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Ribbon Hopf Superalgebras and Drinfel'd Double*

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Abstract Finite dimensional ribbon Hopf (super) algebras play an important role in constructing invariants of 3-manifolds. In the present paper, the authors give a necessary and sufficient condition for the Drinfel'd double of a finite dimensional Hopf superalgebra to have a ribbon element. The criterion can be seen as a generalization of Kauffman and Radford's result in the non-super situation to the $\mathbb{Z}_2$-graded situation, however, the derivation of the result in the $\mathbb{Z}_2$-graded case will be much more complicated.

Keywords Hopf superalgebras, Drinfel'd double, Ribbon element
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1 Introduction

The Drinfel'd quantum double construction is very important not only because it produces new Hopf algebras but also that it provides a systematic method to produce quasitriangular Hopf algebras, which provide solutions to the quantum Yang-Baxter equation. Many results about quasitriangular Hopf algebras are obtained, see [1–2] for example. In [3], Gould, Zhang and Bracken generalized the double construction to the $\mathbb{Z}_2$-graded case, and described the corresponding graded universal $\mathscr{R}$-matrix explicitly. As is known to all, quasitriangular Hopf (super) algebras which possess an invertible central element known as the ribbon element are called ribbon Hopf (super) algebras, and finite dimensional ribbon Hopf (super) algebras play an important role in constructing invariants of 3-manifolds (see [4]). Thus, it is interesting to decide when the Drinfel'd quantum double is ribbon. In [5], Kauffman and Radford give a necessary and sufficient condition for the Drinfel'd double of a finite dimensional Hopf algebra to have a ribbon element. In recent years, two parameter quantum groups get rapid development and with the criterion given in [5], one can determine when a two parameter quantum group is ribbon (see [6–8] for example). However, in the $\mathbb{Z}_2$-graded case, the corresponding criterion, as far as we know, is not obtained.

In the present paper, we try to give a criterion as in [5] to determine when the $\mathbb{Z}_2$-graded Drinfel'd double constructed in [3] has a ribbon element. By studying the connection between grouplike elements and ribbon elements in finite dimensional quasitriangular Hopf superalgebra H, we give a necessary and sufficient condition for H to have a ribbon element. Then we

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prove that the Drinfel'd double $D(H)$ of a finite dimensional Hopf superalgebra H is always unimodular, and get a criterion to decide when $D(H)$ is ribbon. As an application, we apply the criterion to the Drinfel'd double $D(A_\ell)$ of the Taft superalgebra A_ℓ ($\mathbb{Z}_2$-graded Taft algebra). Different from the non-super case in [5], in the graded case, the quasi-ribbon element is no longer consistent with the ribbon element. At last, we study the quantum superalgebra $u_q(\mathrm{osp}(1,2,\mathbf{c}))$, which is proved to be isomorphic to the Drinfel'd double $D(A_\ell)$ of the Taft superalgebra A_ℓ, and describe its universal $\mathscr{R}$-matrix explicitly.

The paper is organized as follows. In Section 2, we recall the definition of quasitriangular Hopf superalgebras and introduce some notations. Some useful identities are given for finite dimensional Hopf superalgebras. In Section 3, we give a necessary and sufficient condition for a finite dimensional quasitriangular Hopf superalgebra to have a ribbon element. In Section 4, we give a criterion to decide when the Drinfel'd double $D(H)$ of a finite dimensional Hopf superalgebra H is ribbon. In Section 5, we apply the results obtained in Section 4 to the Drinfel'd double $D(A_\ell)$ of the Taft superalgebra A_ℓ, and determine when $D(A_\ell)$ has a ribbon element. In Section 6, we study the quantum superalgebra $u_q(\mathrm{osp}(1,2,\mathbf{c}))$, and describe its universal $\mathscr{R}$-matrix explicitly.

Throughout this paper, we always assume that the ground field k is algebraically closed.

2 Preliminaries

Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0,1\}$, and $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra. The elements in A_0 are called even, while those in A_1 are called odd. For a homogeneous element $a \in A$, we use $[a]$ to denote its grading. For the definition of Hopf superalgebra, one can see [3, 9].

Definition 2.1 (see [3]) *Let H be a Hopf superalgebra. If there exists an invertible even element $R \in H \otimes H$, such that*

$$R\Delta(x) = \Delta'(x)R, \quad \textit{for all } x \in H, \tag{2.1}$$

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \tag{2.2}$$

$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}, \tag{2.3}$$

then H is called a quasi-triangular Hopf superalgebra. Here $\Delta' = \Delta \circ P$ and $P : V \otimes W \to W \otimes V$ is the twist map, which defines for homogeneous elements $v \in V$, $w \in W$ by $P(v \otimes w) = (-1)^{[v][w]} w \otimes v$ and extends to all elements of V and W linearly. Denote $R = \sum R^{(1)} \otimes R^{(2)}$, $R_{12} = \sum R^{(1)} \otimes R^{(2)} \otimes 1$, $R_{13} = \sum R^{(1)} \otimes 1 \otimes R^{(2)}$, and $R_{23} = \sum 1 \otimes R^{(1)} \otimes R^{(2)}$. The element R is called the universal $\mathscr{R}$-matrix of H.

Lemma 2.1 *Let (H, R) be a quasi-triangular Hopf superalgebra. Then the universal $\mathscr{R}$-matrix R satisfies the following equations:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.4}$$

$$(S \otimes S)R = R, \quad (S \otimes \mathrm{id})R = R^{-1}, \quad (\mathrm{id} \otimes S)R^{-1} = R, \tag{2.5}$$

$$(\varepsilon \otimes \mathrm{id})R = (\mathrm{id} \otimes \varepsilon)R = 1. \tag{2.6}$$

Lemma 2.2 (see [10]) *Suppose that (H, R) is a quasi-triangular Hopf superalgebra, and $R = \sum R^{(1)} \otimes R^{(2)}$. Let $u = \sum(-1)^{[R^{(1)}]} S(R^{(2)}) R^{(1)}$. Then u is invertible, and*

$$\Delta(u) = (u \otimes u)(R' R)^{-1}, \quad \varepsilon(u) = 1,$$

where $R' = \sum(-1)^{[R^{(1)}]} R^{(2)} \otimes R^{(1)}$. Furthermore, for all $x \in A$, we have

$$S^2(x) = uxu^{-1}.$$

We introduce some notations which will be used in this paper. For any integer $n > 0$, set

$$(n)_q = 1 + q + \cdots + q^{n-1},$$
$$(n)_q^! = (1)_q (2)_q \cdots (n)_q,$$

and $(0)_q^! = 1$. We define the Gauss binomial number for $0 \le k \le n$ by

$$\binom{n}{k}_q = \frac{(n)_q^!}{(k)_q^! (n-k)_q^!}.$$

As is well known, if $yx = qxy$, then for all $n > 0$, we have

$$(x + y)^n = \sum_{0 \le k \le n} \binom{n}{k}_q x^k y^{n-k}.$$

In the following of this section, we always assume $(H, m, u, \Delta, \varepsilon, S)$ to be a finite dimensional Hopf superalgebra, then its dual space H^* is also a Hopf superalgebra, where the multiplication $m^0 = \Delta^*$, unit $u^0 = \varepsilon^*$, comultiplication $\Delta^0 = m^*$, counit $\varepsilon^0 = u^*$ and the antipode $S^0 = S^*$. With no confusion, we also use S to denote the antipode of H^*.

Define the left and right H^*-module of H as follows:

$$p \rightharpoonup a = \sum(-1)^{[p][a_1]} a_1 \langle p, a_2 \rangle, \quad a \leftharpoonup p = \sum(-1)^{[p][a_2]} \langle p, a_1 \rangle a_2.$$

Similarly, H^* can be made into left and right H-module by the following actions:

$$\langle a \rightharpoonup p, x \rangle = (-1)^{[a]([p]+[x])} \langle p, xa \rangle, \quad \langle p \leftharpoonup a, x \rangle = \langle p, ax \rangle,$$

where $a, x \in H$, $p \in H^*$ are homogeneous elements.

Recall that an element $x \in H$ is called a left (respectively, right) integral of H, if for all $a \in H$, we have $ax = \varepsilon(a)x$ ($xa = \varepsilon(a)x$). Denote by $\int_H^l$ (resp. $\int_H^r$) the space of left integrals (resp. right integrals). Then $\int_H^l$ and $\int_H^r$ are each one-dimensional (see [11, Corollary 5.8]). Integrals play an important role in studying knot invariants of finite dimensional ribbon Hopf (super) algebras.

Lemma 2.3 *Let H be a finite dimensional Hopf superalgebra, with antipode S, and $t \in \int_H^l$. Let α be the distinguished grouplike element of H^*, which means that for all elements $h \in H$, we always have $th = \langle \alpha, h \rangle t$. Then*

$$\sum_t S(h) t_1 \otimes t_2 = \sum_t (-1)^{[t_1][h]} t_1 \otimes h t_2, \tag{2.7}$$

$$\sum_t (-1)^{[t_2][h]} t_1 h \otimes t_2 = \sum_t t_1 \otimes t_2 (\alpha^{-1} \rightharpoonup S(h)). \tag{2.8}$$

Proof Note that $ht = \varepsilon(h)t$, hence $\Delta(ht) = \sum_{h,t}(-1)^{[h_2][t_1]}h_1t_1 \otimes h_2t_2 = \sum_t \varepsilon(h)t_1 \otimes t_2$. Therefore,

$$\begin{aligned}
&\sum_t (-1)^{[h][t_1]} t_1 \otimes ht_2 \\
&= \sum_{h,t} (-1)^{[h_2][t_1]} t_1 \otimes \varepsilon(h_1) h_2 t_2 \\
&= \sum_{h,t} (-1)^{[h_3][t_1]} S(h_1) h_2 t_1 \otimes h_3 t_2 \\
&= \sum_{h,t} S(h_1) \varepsilon(h_2) t_1 \otimes t_2 \\
&= \sum_t S(h) t_1 \otimes t_2.
\end{aligned}$$

Hence (2.7) is obtained. As $th = \langle \alpha, h\rangle t$, then

$$\sum_{h,t} (-1)^{[h_1][t_2]} t_1 h_1 \otimes t_2 h_2 = \sum_t \langle \alpha, h\rangle t_1 \otimes t_2.$$

Therefore,

$$\begin{aligned}
&\sum_t (-1)^{[t_2][h]} t_1 h \otimes t_2 \\
&= \sum_{h,t} (-1)^{[t_2][h_1]} t_1 h_1 \varepsilon(h_2) \otimes t_2 \\
&= \sum_{h,t} (-1)^{[t_2][h_1]} t_1 h_1 \otimes t_2 h_2 S(h_3) \\
&= \sum_{h,t} t_1 \otimes t_2 \langle \alpha, h_1\rangle S(h_2) \\
&= \sum_t t_1 \otimes t_2 (\alpha^{-1} \rightharpoonup S(h)).
\end{aligned}$$

Using the lemma above, we have the following equations.

Lemma 2.4 *Let H, t and α be defined as in Lemma 2.3. Then we have*

$$(\lambda \leftharpoonup h) \rightharpoonup t = (-1)^{[\lambda][h]} S(h), \tag{2.9}$$

$$((\alpha^{-1} \rightharpoonup S(h)) \rightharpoonup \lambda) \rightharpoonup t = h, \tag{2.10}$$

$$\lambda \leftharpoonup h = (-1)^{[\lambda][h]} (\alpha^{-1} \rightharpoonup S^2(h)) \rightharpoonup \lambda, \tag{2.11}$$

$$\sum_\lambda \lambda_1 \otimes \lambda_2 = \sum_\lambda (-1)^{[\lambda_1][\lambda_2]} S^2(\lambda_2)\alpha^{-1} \otimes \lambda_1, \tag{2.12}$$

where $\lambda \in \int_{H^*}^l$ and $\langle \lambda, t\rangle = 1$.

Proof For any $\beta \in H^*$, we have

$$\begin{aligned}
&\langle \beta, (\lambda \leftharpoonup h) \rightharpoonup t\rangle = \langle \beta(\lambda \leftharpoonup h), t\rangle \\
&= \sum_t (-1)^{([\lambda]+[h])[t_1]} \langle \beta, t_1\rangle \langle \lambda, ht_2\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_t (-1)^{[h][t_1]} \langle \beta \otimes \lambda, t_1 \otimes h t_2 \rangle \\
&= \sum_t \langle \beta \otimes \lambda, S(h) t_1 \otimes t_2 \rangle \\
&= \sum_t (-1)^{[\lambda]([h]+[t_1])} \langle \beta, S(h) t_1 \rangle \langle \lambda, t_2 \rangle \\
&= (-1)^{[\lambda][h]} \langle (\beta \leftharpoonup S(h)) \lambda, t \rangle \\
&= (-1)^{[\lambda][h]} \langle \beta \leftharpoonup S(h), 1 \rangle \langle \lambda, t \rangle \\
&= \langle \beta, (-1)^{[\lambda][h]} S(h) \rangle,
\end{aligned}
$$

where we use (2.7) in the fourth equation. Therefore, (2.9) is obtained. Note that

$$
\begin{aligned}
&\langle \beta, ((\alpha^{-1} \rightharpoonup S(h)) \rightharpoonup \lambda) \rightharpoonup t \rangle \\
&= \langle \beta((\alpha^{-1} \rightharpoonup S(h)) \rightharpoonup \lambda), t \rangle \\
&= \sum_t (-1)^{([\lambda]+[h])[t_1]} \langle \beta, t_1 \rangle \langle (\alpha^{-1} \rightharpoonup S(h)) \rightharpoonup \lambda, t_2 \rangle \\
&= \sum_t (-1)^{([\lambda]+[h])[t_1]+[h]([\lambda]+[t_2])} \langle \beta, t_1 \rangle \langle \lambda, t_2(\alpha^{-1} \rightharpoonup S(h)) \rangle \\
&= \sum_t (-1)^{[h][t_1]+[h]([\lambda]+[t_2])} \langle \beta \otimes \lambda, t_1 \otimes t_2(\alpha^{-1} \rightharpoonup S(h)) \rangle \\
&= \sum_t (-1)^{[h][t_1]+[h][\lambda]} \langle \beta \otimes \lambda, t_1 h \otimes t_2 \rangle \\
&= \sum_t (-1)^{[h][t_1]+[\lambda][t_1]} \langle \beta, t_1 h \rangle \langle \lambda, t_2 \rangle \\
&= \sum_t (-1)^{[\lambda][t_1]]+[h][\beta]} \langle h \rightharpoonup \beta, t_1 \rangle \langle \lambda, t_2 \rangle \\
&= (-1)^{[h][\beta]} (h \rightharpoonup \beta) \lambda, t \rangle \\
&= (-1)^{[h][\beta]} \langle (h \rightharpoonup \beta), 1 \rangle \langle \lambda, ts \rangle \\
&= \langle \beta, h \rangle,
\end{aligned}
$$

where we use (2.8) in the fifth equation, and (2.10) is obtained. Using (2.9)–(2.10), we have

$$
\begin{aligned}
\langle \lambda \leftharpoonup h, t \leftharpoonup \beta \rangle &= \langle \beta(\lambda \leftharpoonup h), t \rangle \\
&= \langle \beta, (\lambda \leftharpoonup h) \rightharpoonup t \rangle \\
&= (-1)^{[\lambda][h]} \langle \beta, S(h) \rangle \\
&= (-1)^{[\lambda][h]} \langle \beta, ((\alpha^{-1} \rightharpoonup S^2(h)) \rightharpoonup \lambda) \rightharpoonup t \rangle \\
&= (-1)^{[\lambda][h]} \langle \beta((\alpha^{-1} \rightharpoonup S^2(h)) \rightharpoonup \lambda), t \rangle \\
&= (-1)^{[\lambda][h]} \langle (\alpha^{-1} \rightharpoonup S^2(h)) \rightharpoonup \lambda, t \leftharpoonup \beta \rangle.
\end{aligned}
$$

Hence we get (2.11). Now we prove (2.12). On one hand,

$$\langle \lambda \leftharpoonup h, x \rangle = \langle \lambda, hx \rangle = \Big\langle \sum_\lambda \lambda_1 \otimes \lambda_2, h \otimes x \Big\rangle.$$

On the other hand,

$$\langle (-1)^{[\lambda][h]} (\alpha^{-1} \rightharpoonup S^2(h)) \rightharpoonup \lambda, x \rangle$$

$$
\begin{aligned}
&= (-1)^{[x][h]}\langle \lambda_1, x(\alpha^{-1} \rightharpoonup S^2(h))\rangle \\
&= \sum_{\lambda}(-1)^{[x][h]+[\lambda_2][x]}\langle \lambda_1, x\rangle\langle \lambda_2, \alpha^{-1} \rightharpoonup S^2(h)\rangle \\
&= \sum_{\lambda,h}(-1)^{[x][h]+[\lambda_2][x]}\langle \lambda_1, x\rangle\langle \lambda_2, S^2(h_1)\rangle\langle \alpha^{-1}, S^2(h_2)\rangle \\
&= \sum_{\lambda,h}(-1)^{[x][h]+[\lambda_2][x]}\langle \lambda_1, x\rangle\langle S^2(\lambda_2), h_1\rangle\langle \alpha^{-1}, h_2\rangle \\
&= \sum_{\lambda}(-1)^{[x][h]+[\lambda_2][x]}\langle \lambda_1, x\rangle\langle S^2(\lambda_2)\alpha^{-1}, h\rangle \\
&= \sum_{\lambda}(-1)^{[\lambda_2][\lambda_2]}\langle S^2(\lambda_2)\alpha^{-1} \otimes \lambda_1, h \otimes x\rangle.
\end{aligned}
$$

By (2.11), we have $\lambda \leftharpoonup h = (-1)^{[\lambda][h]}(\alpha^{-1} \rightharpoonup S^2(h)) \rightharpoonup \lambda$, hence

$$\sum_{\lambda} \lambda_1 \otimes \lambda_2 = \sum_{\lambda}(-1)^{[\lambda_1][\lambda_2]}S^2(\lambda_2)\alpha^{-1} \otimes \lambda_1.$$

Equivalently,

$$\sum_{\lambda} \lambda_2 \otimes \lambda_1 = \sum_{\lambda}(-1)^{[\lambda_1][\lambda_2]}\lambda_1 \otimes S^2(\lambda_2)\alpha^{-1}.$$

Remark 2.1 Suppose $t \in \int_H^l$ and $T \in \int_{H^*}^r$. Let α and g be the distinguished grouplike element of H^* and H respectively. Then for any $h \in H$ and $p \in H^*$, we have $th = \langle \alpha, h\rangle t$, $pT = \langle p, g\rangle T$, and

$$\sum_{t} t_1 \otimes t_2 = \sum_{t}(-1)^{[t_1][t_2]}S^2(t_2)g \otimes t_1. \tag{2.13}$$

(2.13) is in fact the duality of (2.12). Note that t is the left integral of $H = H^{**}$, $S(T)$ is the left integral of H^*, and satisfying $p \in H^*$, $S(T)p = \langle p, g^{-1}\rangle S(T)$, hence (2.13) is obtained.

3 Ribbon Hopf Superalgebra

First we recall the definition of ribbon Hopf superalgebra. Let (H, R) be a quasitriangular Hopf superalgebra. An invertible even element $v \in H$ is a quasi-ribbon element, if the following conditions are satisfied:

$$v^2 = c := uS(u), \tag{3.1}$$

$$S(v) = v, \tag{3.2}$$

$$\varepsilon(v) = 1, \tag{3.3}$$

$$\Delta(v) = (v \otimes v)(R'R)^{-1}. \tag{3.4}$$

Furthermore, if v is in the center of H, then v is a ribbon element, and (H, R, v) is called a ribbon Hopf superalgebra.

Lemma 3.1 *Let (H, R) be a quasitriangular Hopf superalgebra. Denote $R = \sum R^{(1)} \otimes R^{(2)}$. For $\eta \in G(H^*)$, define $g_\eta = \sum R^{(1)}\eta(R^{(2)})$. Then g_η lies in the center of $G(H)$, and $g_\eta g_\rho = g_{\eta\rho}$, where $\rho \in G(H^*)$.*

Proof By (2.2), $\Delta(g_\eta) = \sum \Delta(R^{(1)})\eta(R^{(2)}) = \sum(-1)^{[r^{(1)}][R^{(2)}]} R^{(1)} \otimes r^{(1)}\eta(R^{(2)}r^{(2)}) = g_\eta \otimes g_\eta$. For any $a \in G(H)$, by (2.1), $\sum aR^{(1)} \otimes aR^{(2)} = \sum R^{(1)}a \otimes R^{(2)}a$. Applying $\mathrm{id} \otimes \eta$ to the two sides of the above equation, we get $ag_\eta\eta(a) = g_\eta a\eta(a)$, hence g_η is in the center of $G(H)$. Let $\rho \in G(H^*)$. Then $g_{\eta\rho} = \sum R^{(1)}(\eta\rho)(R^{(2)}) = \sum R^{(1)}\eta((R^{(2)})_1)\rho((R^{(2)})_2) = \sum R^{(1)}r^{(1)}\eta(r^{(2)})\rho(R^{(2)}) = g_\rho g_\eta = g_\eta g_\rho$.

Theorem 3.1 *Let (H, R) be a quasitriangular Hopf superalgebra, and $R = \sum R^{(1)} \otimes R^{(2)}$. Denote $u = \sum(-1)^{[R^{(1)}]}S(R^{(2)})R^{(1)}$, and $w = S(u)^{-1}$. Let g and α be the distinguished group-like elements of H and H^* respectively, $h = g_\alpha g^{-1}$. Then we have*

$$g = w\Big(\sum(-1)^{[R^{(1)}][R^{(2)}]}S(R^{(2)} \leftharpoonup \alpha)R^{(1)}\Big), \tag{3.5}$$

$$g = wug_\alpha, \tag{3.6}$$

$$h^{-1} = wu. \tag{3.7}$$

Proof Let $t \in H$ be a left integral. By (2.7) and $\alpha^{-1} \rightharpoonup S(x) = S(x \leftharpoonup \alpha)$, we have

$$\sum_t(-1)^{[t_2][x]}t_1x \otimes t_2 = \sum_t t_1 \otimes t_2(S(x \leftharpoonup \alpha)). \tag{3.8}$$

Therefore

$$\begin{aligned}
&\sum_t(-1)^{[t_1]([t_2]+[R^{(1)}]+[R^{(2)}])}t_2(S(R^{(2)} \leftharpoonup \alpha)R^{(1)} \otimes t_1 \\
=&\sum_t(-1)^{[t_1][t_2]+[t_1][R^{(1)}]+[R^{(1)}][R^{(2)}]}t_2R^{(1)} \otimes t_1R^{(2)} \\
=&\sum_t(-1)^{[R^{(1)}][R^{(2)}]+[t_1][R^{(2)}]}R^{(1)}t_1 \otimes R^{(2)}t_2 \\
=&\sum_t(-1)^{[R^{(1)}][R^{(2)}]}R^{(1)}S(R^{(2)})t_1 \otimes t_2 \\
=&\sum_t(-1)^{[R^{(1)}][R^{(2)}]+[t_1][t_2]}R^{(1)}S(R^{(2)})S^2(t_2)g \otimes t_1,
\end{aligned}$$

where we used (2.1) in the second equation, (2.7) in the third equation, and (2.13) in the last equation. Equivalently, we have

$$\sum_t(-1)^{[t_1][t_2]}t_2(S(R^{(2)} \leftharpoonup \alpha)R^{(1)} \otimes t_1 = \sum_t(-1)^{[R^{(1)}][R^{(2)}]+[t_1][t_2]}R^{(1)}S(R^{(2)})S^2(t_2)g \otimes t_1.$$

Taking sums over R on the two sides above, we have

$$\sum_{t,R}(-1)^{[t_1][t_2]+[R^{(1)}][R^{(2)}]}t_2(S(R^{(2)} \leftharpoonup \alpha)R^{(1)} \otimes t_1 = \sum_{t,R}(-1)^{[t_1][t_2]}R^{(1)}S(R^{(2)})S^2(t_2)g \otimes t_1.$$

Since $(H, \rightharpoonup)$ is a free left H^*-module with basis t, there is some $p \in H^*$ such that $p \rightharpoonup t = \sum_t(-1)^{[p][t_1]}t_1\langle p, t_2\rangle = 1$. Applying $p \otimes \mathrm{id}$ to the two sides of the above equation, we get

$$\sum_R(-1)^{[R^{(1)}][R^{(2)}]}(S(R^{(2)} \leftharpoonup \alpha)R^{(1)} = \sum_R R^{(1)}S(R^{(2)})g.$$

By (2.5),

$$\sum_R R^{(1)}S(R^{(2)}) = \sum_R S(R^{(1)})S^2(R^{(2)})$$

$$= S\Big(\sum_R (-1)^{[R^{(1)}][R^{(2)}]} S(R^{(2)}) R^{(1)}\Big)$$
$$= S(u) = w^{-1}.$$

Therefore $g = w\big(\sum_R (-1)^{[R^{(1)}][R^{(2)}]} S(R^{(2)} \leftharpoonup \alpha) R^{(1)}\big)$, and we get (3.5).

To prove (3.6), just to prove $\sum_R (-1)^{[R^{(1)}]} S(R^{(2)} \leftharpoonup \alpha) R^{(1)} = u g_\alpha$. By (2.3),

$$\sum_R R^{(1)} \otimes (R^{(2)})_1 \otimes (R^{(2)})_2 = \sum_R R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}.$$

Hence,

$$\begin{aligned}\sum_R R^{(1)} \otimes (R^{(2)} \leftharpoonup \alpha) &= \sum_R R^{(1)} \otimes \langle \alpha, (R^{(2)})_1 \rangle (R^{(2)})_2 \\ &= \sum_R R^{(1)} r^{(1)} \otimes \langle \alpha, r^{(2)} \rangle R^{(2)} \\ &= \sum_R R^{(1)} g_\alpha \otimes R^{(2)}.\end{aligned}$$

By this equation and $\sum_R (-1)^{[R^{(1)}]} S(R^{(2)} \leftharpoonup \alpha) R^{(1)} = \sum_R (-1)^{[R^{(1)}]} S(R^{(2)} R^{(1)} g_\alpha = u g_\alpha$, we get (3.6).

To prove (3.7), just note that by (3.6) and Lemma 3.1, $wu = g g_\alpha^{-1} = h^{-1}$.

Lemma 3.2 *Let (H, R) be a quasitriangular Hopf superalgebra, v be the quasi-ribbon element of H, $u = \sum (-1)^{[R^{(1)}]} S(R^{(2)}) R^{(1)}$, $h = g_\alpha g^{-1}$, $c = uS(u)$, $l = u^{-1} v$. Then $l^2 = h$, and $l \in G(H)$.*

Proof Firstly, $S^2(v) = uvu^{-1}$. Since v is the quasi-ribbon element, $S^2(v) = v$, hence $uv = vu$. As $v^2 = c$, by Theorem 3.1, $c = u^2 h$, therefore $v^2 = u^2 h$, $l^2 = (u^{-1}v)^2 = u^{-2} v^2 = h$.

Note that $\Delta(l) = \Delta(u)^{-1} \Delta(v) = ((R'R)^{-1}(u \otimes u))^{-1}((R'R)^{-1}(v \otimes v)) = l \otimes l$, hence $l \in G(H)$.

Theorem 3.2 *Let (H, R) be a quasitriangular Hopf superalgebra, u and h be defined as in Lemma 3.2. Then*

(1) *$l \longmapsto ul$ defines a one-one correspondence*

$$\{l \in G(H) \mid l^2 = h\} \leftrightarrow \{\textit{quasi-ribbon elements of (H,R)}\}.$$

(2) *Suppose that $l \in G(H)$ satisfies $l^2 = h$. Then $v = ul$ is a ribbon element of (H, R) if and only if $S^2(x) = l^{-1} x l$ for all $x \in H$.*

The proof is similar to the non-super case in [5].

If H is unimodular, then $\alpha = \varepsilon$, hence $h = g_\alpha g^{-1} = g^{-1}$. Then by Theorem 3.2, we have the following proposition.

Proposition 3.1 *Let (H, R) be a finite dimensional quasitriangular Hopf superalgebra, with antipode S. Suppose that H is unimodular and g is the distinguished grouplike element of H. Then*

(1) *(H, R) has a quasi-ribbon element if and only if there exists $l \in G(H)$, such that $l^2 = g$.*

(2) (H, R) *has a ribbon element if and only if there exists* $l \in G(H)$, *such that* $l^2 = g$, *and for all* $x \in H$, *we have* $S^2(x) = lxl^{-1}$.

4 Drinfel'd Double of Hopf Superalgebra

Firstly we introduce the construction of the Drinfel'd double of a finite dimensional Hopf superalgebra (see [3]). Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional Hopf superalgebra. Then $H^{*\mathrm{cop}}$ is also a Hopf superalgebra, with multiplication $m_0 = m^0 = \Delta^*$, unit $u_0 = u^0 = \varepsilon^*$, comultiplication $\Delta_0 = P \circ m^*$, counit $\varepsilon_0 = \varepsilon^0 = u^*$, and antipode $S_0 = (S^{-1})^*$. More explicitly, see [3]. The Drinfel'd double $D(H)$ of H is defined as follows: As a $\mathbb{Z}_2$-graded space, $D(H) := H \otimes H^{*\mathrm{cop}}$. For $a \in H$, $b^* \in H^*$, denote $a \otimes b^*$ by ab^* simply. Then $D(H)$ is a Hopf superalgebra with multiplication M_D, unit u_D, comultiplication Δ_D, and counit ε_D defined as follows.

The multiplication is

$$M_D(ba^* \otimes dc^*) = \sum_{a^*, d} (-1)^{([a^*]+[d_1])([d]+[d_3])} \langle S_0(a_1^*), d_1 \rangle \langle a_3^*, d_3 \rangle (bd_2)(a_2^* c^*);$$

the unit is

$$u_D(k) = u(k)\varepsilon;$$

the comultiplication is

$$\Delta_D(ba^*) = \Delta(b)\Delta_0(a^*) = \sum_{a^*, b} (-1)^{([a_1^*]+[b_2])[a_2^*]} b_1 a_2^* \otimes b_2 a_1^*;$$

the counit is

$$\varepsilon_D(ba^*) = \varepsilon(b)\varepsilon_0(a^*);$$

the antipode is

$$S_D(ba^*) = (-1)^{[a^*][b]} 1_H S_0(a^*) 1_{H^*} S(b).$$

Remark 4.1 Here we follow the expressions in [3], and denote $\Delta_0(a^*) = \Sigma a_1^* \otimes a_2^*$.

Theorem 4.1 *Let* H *be a finite dimensional Hopf superalgebra,* T *the right integral of* $H^{*\mathrm{cop}}$, *and* t *the left integral of* H. *Then* tT *is the left and right integral of* $D(H)$. *In particular,* $D(H)$ *is unimodular.*

Proof Let α and g be the distinguished grouplike elements of H^* and H respectively. First we prove that tT is the right integral of $D(H)$. Let $d \in H$, and $c^* \in H^{*\mathrm{cop}}$, with the multiplication in $D(H)$, we have

$$\begin{aligned}
(tT)(dc^*) &= \sum_{T,d} (-1)^{([T]+[d_1])([d]+[d_3])} \langle S_0(T_1), d_1 \rangle \langle T_3, d_3 \rangle (td_2)(T_2 c^*) \\
&= \sum_{T,d} (-1)^{([T]+[d_1])([d]+[d_3])} \langle S_0(T_1), d_1 \rangle \langle T_3, d_3 \rangle \langle \alpha, d_2 \rangle t T_2 c^* \\
&= \sum_{T} (-1)^{[T_1][T_2]} \langle S_0(T_1) \alpha T_3, d \rangle t T_2 c^*.
\end{aligned}$$

We claim

$$\sum_T (-1)^{[T_2][T_3]} S_0(T_1)\alpha T_3 \otimes T_2 = \varepsilon \otimes T. \tag{4.1}$$

Then applying $d \otimes \mathrm{id}$ to the two sides of (4.1), we have

$$\sum_T (-1)^{[T_1][T_2]} \langle S_0(T_1)\alpha T_3, d\rangle T_2 = \varepsilon(d)T,$$

hence

$$\begin{aligned}(tT)(dc^*) &= \sum_{T,d} (-1)^{[T_1][T_2]} \langle S_0(T_1)\alpha T_3, d\rangle tT_2 c^* \\ &= \varepsilon(d)tTc^* = \varepsilon(d)\langle c^*, 1\rangle tT.\end{aligned}$$

As we know, $\varepsilon_D(dc^*) = \varepsilon(d)\langle c^*, 1\rangle$, hence tT is a right integral of $D(H)$.

Now we prove (4.1). By (2.12), we have

$$\sum_\lambda \lambda_1 \otimes \lambda_2 = \sum_\lambda (-1)^{[\lambda_1][\lambda_2]} S^2(\lambda_2)\alpha^{-1} \otimes \lambda_1.$$

Applying $S \otimes S$ to the two sides of the equation above, we get

$$\begin{aligned}\sum_\lambda S(\lambda_1) \otimes S(\lambda_2) &= \sum_\lambda (-1)^{[\lambda_1][\lambda_2]} \alpha S^2(S(\lambda_2)) \otimes S(\lambda_1) \\ &= \sum_\lambda \alpha S^2(S(\lambda)_1) \otimes S(\lambda)_2.\end{aligned}$$

Let $p = S(\lambda) \in \int_{H^*}^r$, then

$$\sum_p p_1 \otimes p_2 = \sum_p (-1)^{[p_1][p_2]} p_2 \otimes \alpha S^2(p_1).$$

Noting that $H^{*\mathrm{cop}} \cong (H^{\mathrm{op}})^*$, together with the equation above, we have

$$\sum_T T_1 \otimes T_2 = \sum_T (-1)^{[T_1][T_2]} T_2 \otimes \beta S_0^2(T_1).$$

Equivalently,

$$\sum_T (-1)^{[T_1][T_2]} T_2 \otimes T_1 = \sum_T \beta S_0^2(T_1) \otimes T_2,$$

where β is the distinguished grouplike element of $H^{*\mathrm{cop}}$, and for $t' \in \int_{H^{\mathrm{op}}}^l$, $t' \circ h' = \langle \beta, h'\rangle t'$ holds for all $h' \in H^{\mathrm{op}}$. We claim that $\beta = \alpha^{-1}$. Note that $t \in \int_H^l$, so $S^{-1}(t) \in \int_{H^{\mathrm{op}}}^l$, and

$$S^{-1}(t) \circ S^{-1}(h) = \langle \beta, S^{-1}(h)\rangle S^{-1}(t) = \langle S_0(\beta), h\rangle S^{-1}(t),$$

here we use "$\circ$" to denote the multiplication of H^{op}. On the other hand,

$$S^{-1}(t) \circ S^{-1}(h) = (-1)^{[h][t]} S^{-1}(h) S^{-1}(t) = S^{-1}(th) = S^{-1}(\langle \alpha, h\rangle t) = \langle \alpha, h\rangle S^{-1}(t),$$

hence $S_0(\beta) = \alpha$, $\beta = \alpha^{-1}$, and

$$\sum_T (-1)^{[T_1][T_2]} T_2 \otimes T_1 = \sum_T \alpha^{-1} S_0^2(T_1) \otimes T_2. \tag{4.2}$$

Applying $\mathrm{id} \otimes \Delta_0$ to the two sides of (4.2), we have

$$\sum_T (-1)^{[T_3]([T_1]+[T_2])} T_3 \otimes T_1 \otimes T_2 = \sum_T \alpha^{-1} S_0^2(T_1) \otimes T_2 \otimes T_3.$$

Applying $P \otimes \mathrm{id}$ to the two sides above, we have

$$\sum_T (-1)^{[T_3]([T_1]+[T_2])+[T_3][T_1]} T_1 \otimes T_3 \otimes T_2 = \sum_T (-1)^{[T_1][T_2]} T_2 \otimes \alpha^{-1} S_0^2(T_1) \otimes T_3.$$

Furthermore,

$$\sum_T (-1)^{[T_3][T_2]} S_0(T_1)\alpha T_3 \otimes T_2 = \sum_T S_0(S_0(T_1)T_2) \otimes T_3 = \varepsilon \otimes T,$$

hence we get (4.1).

Next we prove tT is also a left integral of $D(H)$. Note that

$$\begin{aligned}
(ba^*)(tT) &= \sum_{a^*,t} (-1)^{([a^*]+[t_1])([t]+[t_3])} \langle S_0(a_1^*), t_1\rangle \langle a_3^*, t_3\rangle (bt_2)(a_2^* T) \\
&= \sum_{a^*,t} (-1)^{([a^*]+[t_1])([t]+[t_3])} \langle a_1^*, S^{-1}(t_1)\rangle \langle a_3^*, t_3\rangle \langle a_2^*, g\rangle (bt_2)T \\
&= \sum_{a^*,t} (-1)^{[t_3]([t_1]+[t_2])} \langle a^*, t_3 g S^{-1}(t_1)\rangle (bt_2)T \\
&= b\langle \mathrm{id} \otimes a^*, \sum_t (-1)^{[t_1]([t_2]+[t_3])} t_2 \otimes t_3 g S^{-1}(t_1)\rangle T \\
&= b\langle \mathrm{id} \otimes a^*, t \otimes 1\rangle T \\
&= (-1)^{[a^*][t]} bt \otimes \langle a^*, 1\rangle T \\
&= \varepsilon(b)\langle a^*, 1\rangle tT,
\end{aligned}$$

hence tT is also a left integral of $D(H)$. In the fifth equation, we used the following result:

$$\sum_t (-1)^{[t_1]([t_2]+[t_3])} t_2 \otimes t_3 g S^{-1}(t_1) = t \otimes 1. \tag{4.3}$$

Now we prove it. By (2.13), we have

$$\sum_t t_1 \otimes t_2 = \sum_t (-1)^{[t_1][t_2]} S^2(t_2) g \otimes t_1.$$

Applying $S^{-1} \otimes \Delta$ to the two sides above, we have

$$\sum_t S^{-1}(t_1) \otimes t_2 \otimes t_3 = \sum_t (-1)^{([t_1]+[t_2])[t_3]} g^{-1} S(t_3) \otimes t_1 \otimes t_2.$$

Applying $(P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes P)$ to the two sides above, we have

$$\sum_t (-1)^{([t_1]+[t_2])[t_3]} t_3 \otimes S^{-1}(t_1) \otimes t_2 = \sum_t (-1)^{([t_3]+[t_2])[t_1]} t_2 \otimes g^{-1} S(t_3) \otimes t_1.$$

Hence

$$\sum_t (-1)^{([t_1]+[t_2])[t_3]} t_3 g S^{-1}(t_1) \otimes t_2 = \sum_t (-1)^{([t_3]+[t_2])[t_1]} t_2 S(t_3) \otimes t_1.$$

Furthermore,

$$\sum_t (-1)^{([t_3]+[t_2])[t_1]} t_2 \otimes t_3 g S^{-1}(t_1) = \sum_t t_1 \otimes t_2 S(t_3) = t \otimes 1,$$

hence we obtain (4.3).

So far we have proved that $D(H)$ is unimodular.

It is easy to know that $g \otimes \alpha$ is the distinguished grouplike element of $D(H)$. Applying Proposition 3.1 to the double $D(H)$, we have the following result.

Theorem 4.2 *Let H be a finite dimensional Hopf superalgebra, with antipode S. Let g and α be the distinguished grouplike elements of H and H^* respectively. Then*

(1) $(D(H), R)$ has a quasi-ribbon element if and only if there exist $l \in G(H)$ and $\beta \in G(H^*)$ such that $l^2 = g$ and $\beta^2 = \alpha$.

(2) $(D(H), R)$ has a ribbon element if and only if there exist l and β satisfying the conditions in (1), and for any $x \in H$, we have $S^2(x) = l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1}$.

Proof (1) Note that $(x, \eta) \longmapsto x \otimes \eta$ gives an isomorphism of groups $G(H) \times G(H^*)$ and $G(D(H))$ (see [12]). Therefore part (1) is obtained by Proposition 3.1(1).

(2) By Proposition 3.1(2), $(D(H), R)$ has a ribbon element if and only if there exist l and β satisfying the conditions in (1), such that

$$S^2(x) \otimes S_0^2(p) = (l \otimes \beta)(x \otimes p)(l \otimes \beta)^{-1} \tag{4.4}$$

holds for all $x \in H$, and $p \in H^{*\mathrm{cop}}$. As $(l \otimes \beta)^{-1} = l^{-1} \otimes \beta^{-1}$, considering the multiplication in $D(H)$, we have

$$\begin{aligned}
&(l \otimes \beta)(x \otimes p)(l \otimes \beta)^{-1} \\
&= \sum_x (-1)^{([\beta]+[x_1])([x]+[x_3])} \langle \beta^{-1}, x_1\rangle \langle \beta, x_3\rangle (l x_2 \otimes \beta p)(l^{-1} \otimes \beta^{-1}) \\
&= \sum_p (-1)^{([\beta p]+[l_1])([l]+[l_3])} \langle \beta^{-1}, x_1\rangle \langle \beta, x_3\rangle \langle S_0((\beta p)_1), l^{-1}\rangle \langle (\beta p)_3, l^{-1}\rangle l x_2 l^{-1} \otimes (\beta p)_2 \beta^{-1} \\
&= \sum_p \langle \beta^{-1}, x_1\rangle \langle \beta, x_3\rangle \langle \beta, l\rangle \langle p_1, l\rangle \langle \beta, l^{-1}\rangle \langle p_3, l^{-1}\rangle l x_2 l^{-1} \otimes \beta p_2 \beta^{-1} \\
&= l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1} \otimes \beta(l^{-1} \rightharpoonup p \leftharpoonup l)\beta^{-1},
\end{aligned}$$

hence (4.4) holds if and only if

$$S^2(x) \otimes S_0^2(p) = l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1} \otimes \beta(l^{-1} \rightharpoonup p \leftharpoonup l)\beta^{-1}. \tag{4.5}$$

Let $p = \varepsilon$. Then by (4.5), for any $x \in H$, we have $S^2(x) = l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1}$. On the other hand, it is easy to prove that $S^2(x) = l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1}$ implies (4.5).

5 Taft Superalgebras

As is well known, the Taft n^2-dimensional Hopf algebras A_n are an interesting class of pointed Hopf algebras. In [5], Kauffman and Radford extended Henning's result (see [13]) and proved that the Drinfel'd double $D(A_n)$ of A_n has a quasi-ribbon element if and only if $D(A_n)$

has unique ribbon element if and only if n is odd. Furthermore, when n is odd, $D(A_n)$ provides an invariant of three-manifolds. In this section we study the Taft superalgebra A_ℓ ($\mathbb{Z}_2$-graded Taft algebra), construct its Drinfel'd double, and determine when the double has a ribbon element.

Suppose that $\ell > 2$, and $q \in k$ is a primitive ℓ-th root of unity. Define

$$\ell' = \begin{cases} \ell, & \text{if } \ell \in 4\mathbb{N}, \\ \dfrac{\ell}{2}, & \text{if } \ell \in 4\mathbb{N}+2, \\ 2\ell, & \text{if } \ell \in 2\mathbb{N}+1. \end{cases} \tag{5.1}$$

The Taft superalgebra A_ℓ is generated by a and x, subject to the relations $a^\ell = 1$, $x^{\ell'} = 0$ and $ax = qxa$. The $\mathbb{Z}_2$-grading is $[a] = 0$, and $[x] = 1$. The Hopf superalgebra structure is given by

$$\Delta(a) = a \otimes a, \quad \Delta(x) = x \otimes a + 1 \otimes x,$$
$$S(x) = -xa^{-1}, \quad S(a) = a^{-1}, \quad \varepsilon(x) = 0, \quad \varepsilon(a) = 1.$$

Next we construct the Drinfel'd double $D(A_\ell)$ of A_ℓ. Take the basis $\{x^m a^n\}$ of A_ℓ, and define the linear forms α and η on the basis as

$$\langle \alpha, x^m a^n \rangle = \delta_{m0} q^n, \quad \langle \eta, x^m a^n \rangle = \delta_{m1}. \tag{5.2}$$

Then we have the following lemma.

Lemma 5.1 *For all integrals i, j, m, n, the following equation holds:*

$$\langle \eta^i \alpha^j, x^m a^n \rangle = \delta_{mi} (i)!_{(-q)} q^{j(i+n)}.$$

Proof Set $X = (A_\ell^{\mathrm{op}})^*$, $\lambda, \gamma \in X$. Then for any $h \in A_\ell$, we have

$$\langle \lambda\gamma, h \rangle = \langle \lambda \otimes \gamma, \Delta(h) \rangle = \sum (-1)^{[\gamma][h_1]} \langle \lambda, h_1 \rangle \langle \gamma, h_2 \rangle. \tag{5.3}$$

Note that

$$\Delta(x)^m = \sum_{r=0}^{m} (-1)^{r(m-r)} \binom{m}{r}_{-q} (x^{m-r} \otimes x^r a^{m-r}), \tag{5.4}$$

hence by (5.2) and induction on i and j, we have

$$\langle \eta^i, x^m a^n \rangle = \delta_{mi} (i)!_{(-q)}, \quad \langle \alpha^j, x^m a^n \rangle = \delta_{m0} q^{jn}.$$

Therefore

$$\begin{aligned} \langle \eta^i \alpha^j, x^m a^n \rangle &= \sum_{r=0}^{m} (-1)^{r(m-r)} \binom{m}{r}_{-q} \langle \eta^i, x^{m-r} a^n \rangle \langle \alpha^j, x^r a^{m+n-r} \rangle \\ &= \sum_{r=0}^{m} (-1)^{r(m-r)} \delta_{m-r,i} (i)!_{(-q)} \delta_{r0} q^{j(m+n-r)} \\ &= \delta_{m,i} (i)!_{(-q)} q^{j(i+n)}. \end{aligned}$$

Now we are ready to prove the following proposition.

Proposition 5.1 *The following relations hold in X:*

$$\begin{aligned}
&\alpha^\ell = 1, \quad \eta^{\ell'} = 0, \quad \alpha\eta = q^{-1}\eta\alpha,\\
&\Delta(\alpha) = \alpha\otimes\alpha, \quad \Delta(\eta) = 1\otimes\eta + \eta\otimes\alpha,\\
&\varepsilon(\alpha) = 1, \quad \varepsilon(\eta) = 0,\\
&S(\alpha) = \alpha^{\ell-1}, \quad S(\eta) = -\eta\alpha^{\ell-1}.
\end{aligned}$$

Proof (1) By Lemma 5.1 and $q^\ell = 1$, we have

$$\langle \alpha^\ell, x^m a^n\rangle = \delta_{m0} = \langle \varepsilon, x^m a^n\rangle.$$

Hence, $\alpha^\ell = \varepsilon$. Similarly, we have

$$\langle \eta^{\ell'}, x^m a^n\rangle = \delta_{m\ell'}(\ell')!_{(-q)} = 0,$$

where we have used $(\ell')!_{(-q)} = (-q)^{\ell'} - \frac{1}{-q-1} = 0$. By Lemma 5.1, we get

$$\langle \eta\alpha, x^m a^n\rangle = \delta_{m1}q^{n+1}.$$

But

$$\langle \alpha\eta, x^m a^n\rangle = \delta_{m1}q^n,$$

so $\alpha\eta = q^{-1}\eta\alpha$.

(2) Now we deal with the comultiplication of X. By [3], for any $\lambda\in X$, $h, g\in A_\ell$, $\Delta(\lambda)(h\otimes g) = (-1)^{[h][g]}\langle\lambda, gh\rangle$. Therefore,

$$\begin{aligned}
\Delta(\alpha)(x^i a^j\otimes x^m a^n) &= (-1)^{[x^i][x^m]}q^{ni}\langle \alpha, x^{i+m}a^{j+n}\rangle\\
&= (-1)^{[x^i][x^m]}q^{ni}\delta_{i+m,0}q^{n+j}\\
&= \delta_{i0}\delta_{m0}q^n q^j\\
&= \langle \alpha, x^i a^j\rangle\langle \alpha, x^m a^n\rangle,
\end{aligned}$$

hence $\Delta(\alpha) = \alpha\otimes\alpha$. Similarly, we have

$$\begin{aligned}
\Delta(\eta)(x^i a^j\otimes x^m a^n) &= (-1)^{[x^i][x^m]}q^{ni}\langle \eta, x^{i+m}a^{j+n}\rangle\\
&= (-1)^{[x^i][x^m]}q^{ni}\delta_{i+m,1}\\
&= \delta_{i0}\delta_{m1} + \delta_{i1}\delta_{m0}q^n\\
&= \langle 1\otimes\eta + \eta\otimes\alpha, x^i a^j\otimes x^m a^n\rangle,
\end{aligned}$$

hence $\Delta(\eta) = 1\otimes\eta + \eta\otimes\alpha$.

(3) As for the counit and antipode, we have

$$\begin{aligned}
&\varepsilon(\alpha) = \langle\alpha, 1\rangle = 1, \quad \varepsilon(\eta) = \langle \eta, 1\rangle = 0,\\
&\langle S(\alpha), x^i a^j\rangle = \langle \alpha, S^{-1}(x^i a^j)\rangle = \delta_{i0}\langle\alpha, a^{-j}\rangle = \delta_{i0}q^{-j},\\
&\langle \alpha^{\ell-1}, x^i a^j\rangle = \delta_{i0}q^{(\ell-1)j}.
\end{aligned}$$

As $q^\ell = 1$, hence $S(\alpha) = \alpha^{\ell-1}$. Similarly, we have $S(\eta) = -\eta\alpha^{\ell-1}$.

According to the multiplication given in Section 4, it is easy to get the following proposition.

Proposition 5.2 *The following relations hold in $D(A_\ell)$:*

$$\alpha a = a\alpha, \quad \alpha x = qx\alpha, \quad \eta a = qa\eta, \quad \eta x = 1 - qx\eta - a\alpha.$$

Obviously, $t = (1 + a + \cdots + a^{\ell-1})x^{\ell'-1}$ is a non-zero left integral of A_ℓ, and $T = \eta^{\ell'-1}(1 + \alpha + \cdots + \alpha^{\ell-1})$ is a non-zero right integral of A_ℓ^*. Let γ and g be the distinguished grouplike elements of A_ℓ^* and A_ℓ respectively. Note that

$$ta = \langle \gamma, a \rangle t = \begin{cases} qt, & \ell' \neq \dfrac{\ell}{2}, \\ -qt, & \ell' = \dfrac{\ell}{2}. \end{cases}$$

By Lemma 5.1, we have $\gamma = \alpha$ when $\ell' \neq \frac{\ell}{2}$, and $\gamma = \alpha^{\frac{\ell}{2}+1}$ when $\ell' = \frac{\ell}{2}$. Similarly, we have $g = a$ if $\ell' \neq \frac{\ell}{2}$ and $g = a^{\frac{\ell}{2}+1}$ if $\ell' = \frac{\ell}{2}$. In each case, we have $G(A_\ell^*) = (\alpha)$ and $G(A_\ell) = (a)$.

Applying Theorem 4.2 to the double $D(A_\ell)$, we get the following proposition.

Proposition 5.3 *Let A_ℓ be the Taft superalgebra. The following results hold:*
(1) $(D(A_\ell), R)$ has a quasi-ribbon element if and only if ℓ cannot be divided by 4;
(2) $(D(A_\ell), R)$ has a ribbon element if and only if ℓ is odd.

Proof (1) Suppose that $(D(A_\ell), R)$ has a quasi-ribbon element. Then by Theorem 4.2(1), there exist $l \in G(A_\ell)$ and $\beta \in G(A_\ell^*)$, such that $l^2 = g$, and $\beta^2 = \gamma$. If $\ell = 4s$, where s is an integer, then $\gamma = \alpha$, and $\beta^2 = \gamma = \alpha^{4s+1}$. This is a contradiction. Therefere ℓ cannot be divided by 4. Conversarly, assume that ℓ can't be divided by 4. If ℓ is odd, denote $\ell = 2s - 1$, then take $\beta = \alpha^s$, $l = a^s$, and we have $\beta^2 = \alpha$, $l^2 = a$; if $\ell = 4s + 2$, then $\gamma = \alpha^{2s+2}$, and $g = a^{2s+2}$, obviously $\beta = \alpha^{s+1}$ and $l = a^{s+1}$ are the square roots of α^{2s+2} and a^{2s+2} respectively. By Theorem 4.2(1), $(D(A_\ell), R)$ has a ribbon element.

(2) By Theorem 4.2(2), $(D(A_\ell), R)$ has a ribbon element if and only if $S^2(y) = l(\beta \rightharpoonup y \leftharpoonup \beta^{-1})l^{-1}$, where $y = a$ or $y = x$. Note that $S^2(a) = a$, $S^2(x) = qx$. Let l and β be defined as in (1). Then when $\ell = 4s + 2$, $l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1} = q^{2s+2}x \neq S^2(x)$. Therefore, $(D(A_\ell), R)$ does not have a ribbon element. When ℓ is odd, $S^2(y) = l(\beta \rightharpoonup y \leftharpoonup \beta^{-1})l^{-1}$ always holds no matter $y = x$ or $y = a$, hence $(D(A_\ell), R)$ has a ribbon element.

Compared to the non-super case (see [5]), the quasi-ribbon element of $D(A_\ell)$ may not be the ribbon element.

6 Quantum Superalgebra $u_q(\mathrm{osp}(1, 2, \mathrm{c}))$

In [14], corresponding to the simple Lie algebra $\mathfrak{sl}_2$, Liu constructed a new class of quantum algebra $\mathfrak{sl}_{q,z}^t(2)$. It is generated by E, F, $K^{\pm 1}$ and the central elements $z^{\pm 1}$, subject to the following relations:

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \quad zz^{-1} = z^{-1}z = 1, \\ &KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \\ &EF - FE = \frac{K - K^{-1}z^{-1}}{q - q^{-1}}, \\ &E^t = F^t = 0, \quad K^t = z^t = 1, \end{aligned}$$

where q^2 is a primitive t-th root of unity ($t \geq 2$). Liu also studied the left and right universal $\mathscr{R}$-matrix of $\mathfrak{sl}^t_{q,z}(2)$, and obtained a new class of universal $\mathscr{R}$-matrix. Particularly, when t is odd, $\mathfrak{sl}^t_{q,z}(2)$ is a charmed Hopf algebra. By the relations between charmed element and ribbon element given in [15], we know that when t is odd, $\mathfrak{sl}^t_{q,z}(2)$ is a ribbon Hopf algebra.

Inspired by this, corresponding to the Lie superalgebra $\mathrm{osp}(1,2)$, in this section we construct a new class of quantum algebra $u_q(\mathrm{osp}(1,2,\mathbf{c}))$, and prove that it is isomorphic to the Drinfel'd double of the Taft superalgebra given in Section 5. We describe its universal $\mathscr{R}$-matrix explicitly. By the results in Section 5, we know that $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ is a ribbon Hopf superalgebra if and only if ℓ is odd, here q is a primitive ℓ-th root of unity.

Definition 6.1 *The superalgebra $U = U_q(\mathrm{osp}(1,2,\mathbf{c}))$ is a $\mathbb{Z}_2$-graded algebra generated by E, F, $K^{\pm 1}$ and the central elements $\mathbf{c}$ and $\mathbf{c}^{-1}$, satisfying the following relations:*

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \quad \mathbf{c}\mathbf{c}^{-1} = \mathbf{c}^{-1}\mathbf{c} = 1, \\ &KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \\ &EF + FE = \frac{K - K^{-1}\mathbf{c}^{-1}}{q - q^{-1}}. \end{aligned}$$

The $\mathbb{Z}_2$-grading is given as

$$[K^{\pm 1}] = [\mathbf{c}^{\pm 1}] = 0, \quad [E] = [F] = 1.$$

U is a Hopf superalgebra, with comultiplication Δ, counite ε, and the antipode S defined as

$$\begin{aligned} &\Delta(E) = E \otimes K + 1 \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -EK^{-1}, \\ &\Delta(F) = F \otimes 1 + \mathbf{c}^{-1}K^{-1} \otimes F, \quad \varepsilon(F) = 0, \quad S(F) = -K\mathbf{c}F, \\ &\Delta(X) = X \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = X^{-1}, \end{aligned}$$

where $X = K^{\pm 1}$, $\mathbf{c}^{\pm 1}$.

Lemma 6.1 *For all $a, b \in \mathbb{Z}^+$, the following relations hold in U:*

$$\begin{aligned} FE^a &= (-1)^a E^a F + E^{a-1}\Big(\frac{q^a + (-1)^{a-1}}{q+1}K - \frac{q^{-a} + (-1)^{a-1}}{q^{-1}+1}K^{-1}\mathbf{c}^{-1}\Big)(q - q^{-1})^{-1}, \\ EF^b &= (-1)^b F^b E + F^{b-1}\Big(\frac{q^{-b} + (-1)^{b-1}}{q^{-1}+1}K - \frac{q^{b} + (-1)^{b-1}}{q+1}K^{-1}\mathbf{c}^{-1}\Big)(q - q^{-1})^{-1}. \end{aligned}$$

Suppose that q is a primitive ℓ-th root of unity. ℓ' is defined as in (5.1). By Lemma 6.1, it is easy to see that $EF^{\ell'} = (-1)^{[E][F^{\ell'}]}F^{\ell'}E$, $FE^{\ell'} = (-1)^{[F][E^{\ell'}]}E^{\ell'}F$, $K^{\ell'}E = EK^{\ell'}$, $K^{\ell'}F = FK^{\ell'}$. Let I be the ideal generated by $E^{\ell'}$, $F^{\ell'}$, $K^{\ell} - 1$ and $\mathbf{c}^{\ell} - 1$.

Definition 6.2 *The restricted quantum superalgebra $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ is defined as*

$$u_q(\mathrm{osp}(1,2,\mathbf{c})) := U_q(\mathrm{osp}(1,2,\mathbf{c}))/I.$$

It is easy to see that I is the Hopf ideal of $U_q(\mathrm{osp}(1,2,\mathbf{c}))$, hence $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ is a finite dimensional Hopf superalgebra with basis $E^iF^jK^k\mathbf{c}^l$, where $i, j \in \{0, 1, 2, \cdots, \ell' - 1\}$, $k, l \in \{0, 1, 2, \cdots, \ell - 1\}$.

Let b_q be the sub Hopf superalgebra of $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ generated by E and K with the following relations:

$$KE = qEK, \quad E^{\ell'} = 0, \quad K^{\ell} = 1.$$

In fact, b_q is the Taft superalgebra defined in Section 5. Constructing the Drinfel'd double $D(b_q)$ as in Section 5, we have the following proposition.

Proposition 6.1 *Define* $\chi : D(b_q) \to u_q(\mathrm{osp}(1,2,\mathbf{c}))$ *by*

$$\chi(E^i K^j \eta^k \alpha^l) = \Big(\frac{q^{-1}-q}{q}\Big)^k q^{-jk-\frac{k(k-1)}{2}} E^i F^k K^{j+k+l} \mathbf{c}^{k+l}, \tag{6.1}$$

where $0 \le i,k \le \ell'-1$, $0 \le j,l \le \ell$. *Then* χ *is an isomorphism of Hopf superalgebra.*

Proof Firstly, by the definition of χ, it is easy to see that each generater of $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ has a preimage, so χ is full. Comparing the dimension one knows that χ is bijective. Next we prove that χ is a homomorphism of superalgebra. By (6.1) we know that

$$\chi(E) = E, \quad \chi(K) = K, \quad \chi(\alpha) = K\mathbf{c}, \quad \chi(\eta) = \frac{q^{-1}-q}{q} FK\mathbf{c}.$$

Clearly

$$\begin{aligned}
\chi(K)\chi(\alpha) &= \chi(\alpha)\chi(K),\\
\chi(K)\chi(\eta) &= \frac{q^{-1}-q}{q} KFK\mathbf{c} = q^{-1}\chi(\eta)\chi(K),\\
\chi(\alpha)\chi(E) &= K\mathbf{c}E = qEK\mathbf{c} = \chi(E)\chi(\alpha),\\
\chi(\eta)\chi(E) &= \frac{q^{-1}-q}{q} FK\mathbf{c}E = (q^{-1}-q)FEK\mathbf{c}\\
&= (q^{-1}-q)\Big(-EF + \frac{K-K^{-1}\mathbf{c}^{-1}}{q-q^{-1}}\Big)K\mathbf{c}\\
&= -(q^{-1}-q)EFK\mathbf{c} - K^2\mathbf{c} + 1\\
&= \chi(1) - q\chi(E)\chi(\eta) - \chi(K)\chi(\alpha).
\end{aligned}$$

Hence χ is a homomorphism of superalgebra. At last we prove that χ keeps the comultiplication and the antipode. Note that

$$\begin{aligned}
\Delta(\chi(\eta)) &= \frac{q^{-1}-q}{q}\Delta(FK\mathbf{c}) = \frac{q^{-1}-q}{q}(FK\mathbf{c}\otimes K\mathbf{c} + FK\mathbf{c})\\
&= \chi(\eta)\otimes\chi(\alpha) + \chi(1)\otimes\chi(\eta)\\
&= (\chi\otimes\chi)\Delta(\eta)
\end{aligned}$$

and

$$\begin{aligned}
\chi(S(\eta)) &= -\chi(\eta\alpha^{-1}) = \frac{q-q^{-1}}{q}F\\
&= \frac{q^{-1}-q}{q}S(K\mathbf{c})S(F) = \frac{q^{-1}-q}{q}S(FK\mathbf{c}) = S(\chi(\eta)).
\end{aligned}$$

One can check it on the other generators E, K and α similarly.

Proposition 6.1 tells us that $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ is quasitriangular. If we set R_D be the universal $\mathscr{R}$-matrix of $D(b_q)$, then the universal $\mathscr{R}$-matrix of $U_q(\mathrm{osp}(1,2,\mathbf{c}))$ $R =: (\chi \otimes \chi)(R_D)$. Let E^iK^j be the basis of b_q. Now we seek for its dual basis. By Lemma 5.1, $\langle \eta^m e_k, E^iK^j\rangle = \delta_{i,m}(i)!_{(-q)}\delta_{k,i+j}$, where $e_k = \frac{1}{\ell}\sum\limits_{s=0}^{N-1} q^{-ks}\alpha^s$, and we use the equation $\sum\limits_{s=0}^{N-1} q^{-ks} = \delta_{k,0}$. Hence the dual basis of E^iK^j is

$$\frac{1}{(i)!_{(-q)}}\eta^i e_{i+j} = \frac{1}{(i)!_{(-q)}}\eta^i \frac{1}{\ell}\sum_{s=0}^{\ell-1} q^{-s(i+j)}\alpha^s.$$

By Proposition 6.1, we have the following theorem.

Theorem 6.1 $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ *is a quasitriangular Hopf superalgebra, and the universal $\mathscr{R}$-matrix is*

$$R = \frac{1}{\ell}\sum_{i=0}^{\ell'-1}\sum_{j,s=0}^{\ell-1} \frac{q^{-s(i+j)}}{(i)!_{(-q)}}\Big(\frac{q^{-1}-q}{q}\Big)^i q^{-\frac{i(i-1)}{2}} E^iK^j \otimes F^iK^{i+s}\mathbf{c}^{i+s}.$$

By Proposition 5.3, it is easy to see that $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ has a ribbon element if and only if ℓ is odd, and $u_q(\mathrm{osp}(1,2,\mathbf{c}))$ has a quasi-ribbon element if and only if ℓ cannot be divided by 4.

Clearly the universal $\mathscr{R}$-matrix is not unique. For example, if we set $\overline{R} = \frac{1}{\ell}\sum\limits_{a,b=0}^{\ell-1} q^{-ab}\mathbf{c}^a \otimes \mathbf{c}^b$, then $R\overline{R}$ also satisfies (2.1)–(2.3), hence $R\overline{R}$ is also the universal $\mathscr{R}$- matrix of $u_q(\mathrm{osp}(1,2,\mathbf{c}))$.

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