $x \in B = B(x_0, r)$ and $t > 0$,

$$\begin{aligned}
|\Delta e^{-t\Delta}(I-e^{-r^2\Delta})^n f_j(x,t)| &= |\Delta e^{-\frac{t}{2}\Delta}((I-e^{-r^2\Delta})e^{-\frac{t}{2n}\Delta})^n f_j(x,t)| \\
&\leq \frac{C}{tV(x_0,\sqrt{t})}\|((I-e^{-r^2\Delta})e^{-\frac{t}{2n}\Delta})^n f_j\|_{L^1(\mu)} \\
&\leq \frac{C}{tV(x_0,\sqrt{t})}\Big(\frac{r^2}{t}\Big)^n\int_{2^{j+1}B}|f(x,t)| d\mu.
\end{aligned}$$

We have used the fact that $B(x_0,\sqrt{t}) \subset B(x,2\sqrt{t})$, and thus $V(x_0,\sqrt{t}) \leq cV(x,\sqrt{t})$ in the second line. Substituting the above estimate into I_2 and using Hölder inequality, one gets

$$
\begin{aligned}
\mathrm{I}_2 &\le C \int_{2^{j+1}r^2 n}^{\infty} \frac{1}{tV(x_0,\sqrt{t})} \Big(\frac{r^2}{t}\Big)^n \int_{2^{j+1}B} |f| \mathrm{d}\mu \mathrm{d}t \\
&= C \int_{2^{j+1}B} \int_{2^{j+1}r^2 n}^{\infty} \frac{|f|}{\sqrt{t}} \frac{1}{\sqrt{t}V(x_0,\sqrt{t})} \Big(\frac{r^2}{t}\Big)^n \mathrm{d}t\mathrm{d}\mu \\
&\le C \int_{2^{j+1}B} \Big(\int_0^\infty |f|^2 \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}} \Big(\int_{2^{j+1}r^2 n}^{\infty} \frac{1}{tV(x_0,\sqrt{t})^2} \Big(\frac{r^2}{t}\Big)^{2n} \mathrm{d}t\Big)^{\frac{1}{2}} \mathrm{d}\mu \\
&= \frac{C}{V(2^{j+1}B)} \int_{2^{j+1}B} \Big(\int_0^\infty |f|^2 \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}} \Big(\int_{2^{j+1}r^2 n}^{\infty} \frac{1}{t} \Big[\frac{V(x_0,2^{j+1}r)}{V(x_0,\sqrt{t})}\Big]^2 \Big(\frac{r^2}{t}\Big)^{2n} \mathrm{d}t\Big)^{\frac{1}{2}} \mathrm{d}\mu \\
&\le CM(|f|_{\mathbb{H}}^{p_0})^{\frac{1}{p_0}} \Big(\int_{2^{j+1}r^2 n}^{\infty} \Big(\frac{2^{j+1}r}{\sqrt{t}}\Big)^{2\nu} \Big(\frac{r^2}{t}\Big)^{2n} \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}} \\
&\le C 2^{-j(n-\frac{\nu}{2})} M(|f|_{\mathbb{H}}^{p_0})^{\frac{1}{p_0}}.
\end{aligned}
$$

Combining the above estimate and the estimate of I_1, we obtain that (1.5) holds provided $n > \frac{\nu}{2}$.

Proof of (1.6) in Theorem 1.2 Take $p_0 = 1$, $q_0 = \infty$, $\mathcal{A}_r = I - (I - \mathrm{e}^{-r^2\Delta})^n$. We will check the conditions (a),(b),(c) and (d) required by Lemma 2.3.

Fix $w \in A_1(\mu)$. (a) follows from (1.5) as the operator S is bounded from $L^q_{\mathbb{H}}(w)$ to $L^q(w)$ for $1 < q < \infty$.

By expanding $\mathcal{A}_r$, it is sufficient to show (b) for $\mathrm{e}^{-lr^2\Delta}$ with $1 \le l \le n$. Fixing any l, we have for $j \ge 2$,

$$
\begin{aligned}
\sup_{x\in C_j(B)} |\mathrm{e}^{-lr^2\Delta} f|_{\mathbb{H}} &\le C \sup_{x\in C_j(B)} \int_B P_{lr_2}(x,y)|f(x,\cdot)|_{\mathbb{H}} \,\mathrm{d}\mu \\
&\le C\mathrm{e}^{-c4^j} \frac{1}{V(B)} \int_B |f|_{\mathbb{H}} \,\mathrm{d}\mu.
\end{aligned}
$$

For $j = 1$, it follows by the Gaussian upper bounds of the heat kernel and (1.2).

To see (c), we use the same method as in proving (1.5) and identify $\alpha_j = 2^{-j(n-\frac{\nu}{2})}$ for $j \ge 2$. In fact, take $\beta = \infty$. Then for $j \ge 2$,

$$
\begin{aligned}
&\sup_{x\in C_j(B)} \int_0^\infty |\Delta \mathrm{e}^{-t\Delta}(I - \mathrm{e}^{-r^2\Delta})^n f| \mathrm{d}t \\
&\le \sup_{x\in C_j(B)} \int_0^{2^{j+1}r^2 n} |\Delta \mathrm{e}^{-t\Delta}(I - \mathrm{e}^{-r^2\Delta})^n f| \mathrm{d}t + \sup_{x\in C_j(B)} \int_{2^{j+1}r^2 n}^\infty |\Delta \mathrm{e}^{-t\Delta}(I - \mathrm{e}^{-r^2\Delta})^n f| \mathrm{d}t \\
&= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
$$

For I_1, using Remark 2.1, we obtain

$$
\begin{aligned}
\mathrm{I}_1 &\le C \sum_{l=0}^n \frac{1}{V(B)} \int_B \Big(\int_0^\infty |f|^2 \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}} \Big(\int_0^{2^{j+1}r^2 n} \frac{t}{(t+lr^2)^2} \mathrm{e}^{-c\frac{4^j r^2}{t+lr^2}} \mathrm{d}t\Big)^{\frac{1}{2}} \mathrm{d}\mu \\
&\le \mathrm{e}^{-c2^j} \frac{C}{V(B)} \int_B |f|_{\mathbb{H}} \,\mathrm{d}\mu.
\end{aligned}
$$

For I_2, since the following inequality holds with every $t > 0$:

$$|\Delta \mathrm{e}^{-t\Delta}(I - \mathrm{e}^{-r^2\Delta})^n f(x,t)| \le \frac{C2^{j\frac{\nu}{2}}}{tV(x_0,\sqrt{t})}\Big(\frac{r^2}{t}\Big)^n \int_B |f(x,t)|\mathrm{d}\mu,$$

we have

$$\mathrm{I}_2 \le 2^{-j(n-\frac{\nu}{2})}\frac{C}{V(B)}\int_B |f|_{\mathbb{H}}\,\mathrm{d}\mu.$$

As a result, we can conclude that (d) holds if $n > \nu_w + \frac{\nu}{2}$, where ν_w is the doubling constant of the measure $w\mathrm{d}\mu$.

Thus we have finished the proof of Theorem 1.2.

3 Proof of Theorem 1.1

In order to prove (1.4), we need a weighted version of the Calderón-Zygmund decomposition.

Lemma 3.1 *Let M be a complete non-compact Riemannian manifold satisfying the volume doubling condition, and it supports 1-Poincaré inequality. Let $w \in A_1(\mu)$. For any function f such that $|\nabla f| \in L^1(w)$ and $\lambda > 0$, one can find a collection of balls $(B_i)_i$, functions $(b_i)_i$ and g, such that*

$$\begin{aligned}
&f = g + \sum_i b_i,\\
&|\nabla g(x)| \le C\lambda \quad \textit{for a.e. } x \in M,\\
&\operatorname{supp} b_i \subset B_i \quad \textit{and} \quad \int_{B_i} |\nabla b_i| w\mathrm{d}\mu \le C\lambda w(B_i),\\
&\int_{B_i} |b_i| w\mathrm{d}\mu \le C\lambda r(B_i) w(B_i),\\
&\sum_i w(B_i) \le C\lambda^{-1}\int_M |\nabla f| w\mathrm{d}\mu,\\
&\sum_i \chi_{B_i} \le N,
\end{aligned}$$

where C and N depend on the constants in Poincaré inequality, ν and w.

Proof According to [3, Proposition 9.1], the results hold when we have the following weighted Poincaré inequality:

$$\int_B |f - f_{B,w}| w\mathrm{d}\mu \le Cr(B)\int_B |\nabla f| w\mathrm{d}\mu,$$

where $f_{B,w} = \frac{1}{w(B)}\int_B f w\mathrm{d}\mu$. However, it is a consequence of Corollary 3.2 in [12]. See also [17, Chapter 15] for more results about the p-admissible weights. Thus we have proved the lemma.

Proof of (1.3) in Theorem 1.1 We will use Theorem 3.7 in [3] again. Since $p \in W_w(r_-, \infty)$, there exist $r_- < p_0 < p < q_0 < \infty$ such that $w \in A_{\frac{p}{p_0}} \cap \mathrm{RH}(\frac{q_0}{p})'$. Take $\mathcal{A}_r = I - (I - \mathrm{e}^{-r^2\Delta})^n$, where r is the radius of the ball B and n is large enough to be determined later. Then it is sufficient to prove the following inequalities for any $f \in L^\infty_{c,\mathbb{H}}(\mu)$ and $x \in B$:

(I′) $\|\Delta^{\frac{1}{2}}(I-\mathrm{e}^{-r^2\Delta})^n f\|_{L^{p_0}(B)} \le CM(|\nabla f|^{p_0})^{\frac{1}{p_0}}(x)\mu(B)^{\frac{1}{p_0}}$,

(II′) $\|\Delta^{\frac{1}{2}}(I-(I-\mathrm{e}^{-r^2\Delta})^n)f\|_{L^{q_0}(B)} \le CM(|\Delta^{\frac{1}{2}} f|^{p_0})^{\frac{1}{p_0}}(x)\mu(B)^{\frac{1}{q_0}}$.

Note first that

$$\mu(B)^{-\frac{1}{q_0}}\|\Delta^{\frac{1}{2}}(I-(I-\mathrm{e}^{-r^2\Delta})^n)f\|_{L^{q_0}(B)} \le \sup_{x\in B}\Delta^{\frac{1}{2}}(I-(I-\mathrm{e}^{-r^2\Delta})^n)f.$$

Since $\Delta^{\frac{1}{2}}$ and $\mathcal{A}_r$ commute, expand $(I-\mathrm{e}^{-r^2\Delta})^n$ and set $h_j=\Delta^{\frac{1}{2}} f\chi_{C_j(B)}$. The proof of (II′) is similar to that of (II).

Now we deal with (I′). Given $B=B(x_0,r)$, let $h=f-f_{4B}$, $h_j=h\chi_{C_j(B)}$ for $j\ge 3$, $h_2=h(\chi_{8B}-\phi_1)$, $h_1=h\phi_1$ and

$$\phi_1(x)=\begin{cases}1, & d(x,x_0)\le 2r,\\ \dfrac{4r-d(x,x_0)}{2r}, & 2r\le d(x,x_0)\le 4r,\\ 0, & d(x,x_0)\ge 4r,\end{cases}$$

where d is the geodesic distance. Note that $|\nabla\phi_1|$ is bounded with compact support.

According to [2], one has the following equalities:

$$\begin{aligned}\Delta^{\frac{1}{2}}(I-\mathrm{e}^{-r^2\Delta})^n f &= \Delta^{\frac{1}{2}}(I-\mathrm{e}^{-r^2\Delta})^n h\\ &= \Delta\Delta^{-\frac{1}{2}}(I-\mathrm{e}^{-r^2\Delta})^n h\\ &= \int_0^\infty g_r(t)\Delta\mathrm{e}^{-t\Delta}h\mathrm{d}t,\end{aligned}$$

where we have used the stochastic complete property of M in the first equality and

$$g_r(t)=\sum_{l=0}^n\binom{n}{l}(-1)^l\frac{\chi_{\{t>lr^2\}}}{\sqrt{t-lr^2}}.$$

We have the following estimates of $g_r(t)$ (see [2]):

$$|g_r(t)|\le\frac{C_n}{\sqrt{t-lr^2}},\quad \text{if } 0\le lr^2<t\le(l+1)r^2\le(n+1)r^2$$

and

$$|g_r(t)|\le C_n r^{2n}t^{-n-\frac{1}{2}},\quad \text{if } t>(n+1)r^2.$$

For any $c,r>0$, $j\ge 2$,

$$\begin{aligned}&\int_0^\infty|g_r(t)|\mathrm{e}^{-\frac{c4^jr^2}{t}}\frac{\mathrm{d}t}{t}\\ &=\int_0^{(n+1)r^2}|g_r(t)|\mathrm{e}^{-\frac{c4^jr^2}{t}}\frac{\mathrm{d}t}{t}+\int_{(n+1)r^2}^\infty|g_r(t)|\mathrm{e}^{-\frac{c4^jr^2}{t}}\frac{\mathrm{d}t}{t}\\ &=\mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

We first estimate I_1 and one has

$$\begin{aligned}
\mathrm{I}_1 &\leq \sum_{l=0}^{n} \int_{lr^2}^{(n+1)r^2} \frac{1}{\sqrt{t-lr^2}} \mathrm{e}^{-\frac{c4^j r^2}{t}} \frac{\mathrm{d}t}{t} \\
&= \sum_{l=0}^{n} \int_{l}^{n+1} \frac{1}{r\sqrt{s-l}} \mathrm{e}^{-\frac{c4^j}{s}} \frac{\mathrm{d}s}{s} \\
&\leq \frac{1}{r} \mathrm{e}^{-\frac{c4^j}{2(n+1)}} \sum_{l=0}^{n} \int_{l}^{n+1} \frac{1}{\sqrt{s-l}} \mathrm{e}^{-\frac{8c}{s}} \frac{\mathrm{d}s}{s} \\
&\leq \frac{C}{r} 4^{-jn},
\end{aligned}$$

where C depends on c, n. On the other hand,

$$\mathrm{I}_2 = \frac{1}{r} \int_{n+1}^{\infty} s^{-n-\frac{1}{2}} \mathrm{e}^{-\frac{c4^j}{s}} \frac{\mathrm{d}s}{s} \leq \frac{1}{r} 4^{-jn-\frac{1}{2}j} \int_{0}^{\infty} t^{-n-\frac{3}{2}} \mathrm{e}^{-\frac{c}{t}} \mathrm{d}t \leq \frac{C}{r} 4^{-jn},$$

where C depends on c, n. Then, combining the estimates of I_1 and I_2, we get that there exists $C > 0$ depending on c, n, such that for every $r > 0$,

$$\int_0^\infty |g_r(t)| \mathrm{e}^{-\frac{c4^j r^2}{t}} \frac{\mathrm{d}t}{t} \leq \frac{C}{r} 4^{-jn}.$$

With the help of Poincaré inequality, we obtain

$$\begin{aligned}
\Delta^{\frac{1}{2}} (I - \mathrm{e}^{-r^2\Delta})^n h_j &\leq C \int_0^\infty |g_r(t)| |\Delta \mathrm{e}^{-t\Delta} h_j| \mathrm{d}t \\
&\leq C \int_0^\infty |g_r(t)| \mathrm{e}^{-\frac{c4^j r^2}{t}} \frac{\mathrm{d}t}{t} \frac{1}{V(2^{j+1}B)} \int_{C_j(B)} |h| \mathrm{d}\mu \\
&\leq C 4^{-jn} r^{-1} \frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{4B}| \mathrm{d}\mu \\
&\leq C 4^{-jn} r^{-1} \Big(1 + \frac{V(2^{j+1}B)}{V(4B)}\Big) \frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{2^{j+1}B}| \mathrm{d}\mu \\
&\leq C 2^{(\nu-2n)j} r^{-1} \Big(\frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{2^{j+1}B}|^{p_0} \mathrm{d}\mu \Big)^{\frac{1}{p_0}} \\
&\leq C 2^{(\nu-2n)j} \Big(\frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |\nabla f|^{p_0} \mathrm{d}\mu \Big)^{\frac{1}{p_0}}.
\end{aligned}$$

In the forth inequality, we have used the fact

$$\begin{aligned}
&\frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{4B}| \mathrm{d}\mu \\
\leq\ & \frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{2^{j+1}B}| \mathrm{d}\mu + |f_{4B} - f_{2^{j+1}B}| \\
\leq\ & \frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{2^{j+1}B}| \mathrm{d}\mu + \frac{V(2^{j+1}B)}{V(4B)} \frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} |f - f_{2^{j+1}B}| \mathrm{d}\mu.
\end{aligned}$$

Thus we have

$$\| \Delta^{\frac{1}{2}} (I - \mathrm{e}^{-r^2\Delta})^n h_j \|_{L^{p_0}(B)} \leq C 2^{(\nu-2n)j} M(|\nabla f|^{p_0})^{\frac{1}{p_0}}(x) \mu(B)^{\frac{1}{p_0}}.$$

The sum of the right-hand side converges when $n > \frac{\nu}{2}$.

For $j = 1$, we have

$$\|\Delta^{\frac{1}{2}}(I - A_r)h_1\|_{L^{p_0}(B)} \leq C\|\Delta^{\frac{1}{2}}h_1\|_{L^{p_0}(\mu)} \leq C\||\nabla h_1|\|_{L^{p_0}(\mu)}.$$

Note that

$$|\nabla h_1| = |\nabla(h\phi_1)| \leq |\nabla f||\phi_1| + |\nabla\phi_1||f - f_{4B}|,$$

and ϕ_1 is bounded with compact support. By Poincaré inequality, we have

$$\|\Delta^{\frac{1}{2}}(I - \mathrm{e}^{-r^2\Delta})^n h_1\|_{L^{p_0}(B)} \leq CM(|\nabla f|^{p_0})^{\frac{1}{p_0}}(x)\mu(B)^{\frac{1}{p_0}}.$$

For $j = 2$, notice that h_2 is supported in $8B/2B$ and we have a similar estimate for h_2 in the spirit of Lemma 2.1. Using the similar method as in the proof for $j \geq 3$, the result for $j \geq 3$ also holds for $j = 2$ and this leads to (I$'$). Then we have proved (1.3).

Proof of (1.4) in Theorem 1.1 We follow the method in [1]. For f smooth with compact support, apply Lemma 3.1. Since $w \in A_1(\mathrm{d}\mu)$, we have $w \in A_q(\mathrm{d}\mu)$ for any $q > 1$. By (1.3) and the property of g, we have the following:

$$\begin{aligned} w\Big(\Big\{x \in M : |\Delta^{\frac{1}{2}}g| > \frac{\lambda}{3}\Big\}\Big) &\leq \frac{C}{\lambda^q}\||\nabla g|\|^q_{L^q(w)} \\ &\leq Cw\Big(\bigcup_i 4B_i\Big) + \frac{C}{\lambda}\int_{M\setminus\bigcup_i 4B_i} |\nabla g| w\mathrm{d}\mu \\ &\leq \frac{C}{\lambda}\int_M |\nabla f| w\mathrm{d}\mu. \end{aligned}$$

Let $r_i = 2^k$ if $2^k \leq r(B_i) < 2^{k+1}$, and set

$$T_i = \int_0^{r_i^2} \Delta\mathrm{e}^{-t\Delta}\frac{\mathrm{d}t}{\sqrt{t}}, \quad U_i = \int_{r_i^2}^{\infty} \Delta\mathrm{e}^{-t\Delta}\frac{\mathrm{d}t}{\sqrt{t}}.$$

Then it is enough to estimate

$$\mathrm{I}_1 = w\Big(\Big\{x \in M : \Big|\sum_i T_i b_i\Big| > \frac{\lambda}{3}\Big\}\Big), \quad \mathrm{I}_2 = w\Big(\Big\{x \in M : \Big|\sum_i U_i b_i\Big| > \frac{\lambda}{3}\Big\}\Big).$$

For I_1, we get

$$\mathrm{I}_1 \leq Cw\Big(\bigcup_i 4B_i\Big) + Cw\Big(\Big\{x \in M\Big\backslash\bigcup_i 4B_i : \Big|\sum_i T_i b_i\Big| > \frac{\lambda}{3}\Big\}\Big).$$

We have

$$\begin{aligned} w\Big(\Big\{x \in M\Big\backslash\bigcup_i 4B_i : \Big|\sum_i T_i b_i\Big| > \frac{\lambda}{3}\Big\}\Big) &\leq \frac{3}{\lambda}\int_M \Big|\sum_i T_i b_i \chi_{(4B_i)^c}\Big| w\mathrm{d}\mu \\ &\leq \frac{C}{\lambda}\sum_i \int_{(4B_i)^c} T_i b_i w\mathrm{d}\mu \\ &\leq \frac{C}{\lambda}\sum_i\sum_{j\geq 2}\int_{C_j(B_i)} |T_i b_i| w\mathrm{d}\mu. \end{aligned}$$

By Remark 2.1 and the properties of b_i, we obtain

$$\sup_{x\in C_j(B_i)} |\Delta \mathrm{e}^{-t\Delta} b_i(x)| \leq \frac{C}{t}\mathrm{e}^{-c\frac{4^j r_i^2}{t}} \frac{1}{V(B_i)} \int_{B_i} |b_i| \mathrm{d}\mu \quad \text{for } j \geq 2.$$

As a result, we have

$$\begin{aligned}
\int_{C_j(B_i)} |T_i b_i| w \mathrm{d}\mu &\leq \int_0^{r_i^2} \frac{C}{t}\mathrm{e}^{-c\frac{4^j r_i^2}{t}} \frac{w(2^{j+1}B_i)}{V(B_i)} \int_{B_i} |b_i| \mathrm{d}\mu \frac{\mathrm{d}t}{\sqrt{t}} \\
&\leq \int_0^{r_i^2} \frac{C}{t}\mathrm{e}^{-c\frac{4^j r_i^2}{t}} \frac{V(2^{j+1}B_i)}{V(B_i)} \int_{B_i} |b_i| \frac{w(2^{j+1}B_i)}{V(2^{j+1}B_i)} \mathrm{d}\mu \frac{\mathrm{d}t}{\sqrt{t}} \\
&\leq \int_0^{r_i^2} \frac{C}{t}\mathrm{e}^{-c\frac{4^j r_i^2}{t}} 2^{j\nu} \int_{B_i} |b_i| w \mathrm{d}\mu \frac{\mathrm{d}t}{\sqrt{t}} \\
&\leq C \int_0^{r_i^2} \lambda 2^{j\nu} \frac{r_i}{t} \mathrm{e}^{-c\frac{4^j r_i^2}{t}} w(B_i) \frac{\mathrm{d}t}{\sqrt{t}} \\
&\leq C\lambda 2^{j\nu} \mathrm{e}^{-c4^j} w(B_i),
\end{aligned}$$

where we have used the definition of $A_1(\mu)$ in the third inequality.

Hence,

$$\begin{aligned}
\mathrm{I}_1 &\leq Cw\Big(\bigcup_i 4B_i\Big) + C\sum_i \sum_{j\geq 2} 2^{j\nu} \mathrm{e}^{-c4^j} w(B_i) \\
&\leq C\sum_i w(B_i) \leq \frac{C}{\lambda} \int_M |\nabla f| w \mathrm{d}\mu.
\end{aligned}$$

It remains to handle I_2. Define

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i}, \quad k \in \mathbb{Z}.$$

Then we have

$$\sum_i U_i b_i = \sum_{k\in\mathbb{Z}} \int_{4^k}^{\infty} \Big(\frac{2^k}{\sqrt{t}}\Big) t\Delta \mathrm{e}^{-t\Delta} \beta_k \frac{\mathrm{d}t}{t} = \int_0^\infty t\Delta \mathrm{e}^{-t\Delta} f_t \frac{\mathrm{d}t}{t} = Sf_t,$$

where

$$f_t = \sum_{k, 4^k \leq t} \Big(\frac{2^k}{\sqrt{t}}\Big) \beta_k.$$

By Cauchy-Schwarz inequality, it gives

$$\int_0^\infty |f_t|^2 \frac{\mathrm{d}t}{t} \leq C \int_0^\infty \sum_{k; 4^k \leq t} \Big(\frac{2^k}{\sqrt{t}}\Big)^2 \beta_k^2 \frac{\mathrm{d}t}{t} \leq C \sum_{k\in\mathbb{Z}} \beta_k^2 \int_{4^k}^\infty \frac{4^k}{t} \frac{\mathrm{d}t}{t} \leq C\Big(\sum_{k\in\mathbb{Z}} |\beta_k|\Big)^2.$$

Thus we have

$$\|f_t\|_{L^1_{\mathbb{H}}(w)} \leq C \int_M \sum_i \Big|\frac{b_i}{r_i}\Big| w \mathrm{d}\mu \leq C\lambda \sum_i w(B_i) \leq C \int_M |\nabla f| w \mathrm{d}\mu < \infty.$$

Since S is a linear operator and $L^\infty_{c,\mathbb{H}}(\mu)$ is dense in $L^1_{\mathbb{H}}(w)$, (1.6) holds for all $h \in L^1_{\mathbb{H}}(w)$. Then by Theorem 1.2, we get

$$\mathrm{I}_2 \leq \frac{C}{\lambda} \sum_i \int_{B_i} \Big|\frac{b_i}{r_i}\Big| w \mathrm{d}\mu \leq \frac{C}{\lambda} \int_M |\nabla f| w \mathrm{d}\mu,$$

which, together with the estimate of I_1, implies (1.4). Thus we have proved Theorem 1.1.

4 Appendix

In this section, we give [3, Theorem 3.7] which plays an important role in this paper.

Given two vector spaces of measurable functions A and B, we say that an operator T acts from A into B, if T is a map defined on A and valued in B. An operator T acting from A to B is sublinear, if

$$|T(f+g)| \le |Tf| + |Tg| \quad \text{and} \quad |T(\lambda f)| = |\lambda||Tf|$$

for all $f, g \in A$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$. Let M be the Hardy-Littlewood maximal operator. [3, Theorem 3.7] is stated as follows.

Theorem 4.1 (see [3, p. 234, Theorem 3.7]) *Let $1 \le p_0 < q_0 \le \infty$. Let $\mathcal{E}$ and $\mathcal{D}$ be vector spaces, such that $\mathcal{D} \subset \mathcal{E}$. Let T, S be operators such that S acts from $\mathcal{D}$ into the set of measurable functions and T is sublinear acting from $\mathcal{E}$ into L^{p_0}. Let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from $\mathcal{D}$ into $\mathcal{E}$. Assume that*

$$\Big(\fint_B |T(I - \mathcal{A}_{r(B)})f|^{p_0}\Big)^{\frac{1}{p_0}} \le CM(|Sf|^{p_0})^{\frac{1}{p_0}}(x) \tag{4.1}$$

and

$$\Big(\fint_B |T\mathcal{A}_{r(B)}f|^{q_0}\Big)^{\frac{1}{q_0}} \le CM(|Sf|^{p_0})^{\frac{1}{p_0}}(x) \tag{4.2}$$

for all $f \in \mathcal{D}$, for all ball B, where $r(B)$ denotes its radius and all $x \in B$. Let $p_0 < p < q_0$ (or $p = q_0$ when $q_0 < \infty$) and $w \in A_{\frac{p}{p_0}} \cap \mathrm{RH}_{(\frac{q_0}{p})'}$. There is a constant C such that

$$\|Tf\|_{L^p(w)} \le C\|Sf\|_{L^p(w)} \tag{4.3}$$

for all $f \in \mathcal{D}$. Furthermore, for all $p_0 < r < q_0$, there is a constant C such that

$$\Big\|\Big(\sum_j |Tf_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(w)} \le C\Big\|\Big(\sum_j |Sf_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(w)} \tag{4.4}$$

for all $f_j \in \mathcal{D}$.

As emphasized in [3], (4.1)–(4.2) are unweighted assumptions by which one can obtain the weighted inequalities as well as the vector valued estimates.

Acknowledgements This work is done during the author's visit to Macquarie University. The author thanks Professor Duong Xuan Thinh and Professor Li Hong-quan for their generous help. The author also thanks the referees for their careful reading of the manuscript and for their constructive and detailed comments.

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