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On the Waring-Goldbach Problem for Six Cubes and Two Biquadrates*

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Abstract Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. In the present paper, it is proved that for any sufficiently large even integer n, the equation

$$n = x^3 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4 + p_7^4$$

has solutions in primes p_i with x being a P_6. This result constitutes a refinement upon that of Hooley C.

Keywords Waring-Goldbach problem, Hardy-Littlewood method, Sieve theory
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1 Introduction

Waring problem for sums of mixed type concerns the representation of a natural number n as the form

$$n = x_1^{k_1} + \cdots + x_s^{k_s}, \quad n > s, \tag{1.1}$$

where $n, k_1, k_2, \cdots, k_s$ are natural integers satisfying $2 \leq k_1 \leq k_2 \leq \cdots \leq k_s$. Not very much is known about results of this kind. For historical references, the reader should consult P12 of LeVeque's reviews in number theory and the bibliography in [11].

The circle method of Hardy and Littlewood provides a technique for problems of this sort, but one has to overcome various difficulties not experienced in the pure waring problem (1.1) with $k_1 = k_2 = \cdots = k_s$. In particular, the choice of the relevant parameters in the definition of major and minor arcs tends to become complicated if a deeper representation problem (1.1) is under consideration.

Hooley [8] established an asymptotic formula for the number of representations of a natural number n as the sum of six cubes and two biquadrates of a natural number, the condition being that the Riemann hypothesis is true for a certain Hasse-Weil L-function.

In view of Hooley's result it is reasonable to propose the conjecture that for every sufficiently large even integer n the equation

$$n = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^4 + p_8^4$$

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is solvable. But this conjecture is perhaps out of the grasp of modern number theory techniques. However, Motivated by [2–4], the Hardy-Littlewood method and the sieve theory enable us to obtain the following approximation to it.

Theorem 1.1 *For any sufficiently large even integer n, let $\nu(n)$ be the number of solutions of the equation*

$$n = x^3 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4 + p_7^4 \tag{1.2}$$

with x being an almost prime P_6 and primes p_i. Then we have

$$\nu(n) \gg \frac{n^{\frac{85}{72}}}{\log^8 n}.$$

2 Notation and Some Preliminary Lemmas

In this paper, $\varepsilon \in (0, 10^{-10})$ and n denotes a sufficiently large even integer. The constants $\ll$-symbol and O-term depend at most on ε. For positive A and B, by $A \asymp B$ we mean that $A \ll B$ and $B \ll A$, and by $x \sim X$ we denote $X < x \le 2X$. The letter p, with or without subscript, is reserved for a prime number. We denote by (m, k) the greatest common divisor of m and k. As usual, $\mu(m)$ and $\phi(m)$ denote Möbius function and Euler's function respectively. By $\tau(m)$ we denote the divisor function, and by $a(d)$ we denote the arithmetical function bounded above by $\tau(m)$. We use $e(\alpha)$ to denote $\mathrm{e}^{2\pi \mathrm{i}\alpha}$ and $e_q(\alpha) = e\big(\frac{\alpha}{q}\big)$. We denote by $\sum\limits_{x(q)}$ and $\sum\limits_{x(q)*}$ sums with x running over a complete system and a reduced system of residues modulo q respectively. We always denote by χ a Dirichlet character $\pmod q$, and by χ^0 the principal Dirichlet characters $\pmod q$. Let

$$\begin{aligned}
&C = 10^{10}, \quad Q_0 = \log^{20C} n, \quad Q_1 = n^{\frac{11}{48}+9\varepsilon}, \quad Q_2 = n^{\frac{1}{2}},\\
&D = n^{\frac{1}{36}-14\varepsilon}, \quad z = D^{\frac{1}{3}}, \quad X_1 = 0.5n^{\frac{1}{3}}, \quad X_2 = 0.5n^{\frac{5}{18}}, \quad Y = 0.5n^{\frac{25}{144}},\\
&\mathcal{M}_r = \{m \mid m \sim X_1,\ m = p_1 p_2 \cdots p_r,\ z \le p_1 \le \cdots \le p_r\},\\
&\mathcal{N}_r = \{m \mid m \sim X_1,\ m = p_1 p_2 \cdots p_{r-1},\\
&\qquad z \le p_1 \le \cdots \le p_{r-1},\ p_1 \cdots p_{r-2} p_{r-1}^2 \le 2X_1\} \quad (7 \le r \le 36),\\
&S_k^*(q,a) = \sum_{r(q)*} e_q(ar^k), \quad S_k(q,a) = \sum_{r(q)} e_q(ar^k),\\
&F_i(\alpha) = \sum_{n \sim X_i} e(\alpha n^3), \quad f_i(\alpha) = \sum_{p \sim X_i} (\log p) e(\alpha p^3)\, u_i(\lambda) = \int_{X_i}^{2X_i} e(\lambda u^3)\mathrm{d}u,\\
&G(\alpha) = \sum_{n \sim Y} e(\alpha n^4), \quad g(\alpha) = \sum_{p \sim Y} (\log p) e(\alpha p^4), \quad v(\lambda) = \int_Y^{2Y} e(\lambda u^4)\mathrm{d}u,\\
&f_{3,r}(\alpha) = \sum_{\substack{n \in \mathcal{N}_r\\ np \sim X_1}} e(\alpha (np)^3)\Bigg(\frac{\log p}{\log\Big(\dfrac{X_1}{n}\Big)}\Bigg),\\
&B_d(q,n) = \sum_{a(q)*} S_3(q, ad^3) S_3^{*5}(q,a) S_4^{*2}(q,a) e_q(-an),\\
&A_d(q,n) = \frac{B_d(q,n)}{q\phi^7(q)}, \quad A(q,n) = A_1(q,n), \quad \mathfrak{S}_d(n) = \sum_{q=1}^{\infty} A_d(q,n),
\end{aligned}$$

$$\mathfrak{S}(n)=\mathfrak{S}_1(n),\quad \mathcal{J}(n)=\int_{-\infty}^{\infty}u_1^3(\lambda)u_2^3(\lambda)v^2(\lambda)e(-\lambda n)\mathrm{d}\lambda.$$

Lemma 2.1 *For $(q,a)=1$, we have*

(i) $S_j(q,a)\le q^{1-\frac{1}{j}}$; (ii) $S_j^*(q,a)\ll q^{\frac12+\varepsilon}$.

In particular, for $(p,a)=1$, we have

(iii) $|S_j(p,a)|\le((j,p-1)-1)p^{\frac12}$;

(iv) $|S_j^*(p,a)|\le((j,p-1)-1)p^{\frac12}+1$;

(v) $S_j^*(p^l,a)=0$ *for* $l\ge\gamma(p)$,

where

$$\gamma(p)=\begin{cases}\theta+2, & \text{if } p^\theta\|j,\ p\ne 2 \text{ or } p=2,\ \theta=0,\\ \theta+3, & \text{if } p^\theta\|j,\ p=2,\ \theta>0.\end{cases}$$

Proof For (i) and (iii)–(iv), see Theorem 4.2 and Lemma 4.3 in [11], respectively. For (ii), see Chapter VI, Problem 14 in [13]. For (v), see Lemma 8.3 in [7].

Lemma 2.2 *We have*

(i) $\displaystyle\int_0^1|F_1(\alpha)F_2^2(\alpha)|^2\mathrm{d}\alpha\ll n^{\frac89+\varepsilon}$,

(ii) $\displaystyle\int_0^1|f_1(\alpha)f_2^2(\alpha)|^2\mathrm{d}\alpha\ll n^{\frac89+\varepsilon}$.

Proof This is the theorem in Vaughan [10] and the inequality (ii) follows from (i) by considering the number of solutions of the underlying Diophantine equations.

Lemma 2.3 *We have*

(i) $\displaystyle\int_0^1|F_1(\alpha)F_2(\alpha)G^2(\alpha)|^2\mathrm{d}\alpha\ll n^{\frac{23}{24}+\varepsilon}$,

(ii) $\displaystyle\int_0^1|f_1(\alpha)f_2(\alpha)g^2(\alpha)|^2\mathrm{d}\alpha\ll n^{\frac{23}{24}+\varepsilon}$.

Proof The inequality (i) follows easily from Theorem 4 in [1], and the inequality (ii) follows from (i) by considering the number of solutions of the underlying Diophantine equations.

Lemma 2.4 *We have*

(i) $\displaystyle\int_0^1|F_1(\alpha)F_2^3(\alpha)G^2(\alpha)|^2\mathrm{d}\alpha\ll n^{\frac{73}{36}}$,

(ii) $\displaystyle\int_0^1|f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|^2\mathrm{d}\alpha\ll n^{\frac{73}{36}}(\log n)^{12}$.

Proof For $(q,a)=1$, put

$$\mathcal{N}=\bigcup_{1\le q\le n^{\frac{1}{12}}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\Big(\frac aq-\frac{1}{100qn^{\frac34}},\frac aq+\frac{1}{100qn^{\frac34}}\Big],$$

$$\mathcal{N}_0=\Big(-\frac{1}{10n^{\frac34}},1-\frac{1}{10n^{\frac34}}\Big]\setminus\mathcal{N}.$$

Then we have

$$\int_0^1|F_1(\alpha)F_2^3(\alpha)G^2(\alpha)|^2\mathrm{d}\alpha=\int_{-\frac{1}{10}n^{-\frac34}}^{1-\frac{1}{10}n^{-\frac34}}|F_1(\alpha)F_2^3(\alpha)G^2(\alpha)|^2\mathrm{d}\alpha$$

$$= \Big(\int_{\mathcal{N}} + \int_{\mathcal{N}_0} \Big) |F_1(\alpha)F_2^3(\alpha)G^2(\alpha)|^2 \mathrm{d}\alpha. \tag{2.1}$$

For $\alpha \in \mathcal{N}_0$, by Weyl's inequality (see [11, Lemma 2.4]), we have

$$F_2(\alpha) \ll n^{\frac{37}{144}+\varepsilon},$$

from which and Lemma 2.3 (i), we get

$$\begin{aligned} \int_{\mathcal{N}_0} |F_1(\alpha)F_2^3(\alpha)G^2(\alpha)|^2 \mathrm{d}\alpha &\ll n^{\frac{37}{36}+4\varepsilon} \int_0^1 |F_1(\alpha)F_2(\alpha)G^2(\alpha)|^2 \mathrm{d}\alpha \\ &\ll n^{\frac{143}{72}+5\varepsilon} \ll n^{\frac{73}{36}}. \end{aligned} \tag{2.2}$$

For $\alpha = \frac{a}{q} + \lambda \in \mathcal{N}$, by Theorem 4.1 in [11] and Lemma 4.2 in [12], we have

$$F_1(\alpha) = \frac{S_3(q,a)}{q} u_1(\lambda) + O(q^{\frac{1}{2}+\varepsilon}) \ll \frac{X_1}{q^{\frac{1}{3}}(1+|\lambda|n)}, \tag{2.3}$$

$$F_2(\alpha) = \frac{S_3(q,a)}{q} u_2(\lambda) + O(q^{\frac{1}{2}+\varepsilon}) \ll \frac{X_2}{q^{\frac{1}{3}}}, \tag{2.4}$$

$$G(\alpha) = \frac{S_4(q,a)}{q} v(\lambda) + O(q^{\frac{1}{2}+\varepsilon}) \ll \frac{Y}{q^{\frac{1}{4}}}, \tag{2.5}$$

where Lemma 2.1(i) and the bound

$$u_1(\lambda) \ll \frac{X_1}{1+|\lambda|n}$$

are used.

By (2.3)–(2.5), we get

$$\begin{aligned} &\int_{\mathcal{N}} |F_1(\alpha)F_2^3(\alpha)G^2(\alpha)|^2 \mathrm{d}\alpha \\ &\ll \sum_{q \le n^{\frac{1}{12}}} \sum_{\substack{a=1 \\ (q,a)=1}}^{q} \int_{|\lambda| \le n^{-\frac{3}{4}}} \frac{X_1^2 X_2^6 Y^4}{q^{\frac{11}{3}}(1+n|\lambda|)^2} \mathrm{d}\lambda \\ &= \sum_{1 \le q \le n^{\frac{1}{12}}} \sum_{\substack{a=1 \\ (q,a)=1}}^{q} q^{-\frac{11}{3}} \int_{|\lambda| \le n^{-\frac{3}{4}}} \frac{n^{\frac{109}{36}}}{(1+n|\lambda|)^2} \mathrm{d}\lambda \\ &\ll \sum_{1 \le q \le n^{\frac{1}{12}}} q^{-\frac{8}{3}} \Big(\int_0^{n^{-1}} n^{\frac{109}{36}} \mathrm{d}\lambda + \int_{n^{-1}}^{n^{-\frac{3}{4}}} \frac{n^{\frac{109}{36}}}{(n\lambda)^2} \mathrm{d}\lambda \Big) \\ &\ll n^{\frac{73}{36}}. \end{aligned} \tag{2.6}$$

Thus by (2.1)–(2.2) and (2.6), the inequality (i) is proved. The inequality (ii) follows from (i) by considering the number of solutions of the underlying Diophantine equations. Then Lemma 2.4 is established.

Lemma 2.5 *For $\alpha = \frac{a}{q} + \lambda$, let*

$$h^*(\alpha) = \sum_{d \le D} \frac{a(d)}{dq} S_3(q, ad^3) u_1(\lambda), \tag{2.7}$$

$$\Delta(\alpha) = f_1(\alpha) - \frac{S_3^*(q,a)}{\phi(q)} \sum_{n\sim X_1} e(\lambda n^3), \tag{2.8}$$

$$\mathfrak{I} = \bigcup_{q\le Q_0} \bigcup_{\substack{a=-q\\(q,a)=1}}^{2q} \Big(\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0}\Big]. \tag{2.9}$$

Then we have

$$\int_{\mathfrak{I}} |h^*(\alpha)\Delta(\alpha)|^2 \mathrm{d}\alpha \ll n^{\frac{1}{3}} (\log n)^{-100C}.$$

Proof The proof is similar to that of Lemma 2.5 in [4].

Lemma 2.6 *For $i = 1, 2$, let*

$$f_i^*(\alpha) = \phi(q)^{-1} S_3^*(q,a) u_i(\lambda). \tag{2.10}$$

Then we have

(i) $\displaystyle\int_{\mathfrak{I}} |f_1^*(\alpha)|^2 \mathrm{d}\alpha \ll n^{-\frac{1}{3}} (\log n)^{21C}$,

(ii) $\displaystyle\int_{\mathfrak{I}} |h^*(\alpha)|^2 \mathrm{d}\alpha \ll n^{-\frac{1}{3}} (\log n)^{27C}$,

where $h^*(\alpha)$ and $\mathfrak{I}$ are defined by (2.7) *and* (2.9)*, respectively.*

Proof The proof is similar to that of Lemma 2.6 in [4].

For $(a,q) = 1, 1 \le a \le q$, let

$$\mathfrak{M}_0(q,a) = \Big(\frac{a}{q} - \frac{Q_0}{n}, \frac{a}{q} + \frac{Q_0}{n}\Big], \quad \mathfrak{M}_0 = \bigcup_{1\le q\le Q_0^5} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}_0(q,a),$$

$$\mathfrak{M}(q,a) = \Big(\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\Big], \quad \mathfrak{M} = \bigcup_{1\le q\le Q_0^5} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(q,a),$$

$$\mathfrak{M}_1(q,a) = \Big(\frac{a}{q} - \frac{1}{10qn^{\frac{25}{36}+14\varepsilon}}, \frac{a}{q} + \frac{1}{10qn^{\frac{25}{36}+14\varepsilon}}\Big],$$

$$\mathfrak{m}_1 = \bigcup_{Q_0^5\le q\le Q_1} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}_1(q,a), \quad \mathfrak{J}_0 = \Big(-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\Big], \quad \mathfrak{m}_0 = \mathfrak{M} \setminus \mathfrak{M}_0,$$

$$\mathfrak{m} = \bigcup_{Q_0^5< q\le Q_1} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(q,a), \quad \mathfrak{m}_3 = \mathfrak{m} \setminus \mathfrak{m}_1, \quad \mathfrak{m}_2 = \mathfrak{J}_0 \setminus (\mathfrak{M} \cup \mathfrak{m}).$$

Then we have the Farey dissection

$$\mathfrak{J}_0 = \mathfrak{M}_0 \cup \mathfrak{m}_0 \cup \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3. \tag{2.11}$$

Lemma 2.7 *For $\alpha = \frac{a}{q} + \lambda \in \mathfrak{M}_0$, let*

$$g^*(\alpha) = \phi(q)^{-1} S_4^*(q,a) v(\lambda).$$

Then for $i = 1, 2$ and $7 \le r \le 36$ we have

$$f_i(\alpha) = f_i^*(\alpha) + O(X_i \exp(-\log^{\frac{1}{3}} n)), \tag{2.12}$$

$$g(\alpha) = g^*(\alpha) + O(Y \exp(-\log^{\frac{1}{3}} n)), \tag{2.13}$$

$$f_{3,r}(\alpha) = \frac{c_r f_1^*(\alpha)}{\log X_1} + O(X_1 \exp(-\log^{\frac{1}{3}} n)), \tag{2.14}$$

where $f_i^*(\alpha)$ *is defined by* (2.10), *and*

$$c_r = (1 + O(\varepsilon)) \int_{r-1}^{35} \frac{\mathrm{d}t_1}{t_1} \cdots \int_3^{t_{r-4}-1} \frac{\mathrm{d}t_{r-3}}{t_{r-3}} \int_2^{t_{r-3}-1} \frac{\log(t_{r-2}-1)\mathrm{d}t_{r-2}}{t_{r-2}}. \tag{2.15}$$

Proof By some routine arrangements and summation by parts, formulae (2.12)–(2.13) follow from Siegel-Walfisz theorem and prime number theorem. The detailed proof of (2.14) is similar to that of Lemma 2.6 in [5], hence we omit it here.

Lemma 2.8 (see [3, Lemma 6]) *Suppose that* $|a(d)| \le \tau(d)$ *and*

$$h(\alpha) = \sum_{d \le D} a(d) \sum_{\frac{X_1}{d} < l \le \frac{2X_1}{d}} e(\alpha d^3 l^3). \tag{2.16}$$

Then for $\alpha = \frac{a}{q} + \lambda \in [0,1]$, $(q,a) = 1$, $q \le Q_2$, $|\lambda| \le \frac{1}{qQ_2}$, *we have*

$$h(\alpha) \ll \frac{n^{\frac{1}{3}+\varepsilon}}{q^{\frac{1}{3}}(1+n|\lambda|)^{\frac{1}{3}}} + n^{\frac{1}{4}+\varepsilon} D^{\frac{1}{4}}.$$

3 Mean Value Theorems

In this section, we shall prove the mean value theorems for the proof of the theorem.

Proposition 3.1 *Let*

$$J_d(n) = \sum_{\substack{(dl)^3 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4 + p_7^4 = n \\ dl, p_1, p_4 \sim X_1,\, p_2, p_3, p_5 \sim X_2,\, p_6, p_7 \sim Y}} (\log p_1) \cdots (\log p_7).$$

Then for $|a(d)| \le \tau(d)$, *we have*

$$\sum_{d \le D} a(d) c_m d_k \Big(J_d(n) - \frac{\mathfrak{S}_d(n)}{d} \mathcal{J}(n) \Big) \ll n^{\frac{85}{72}} (\log n)^{-C}.$$

Proof Let

$$\mathcal{K}(\alpha) = h(\alpha) f_1^2(\alpha) f_2^3(\alpha) g^2(\alpha) e(-\alpha n),$$

where $h(\alpha)$ is defined by (2.16). Then by the farey dissection (2.11), we get

$$\begin{aligned} &\sum_{d \le D} a(d) c_m d_k J_d(n) \mathcal{K}(\alpha) \mathrm{d}\alpha \\ &= \int_{\mathfrak{J}_0} = \Big(\int_{\mathfrak{M}_0} + \int_{\mathfrak{m}_0} + \int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2} \Big) \mathcal{K}(\alpha) \mathrm{d}\alpha. \end{aligned} \tag{3.1}$$

By Cauchy's inequality, we have

$$\int_0^1 \left| f_1^2(\alpha) f_2^3(\alpha) g^2(\alpha) \right| \mathrm{d}\alpha$$

$$\ll \Big(\int_0^1 \big|f_1(\alpha)f_2^2(\alpha)\big|^2 \mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1 \big|f_1(\alpha)f_2(\alpha)g^2(\alpha)\big|^2 \mathrm{d}\alpha\Big)^{\frac{1}{2}}$$
$$\ll n^{\frac{1}{2}(\frac{8}{9}+\varepsilon)}n^{\frac{1}{2}(\frac{23}{24}+\varepsilon)} \ll n^{\frac{133}{144}+\varepsilon}. \tag{3.2}$$

For $\alpha \in \mathfrak{m}_2$, by Lemma 2.8 we get

$$h(\alpha) \ll n^{\frac{37}{144}-2\varepsilon},$$

from which and (3.2), we have

$$\int_{\mathfrak{m}_2} \mathcal{K}(\alpha)\mathrm{d}\alpha \ll \max_{\alpha\in\mathfrak{m}_2} |h(\alpha)|\Big(\int_0^1 \big|f_1^2(\alpha)f_2^3(\alpha)g^2(\alpha)\big| \mathrm{d}\alpha\Big) \ll n^{\frac{85}{72}-\varepsilon}. \tag{3.3}$$

Similarly, Lemma 2.8 and (3.2) imply

$$\int_{\mathfrak{m}_3} \mathcal{K}(\alpha)\mathrm{d}\alpha \ll \max_{\alpha\in\mathfrak{m}_3} |h(\alpha)|\Big(\int_0^1 \big|f_1^2(\alpha)f_2^3(\alpha)g^2(\alpha)\big| \mathrm{d}\alpha\Big) \ll n^{\frac{85}{72}-\varepsilon}. \tag{3.4}$$

Write

$$a(d) = \sum_{\substack{m\le D^{\frac{2}{3}}\\ k\le D^{\frac{1}{3}}\\ mk=d}} c_m d_k, \quad h(\alpha) = \sum_{d\le D} a(d) \sum_{l\sim \frac{X_1}{d}} e(\alpha(dl)^3).$$

Then by Theorem 4.1 in [11], for $\alpha \in \mathfrak{m}_1$, we have

$$h(\alpha) = h^*(\alpha) + O(DQ_1^{\frac{1}{2}+\varepsilon}) = h^*(\alpha) + O(N^{\frac{21}{144}}), \tag{3.5}$$

where $h^*(\alpha)$ is defined by (2.7). Let $\mathcal{K}_1(\alpha) = h^*(\alpha)f_1^2(\alpha)f_2^3(\alpha)g^2(\alpha)e(-\alpha n)$. Then from (3.2) and (3.5), we get

$$\int_{\mathfrak{m}_1} \mathcal{K}(\alpha)\mathrm{d}\alpha = \int_{\mathfrak{m}_1} \mathcal{K}_1(\alpha)\mathrm{d}\alpha + O(n^{\frac{85}{72}-\varepsilon}). \tag{3.6}$$

Let

$$\mathfrak{I}_0 = \bigcup_{1\le q\le Q_0} \bigcup_{\substack{a=-q\\ (q,a)=1}}^{2q} \Big(\frac{a}{q} - \frac{1}{n^{\frac{11}{12}}}, \frac{a}{q} + \frac{1}{n^{\frac{11}{12}}}\Big], \quad \mathfrak{I}_1 = \mathfrak{I} \setminus \mathfrak{I}_0,$$

where $\mathfrak{I}$ is defined by (2.9). Then we have $\mathfrak{m}_1 \subset \mathfrak{I}_0 \subset \mathfrak{I}$. By the rational approximation theorem of Dirichlet, we get

$$\int_{\mathfrak{m}_1} |\mathcal{K}_1(\alpha)| \,\mathrm{d}\alpha \le \int_{\mathfrak{m}_1\cap\mathfrak{I}_0} |\mathcal{K}_1(\alpha)|\mathrm{d}\alpha + \int_{\mathfrak{m}_1\cap\mathfrak{I}_1} |\mathcal{K}_1(\alpha)|\mathrm{d}\alpha. \tag{3.7}$$

By (2.7), Lemma 2.1(i), the inequalities $(q, d^3) \le (q, d)^3$, $\tau(dl) \le \tau(d)\tau(l)$ and $\sum_{d\le x} \frac{\tau(d)}{d} \ll \log^2 x$, we have

$$h^*(\alpha) \ll \sum_{d\le D} \frac{\tau(d)}{d}(q, d^3)^{\frac{1}{3}} q^{-\frac{1}{3}} |u_1(\lambda)|$$

$$\ll \sum_{\substack{t|q\\ t\le D}} \tau(t)\Big(\sum_{d\le \frac{D}{t}} \frac{\tau(d)}{d}\Big) q^{-\frac{1}{3}}|u_1(\lambda)|$$
$$\ll q^{-\frac{1}{3}+\varepsilon}|u_1(\lambda)|\log^2 n. \tag{3.8}$$

For $\alpha \in \mathfrak{I}_1(q,a)$, we obtain

$$h^*(\alpha) \ll \frac{n^{\frac{1}{3}+\varepsilon}}{1+|\lambda|n} \ll n^{\frac{1}{4}+\varepsilon}. \tag{3.9}$$

From (3.2) and (3.9), we get

$$\int_{\mathfrak{m}_1\cap\mathfrak{I}_1} |\mathcal{K}_1(\alpha)|\mathrm{d}\alpha \ll n^{\frac{1}{4}+\varepsilon}\int_0^1 \big|f_1^2(\alpha)f_2^3(\alpha)g^2(\alpha)\big|\,\mathrm{d}\alpha$$
$$\ll n^{\frac{169}{144}+2\varepsilon} \ll n^{\frac{85}{72}-\varepsilon}. \tag{3.10}$$

By Lemma 4.8 in [12], we have

$$\begin{aligned}\int_{\mathfrak{m}_1\cap\mathfrak{I}_0} |\mathcal{K}_1(\alpha)|\mathrm{d}\alpha &= \int_{\mathfrak{m}_1\cap\mathfrak{I}_0} |h^*(\alpha)\Delta(\alpha)f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|\mathrm{d}\alpha\\ &\quad+\int_{\mathfrak{m}_1\cap\mathfrak{I}_0} |h^*(\alpha)f_1^*(\alpha)f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|\mathrm{d}\alpha\\ &\quad+O\Big(\int_{\mathfrak{m}_1\cap\mathfrak{I}_0} |h^*(\alpha)f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|\mathrm{d}\alpha\Big)\\ &=: \mathrm{I}_1+\mathrm{I}_2+O(\mathrm{I}_3),\end{aligned} \tag{3.11}$$

where $\Delta(\alpha), f_1^*(\alpha)$ are defined by (2.8) and (2.10) respectively.

By Cauchy's inequality, Lemma 2.4(ii) and Lemma 2.5, we obtain

$$\begin{aligned}\mathrm{I}_1 &\ll \Big(\int_{\mathfrak{I}_0} |h^*(\alpha)\Delta(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1 |f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}\\ &\ll (n^{\frac{1}{3}}\log^{-100C} n)^{\frac{1}{2}}(n^{\frac{73}{36}}(\log n)^{12})^{\frac{1}{2}} \ll n^{\frac{85}{72}}L^{-10C}.\end{aligned} \tag{3.12}$$

By (3.8), we have for $\alpha\in\mathfrak{m}_1$,

$$h^*(\alpha) \ll q^{-\frac{1}{3}+\varepsilon}|u_1(\lambda)|\log^2 n \ll Q_0^{-\frac{1}{3}+\varepsilon} n^{\frac{1}{3}}\log^2 n \ll n^{\frac{1}{3}}L^{-30C},$$

from which, Cauchy's inequality, Lemma 2.4(ii) and Lemma 2.6(i), we obtain

$$\begin{aligned}\mathrm{I}_2 &\ll n^{\frac{1}{3}}L^{-30C}\Big(\int_{\mathfrak{I}_0} |f_1^*(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1 |f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}\\ &\ll n^{\frac{1}{3}}L^{-30C}(n^{\frac{2}{3}-1}L^{21C})^{\frac{1}{2}}(n^{\frac{73}{36}}(\log n)^{12})^{\frac{1}{2}}\\ &\ll n^{\frac{85}{72}}L^{-10C}.\end{aligned} \tag{3.13}$$

By Cauchy's inequality, Lemma 2.4(ii) and Lemma 2.6(ii), we obtain

$$\mathrm{I}_3 \ll \Big(\int_{\mathfrak{I}_0} |h^*(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1 |f_1(\alpha)f_2^3(\alpha)g^2(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}$$

$$\ll (n^{-\frac{1}{3}} \log^{27C} n)^{\frac{1}{2}} (n^{\frac{73}{36}} (\log n)^{12})^{\frac{1}{2}}$$
$$\ll n^{\frac{85}{72}} L^{-10C}. \tag{3.14}$$

From (3.11)–(3.14), we get

$$\int_{\mathfrak{m}_1 \cap \mathfrak{I}_0} |\mathcal{K}_1(\alpha)| \mathrm{d}\alpha \ll n^{\frac{85}{72}} L^{-10C}. \tag{3.15}$$

It follows from (3.6)–(3.7), (3.10) and (3.15) that

$$\int_{\mathfrak{m}_1} \mathcal{K}(\alpha) \mathrm{d}\alpha \ll n^{\frac{85}{72}} L^{-10C}. \tag{3.16}$$

By arguments similar to but simpler than that leading to (3.16), we have

$$\int_{\mathfrak{m}_0} \mathcal{K}(\alpha) \mathrm{d}\alpha \ll n^{\frac{85}{72}} L^{-10C}. \tag{3.17}$$

For $\alpha \in \mathfrak{M}_0$, let

$$\mathcal{K}_0(\alpha) = h^*(\alpha) f_1^{*2}(\alpha) f_2^{*3}(\alpha) g^{*2}(\alpha) e(-\alpha n). \tag{3.18}$$

By Theorem 4.1 in [11], for $\alpha \in \mathfrak{M}_0$, we have

$$h(\alpha) = h^*(\alpha) + O(D(Q_0^5)^{\frac{1}{2}+\varepsilon}) = h^*(\alpha) + O(n^{\frac{1}{36}-9\varepsilon}),$$

from which and Lemma 2.7, we get

$$\mathcal{K}(\alpha) - \mathcal{K}_0(\alpha) \ll n^{\frac{157}{72}} \exp(-\log^{\frac{1}{4}} n). \tag{3.19}$$

By (3.19), we have

$$\int_{\mathfrak{M}_0} \mathcal{K}(\alpha) \mathrm{d}\alpha = \int_{\mathfrak{M}_0} \mathcal{K}_0(\alpha) \mathrm{d}\alpha + O(n^{\frac{85}{72}} L^{-C}). \tag{3.20}$$

By the well-known standard endgame technique in the Hardy-Littlewood method, we obtain

$$\int_{\mathfrak{M}_0} K_0(\alpha) \mathrm{d}\alpha = \sum_{\substack{m \le D^{\frac{2}{3}} \\ k \le D^{\frac{1}{3}}}} c_m d_k \frac{\mathfrak{S}_{mk}(n)}{mk} \mathcal{J}(n) + O(n^{\frac{85}{72}} L^{-C}), \tag{3.21}$$

$$\mathcal{J}(n) \asymp n^{\frac{85}{72}}. \tag{3.22}$$

From (3.1), (3.3), (3.16)–(3.17) and (3.20)–(3.22), Proposition 3.1 is proved.

By the same method, we obtain the following proposition.

Proposition 3.2 *For $7 \le r \le 36$, let*

$$J_d^{(r)}(n) = \sum_{\substack{(dl)^3 + (mp)^3 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^4 + p_6^4 = n \\ dl, mp, p_3 \sim X_1, m \in \mathcal{N}_r, , p_1, p_2, p_4 \sim X_2, p_5, p_6 \sim Y}} (\log p_1) \cdots (\log p_6) \Big(\frac{\log p}{\log(\frac{X_1}{n})} \Big).$$

Then for $|c_m| \le 1$, $|d_k| \le 1$, we have

$$\sum_{\substack{m \le D^{\frac{2}{3}} \\ k \le D^{\frac{1}{3}}}} c_m d_k \Big(J_{mk}^{(r)}(n) - c_r \frac{\mathfrak{S}_{mk}(n)}{mk \log X_1} \mathcal{J}(n) \Big) \ll n^{\frac{85}{72}} L^{-C},$$

where c_r is defined by (2.15).

4 On the Function $\omega(d)$

In order to prove our theorem, in this section, we consider the function $\omega(d)$ which is defined by (4.9).

Lemma 4.1 *Let $K(q,n)$ and $H(q,n)$ denote the number of solutions to the congruences*

$$y^3+\sum_{i=1}^{4}x_i^3+\sum_{j=1}^{2}y_j^4\equiv n \ (\mathrm{mod}\, q),\quad 1\le y,x_i,y_j\le q,\ (yx_iy_j,q)=1$$

and

$$x^3+y^3+\sum_{i=1}^{4}x_i^3+\sum_{j=1}^{2}y_j^4\equiv n \ (\mathrm{mod}\, q),\quad 1\le x,y,x_i,y_j\le q,\ (yx_iy_j,q)=1$$

respectively. Then we have $H(p,n)>K(p,n)$. Moveover,

$$H(p,n)=p^7+O\left(p^6\right), \tag{4.1}$$

$$K(p,n)=p^6+O\left(p^5\right). \tag{4.2}$$

Proof Let us consider the congruence

$$x^3+y^3+\sum_{i=1}^{4}x_i^3+\sum_{j=1}^{2}y_j^4\equiv n \ (\mathrm{mod}\, p),\quad 1\le x,y,x_i,y_j\le p-1. \tag{4.3}$$

Let $H^*(p,n)$ denote the number of solutions to the congruence (4.3). Then we have $H(p,n)=H^*(p,n)+K(p,n)$. Therefore it suffices to prove $H^*(p,n)>0$. We have

$$\begin{aligned} pH^*(p,n)&=\sum_{a=1}^{p}S_3^{*6}(p,a)S_4^{*2}(p,a)e_p(-\alpha n)\\ &=(p-1)^8+\delta_p, \end{aligned} \tag{4.4}$$

where

$$\delta_p=\sum_{a=1}^{p-1}S_3^{*6}(p,a)S_4^{*2}(p,a)e_p(-\alpha n).$$

By Lemma 2.1(iv), we have

$$|\delta_p|\le(p-1)\left(2\sqrt{p}+1\right)^6\left(3\sqrt{p}+1\right)^2. \tag{4.5}$$

It is easy to show that $|\delta_p|<(p-1)^8$ for $p>13$, hence we have $H^*(p,n)>0$. On the other hand, for $p=2,3,5,7,11,13$ it can be checked by hand that $H^*(p,n)>0$.

By (4.4)–(4.5) we have

$$H^*(p,n)=p^8+O(p^7),$$

and (4.1)–(4.2) follow from similar arguments.

Lemma 4.2 *The series $\mathfrak{S}(n)$ is convergent and $\mathfrak{S}(n)>0$.*

Proof The convergence of $\mathfrak{S}(n)$ immediately follows from Lemma 2.1(i)–(ii). Note that $A(q,n)$ is multiplicative in q and by Lemma 2.1(v), we get

$$\mathfrak{S}(n)=\prod_{p}(1+A(p,n)). \tag{4.6}$$

From Lemma 2.1(iii)–(iv), for $p > 11$ we get

$$|A(p,n)| \le \frac{(p-1)\sqrt{p}\left(2\sqrt{p}+1\right)^5\left(3\sqrt{p}+1\right)^2}{p(p-1)^7} \le \frac{100}{p^2}.$$

Therefore we have

$$\prod_{p>11}((1+A(p,n)) > \prod_{p>11}\Big(1-\frac{100}{p^2}\Big) > c > 0. \tag{4.7}$$

It is easy to show that

$$1 + A(p,n) = \frac{H(p,n)}{(p-1)^7}. \tag{4.8}$$

From Lemma 4.1 and (4.6)–(4.8), we get $\mathfrak{S}(n) > 0$. Hence Lemma 4.2 is proved.

In view of Lemma 4.2, for a natural number d, we may define

$$\omega(d) = \frac{\mathfrak{S}_d(n)}{\mathfrak{S}(n)}.$$

Lemma 4.3 *The function $\omega(d)$ is multiplicative and*

$$0 \le \omega(p) < p, \quad \omega(p) = 1 + O(p^{-1}) \tag{4.9}$$

for each prime p.

Proof Similarly to (4.6), we have

$$\mathfrak{S}_d(n) = \prod_{p\nmid d}(1 + A_d(p,n)) \prod_{p\mid d}(1 + A_d(p,n)). \tag{4.10}$$

By the facts that $S_k(q, ad^k) = S_k(q,a)$ for $(d,q) = 1$, $A_d(p,n) = A_p(p,n)$ for $p \mid d$ and (4.6)–(4.10), we get

$$\omega(p) = \frac{1 + A_p(p,n)}{1 + A(p,n)}. \tag{4.11}$$

It is easy to show that

$$1 + A_p(p,n) = \frac{pK(p,n)}{\phi^7(p)}. \tag{4.12}$$

From (4.8), (4.11)–(4.12) we drive that

$$\omega(p) = \frac{pK(p,n)}{H(p,n)}. \tag{4.13}$$

By (4.13) and Lemma 4.1, the proof of Lemma 4.3 is completed.

5 Proof of Theorem 1.1

In this section, $f(s)$ and $F(s)$ denote the classical functions in the linear sieve theory, γ denotes Euler's constant. Then by (8.2.8) and (8.2.9) in [6] we have

$$f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, \quad 2 \le s \le 4; \tag{5.1}$$

$$F(s) = \frac{2e^{\gamma}}{s}, \qquad 1 \le s \le 3. \tag{5.2}$$

In the proof of Theorem 1.1 we adopt the following notations:

$$\mathcal{P} = \prod_{2<p<z} p, \quad \log 2\mathbf{X} = (\log 2X_1)^2(\log 2X_2)^3(\log^2 2Y),$$
$$\log \mathbf{X} = (\log X_1)^2(\log X_2)^3(\log^2 Y)$$

and let $\lambda^{\pm}(d)$ denote Rosser's weights of order D. Put

$$V(z) = \prod_{p|\mathcal{P}}\Big(1 - \frac{\omega(p)}{p}\Big).$$

Then by Lemma 4.3 and Merten's prime number theorem we get

$$V(z) \asymp \frac{1}{\log n}. \tag{5.3}$$

Let $\nu(n)$ denote the number of solutions of the equation (1.2) with x being a P_6. Then we have

$$\begin{aligned}
\nu(n) &\ge \sum_{\substack{l^3+p_1^3+p_2^3+p_3^3+p_4^3+p_5^3+p_6^4+p_7^4=n \\ (l,\mathcal{P})=1, l, p_1, p_4 \sim X_1 \\ p_2, p_3, p_5 \sim X_2, p_6, p_7 \sim Y}} 1 - \sum_{r=7}^{36} \sum_{\substack{h^3+p_1^3+p_2^3+p_3^3+p_4^3+p_5^3+p_6^4+p_7^4=n \\ h\in\mathcal{M}_r,\, p_1, p_4 \sim X_1 \\ p_2, p_3, p_5 \sim X_2, p_6, p_7 \sim Y}} 1 \\
&\ge \sum_{\substack{l^3+p_1^3+p_2^3+p_3^3+p_4^3+p_5^3+p_6^4+p_7^4=n \\ (l,\mathcal{P})=1, l, p_1, p_4 \sim X_1 \\ p_2, p_3, p_5 \sim X_2, p_6, p_7 \sim Y}} 1 - \sum_{r=7}^{36} \sum_{\substack{(mp)^3+p_1^3+p_2^3+p_3^3+p_4^3+p_5^3+p_6^4+p_7^4=n \\ m\in\mathcal{M}_r,\, mp, p_1, p_4 \sim X_1 \\ p_2, p_3, p_5 \sim X_2, p_6, p_7 \sim Y}} 1 \\
&=: \nu_0(n) - \sum_{r=7}^{36} \nu_r(n).
\end{aligned} \tag{5.4}$$

Next we shall give a nontrivial lower bound for $\nu(n)$. The facts are required:

$$\sum_{d|\mathcal{P}} \frac{\lambda^-(d)\omega(d)}{d} \ge V(z)\Big(f\Big(\frac{\log D}{\log z}\Big) + O(\log^{-\frac{1}{3}} D)\Big), \tag{5.5}$$

$$\sum_{d|\mathcal{P}} \frac{\lambda^+(d)\omega(d)}{d} \le V(z)\Big(F\Big(\frac{\log D}{\log z}\Big) + O(\log^{-\frac{1}{3}} D)\Big), \tag{5.6}$$

which follow from (12)–(13) in [9] and (4.9).

(1) The lower bound for $\nu_0(n)$. Let

$$\mathcal{R}(l) = \sum_{\substack{l^3+p_1^3+p_2^3+p_3^3+p_4^3+p_5^3+p_6^4+p_7^4=N \\ p_1, p_4 \sim X_1 \\ p_2, p_3, p_5 \sim X_2, p_6, p_7 \sim Y}} \prod_{i=1}^{7} \log p_i.$$

Then by the property for $\lambda^-(d)$, Proposition 3.1 and (5.5), we have

$$\begin{aligned}\nu_0(n) &\geq \frac{1}{\log 2\mathbf{X}} \sum_{\substack{l\sim X_1\\(l,\mathcal{P})=1}} \mathcal{R}(l) = \frac{1}{\log 2\mathbf{X}} \sum_{l\sim X_1} \mathcal{R}(l) \sum_{d|(l,\mathcal{P})} \mu(d)\\ &\geq \frac{1}{\log \mathbf{X}} \sum_{l\sim X_1} \mathcal{R}(l) \sum_{d|(l,\mathcal{P})} \lambda^-(d) = \frac{1}{\log \mathbf{X}} \sum_{d|\mathcal{P}} J_d(n)\\ &= \Big(1+O\Big(\frac{1}{\log n}\Big)\Big)\frac{\mathfrak{S}(n)\mathcal{J}(n)}{\log \mathbf{X}} \sum_{d|\mathcal{P}} \frac{\lambda^-(d)\omega(d)}{d} + O(n^{\frac{85}{72}}L^{-100})\\ &\geq (1+O(\log^{-\frac{1}{3}} D))\frac{f(3)\mathfrak{S}(n)\mathcal{J}(n)V(z)}{\log \mathbf{X}} + O(n^{\frac{85}{72}}L^{-100}). \qquad (5.7)\end{aligned}$$

(2) The upper bound for $\nu_r(n)$. Let

$$\mathcal{R}_r(l) = \sum_{\substack{l^3+(mp)^3+p_1^3+p_2^3+p_3^3+p_4^3+p_5^4+p_6^4=n\\ m\in\mathcal{M}_r, l,mp,p_3\sim X_1,\, p_1,p_2,p_4\sim X_2, p_5,p_6\sim Y}} \prod_{i=1}^{6} \log p_i \Big(\frac{\log p}{\log(\frac{X_1}{n})}\Big).$$

Then by Proposition 3.2 and (5.6) the upper bound for $\nu_r(n)$ is obtained along the similar arguments that lead to $\nu_0(n)$. We have

$$\begin{aligned}\nu_r(n) &\leq \frac{\log X_1}{\log \mathbf{X}} \sum_{\substack{l\sim X_1\\(l,\mathcal{P})=1}} \mathcal{R}_r(l)\\ &\leq (1+O(\log^{-\frac{1}{3}} D))\frac{F(3)c_r\mathfrak{S}(n)\mathcal{J}(n)V(z)}{\log \mathbf{X}} + O(n^{\frac{85}{72}}L^{-100}). \qquad (5.8)\end{aligned}$$

(3) The proof of Theorem 1.1.

By numerical integration we get

$$\begin{aligned}&c_7<0.4487, \quad c_8<0.1136, \quad c_9<0.0226, \quad c_{10}<0.0036, \quad c_{11}<0.0005,\\ &c_k<0.0001, \quad \text{for } 12\leq k\leq 36, \quad \sum_{r=7}^{36} c_r < 0.5915. \qquad (5.9)\end{aligned}$$

By (5.4), (5.7)–(5.9) we obtain

$$\nu(n) \geq (0.6931-0.5915)\frac{2e^\gamma}{3\log \mathbf{X}}\mathfrak{S}(n)\mathcal{J}(n)V(z) \gg \frac{n^{\frac{85}{72}}}{\log^8 n},$$

where (3.22) and (5.1)–(5.3) and Lemma 4.2 are used. Then Theorem 1.1 is proved.

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