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Mathematical Analysis of the Jin-Neelin Model of El Niño-Southern-Oscillation*

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Abstract The Jin-Neelin model for the El Niño–Southern Oscillation (ENSO for short) is considered for which the authors establish existence and uniqueness of global solutions in time over an unbounded channel domain. The result is proved for initial data and forcing that are sufficiently small. The smallness conditions involve in particular key physical parameters of the model such as those that control the travel time of the equatorial waves and the strength of feedback due to vertical-shear currents and upwelling; central mechanisms in ENSO dynamics.

From the mathematical view point, the system appears as the coupling of a linear shallow water system and a nonlinear heat equation. Because of the very different nature of the two components of the system, the authors find it convenient to prove the existence of solution by semi-discretization in time and utilization of a fractional step scheme. The main idea consists of handling the coupling between the oceanic and temperature components by dividing the time interval into small sub-intervals of length k and on each sub-interval to solve successively the oceanic component, using the temperature T calculated on the previous sub-interval, to then solve the sea-surface temperature (SST for short) equation on the current sub-interval. The passage to the limit as k tends to zero is ensured via a priori estimates derived under the aforementioned smallness conditions.

Keywords El Niño–Southern Oscillation, Coupled nonlinear hyperbolic-parabolic systems, Fractional step method, Semigroup theory
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1 Introduction

A long tradition of mathematical analysis of various systems of partial differential equations (PDEs for short) relevant to climate modeling, has been followed over the last three decades. Among the numerous references on the topic we may, for instance, refer the reader to [10, 19, 23–26, 34, 36, 39–40]. Nevertheless, systems of PDEs used by physicists to understand and to model the El Niño-Southern Oscillation (ENSO for short) — a major large-scale phenomenon affecting global climate and weather events such as drought/flooding (see [41, 47]) and tropical

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storms (see [6]) or typhoons (see [62]) — have not been yet analyzed mathematically. The present study proposes to perform such an analysis regarding the existence and uniqueness of solutions to the Jin and Neelin (JN for short) model of ENSO (see [28–29, 52]); a model that we describe hereafter.

ENSO is a dominant mode of climate variability on seasonal-to-interannual time scales and affects the climate over a great portion of the globe on interdecadal and longer time scales (see [33, 47]). A major aspect of ENSO is the strong coupling between the Tropical Pacific ocean and the atmosphere above, and the physical mechanisms that give rise to ENSO are fairly well understood (see [50, 55–56, 61]).

A key mechanism, originally proposed in [4], is the positive atmospheric feedback on the equatorial sea surface temperature (SST for short) field via the surface wind stress. Still, ENSO's unstable, recurrent but irregular behavior implies challenges for prediction (see [3, 8, 22, 47]), even at subannual lead times. Conceptual as well as statistical modeling plays a prominent role in understanding ENSO variability and developing prediction methods for it; see e.g. [14, 16–17, 27, 35, 48, 50, 54, 61].

Among the models derived from first principles (see e.g. [7, 20, 56, 66]), the intermediate coupled model (ICM for short) of Cane and Zebiak (CZ for short) has proven influential in ENSO studies and has provided the first successful ENSO forecasts with a coupled model. A version of the ocean component is described hereafter. One of several simple atmosphere models which attempt to improve on that of Gill [20, Chapter 7] appears in [66] (see also [64–65]), but its drawbacks include the lack of a moisture budget and a formulation with discontinuous derivatives. In a series of papers [28–29, 52] Jin and Neelin have shown though that similar dynamical behaviors can be obtained with different atmospheric models (see [28]), which are more amenable to analysis.

We focus thus on the JN model of ENSO, which can be considered as a "stripped-down" version of the CZ model, as a basis for deriving simpler models and discussing flow regimes and dynamical behavior. The JN model's main ingredients are the following. Its oceanic component is made up of two parts. The vertical-mean motions above the thermocline are governed by linearized shallow-water inviscid equations (SWEs for short) and with a lower-order damping — forced by the wind stress — on an equatorial β-plane following the Matsuno's linear theory (see [43]); see (1.1b)–(1.1d) below. The resulting currents drive an advection-diffusion equation that describes the SST dynamics; see (1.1a). The atmospheric component is a Gill-type model for the wind stress anomaly field, which establishes diagnostic relations (i.e., equations with no time derivative present) between the latter and the SST anomalies; see (1.6) below. The magnitude of the wind stress anomalies controls the coupling between the oceanic and atmospheric components.

After rescaling (see [20, Chapter 7]), the model can be described as the following dimensionless system of equations, in which the ocean dynamics is described by linear SWEs for the oceanic currents and a nonlinear equation for the SST, namely

$$\frac{\partial T}{\partial t} + w_s \mathscr{H}(w_s)(T - T_s(h)) - \epsilon_T \Delta(T - T_e) + (u + u_s)\frac{\partial T}{\partial x} + (v + v_s)\frac{\partial T}{\partial y} - \mathscr{H}(-v_N)v_N(T - T_N) = 0, \tag{1.1a}$$

$$\delta \frac{\partial u}{\partial t} - yv + \frac{\partial h}{\partial x} + \epsilon_0 u = \frac{L_1}{c_0^2}\frac{\tau^x(T)}{\rho H}, \tag{1.1b}$$

$$\mathcal{S}_0^2 \frac{\partial v}{\partial t} + yu + \frac{\partial h}{\partial y} + \epsilon_0 \mathcal{S}_0 v = 0, \tag{1.1c}$$

$$\delta \frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \epsilon_0 h = 0. \tag{1.1d}$$

Here, $\mathscr{H}(x)$ denotes a smooth version of the Heaviside function. The presence of an Heaviside-type function in (1.1a) is justified because the mixed layer temperature is mainly affected by net upwelling (see [28, 52]), while its smooth features are convenient for mathematical analysis such as the study of (numerical) bifurcations (see [51]) as well as for the present existence and uniqueness study. The dynamical variables are respectively the zonal and meridional velocity components, u and v; the thermocline depth anomaly h; and the SST, denoted by T. The expressions of the surface layer velocities (resp. meridional velocity at the northern boundary of a strip[1]), u_s, v_s and w_s (resp. v_N), present in the coefficients of (1.1a) are described in (1.5) (resp. (1.7)) below. We discussed hereafter the main parameters of the model.

The non-dimensionalization is aimed at bringing out a few primary parameters from among the many lurking in the original coupled system. These parameters are described in Table 1 below. Of primary interest is the parameter δ which affects the travel time[2] of the equatorially trapped waves (see [43]) produced by the SWEs (1.1b)–(1.1d) when $\tau^x \equiv 0$. Its variation, even within a small range, may result into a rich set of statistical and dynamical behaviors when seasonal cycle effects are included (see [15]).

Table 1 Glossary of the JN model primary parameters.

Symbol	Interpretation
δ	The relative adjustment time coefficient. It measures the ratio of the time scale of oceanic adjustment by wave dynamics to the time scale of adjustment of SST by coupled feedback and damping processes.
δ_s	Surface-layer coefficient. This parameter governs the strength of feedbacks due to vertical-shear currents and upwelling. It governs also the decay of correlations as numerically illustrated in [15].
μ	The relative coupling coefficient. It governs the strength of the wind stress feedback from the atmosphere per unit SST anomaly, scaled to be order unity for the strongest realistic coupling; for μ= 0 the model is uncoupled.

The parameter $\mathcal{S}_0 = \frac{\lambda_0}{L}$, with L denoting the zonal basin length, while λ_0 denotes a characteristic meridional length scale (see [49]). When $S_0 = 0$, only the low-frequency Rossby waves remain; see e.g. [32] and [20, Chapter 7.2.4]. Our study is not limited to this case. Other parameters are more secondary in the sense that they do not affect substantially the dynamics. Those are: T_N which denotes a constant off-equatorial temperature, c_0, a positive constant and, $T_e(x, y)$, which denotes a reference temperature field; see [20, Chapter 7] for more details. The parameter ϵ_0 is a linear damping parameter, while ϵ_T is a diffusion parameter. Usually ϵ_T represents a Newtonian damping time and is thus not factor of a diffusion operator, but

[1]Strip centered at the equator ($y = 0$) that we consider to be of meridional half-width equal to unity, that is, $y \in (-1, 1)$ in adimensionalized units. The corresponding term, $\mathscr{H}(-v_N)v_N(T - T_N)$, was initially introduced in [49] in the simplification of the full SST equation of [66] to such an equatorial strip.

[2]Note that δ is scaled to be order unity at standard values of dimensional coefficients (see [28]).

rather of $T - T_e$. This modification compared to [28, 52] is in part motivated by the mathematical analysis conducted hereafter but not only. There exist indeed versions of the JN model that include both diffusion and damping terms in the SST equation from the physics literature (see [52]).

Compared to the JN model as originally formulated in [28,52], the SST equation includes also the meridional advection term $(v + v_s)\partial_y T$, and is thus not limited to the equator. The ocean dynamics is as in [49] also not restricted to the case $\mathcal{S}_0 = 0$, while the atmospheric component follows [28, 52] (see hereafter), and our mathematical treatment allows for considering more general wind-stress forcing; see Sections 2–4 below.

When $\mathcal{S}_0 \neq 0$, the boundary conditions conventionally used for the oceanic part (see [9]) are $u = 0$, at $x = 0$ and $x = 1$ and $u, v, h \to 0$, as $y \to \pm\infty$, for a channel domain given by

$$\mathcal{M} := (0, 1)_x \times (-\infty, +\infty)_y, \tag{1.2}$$

after adimentionalization. Here due to the presence of the term $\epsilon_T \Delta(T - T_e)$ compared to [28], Robin boundary conditions are added for the temperature equation; see (3.7) below. Work on models of intermediate complexity has largely been carried out when the ocean model is confined in such a zonally bounded basin. A few studies, however, have allowed the ocean domain to be zonally cyclic, thereby eliminating eastern and western boundaries (see [44–46]). We consider such periodic boundary conditions (in the x-direction) for the ocean in order to study the fully coupled system (Section 4) while the oceanic and SST equations are treated separately in the more classical, zonally bounded case; see Sections 2–3 below.

The coupling between the SWEs (1.1b)–(1.1d) and the SST equation (1.1a) is articulated in three parts. First, the subsurface temperature field, $T_s(h)$ in (1.1a), characterizes temperature values upwelled from the underlying shallow-water layer and is parameterized nonlinearly as a function of the thermocline depth; deeper thermocline resulting into warmer $T_s(h)$. The functional form of $T_s(h)$ is given by

$$T_s(h) = T_{s0} + (T_e - T_{s0}) \tanh(\eta_1 h + \eta_2) \tag{1.3}$$

with T_{s0}, η_1 and η_2 that are positive parameters; see [20, Chapter 7] for a physical interpretation.

The second key ingredient in the coupling between (1.1b)–(1.1d) and (1.1a), is the wind stress τ^x in (1.1b); coupling expressed through the relation

$$\frac{L_1}{c_0^2} \frac{\tau^x(T)}{\rho H} = F_0 \tau_z^x(x) \tau_m^x(y) + \mu \mathcal{C}(T - T_r) \tag{1.4}$$

with τ_z^x and τ_m^x denoting respectively, the zonal and meridional component of the wind stress τ; the parameter ρ denoting the oceanic density, and H a layer depth parameter. The parameter μ is a coupling parameter; see Table 1 below.

In (1.4), T_r denotes typically the steady state of the SST equation (1.1a) obtained for $\mu = 0$; i.e., without feedback of the SST equation into the ocean model (1.1b)–(1.1d). The coupling term $\mathcal{C}(T - T_r)$ models the zonal wind response to a temperature anomaly $\widetilde{T} = T - T_r$ and is given typically by the integral operator

$$\mathcal{C}(\widetilde{T})(x,y) := \frac{3}{2}\int_x^1 \exp(3\epsilon_a(x-x'))\widetilde{T}(x',y)\mathrm{d}x' \\ -\frac{1}{2}\int_0^x \exp(\epsilon_a(x'-x))\widetilde{T}(x',y)\mathrm{d}x' \quad \text{for all } y\in\mathbb{R}, \tag{1.5}$$

where the dependence on time for $\widetilde{T}$ has been dropped for writing convenience; see [20, Chapter 7] for more details.

Finally, the last key ingredient in the coupling between (1.1b)–(1.1d) and (1.1a) are the surface layer velocities, u_s, v_s and w_s, present in particular in the advective terms of (1.1a). These velocities are typically given through the following diagnostic relations

$$\begin{aligned} u_s(x,y) &= \delta_u(F_0\tau_z^x(x)\tau_m^x(y) + \mu\mathcal{C}(T-T_r)(x,y)), && (x,y)\in\mathcal{M},\\ v_s(x,y) &= -y(\delta_s F_0\tau_z^x(x)\tau_m^x(y) + \mu\delta_s\mathcal{C}(T-T_r)(x,y)), && v(x,y)\in\mathcal{M},\\ w_s(x,y) &= -\frac{\partial u_s}{\partial x} + (\delta_s F_0\tau_z^x(x)\tau_m^x(y) + \mu\delta_s\mathcal{C}(T-T_r)(x,y)), && (x,y)\in\mathcal{M}. \end{aligned} \tag{1.6}$$

The meridional velocity v_N at the northern boundary of the strip $y\in(-1,1)$, also arises in the coupling and is given in a similar fashion as

$$v_N(x,y) = \delta_s F_0\tau_z^x(x)\tau_m^x(y) + \mu\delta_s\mathcal{C}(T-T_r)(x,y), \quad (x,y)\in\mathcal{M}. \tag{1.7}$$

Here again we refer to [20, Chapter 7] for physical details about the above parameters δ_u, F_0, as well as ϵ_a arising in (1.5). Note that although not formulated explicitly above, our analysis conducted hereafter allows for a time-dependent wind stress including seasonal forcing which has been suggested as a crucial ingredient in explaining ENSO's irregularity[3]; see e.g. [12–13, 15, 30–31, 59–60].

From the mathematical point of view the system that we aim to study (1.1a)–(1.7) is composed of the linearized inviscid SWEs (1.1b)–(1.1d), of the advection-diffusion equation (1.1a) and of various linear or nonlinear relations between the variables (1.3)–(1.7), as described above. The linearized SWEs (1.1b)–(1.1d) is studied hereafter by using a linear semigroup approach in line with [23–26]. The advection-diffusion equation (1.1a) and the nonlinear coupling terms arising from (1.3)–(1.7) pertain to a different approach in which (1.1a) is studied first as decoupled from the full system to identify a priori estimates (see (3.19)–(3.22)) that are essential. In a second step, we derive (in Section 4) the formal a priori estimates for the whole coupled system (1.1a) supplemented with (1.3)–(1.7). Then the issue is to use these a priori estimates to actually "construct" a solution to the whole system. The usual approximation methods (Galerkin method, vanishing viscosity method) appear not to be convenient here. Instead we thought of taking advantage of the very different two components of the system and implementing an approximation by a fractional step method, by which the two components of the spatial operator are the heat equation corresponding to (1.1a) and the SWEs corresponding to (1.1b)–(1.1d), see below for the details, and see e.g. [18], [42], [57] and [63], for the fractional step method.

[3] Note that in this case not only the wind stress becomes time-periodic but also the "equilibrium" T_r, as obtained by solving the JN model with $\mu = 0$ (see [31]).

Then uniform estimates on the approximate solution (see (4.62)–(4.66)) are derived for proving the existence and uniqueness of solutions to the fully coupled nonlinear system.

The article is organized as follows. In Section 2 we study the oceanic component of the JN model as taken uncoupled from it. Following [23, 25], the corresponding forced linearized SWEs are recast into an abstract evolution equation in a Hilbert space whose (mild) solutions are obtained by application of the standard Hille-Yosida theory to the underlying strongly continuous semigroup. The latter is generated by an unbounded operator obtained as a bounded perturbation of the positive, skew-adjoint operator accounting for the spatial derivatives in (1.1b)–(1.1d) (see Section 2.3).

In Section 3 we study the diffusion-advection equation for the SST that is dealt with, as uncoupled from the JN model. There, after a reformulation of the SST equation aimed at structuring/identifying the main terms that require an attention for the analysis (see Section 3.1), a priori estimates are established in Section 3.3, which allow for concluding to the existence and uniqueness of global solutions via a Galerkin scheme; see Theorem 3.1.

The main result, Theorem 4.1, establishing the existence and uniqueness of global solutions in time for the coupled JN model supplied with periodic boundary conditions in the x-direction, is finally proved in Section 4 using a fractional step method, where we divide the time interval into small sub-intervals of length k and on each sub-interval we successively solve the SWEs (1.1b)–(1.1d) using the temperature T calculated on the previous sub-interval. The SST equation (1.1a) on the current sub-interval is then solved using the solution u, v, h of the SWEs. For the fractional step method to operate, a priori estimates on the exact equations and uniform estimates on the approximate solutions (with respect to the the sub-interval length k) are derived in Section 4.1 and Section 4.2, respectively. The analysis there benefits in particular from the one conducted in Section 3 for the SST equation, when uncoupled from the JN model. It allows in particular for identifying key estimates from Section 3 that require amendments in the coupled case to arrive at the desired uniform estimates; see again (3.19)–(3.22) and (4.24)–(4.25) below. The passage to the limit for fractional step method is then dealt with in Sections 4.3, and 4.4 concludes about the uniqueness.

The existence and uniqueness result thus obtained is valid for initial data and forcing that are sufficiently small. The smallness conditions involve in particular key physical parameters of the model such as those that control the travel time of the equatorial waves and the strength of feedbacks due to vertical-shear currents and upwelling; those are central mechanisms in the ENSO dynamics (see Remark 4.2).

This article is dedicated with friendship to Philippe Ciarlet on the occasion of his 80th birthday, wishing him continued success and happiness.

2 The Linearized Shallow Water Equations

In this section, we first show the well-posedness of the linearized shallow water equations (SWEs for short) as decoupled from the original system (1.1a)–(1.1d). The fully coupled system is analyzed in Section 4 below. As in [23], a semigroup approach is adopted to study the well-posedness. Our expository deals in this section with the case of a zonally bounded oceanic basin with the standard boundary conditions considered in the physics literature [9]. For the case of a zonally periodic basin used in Section 4, we refer to Remark 2.2 below.

Our focus is thus on the following SWEs

$$\begin{cases} \delta u_t - yv + h_x + \epsilon_0 u = f_u, \\ \mathcal{S}_0^2 v_t + yu + h_y + \epsilon_0 \mathcal{S}_0 v = f_v, \\ \delta h_t + u_x + v_y + \epsilon_0 h = f_h, \end{cases} \tag{2.1}$$

where f_u represents the force term $\frac{L_1 \tau^x}{c_0^2 \rho H}$, caused by the zonal wind stress; the terms f_v and f_h are null in the original equations, but added here for the sake of mathematical generality; the three components of f are given functions. We allow also here for both time- and space-dependence of these forcing terms, whose regularity will be specified hereafter. The other parameters were already explained in the introduction.

Following [9], the domain under consideration is the channel (strip)

$$\mathcal{M} := (0,1)_x \times \mathbb{R}_y = (0,1)_x \times (-\infty, +\infty)_y,$$

and the boundary conditions for (2.1) are

$$\begin{aligned} &u = 0, \quad \text{at } x = 0, 1, \\ &u, v, h \to 0, \quad \text{when } y \to \pm\infty. \end{aligned} \tag{2.2}$$

In what follows we set

$$\begin{aligned} U &:= (u, v, h)^{\mathrm{tr}}, \\ F_U &:= (f_u, f_v, f_h)^{\mathrm{tr}}, \end{aligned}$$

and associate to (2.1) the initial data:

$$U_0(x, y) := (u_0(x, y), v_0(x, y), h_0(x, y)). \tag{2.3}$$

Note that the notation F_U is aimed at emphasizing the components of the forcing acting on the evolution equations of the u-, v-, and h-variables. We introduce such a notation to differentiate with the component of the forcing, F_T, acting on the temperature equation considered later on; see Section 3.

In the following subsections, we first consider a simplified version of the initial boundary value problem (IBVP for short) (2.1)–(2.3) with $\delta = \mathcal{S}_0 = 1$ in Section 2.1 (as in e.g. [9]) and then revert back to the problem (2.1) in Section 2.3. In particular, the phase space in which the well-posedness problem for the IBVP (2.1)–(2.3) is considered (i.e., when $\delta \neq 1$ or $\mathcal{S}_0 \neq 1$) is the Hilbert space

$$\mathcal{H} := L^2(\mathcal{M})^3,$$

endowed with the following inner product

$$\langle U, \widetilde{U} \rangle_{\mathcal{H}} := \int_{\mathcal{M}} (\delta u \widetilde{u} + \mathcal{S}_0^2 v \widetilde{v} + \delta h \widetilde{h}) \, \mathrm{d}x \mathrm{d}y, \quad \forall U, \widetilde{U} \in \mathcal{H}. \tag{2.4}$$

Within this framework we prove hereafter the well-posedness for the IBVP (2.1)–(2.3).

Theorem 2.1 *Let $t_1 > 0$ and $\mathcal{H}$ be the Hilbert space $L^2(\mathcal{M})^3$ endowed with the inner product* (2.4)*. Let U_0 be in $\mathcal{H}$ and F_U in $L^1(0,t_1;\mathcal{H})$. Then there exists a unique weak solution U in $\mathcal{C}([0,t_1];\mathcal{H})$ to the IBVP* (2.1)–(2.3).

Remark 2.1 Existence and uniqueness of strong solutions hold for more regular initial data and forcing; see Theorem 2.3 below for details.

To apply the semigroup theory we need first to determine the differential operator and its domain which we do next, in the case $\delta = \mathcal{S}_0 = 1$.

2.1 The SWE operator and its domain

As mentioned about we consider here the case $\delta = \mathcal{S}_0 = 1$ which thus leads to

$$\begin{cases} u_t + h_x + \epsilon_0 u - yv = f_u, \\ v_t + h_y + \epsilon_0 v + yu = f_v, \\ h_t + u_x + v_y + \epsilon_0 h = f_h. \end{cases} \tag{2.5}$$

By introducing the operators

$$\begin{aligned} B_1 U &:= \epsilon_0 U, \\ B_2 U &:= (-yv, yu, 0)^{\mathrm{tr}} \end{aligned} \tag{2.6}$$

and the matrices

$$\mathcal{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.7}$$

the system (2.5) can be written into the following compact form

$$U_t + \mathcal{E}_1 U_x + \mathcal{E}_2 U_y + B_1 U + B_2 U = F_U. \tag{2.8}$$

This equation is the abstract form of a forced 2D inviscid SWEs linearized at the reference state $(0,0,1)$ from 2D nonlinear (dimensionless) SWEs, that include additional damping terms. As in [23], we introduce the stationary SWE differential operator

$$\mathscr{A}U := \mathcal{E}_1 U_x + \mathcal{E}_2 U_y = \begin{pmatrix} h_x \\ h_y \\ u_x + v_y \end{pmatrix}. \tag{2.9}$$

In [23] the linearized 2D inviscid SWEs linearized at the reference state (u_0, v_0, H) with $u_0, v_0 > 0$ (generic case) have been already analyzed, for which the boundary conditions were chosen for each mode, either hyperbolic mode or elliptic mode. Here, we encounter the non-generic case when u_0 and v_0 both vanish and we are still going to apply the semigroup approach as in [23].

In that respect we define the linear unbounded operator A on $\mathcal{H} = L^2(\mathcal{M})^3$ by

$$\begin{cases} \mathcal{D}(A) = \{U \in \mathcal{H},\ \mathscr{A}U \in \mathcal{H},\ u = 0 \text{ at } x = 0,1\}, \\ AU := \mathscr{A}U, \quad \forall U \in \mathcal{D}(A). \end{cases} \tag{2.10}$$

Note that the boundary conditions formulated in the IBVP (2.1)–(2.3), i.e.,

$$u, v, h \to 0, \quad \text{when } y \to \pm\infty, \tag{2.11}$$

automatically hold when U lies $\mathcal{D}(A)$, since

$$\int_{|y|>R} \int_0^1 |u|^2 + |v|^2 + |h|^2 \mathrm{d}x\mathrm{d}y \to 0, \quad \text{when } R \to \infty.$$

The boundary conditions (2.11) actually holds if U has more regularity, for example, if U belongs to $H^1(\mathcal{M})^3$.

Nevertheless, in order that the boundary conditions in the definition (2.10) of domain $\mathcal{D}(A)$ make sense, we need the following trace result.

Proposition 2.1 *Let* $\mathcal{W} = \{U \in \mathcal{H} \mid \mathscr{A}U \in \mathcal{H}\}$. *If* U *belongs to* $\mathcal{W}$ *then the traces* $u|_{x=0,1}$ *are well defined, and they (at least) belong to the space* $H^{-1}(\mathbb{R}_y)$. *Furthermore, for* $a \in \{0, 1\}$, *the trace operator* $\mathfrak{T}_a$ *defined as*

$$\begin{aligned} \mathfrak{T}_a : \mathcal{W} &\to H^{-1}(\mathbb{R}_y), \\ U &\mapsto U|_{x=a} \end{aligned} \tag{2.12}$$

is linear continuous.

Proof Since v lies in $L^2(\mathcal{M}) = L^2_x(0, 1; L^2(\mathbb{R}_y))$, we have that v_y belongs to $L^2_x(0, 1; H^{-1}(\mathbb{R}_y))$, which, together with the fact that $u_x + v_y$ lies in $L^2(\mathcal{M})$ (since $\mathscr{A}U \in \mathcal{H}$), implies that

$$u_x \in L^2_x(0, 1; H^{-1}(\mathbb{R}_y)).$$

This in combination with $u \in L^2(\mathcal{M})$, ensures that u belongs to $\mathcal{C}_x([0, 1]; H^{-1}(\mathbb{R}_y))$. Hence, the traces $u|_{x=0,1}$ are well-defined. The continuity of $\mathfrak{T}_a$ (for $a = 0, 1$) is also straightforward. The proof is similar for h. We observe that if U lies in $\mathcal{W}$, then h belongs to $H^1(\mathcal{M})$.

With the help of Proposition 2.1, the domain $\mathcal{D}(A)$ given in (2.10) is thus well defined. We show next that each function in $\mathcal{D}(A)$ can be approximated by smooth functions by using a Hörmander's technique. In that respect, we introduce the following space of smooth functions

$$\mathcal{V}(\mathcal{M}) := \{U \in \mathcal{C}_c^\infty(\overline{\mathcal{M}})^3, \text{ and } u \text{ vanishes in a neighborhood of } \{x = 0\} \cup \{x = 1\}\}.$$

The density result reads then the following lemma.

Lemma 2.1 $\mathcal{V}(\mathcal{M}) \cap \mathcal{D}(A)$ *is dense in* $\mathcal{D}(A)$.

Proof First, for any $U = (u, v, h)^{\mathrm{tr}} \in \mathcal{D}(A)$, we have that h belongs to $H^1(\mathcal{M})$ and since $\mathcal{C}_c^\infty(\overline{\mathcal{M}})$ is dense in $H^1(\mathcal{M})$ (see e.g. [1, Chapter 3, 38]), h can be approximated by smooth functions.

In order to approximate u and v by smooth functions, we use a proper covering of $[0, 1]$ given by the finite family of intervals $\{I_0, I_1, I_2\}$, where I_0 is an open interval centered at $x = 0$, which does not include the point $x = 1$, I_2 is an open interval centered at $x = 1$, which does not include the point $x = 0$, and I_1 is a relatively compact open sub-interval of $[0, 1]$.

We consider a partition of unity subordinated to this covering,

$$\psi_0(x) + \psi_1(x) + \psi_2(x) = 1, \quad x \in [0,1]$$

for which $\text{supp}(\psi_k) \subseteq I_k$ for any k in $\{0,1,2\}$.

For u, v, and $u_x + v_y$ belonging to $L^2(\mathcal{M})$, it follows thus that $\psi_k u, \psi_k v$, and $(\psi_k u)_x + (\psi_k v)_y$ also belong to $L^2(\mathcal{M})$ for $k = 0,1,2$. We therefore only need to approximate $\psi_k u$ and $\psi_k v$ by smooth functions.

We first consider the case when $\psi_k = \psi_0$ and for the sake of simplicity, we still write $u := \psi_0 u$ and $v := \psi_0 v$ where u, v are supported in $I_0 \times \mathbb{R}_y$. Let $\rho(x,y)$ be a mollifier such that $\rho \geq 0$, $\int \rho = 1$ and ρ has compact support in the rectangle

$$J_x^+ := \{(x,y) \,|\, x > 0, -x < y < x\}. \tag{2.13}$$

Let $\widetilde{u}$, $\widetilde{v}$ be extensions of u, v by zeros in $\mathbb{R}^2$. Denoting by "$*$" the convolution in $\mathbb{R}^2$, we introduce the regularizations

$$u_\epsilon := (\rho_\epsilon * \widetilde{u})|_{\mathcal{M}}, \quad v_\epsilon := (\rho_\epsilon * \widetilde{v})|_{\mathcal{M}}.$$

It is not difficult to observe that, as $\epsilon \to 0$,

$$u_\epsilon \to (\widetilde{u})|_{\mathcal{M}} = u, \quad v_\epsilon \to (\widetilde{v})|_{\mathcal{M}} = v \quad \text{in } L^2(\mathcal{M}).$$

By the choice (2.13) of the support of ρ, we find that u_ϵ vanishes away from $x = 0$ and $x = 1$. Observe that

$$\widetilde{u}_x + \widetilde{v}_y = \widetilde{u_x} + \widetilde{v_y}\ ,$$

and hence as $\epsilon \to 0$:

$$(\rho_\epsilon * \widetilde{u})_x + (\rho_\epsilon * \widetilde{v})_y = \rho_\epsilon * \widetilde{u_x} + \rho_\epsilon * \widetilde{v_y} \ \to \widetilde{u_x} + \widetilde{v_y} \quad \text{in } L^2(\mathbb{R}^2),$$

and restricting to $\mathcal{M}$, we have thus

$$(u_\epsilon)_x + (v_\epsilon)_y \to u_x + v_y \quad \text{in } L^2(\mathcal{M}).$$

In summary, we have that u_ϵ and v_ϵ belong to $\mathcal{C}_c^\infty(\overline{\mathcal{M}})$, u_ϵ vanishes away from $x = 0$ and $x = 1$ and, as $\epsilon \to 0$,

$$(u_\epsilon,\, v_\epsilon,\, (u_\epsilon)_x + (v_\epsilon)_y) \to (u,\, v,\, u_x + v_y) \quad \text{in } L^2(\mathcal{M}). \tag{2.14}$$

For the case $\psi_k = \psi_2$, the above arguments still work if we choose the compact support of the mollifier ρ to be contained in

$$J_x^- := \{(x,y) \,|\, x < 0, -x < y < x\}, \tag{2.15}$$

For the case $\psi_k = \psi_1$, the arguments above also work only if the mollifier ρ has a compact support, which is the case here.

Finally, for u, v, and $u_x + v_y$ belonging to $L^2(\mathcal{M})$, we may write

$$u = \psi_0 u + \psi_1 u + \psi_2 u =: u_0 + u_1 + u_2,$$

$$v = \psi_0 v + \psi_1 v + \psi_2 v =: v_0 + v_1 + v_2$$

for which each pair (u_k, v_k) can be approximated by pairs of smooth functions $(u_{k,\epsilon}, v_{k,\epsilon})$ such that $u_{k,\epsilon}$ vanishes away from $x = 0$ and $x = 1$ and

$$(u_{k,\epsilon})_x + (v_{k,\epsilon})_y \to (u_k)_x + (v_k)_y \quad \text{in } L^2(\mathcal{M}).$$

Setting $u_\epsilon = u_{0,\epsilon} + u_{1,\epsilon} + u_{2,\epsilon}$ and $v_\epsilon = v_{0,\epsilon} + v_{1,\epsilon} + v_{2,\epsilon}$, we obtain that u_ϵ vanishes away from $x = 0$ and $x = 1$ and as $\epsilon \to 0$,

$$(u_\epsilon,\, v_\epsilon,\, (u_\epsilon)_x + (v_\epsilon)_y) \to (u,\, v,\, u_x + v_y) \quad \text{in } L^2(\mathcal{M}). \tag{2.16}$$

Thus, we have proved that $\mathcal{V}(\mathcal{M}) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$. The proof is complete.

2.2 The SWE semigroup

We endow the space $\mathcal{H} = L^2(\mathcal{M})^3$ with its usual inner product and norm, namely, for any U, $\widetilde{U}$ in $\mathcal{H}$,

$$\langle U, \widetilde{U}\rangle_{\mathcal{H}} := \int_{\mathcal{M}} (u\widetilde{u} + v\widetilde{v} + h\widetilde{h})\mathrm{d}x\mathrm{d}y, \quad \|U\|_{\mathcal{H}} = \langle U, U\rangle_{\mathcal{H}}^{\frac{1}{2}}.$$

Our aim is to prove first that A and its adjoint A^* defined below are positive in the sense that:

$$\begin{cases} \langle AU, U\rangle_{\mathcal{H}} \geq 0, & \forall\, U \in \mathcal{D}(A), \\ \langle A^*U, U\rangle_{\mathcal{H}} \geq 0, & \forall\, U \in \mathcal{D}(A^*). \end{cases}$$

As recalled hereafter, these properties are used for applying the Hille-Yosida-Phillips theorem (see [11, Chapter III-3.8]) ensuring the generation of a strongly continuous semigroup acting on $\mathcal{H}$; see also [21, Chapter 3.8].

The positivity of A is essentially a consequence of Lemma 2.1. Indeed for U in $\mathcal{V}(\mathcal{M}) \cap \mathcal{D}(A)$, thus smooth, we have

$$\begin{aligned} \langle \mathscr{A}U, U\rangle_{\mathcal{H}} &= \int_{\mathcal{M}} [h_x u + h_y v + (u_x + v_y)h]\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathcal{M}} [(uh)_x + (vh)_y]\mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{\infty} (uh)\Big|_{x=0}^{x=1}\mathrm{d}y = 0. \end{aligned} \tag{2.17}$$

Therefore, we conclude that, in particular, $\langle AU, U\rangle_H \geq 0$ for U smooth in $\mathcal{D}(A)$, which is also valid for all U in $\mathcal{D}(A)$ by density thanks to Lemma 2.1.

The formal definitions of A^* and its domain $\mathcal{D}(A^*)$ in the sense of the adjoint of a linear unbounded operator [5, Chapter 2.6] can be treated similarly as in [23, Section 3.1.1], we thus only sketch the main elements here. For $\widetilde{U}$ in $\mathcal{H}$ and U in $\mathcal{D}(A)$, both taken as smooth functions, an integrations by part yields

$$\begin{aligned} \langle \mathscr{A}U, \widetilde{U}\rangle_{\mathcal{H}} &= \int_{\mathcal{M}} [h_x \overline{u} + h_y \overline{v} + (u_x + v_y)\overline{h}]\mathrm{d}x\mathrm{d}y \\ &= \langle U, \mathscr{A}^*\widetilde{U}\rangle_{\mathcal{H}} + \int_{-\infty}^{\infty} (h\overline{u} + \overline{h}u)\Big|_{x=0}^{x=1}\mathrm{d}y, \end{aligned} \tag{2.18}$$

where

$$\mathscr{A}^*\widetilde{U} := \begin{pmatrix} -\widetilde{h_x} \\ -\widetilde{h}_y \\ -\widetilde{u}_x - \overline{v}_y \end{pmatrix}.$$

Taking the boundary conditions for u into account, we find

$$\langle AU, \widetilde{U}\rangle_{\mathcal{H}} = \langle U, \mathscr{A}^*\widetilde{U}\rangle_{\mathcal{H}} + \int_{-\infty}^{\infty} (h\overline{u})\Big|_{x=0}^{x=1} \mathrm{d}y.$$

Hence, in order to guarantee that $U \mapsto \langle AU, \widetilde{U}\rangle_{\mathcal{H}}$ is continuous on $\mathcal{D}(A)$, the following boundary conditions for $\widetilde{U}$ need to be satisfied:

$$\widetilde{u} = 0 \quad \text{at } x = 0, 1.$$

Arguing exactly as in [23, Section 3.1.1], we arrive at the following definition for adjoint of the operator A,

$$\begin{cases} \mathcal{D}(A^*) = \{\widetilde{U} \in \mathcal{H},\ \mathscr{A}^*\widetilde{U} \in \mathcal{H},\ \widetilde{u} = 0 \quad \text{at } x = 0, 1\}, \\ A^*\widetilde{U} = \mathscr{A}^*\widetilde{U}, \quad \forall \widetilde{U} \in \mathcal{D}(A^*). \end{cases} \tag{2.19}$$

Thus $\mathcal{D}(A^*)$ is the same as $\mathcal{D}(A)$ and $A^*\widetilde{U} = -A\widetilde{U}$ for all $\widetilde{U}$ in $\mathcal{D}(A^*)$, and the operator A is therefore skew-adjoint. The positivity of A^* follows now from (2.17) and Lemma 2.1.

We are now in position to formulate the main semigroup generation theorems of this subsection. The first is concerned with the operator A.

Proposition 2.2 *The operator $(-A, \mathcal{D}(A))$ is the infinitesimal generator of a contraction semigroup on $\mathcal{H} = L^2(\mathcal{M})^3$.*

Since A and A^* are both positive, the proof of Proposition 2.2 boils down to showing that A and A^* are closed operators and that their domains are dense in $\mathcal{H}$. The latter properties are ensured by the same arguments than provided for the proof of [23, Theorem 8] and are thus omitted here.

Let us finally introduce the linear operator A_U:

$$\begin{cases} \mathcal{D}(A_U) := \mathcal{D}(A), \\ A_U := A + B_1 + B_2, \end{cases} \tag{2.20}$$

where B_1 and B_2 are given in (2.6).

By observing that the operators B_1 and B_2 are linear continuous operators on $\mathcal{H}$, we are now in position to apply the Bounded Perturbation Theorem [21, Chapter III-1.3] (see also [25, Theorem E.7]) to ensure the following result.

Proposition 2.3 *The operator $(-A_U, \mathcal{D}(A_U))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.*

With Proposition 2.3 at hand, we are able to solve the initial and boundary value problem associated with (2.8) either weakly or classically under suitable assumptions. We do not intend to state these results in this subsection, but instead state these results in the next subsection for the original system (2.1).

2.3 The original linearized SWE problem

We now revert to the original problem (2.1) where $\delta, \mathcal{S}_0 > 0$, which is equivalent to

$$\begin{cases} u_t - \frac{1}{\delta} y v + \frac{1}{\delta} h_x + \frac{\epsilon_0}{\delta} u = \frac{1}{\delta} f_u, \\ v_t + \frac{1}{\mathcal{S}_0^2} y u + \frac{1}{\mathcal{S}_0^2} h_y + \frac{\epsilon_0}{\mathcal{S}_0} v = \frac{1}{\mathcal{S}_0^2} f_v, \\ h_t + \frac{1}{\delta}(u_x + v_y) + \frac{\epsilon_0}{\delta} h = \frac{1}{\delta} f_h. \end{cases} \tag{2.21}$$

We write (2.21) into the following compact form

$$U_t + \mathscr{A} U + B_1 U + B_2 U = F_U, \tag{2.22}$$

where

$$\begin{aligned} B_1 U &= \Big(\frac{\epsilon_0}{\delta} u, \frac{\epsilon_0}{\mathcal{S}_0} v, \frac{\epsilon_0}{\delta} h \Big), \\ B_2 U &= \Big(-\frac{1}{\delta} y v, \frac{1}{\mathcal{S}_0^2} y u, 0 \Big), \\ F_U &= \Big(\frac{1}{\delta} f_u, \frac{1}{\mathcal{S}_0^2} f_v, \frac{1}{\delta} f_h \Big)^{\mathrm{tr}} \end{aligned} \tag{2.23}$$

and

$$\mathscr{A} U := \begin{pmatrix} \frac{1}{\delta} h_x \\ \frac{1}{\mathcal{S}_0^2} h_y \\ \frac{1}{\delta}(u_x + v_y) \end{pmatrix}. \tag{2.24}$$

Note that we have kept the same notations for the operators $B_1, B_2, \mathscr{A}$, and the forcing F_U as in Section 2.1; the meaning being clear from the present context here.

The Hilbert space $\mathcal{H} = L^2(\mathcal{M})^3$ is now endowed with the following inner product

$$\langle U, \widetilde{U} \rangle_{\mathcal{H}} = \int_{\mathcal{M}} (\delta u \widetilde{u} + \mathcal{S}_0^2 v \widetilde{v} + \delta h \widetilde{h}) \,\mathrm{d}x \mathrm{d}y, \quad \forall U, \widetilde{U} \in \mathcal{H}. \tag{2.25}$$

The unbounded operator A on $\mathcal{H}$ is still defined by $AU = \mathscr{A}U$, for U in $\mathcal{D}(A)$, where $\mathcal{D}(A)$ is the same as in Section 2.1. Similarly to Section 2.2, we can easily check that both A and its adjoint A^* are positive and that A generates a contraction semigroup. Since the operators B_1 and B_2 are linear continuous operators on $\mathcal{H}$, the same arguments as in Section 2.2 ensure that $A_U = A + B_1 + B_2$ with $\mathcal{D}(A_U) = \mathcal{D}(A)$ generates a strongly continuous semigroup on $\mathcal{H}$.

Theorem 2.2 *The operator $(-A_U, \mathcal{D}(A_U))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.*

The IBVP (2.1)–(2.3) is equivalent to the following abstract, inhomogeneous, initial value problem posed in $\mathcal{H}$,

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + A_U U = F_U, \\ &U(0) = U_0 \in \mathcal{H}. \end{aligned} \tag{2.26}$$

Thanks to Theorem 2.2, this problem is now solved by application of standard tools from the Hille-Yosida theory (see e.g. [53, Chapter 4]) which leads to the following result.

Theorem 2.3 *Let $\mathcal{H}, A_U$ and $\mathcal{D}(A_U)$ be defined as above. Then the initial value problem (2.26) is well-posed. That is, for every U_0 in $\mathcal{H}$ and F_U in $L^1(0,t_1;\mathcal{H})$, the problem (2.26) admits a unique weak solution U in $\mathcal{C}([0,t_1];\mathcal{H})$ that satisfies*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F_U(s)\mathrm{d}s, \quad \forall t \in [0,t_1],$$

where $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by the operator $-A_U$.

Furthermore, if U_0 lies in $D(A_U)$, and F_U' belongs to $L^1(0,t_1;\mathcal{H})$, then the problem (2.26) admits a unique strong solution in $C^1([0,t_1];\mathcal{H}) \cap C([0,t_1];D(A_U))$.

Theorem 2.1 is now just a restatement of Theorem 2.3.

Remark 2.2 The results in Section 2 can be extended to the case where the boundary condition $u = 0,$ at $x = 0, 1$ in (2.2) is replaced by the following periodic boundary conditions

$$\begin{aligned} &u(0) = u(1), \quad u_x(0) = u_x(1), \\ &h(0) = h(1), \quad h_x(0) = h_x(1). \end{aligned} \tag{2.27}$$

In the periodic case, $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ become

$$\begin{aligned} &\mathcal{D}(A) = \{U \in \mathcal{H},\ \mathscr{A}U \in \mathcal{H},\ u(0) = u(1),\ h(0) = h(1)\} \\ &\mathcal{D}(A^*) = \{\widetilde{U} \in \mathcal{H},\ \mathscr{A}^*\widetilde{U} \in \mathcal{H},\ \widetilde{u}(0) = \widetilde{u}(1),\ \widetilde{h}(0) = \widetilde{h}(1)\}. \end{aligned} \tag{2.28}$$

We can show that A and its adjoint A^* are positive in the same way as in Section 2.2 and Theorem 2.2–2.3 still hold.

3 The Sea Surface Temperature Equation

In this section, we study the SST equation (1.1a) with prescribed velocity field (u, v) and thermocline depth h. Within this context we first reformulate (1.1a) into an abstract version to show a local well-posedness result.

3.1 SST equation: Abstract formulation

The original SST equation reads

$$\begin{aligned} &\frac{\partial T}{\partial t} + w_s \mathscr{H}(w_s)(T - T_s(h)) - \epsilon_T \Delta(T - T_e) \\ &+ (u + u_s)\frac{\partial T}{\partial x} + (v + v_s)\frac{\partial T}{\partial y} - \mathscr{H}(-v_N)v_N(T - T_N) = 0. \end{aligned} \tag{3.1}$$

We now set $\widetilde{T} = T - T_r$ and write (3.1) in the variable $\widetilde{T}$. The new equation for $\widetilde{T}$, dropping the tilde, reads then

$$\begin{aligned} &\frac{\partial T}{\partial t} + w_s \mathscr{H}(w_s)(T + T_r - T_s(h)) - \epsilon_T \Delta(T + T_r - T_e) \\ &+ (u + u_s)\frac{\partial (T + T_r)}{\partial x} + (v + v_s)\frac{\partial (T + T_r)}{\partial y} - \mathscr{H}(-v_N)v_N(T + T_r - T_N) = 0. \end{aligned}$$

This equation is equivalent to

$$\begin{aligned}
&\partial_t T - \epsilon_T \Delta T + (u+u_s)\partial_x T + (v+v_s)\partial_y T + w_s \mathscr{H}(w_s)T - v_N \mathscr{H}(-v_N)T \\
&= \epsilon_T \Delta(T_r - T_e) - (u+u_s)\partial_x T_r - (v+v_s)\partial_y T_r + v_N \mathscr{H}(-v_N)(T_r - T_N) \\
&\quad - w_s \mathscr{H}(w_s)(T_r - T_{s0} - (T_e - T_{s0})\tanh(\eta_1 h + \eta_2)),
\end{aligned} \tag{3.2}$$

where we replaced $T_s(h)$ by its expression given by (1.3).

To analyze (3.2) we aim at reformulating its right-hand side (RHS for short) in order to identify the main terms that require a particular attention for the existence problem from those that are less of an issue. This is organized in two steps. First, by going back to the expression of u_s, v_s, w_s and v_N given respectively by (1.6)–(1.7), we identify that two class of terms play an important role in the structure of the RHS of (3.2): Those involving the (time-independent) wind stress profile, $\tau_z^x \tau_m^y$, and those involving the zonal wind response to a temperature anomaly given by $\mathcal{C}(T)$ (see (1.5)). The rest of the terms can be grouped into forcing terms, terms expressing a linear dependence on u and v, or terms expressing a nonlinear dependence on h.

With this structuration goal in mind and allowing for the abuse of notation of symbols that may enclose different detailed expression within a same class of terms, we arrive, after simplification, at the following first abstraction of the SST equation:

$$\begin{aligned}
&\partial_t T - \epsilon_T \Delta T + (G_0 + G_1(U) + G_2(T))(\partial_x T + \partial_y T) + (G_0 + G_2(T))\mathscr{H}(T)T \\
&= F_T + (G_0 + G_1(U) + G_2(T))F_T + (G_0 + G_2(T))\mathscr{H}(T)(F_T + \overline{F}_T G_3(U)).
\end{aligned} \tag{3.3}$$

The symbols used in the RHS of (3.3) and their meanings is summarized in Table 2 below.

Table 2 Glossary of notations for the RHS of (3.3).

Symbol	Interpretation
$G_0(x,y)$	Terms involving the wind stress profile, $\tau_z^x \tau_m^y$, appearing in u_s, v_s, w_s and v_N given respectively by (1.6)–(1.7).
$G_1(U)$	Represents linear operator $G_1^1(U)$ and $G_1^2(U)$, where $G_1^1(U) = u$ and $G_1^2(U) = v$. More specifically, $G_1(U)\partial_x T = G_1^1(U)\partial_x T$ and $G_1(U)\partial_y T = G_1^2(U)\partial_y T$.
$G_2(T)$	terms in (1.6) involving the integral operator $\mathcal{C}(T)$ given by (1.5).
$\mathscr{H}(T)$	Represents either $\mathscr{H}(w_s)$ or $\mathscr{H}(-v_N)$; the dependence in T is to emphasize the dependence of w_s and v_N on T (see (1.6) and (1.7)).
F_T	Forcing terms such as $\epsilon_T \Delta(T_r - T_e)$, ∇T_r, $T_r - T_N$, $T_r - T_{s0}$.
$\overline{F}_T$	$-(T_e - T_{s0})$.
$G_3(U)$	$\tanh(\eta_1 h + \eta_2)$.

Pursuing our effort of structural simplification in view of studying the existence problem, we can safely drop the terms $\mathscr{H}(T)$ and $G_3(U)$ since the latter are Lipschitz continuous and uniformly bounded (actually bounded by 1) and thus do not raise any difficulty in our estimates. We arrive then at the following second abstraction of the SST equation

$$\begin{aligned}
&\partial_t T - \epsilon_T \Delta T + (G_0 + G_1(U) + G_2(T))(\partial_x T + \partial_y T) + (G_0 + G_2(T))T \\
&= F_T + (G_0 + G_1(U) + G_2(T))F_T + (G_0 + G_2(T))(F_T + \overline{F}_T).
\end{aligned} \tag{3.4}$$

Now, the RHS of (3.4) can be rewritten as

$$F_T + G_0 F_T + G_1(U)F_T + G_2(T)F_T + G_0(F_T + \overline{F}_T) + G_2(T)(F_T + \overline{F}_T),$$

which is equivalent to

$$\widetilde{F}_T + \widetilde{F}_T G_1(U) + \widetilde{F}_T G_2(T),$$

where the new term $\widetilde{F}_T$ represents terms such as $F_T + G_0(2F_T + \overline{F}_T)$, F_T, or $2F_T + \overline{F}_T$, adopting a similar abuse of notation as above.

Dropping the tildes, we arrive finally at the following abstract expression of the SST equation that we study in the rest of this section,

$$\begin{aligned} &\partial_t T - \epsilon_T \Delta T + (G_0 + G_1(U) + G_2(T))(\partial_x T + \partial_y T) + (G_0 + G_2(T))T \\ &= F_T + F_T G_1(U) + F_T G_2(T). \end{aligned} \tag{3.5}$$

3.2 Setting of the problem

In order to study the abstract SST equation (3.5), we impose reasonable properties on $G_1(U)$ and $G_2(T)$, which allows for encompassing, in particular, the original equation (3.2). In that respect we assume that

(C1) $G_1(U)$ and $G_2(T)$ are linear in U and T, respectively, and

$$\begin{aligned} &|G_1(U)| \le |U|, \quad \text{a.e.}, \\ &\|G_2(T)\|_{L^p(\mathcal{M})} \le C_1 \|T\|_{L^p(\mathcal{M})}, \quad 1 \le p \le \infty, \end{aligned} \tag{3.6}$$

for some constant $C_1 > 0$ independent of p. Hereafter we only use the cases $p = 2, 4$.

We supplement (3.5) with the following

(C2) Robin boundary conditions

$$\begin{aligned} &T_x - \alpha_T T = 0 \quad \text{at } x = 0, \\ &T_x + \alpha_T T = 0 \quad \text{at } x = 1, \end{aligned} \tag{3.7}$$

for some positive parameter $\alpha_T > 0$.

To deal with the Robin boundary condition (3.7) for T, we equip naturally $H^1(\mathcal{M})$ with the following inner product

$$((T, \overline{T})) := \int_{\mathcal{M}} \nabla T \cdot \nabla \overline{T} \mathrm{d}x\mathrm{d}y + \alpha_T \int_{\mathbb{R}_y} [(T\overline{T})|_{x=0} + (T\overline{T})|_{x=1}] \mathrm{d}y \tag{3.8}$$

and norm,

$$\|T\|_{H^1} = ((T, \overline{T}))^{\frac{1}{2}}. \tag{3.9}$$

The generalized Poincaré inequality with trace terms [58], ensures that

$$\|T\|^2 \le c_p \|T\|_{H^1}^2 \tag{3.10}$$

with $c_p > 0$ denoting the corresponding Poincaré constant.

For $T, \overline{T}$ in $H^1(\mathcal{M})$, we define now the bilinear form a_T:

$$a_T(T,\overline{T}) = \epsilon_T \int_{\mathcal{M}} \nabla T \cdot \nabla \overline{T} \mathrm{d}x\mathrm{d}y + \epsilon_T \alpha_T \int_{\mathbb{R}_y} [(T\overline{T})|_{x=0} + (T\overline{T})|_{x=1}]\mathrm{d}y,$$

where ϵ_T is the diffusion coefficient in (3.5).

Clearly, a_T is a bilinear continuous symmetric form on $H^1(\mathcal{M})$ and we have

$$|a_T(T,\overline{T})| \leq \epsilon_T \|T\|_{H^1} \|\overline{T}\|_{H^1}, \quad \forall T, \overline{T} \in H^1(\mathcal{M}).$$

The form a_T is also coercive, that is for all $T \in H^1(\mathcal{M})$, we have

$$a_T(T,T) \geq \epsilon_T \|T\|_{H^1}^2. \tag{3.11}$$

Let us denote by $(H^1(\mathcal{M}))'$, the dual space of $H^1(\mathcal{M})$. We define also the linear continuous operator $A_T : H^1(\mathcal{M}) \mapsto (H^1(\mathcal{M}))'$ that satisfies

$$\langle A_T T, \overline{T} \rangle_{\langle (H^1(\mathcal{M}))', H^1(\mathcal{M}) \rangle} = a_T(T,\overline{T}), \quad \forall T, \overline{T} \in H^1(\mathcal{M}). \tag{3.12}$$

Then the square root $A_T^{\frac{1}{2}}$ of A_T obeys

$$\langle A_T^{\frac{1}{2}} T, A_T^{\frac{1}{2}} \overline{T} \rangle_{L^2} = a_T(T,\overline{T}), \quad \forall T, \overline{T} \in H^1(\mathcal{M}).$$

From standard estimates, the norm $\|A_T^{\frac{1}{2}} T\|$ is equivalent to the norm $\|T\|_{H^1}$ for T in $H^1(\mathcal{M})$ and the norm $\|A_T T\|$ is equivalent to the norm $\|T\|_{H^2}$ for T in $H^2(\mathcal{M})$ satisfying the boundary condition (3.7).

Remark 3.1 Proving the norm equivalence between $\|A_T T\|$ and $\|T\|_{H^2}$, amounts to showing the H^2-regularity for the following boundary value problem

$$-\Delta T = f \quad \text{in } \mathcal{M}, \qquad \frac{\partial T}{\partial \boldsymbol{n}} + \alpha_T T = 0 \quad \text{on } \partial\mathcal{M},$$

where $\boldsymbol{n}$ is the unit normal vector to $\partial\mathcal{M}$. Due to the simplicity of the geometry of the (physical) domain $\mathcal{M}$, we just classically show that T_y is in $H^1(\mathcal{M})$ by noticing that the y-direction is parallel to the boundary of $\mathcal{M}$, so that $T_{yy}, T_{xy} \in L^2(\mathcal{M})$. Finally, we infer from the equation that $T_{xx} \in L^2(\mathcal{M})$, all with the desired equivalence of the norms.

We now state the global well-posedness result related to (3.5) (with prescribed u, v and h), for sufficiently small time-dependent forcing and initial datum

$$T(0,x,y) = T_0(x,y). \tag{3.13}$$

Theorem 3.1 *Suppose that* $t_1 > 0$ *and*

$$G_0 = G_0(x,y) \in L^4(\mathcal{M}), \quad F_T = F_T(t,x,y) \in L^\infty(0,\infty; L^2(\mathcal{M}) \cap L^4(\mathcal{M})),$$

and also

$$T_0 = T_0(x,y) \in H^1(\mathcal{M}), \quad U = U(t,x,y) \in L^\infty(0,t_1; L^4(\mathcal{M})).$$

Let f_1 *and* f_2 *be the constants defined as*

$$\begin{aligned}\mathsf{f}_1 &:= \sup_{t\in[0,t_1]} (\|F_T(t)\|^2 + \|F_T(t)\|_{L^4}^2 \|U(t)\|_{L^4}^2),\\ \mathsf{f}_2 &:= \sup_{t\in[0,t_1]} (\|G_0\|_{L^4}^4 + \|U(t)\|_{L^4}^4 + \|F_T(t)\|_{L^4}^2).\end{aligned} \tag{3.14}$$

Then there exists a positive constant $C_* > 0$ *independent of the data* G_0, F_T, T_0, U *and the time* t_1*, such that if the following smallness conditions are satisfied:*

$$\begin{aligned}&\max(\|A_T^{\frac{1}{2}} T_0\|^2, 4C_* c_p \mathsf{f}_1) \le \frac{1}{4}\Big(\frac{1}{2C_*}\Big)^2,\\ &\mathsf{f}_2 \le \frac{1}{4C_* c_p},\end{aligned} \tag{3.15}$$

where c_p *is the Poincaré inequality arising in* (3.10)*, then the IBVP* (3.5)*,* (3.7) *and* (3.13) *possesses a unique global solution* T *that satisfies*

$$T \in L^\infty(0, t_1; H^1(\mathcal{M})) \cap L^2(0, t_1; H^2(\mathcal{M})). \tag{3.16}$$

The proof of Theorem 3.1 results from standard Galerkin approximations and a priori estimates; the latter are provided in the following subsection.

3.3 The uniform estimates

For the sake of simplicity, we denote by $\|\cdot\|$ the L^2-norm. Taking the inner product of (3.5) with $A_T T$ in $L^2(\mathcal{M})$ and using Hölder's inequality, we classically obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}} T\|^2 + \|A_T T\|^2\\ &\le (\|G_0\|_{L^4} + \|G_1(U)\|_{L^4} + \|G_2(T)\|_{L^4})\|\nabla T\|_{L^4}\|A_T T\|\\ &\quad + (\|G_0\|_{L^4} + \|G_2(T)\|_{L^4})\|T\|_{L^4}\|A_T T\| + \|F_T\|\|A_T T\|\\ &\quad + \|F_T\|_{L^4}\|G_1(U)\|_{L^4}\|A_T T\| + \|F_T\|_{L^4}\|G_2(T)\|_{L^4}\|A_T T\|,\end{aligned}$$

which, due to condition (C1) (see (3.6)), gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}} T\|^2 + \|A_T T\|^2\\ &\le (\|G_0\|_{L^4} + \|U\|_{L^4} + C_1\|T\|_{L^4})\|\nabla T\|_{L^4}\|A_T T\|\\ &\quad + (\|G_0\|_{L^4} + C_1\|T\|_{L^4})\|T\|_{L^4}\|A_T T\| + \|F_T\|\|A_T T\|\\ &\quad + \|F_T\|_{L^4}\|U\|_{L^4}\|A_T T\| + C_1\|F_T\|_{L^4}\|T\|_{L^4}\|A_T T\|.\end{aligned} \tag{3.17}$$

The Ladyzhenskaya's inequality which is still valid for the unbounded domain $\mathcal{M}$, combined with the Poincaré's inequality, gives

$$\|T\|_{L^4} \le C\|T\|^{\frac{1}{2}}\|T\|_{H^1}^{\frac{1}{2}} \le C\|T\|_{H^1}. \tag{3.18}$$

We now use (3.18) and Ladyzhenskaya's and Young's inequalities to estimate the right-hand side of (3.17) term by term. We also use the fact that the norm $\|A_T T\|$ (resp. $\|A_T^{\frac{1}{2}} T\|$) is equivalent to the norm $\|T\|_{H^2}$ (resp. $\|T\|_{H^1}$).

The first two terms are estimated as follows.

$$\begin{aligned}
&(\|G_0\|_{L^4}+\|U\|_{L^4})\|\nabla T\|_{L^4}\|A_T T\| \\
&\le C(\|G_0\|_{L^4}+\|U\|_{L^4})\|T\|_{H^1}^{\frac{1}{2}}\|A_T T\|^{\frac{3}{2}} \\
&\le C(\|G_0\|_{L^4}^4+\|U\|_{L^4}^4)\|A_T^{\frac{1}{2}}T\|^2+\frac{1}{16}\|A_T T\|^2
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
C_1\|T\|_{L^4}\|\nabla T\|_{L^4}\|A_T T\| &\le C\|A_T^{\frac{1}{2}}T\|\|A_T T\|^2, \\
\|G_0\|_{L^4}\|T\|_{L^4}\|A_T T\| &\le C\|G_0\|_{L^4}\|T\|^{\frac{1}{2}}\|T\|_{H^1}^{\frac{1}{2}}\|A_T T\| \\
&\le C\|G_0\|_{L^4}\|A_T T\|^{\frac{1}{2}}\|A_T T\|^{\frac{3}{2}} \\
&\le C\|G_0\|_{L^4}^4\|A_T^{\frac{1}{2}}T\|^2+\frac{1}{16}\|A_T T\|^2.
\end{aligned} \tag{3.20}$$

We also estimate the remaining terms as:

$$\begin{aligned}
C_1\|T\|_{L^4}^2\|A_T T\| &\le C\|A_T^{\frac{1}{2}}T\|^2\|A_T T\| \le C\|A_T^{\frac{1}{2}}T\|\|A_T T\|^2, \\
\|F_T\|\|A_T T\| &\le C\|F_T\|^2+\frac{1}{16}\|A_T T\|^2, \\
\|F_T\|_{L^4}\|U\|_{L^4}\|A_T T\| &\le \|F_T\|_{L^4}^2\|U\|_{L^4}^2+\frac{1}{16}\|A_T T\|^2
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&C_1\|F_T\|_{L^4}\|T\|_{L^4}\|A_T T\| \\
&\le C\|F_T\|_{L^4}\|T\|_{H^1}\|A_T T\| \\
&\le C\|F_T\|_{L^4}^2\|A_T^{\frac{1}{2}}T\|^2+\frac{1}{16}\|A_T T\|^2.
\end{aligned} \tag{3.22}$$

Combining these estimates, we derive from (3.17) the differential inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T\|^2+(1-C_*\|A_T^{\frac{1}{2}}T\|)\|A_T T\|^2 \\
&\le C_*(\|F_T\|^2+\|F_T\|_{L^4}^2\|U\|_{L^4}^2)+C_*(\|G_0\|_{L^4}^4+\|U\|_{L^4}^4+\|F_T\|_{L^4}^2)\|A_T^{\frac{1}{2}}T\|^2
\end{aligned} \tag{3.23}$$

for some positive constant $C_*>0$ independent of the data G_0, U, F_T, T_0 and the time t_1, which we choose to be the constant C^* appearing in condition (3.15).

We recall that this condition implies

$$C_*\|A_T^{\frac{1}{2}}T_0\| \le \frac{1}{4}.$$

Then as long as $C_*\|A_T^{\frac{1}{2}}T\| \le \frac{1}{2}$, we have by the Poincaré inequality and (3.23), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T\|^2+\frac{1}{2c_p}\|A_T^{\frac{1}{2}}T\|^2 \le C_*\mathsf{f}_1+C_*\mathsf{f}_2\|A_T^{\frac{1}{2}}T\|^2, \tag{3.24}$$

where f_1 and f_2 are defined in (3.14).

Using the smallness condition (3.15) on f_2, we infer from (3.24) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T\|^2 + \frac{1}{4c_p}\|A_T^{\frac{1}{2}}T\|^2 \le C_*\mathsf{f}_1,$$

which, by the Gronwall inequality, implies

$$\begin{aligned}\|A_T^{\frac{1}{2}}T(t)\|^2 &\le \exp\Big(-\frac{t}{4c_p}\Big)\|A_T^{\frac{1}{2}}T_0\|^2 + 4c_pC_*\mathsf{f}_1\Big(1-\exp\Big(-\frac{t}{4c_p}\Big)\Big)\\ &\le \exp\Big(-\frac{t}{4c_p}\Big)\|A_T^{\frac{1}{2}}T_0\|^2 + 4c_pC_*\mathsf{f}_1.\end{aligned} \tag{3.25}$$

By the smallness assumption (3.15) again on the initial data and the forcing term f_1, we find finally that

$$\|A_T^{\frac{1}{2}}T(t)\| \le \frac{1}{2C_*}, \quad \forall t\in[0,t_1]. \tag{3.26}$$

Now, going back to (3.23) and using the Poincaré inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T(t)\|^2 + \frac{1}{2}\|A_TT(t)\|^2 \le C_*\mathsf{f}_1 + C_*\mathsf{f}_2\|A_T^{\frac{1}{2}}T(t)\|^2 \le C_*\mathsf{f}_1 + c_pC_*\mathsf{f}_2\|A_TT(t)\|^2,$$

which, by using (3.15), implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T(t)\|^2 + \frac{1}{4}\|A_TT(t)\|^2 \le \mathsf{f}_1. \tag{3.27}$$

Integrating (3.27) from 0 to t_1 in t and using (3.15) again give

$$\int_0^{t_1}\|A_TT(t)\|^2\mathrm{d}t \le 4t_1C_*\mathsf{f}_1 + \|A_T^{\frac{1}{2}}T_0\|^2 \le \Big(\frac{1}{2}+\frac{t_1}{2c_p}\Big)\Big(\frac{1}{2C_*}\Big)^2. \tag{3.28}$$

Finally, the uniform estimates (3.26) and (3.28), together with a standard Galerkin approximation scheme, allow for completing the proof of Theorem 3.1. We omit these details here.

Remark 3.2 The analysis and results in Section 3 are not affected if one replaces the original physical boundary condition $u = 0$, at $x = 0, 1$ by the periodic boundary conditions (2.27) for the oceanic component.

4 The Coupled Equations: Main Result

In this section, we aim to use the fractional step method to investigate the full coupled system of equations (1.1a)–(1.1d) supplied with the periodic boundary conditions in the x-direction. As in Section 3, we are going to use the abstract version (3.5) of the SST instead of (1.1a) and the full coupled SWE-SST equations that we consider then read

$$\begin{cases}\partial_tU + \mathscr{A}U + B_1U + B_2U = F_U, & \text{(4.1a)}\\ \partial_tT - \epsilon_T\Delta T + (G_0+G_1(U)+G_2(T))(\partial_xT+\partial_yT) + (G_0+G_2(T))T & \\ = F_T + F_TG_1(U) + F_TG_2(T). & \text{(4.1b)}\end{cases}$$

Here the U-component of the forcing F_U depends on T, and is given by

$$F_U := (G_0 + G_2(T), 0, 0)^{\mathrm{tr}}. \tag{4.2}$$

The operators $\mathscr{A}$, B_1 and B_2 are those defined in Section 2.3. We associate with (4.1a) and (4.1b) the initial data

$$U(0,x,y) = U_0(x,y), \quad T(0,x,y) = T_0(x,y), \tag{4.3}$$

and the boundary conditions

$$\begin{cases} u(0) = u(1), \quad u_x(0) = u_x(1), \\ h(0) = h(1), \quad h_x(0) = h_x(1). \\ \dfrac{\partial T}{\partial \boldsymbol{n}} + \alpha_T T = 0 \quad \text{at } x = 0, 1, \\ u,\, v,\, h,\, T \to 0, \quad \text{when } y \to \pm\infty, \end{cases} \tag{4.4}$$

where $\alpha_T > 0$.

As in Section 3, we assume that condition (C1) (see (3.6)) holds and that furthermore

$$\|\nabla(G_2(T))\| \le C\|\nabla T\| \quad \text{for some } C > 0. \tag{4.5}$$

For the forcing terms and initial data, let us assume that

$$G_0 = G_0(x,y) \in H^1(\mathcal{M}), \quad F_T = F_T(t,x,y) \in L^\infty(0,\infty; L^2(\mathcal{M}) \cap L^4(\mathcal{M})), \tag{4.6}$$

and

$$U_0 = U_0(x,y) \in H^1(\mathcal{M})^3, \quad T_0 = T_0(x,y) \in H^1(\mathcal{M}). \tag{4.7}$$

We then want to prove a global well-posedness result concerning the coupled system (4.1a)–(4.4) for sufficiently small forcing and small initial data. The full theorem is stated at the end of Section 4.4. Here we first introduce the smallness conditions we need and the a priori estimates we are aiming to derive.

We set $c_1 = \min\left\{\frac{\epsilon_0}{\delta}, \frac{\epsilon_0}{\mathcal{S}_0}\right\}$ and

$$c_2 = \max\left\{\frac{1}{\delta}, \frac{1}{\mathcal{S}_0^2}\right\}, \quad c_3 = \min\left\{\frac{2c_2^2}{c_1^2}, c_1\right\}, \quad c_4 = \min\left(\frac{c_3}{4\max\left\{\frac{4c_2^2}{c_1^2}, c_1\right\}}, \frac{1}{2c_p}\right).$$

Our goal is to show that if for some positive constant $C_* > 0$ independent of the data G_0, F_T, T_0, U_0 and the time t_1, the following smallness conditions are satisfied:

$$\sup_{t\in[0,t_1]} \|F_T(t)\|_{L^4}^4 \le \frac{1}{4}\cdot\frac{c_3\bar{\epsilon}_*^2}{2C_*} = \frac{c_3}{8C_*}\cdot\frac{c_4^2}{16C_*}, \quad \|G_0\|_{L^4}^4 + \sup_{t\in[0,t_1]} \|F_T(t)\|_{L^4}^2 \le \frac{1}{4C_*c_p} \tag{4.8}$$

and

$$\max\left\{Y(0), \frac{C_*}{c_4}\Big(\sup_{t\in[0,t_1]} \|F_T(t)\|^2 + \bar{\epsilon}_*\|G_0\|_{H^1}^2\Big)\right\}$$

$$\leq \min\Big\{\frac{2C_* c_2^2 c_3 \overline{\epsilon}_*^2}{c_1^2}, \Big(\frac{1}{8C_*}\Big)^2\Big\} = \min\Big\{\frac{c_2^2 c_3 c_4^2}{8c_1^2 C_*}, \frac{1}{64 C_*^2}\Big\}, \tag{4.9}$$

where

$$Y(0) := \frac{4c_2^2\overline{\epsilon}_*}{c_1^2}\|U_0\|^2 + c_1\overline{\epsilon}_*\|U_{0,x}\|^2 + c_1\overline{\epsilon}_*\|U_{0,y}\|^2 + \|A_T^{\frac{1}{2}} T_0\|^2 \tag{4.10}$$

with $\overline{\epsilon}_* = \frac{c_4}{4C_*}$ and c_p denoting the Poincaré inequality arising in (3.10), then (U,T) satisfies

$$(U,T) \in L^\infty(0,t_1;H^1(\mathcal{M})^4), \quad T \in L^2(0,t_1;H^2(\mathcal{M})), \tag{4.11}$$

$$(\partial_t U, \partial_t T) \in L^2(0,t_1;L^2(\mathcal{M})^4). \tag{4.12}$$

As mentioned earlier, in order to study the system of coupled equations (4.1a)–(4.1b), we use the fractional step method. The rest of this section is organized as follows. In Section 4.1 we show the motivations behind the smallness conditions (4.8)–(4.9) by deriving natural a priori estimates on the original system. In Section 4.2 we introduce the approximate solutions $U_k(t)$, $T_k(t)$ using the fractional step method, for which we derived a priori estimates similar to those of Section 4.1. The passage to the limit $k \to 0$ is then established in Section 4.3. Finally, the uniqueness is proved in Section 4.4.

4.1 A priori estimates on the original equations

In this section, we aim to derive some formal a priori estimates on the solutions of the full coupled system of equations (4.1a)–(4.1b), under some smallness assumptions on the initial data and forcing terms.

Take the inner product of (4.1a) with U in $L^2(\mathcal{M})$. We arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \langle \mathscr{A}U, U\rangle_{L^2} + \langle B_1 U, U\rangle_{L^2} + \langle B_2 U, U\rangle_{L^2} = \langle F_U, U\rangle_{L^2}.$$

Direct calculations such as integration by parts and the Cauchy-Schwarz inequality, show that

$$\begin{aligned}
&\langle \mathscr{A}U, U\rangle_{L^2} \geq 0,\\
&\langle B_1 U, U\rangle_{L^2} \geq c_1\|U\|^2,\\
&\langle B_2 U, U\rangle_{L^2} = 0,\\
&\langle F_U, U\rangle_{L^2} \leq \frac{1}{2c_1}\|F_U\|^2 + \frac{c_1}{2}\|U\|^2,
\end{aligned}$$

where

$$c_1 = \min\Big\{\frac{\epsilon_0}{\delta}, \frac{\epsilon_0}{\mathcal{S}_0}\Big\}. \tag{4.13}$$

By collecting these equalities and inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + c_1\|U\|^2 \leq \frac{1}{c_1}\|F_U\|^2. \tag{4.14}$$

We then estimate the derivatives $\partial_x U$ and $\partial_y U$. In that respect, we first apply ∂_x to (4.1a) and find

$$\partial_t U_x + \mathscr{A}U_x + B_1 U_x + B_2 U_x = \partial_x F_U.$$

Then by taking the inner product with $\partial_x U$ in $L^2(\mathcal{M})$ and proceeding similarly as for (4.14), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U_x\|^2 + c_1\|U_x\|^2 \le \frac{1}{c_1}\|\partial_x F_U\|^2 \le C(\|G_{0,x}\|^2 + \|T_x\|^2), \tag{4.15}$$

where we have used

$$\begin{aligned}\langle \mathscr{A}U_x, U_x\rangle_{\mathcal{H}} &= \int_{\mathcal{M}} [h_{xx}u_x + h_{yx}v_x + (u_{xx}+v_{yx})h_x]\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathcal{M}} [(u_x h_x)_x + (v_x h_x)_y]\mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{\infty} (u_x h_x)\Big|_{x=0}^{x=1}\mathrm{d}y = 0,\end{aligned} \tag{4.16}$$

thanks to the periodic boundary conditions (4.4) in x. We then apply ∂_y to (4.14) and find

$$\partial_t U_y + \mathscr{A}U_y + B_1 U_y + B_2 U_y + B_3 U = \partial_y F_U, \tag{4.17}$$

where

$$B_3 U := \Big(-\frac{1}{\delta}v, \frac{1}{\mathcal{S}_0^2}u, 0\Big)^{\mathrm{tr}}. \tag{4.18}$$

The extra term B_3U appearing in (4.17) is due to the β-approximation of the Coriolis force $(-yv, yu)$ in dimensionless form. Again, by taking the inner product of (4.17) with $\partial_y U$ in $L^2(\mathcal{M})$ and proceeding similarly as for (4.15) together with the following (additional) inequalities

$$\begin{aligned}\langle \partial_y F_U, U_y\rangle_{L^2} &\le \frac{1}{c_1}\|\partial_y F_U\|^2 + \frac{c_1}{4}\|U_y\|^2, \\ \langle B_3 U, U_y\rangle_{L^2} &\le \frac{c_2^2}{c_1}\|U\|^2 + \frac{c_1}{4}\|U_y\|^2\end{aligned} \tag{4.19}$$

with

$$c_2 = \max\Big\{\frac{1}{\delta}, \frac{1}{\mathcal{S}_0^2}\Big\}, \tag{4.20}$$

we finally arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|U_y\|^2 + c_1\|U_y\|^2 &\le \frac{2c_2^2}{c_1}\|U\|^2 + \frac{2}{c_1}\|\partial_y F_U\|^2 \\ &\le \frac{2c_2^2}{c_1}\|U\|^2 + C(\|G_{0,y}\|^2 + \|T_y\|^2).\end{aligned} \tag{4.21}$$

Multiply (4.14) by $\frac{2c_2^2}{c_1^2}$ and sum it together with (4.15) and (4.21), then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2}{c_1^2}\|U\|^2 + c_1\|U_x\|^2 + c_1\|U_y\|^2\Big) + c_3\|U\|_{H^1}^2 \le C_U(\|G_0\|_{H^1}^2 + \|A_T^{\frac{1}{2}}T\|^2), \tag{4.22}$$

where c_3 appearing in (4.22) is given by

$$c_3 = \min\Big\{\frac{2c_2^2}{c_1^2}, c_1\Big\}, \tag{4.23}$$

and C_U is some constant depending only on $\mathcal{M}$ and ϵ_0, δ, $\mathcal{S}_0$, derived from (4.14)–(4.15) and (4.21).

For the SST equation, as in Section 3.3, by taking the inner product of (4.1b) with $A_T T$ in $L^2(\mathcal{M})$ and by using Hölder's inequality and condition (C1) (see (3.6)), we arrive at (3.17) again.

We still need the estimates in (3.19)–(3.22), however, the estimates involving the variable U need some amendments due to the coupling considered here. We describe below the required modifications.

First, we observe that by using Ladyzhenskaya's and Young's inequalities, we find

$$\begin{aligned}\|U\|_{L^4}\|\nabla T\|_{L^4}\|A_T T\| &\leq C\|U\|^{\frac{1}{2}}\|U\|^{\frac{1}{2}}_{H^1}\|T\|^{\frac{1}{2}}_{H^1}\|A_T T\|^{\frac{3}{2}}\\ &\leq C\|U\|^2\|A_T^{\frac{1}{2}}T\|^2\|U\|^2_{H^1}+\frac{1}{16}\|A_T T\|^2 \end{aligned}\tag{4.24}$$

and

$$\begin{aligned}\|F_T\|_{L^4}\|U\|_{L^4}\|A_T T\| &\leq C\|F_T\|_{L^4}\|U\|^{\frac{1}{2}}\|U\|^{\frac{1}{2}}_{H^1}\|A_T T\|\\ &\leq \frac{C}{\epsilon}\|F_T\|^4_{L^4}\|U\|^2+\frac{c_3\epsilon}{2}\|U\|^2_{H^1}+\frac{1}{16}\|A_T T\|^2 \end{aligned}\tag{4.25}$$

for some ϵ to be determined later.

Notice that the generic constants C in the above estimates (4.24)–(4.25) and (3.19)–(3.22) only depends on $\mathcal{M}$. We then replaced these constants C by a constant C_T, which is larger than all the C above and also only depends on $\mathcal{M}$. Collecting now the estimates (3.19)–(3.22) and using the new estimates (4.24)–(4.25) for the terms involving the oceanic variable, U, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T\|^2+\|A_T T\|^2\\ &\leq C_T\|F_T\|^2+\frac{c_3\epsilon}{2}\|U\|^2_{H^1}+\frac{C_T}{\epsilon}\|F_T\|^4_{L^4}\|U\|^2\\ &\quad+C_T(\|G_0\|^4_{L^4}+\|F_T\|^2_{L^4})\|A_T^{\frac{1}{2}}T\|^2+C_T\|U\|^2\|A_T^{\frac{1}{2}}T\|^2\|U\|^2_{H^1}\\ &\quad+C_T\|A_T^{\frac{1}{2}}T\|\|A_T T\|^2. \end{aligned}\tag{4.26}$$

Multiply (4.22) by ϵ and add it to (4.26) and set $C_* = \max(C_U, C_T)$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2\epsilon}{c_1^2}\|U\|^2+c_1\epsilon\|U_x\|^2+c_1\epsilon\|U_y\|^2+\|A_T^{\frac{1}{2}}T\|^2\Big)\\ &\quad+\Big(\frac{c_3\epsilon}{2}-C_*\|U\|^2\|A_T^{\frac{1}{2}}T\|^2\Big)\|U\|^2_{H^1}+(1-C_*\|A_T^{\frac{1}{2}}T\|)\|A_T T\|^2\\ &\leq C_*(\|F_T\|^2+\epsilon\|G_0\|^2_{H^1})+\frac{C_*}{\epsilon}\|F_T\|^4_{L^4}\|U\|^2\\ &\quad+C_*(\|G_0\|^4_{L^4}+\|F_T\|^2_{L^4}+\epsilon)\|A_T^{\frac{1}{2}}T\|^2. \end{aligned}\tag{4.27}$$

Note that C_* only depends on $\mathcal{M}$ and δ, ϵ_0, $\mathcal{S}_0$.

Apply the generalized Poincaré inequality (3.10) on the RHS of (4.27), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2\epsilon}{c_1^2}\|U\|^2 + c_1\epsilon\|U_x\|^2 + c_1\epsilon\|U_y\|^2 + \|A_T^{\frac{1}{2}}T\|^2\Big) \\
&+ \Big(\frac{c_3\epsilon}{2} - \frac{C_*}{\epsilon}\|F_T\|_{L^4}^4 - C_*\|U\|^2\|A_T^{\frac{1}{2}}T\|^2\Big)\|U\|_{H^1}^2 \\
&+ (1 - C_*c_p(\|G_0\|_{L^4}^4 + \|F_T\|_{L^4}^2 + \epsilon) - C_*\|A_T^{\frac{1}{2}}T\|)\|A_T T\|^2 \\
\le\ & C_*(\|F_T\|^2 + \epsilon\|G_0\|_{H^1}^2).
\end{aligned} \tag{4.28}$$

Now we introduce

$$Y(t) = \frac{4c_2^2\epsilon}{c_1^2}\|U\|^2 + c_1\epsilon\|U_x\|^2 + c_1\epsilon\|U_y\|^2 + \|A_T^{\frac{1}{2}}T\|^2. \tag{4.29}$$

We are aiming to attain a uniform bound on $Y(t)$ so that we can show (U, T) lies in a bounded set in $L^\infty(0, t_1; H^1(\mathcal{M})^4)$. For that purpose, we need to impose appropriate smallness conditions on G_0, F_T, and $Y(0)$ and choose a proper ϵ, which guarantees that

$$\begin{aligned}
&\Big(\frac{c_3\epsilon}{2} - \frac{C_*}{\epsilon}\|F_T\|_{L^4}^4 - C_*\|U\|^2\|A_T^{\frac{1}{2}}T\|^2\Big) > 0, \\
&(1 - C_*c_p(\|G_0\|_{L^4}^4 + \|F_T\|_{L^4}^2 + \epsilon) - C_*\|A_T^{\frac{1}{2}}T\|) > 0
\end{aligned}$$

for all $t \in [0, t_1]$.

Firstly, we set the smallness conditions on G_0, F_T to be

$$\sup_{t\in[0,t_1]} \frac{C_*}{\epsilon}\|F_T(t)\|_{L^4}^4 \le \frac{1}{4}\cdot\frac{c_3\epsilon}{2}, \quad C_*c_p(\|G_0\|_{L^4}^4 + \sup_{t\in[0,t_1]}\|F_T(t)\|_{L^4}^2 + \epsilon) \le \frac{1}{4}. \tag{4.30}$$

At this step, let us set $\epsilon = \epsilon_* = \frac{1}{8C_*c_p}$, then (4.30) becomes

$$\sup_{t\in[0,t_1]} \frac{C_*}{\epsilon}\|F_T(t)\|_{L^4}^4 \le \frac{1}{4}\cdot\frac{c_3\epsilon}{2}, \quad C_*c_p(\|G_0\|_{L^4}^4 + \sup_{t\in[0,t_1]}\|F_T(t)\|_{L^4}^2) \le \frac{1}{8}. \tag{4.31}$$

The uniform bounds we want on $Y(t)$ should make the following bounds hold for all $t \in [0, t_1]$:

$$C_*\|U\|^2\|A_T^{\frac{1}{2}}T\|^2 \le \frac{c_3\epsilon_*}{2}\cdot\frac{1}{4}, \quad C_*\|A_T^{\frac{1}{2}}T\| \le \frac{1}{4}. \tag{4.32}$$

Substituting $\epsilon = \epsilon_* = \frac{1}{8C_*c_p}$ in (4.32) leads to

$$\sup_{t\in[0,t_1]}\|A_T^{\frac{1}{2}}T\| \le \frac{1}{4C_*}, \quad \sup_{t\in[0,t_1]}\|U\|^2 \le 2c_3\epsilon_*C_* = \frac{c_3}{4c_p}. \tag{4.33}$$

The uniform bound desired on $Y(t)$ can be written as

$$\begin{aligned}
\sup_{t\in[0,t_1]} Y(t) &\le \min\Big\{\frac{4c_2^2\epsilon_*}{c_1^2}\cdot 2c_3\epsilon_*C_*, \Big(\frac{1}{4C_*}\Big)^2\Big\} \\
&= \min\Big\{\frac{c_2^2c_3}{8C_*c_1^2c_p^2}, \frac{1}{16C_*^2}\Big\}.
\end{aligned} \tag{4.34}$$

Then for $Y(0)$, we set the smallness condition as follows,

$$\begin{aligned} Y(0) &\le \min\Big\{\frac{c_2^2\epsilon_*}{c_1^2}\cdot 2c_3\epsilon_* C_*, \Big(\frac{1}{8C_*}\Big)^2\Big\} \\ &= \min\Big\{\frac{c_2^2c_3}{32C_*c_1^2c_p^2}, \frac{1}{64C_*^2}\Big\}. \end{aligned} \tag{4.35}$$

Assuming that (4.31) and (4.35) are satisfied by G_0, F_T, $Y(0)$, we aim at proving

$$\sup_{t\in[0,t_1]} Y(t) \le M,$$

M = RHS of (4.34). First we observe that (4.32) and (4.34) hold at $t = 0$, then as long as (4.32) holds, and equation (4.28) gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2\epsilon_*}{c_1^2}\|U\|^2 + c_1\epsilon_*\|U_x\|^2 + c_1\epsilon_*\|U_y\|^2 + \|A_T^{\frac{1}{2}}T\|^2\Big) \\ &+ \frac{c_3\epsilon_*}{4}\|U\|_{H^1}^2 + \frac{1}{2}\|A_TT\|^2 \le C_*(\|F_T\|^2 + \epsilon_*\|G_0\|_{H^1}^2). \end{aligned} \tag{4.36}$$

Recall that here $\epsilon_* = \frac{1}{8C_*c_p}$. Now (4.36) shows $Y(t)$ satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + c_4Y(t) \le C_*(\|F_T\|^2 + \epsilon_*\|G_0\|_{H^1}^2),$$

where

$$c_4 := \min\Big\{\frac{c_3}{4\max\big\{\frac{4c_2^2}{c_1^2}, c_1\big\}}, \frac{1}{2c_p}\Big\}.$$

A direct integration shows that

$$Y(t) \le \exp(-c_4t)Y(0) + \frac{C_*}{c_4}(\|F_T\|^2 + \epsilon_*\|G_0\|_{H^1}^2). \tag{4.37}$$

Here we add one more smallness condition on G_0, F_T so that

$$\frac{C_*}{c_4}\Big(\sup_{t\in[0,t_1]}\|F_T\|^2 + \epsilon_*\|G_0\|_{H^1}^2\Big) \le \frac{1}{4}M = \min\Big\{\frac{c_2^2c_3}{32C_*c_1^2c_p^2}, \frac{1}{64C_*^2}\Big\}. \tag{4.38}$$

Thus, by (4.35) and (4.38), we derive from (4.37) that

$$Y(t) \le \min\Big(\frac{c_2^2c_3}{8C_*c_1^2c_p^2}, \frac{1}{16C_*^2}\Big) = M, \quad \forall\, t\in[0,t_1]. \tag{4.39}$$

Also for all $t\in[0,t_1]$,

$$\|U\|^2 \le \frac{c_3}{4c_p}, \quad \|A_T^{\frac{1}{2}}T\| \le \frac{1}{4C_*}, \quad \|U\|^2\|A_T^{\frac{1}{2}}T\|^2 \le \frac{c_3\epsilon_*}{8C_*} = \frac{c_3}{64C_*^2c_p}, \tag{4.40}$$

so we have proved (4.32) and (4.34).

Moreover, by integrating (4.36), and using the smallness conditions (4.35) and (4.38) we arrive at

$$\int_0^{t_1}\|A_TT\|^2\mathrm{d}t \le \mathcal{Q}(Y(0), t_1) \tag{4.41}$$

for some generic constant $\mathcal{Q}(Y(0), t_1)$ depends on $Y(0)$ and t_1.

In addition, using the equations (4.1a) and (4.1b) on one hand, and estimates (4.34) and (4.41), on the other, it is not difficult to infer that $(\partial_t U, \partial_t T)$ is bounded in $L^2(0, t_1; L^2(\mathcal{M})^4)$.

For the reader's convenience, we summarize below the smallness conditions on G_0, F_T, $Y(0)$ identified from the analysis above, and the corresponding regularity in time and space they induce on (U, T).

For the system of equations (4.1a)–(4.1b) under the boundary conditions (4.4), there exists a positive constant $C_* > 0$ independent of the data G_0, F_T, T_0, U_0 and the time t_1, such that if the following smallness conditions are satisfied:

$$\sup_{t\in[0,t_1]} \|F_T(t)\|_{L^4}^4 \leq \frac{1}{4} \cdot \frac{c_3 \epsilon_*^2}{2C_*} = \frac{c_3}{8C_*} \cdot \frac{1}{64 C_*^2 c_p^2}, \quad \|G_0\|_{L^4}^4 + \sup_{t\in[0,t_1]} \|F_T(t)\|_{L^4}^2 \leq \frac{1}{8C_* c_p} \tag{4.42}$$

and

$$\begin{aligned} &\max\Big\{Y(0), \frac{C_*}{c_4}\Big(\sup_{t\in[0,t_1]} \|F_T(t)\|^2 + \epsilon_* \|G_0\|_{H^1}^2\Big)\Big\} \\ &\leq \min\Big\{\frac{2C_* c_2^2 c_3 \epsilon_*^2}{c_1^2}, \Big(\frac{1}{8C_*}\Big)^2\Big\} = \min\Big\{\frac{c_2^2 c_3}{32 C_* c_1^2 c_p^2}, \frac{1}{64 C_*^2}\Big\}, \end{aligned} \tag{4.43}$$

where

$$\begin{aligned} c_1 &= \min\Big\{\frac{\epsilon_0}{\delta}, \frac{\epsilon_0}{\mathcal{S}_0}\Big\}, \quad c_2 = \max\Big\{\frac{1}{\delta}, \frac{1}{\mathcal{S}_0^2}\Big\}, \\ c_3 &= \min\Big\{\frac{2c_2^2}{c_1^2}, c_1\Big\}, \quad c_4 = \min\Big(\frac{c_3}{4\max\big\{\frac{4c_2^2}{c_1^2}, c_1\big\}}, \frac{1}{2c_p}\Big) \end{aligned}$$

and

$$Y(0) := \frac{4c_2^2 \epsilon_*}{c_1^2} \|U_0\|^2 + c_1 \epsilon_* \|U_{0,x}\|^2 + c_1 \epsilon_* \|U_{0,y}\|^2 + \|A_T^{\frac{1}{2}} T_0\|^2 \tag{4.44}$$

with $\epsilon_* = \frac{1}{8C_* c_p}$ and c_p denoting the Poincaré inequality arising in (3.10), then the following a priori estimates are derived

$$(U, T) \text{ in } L^\infty(0, t_1; H^1(\mathcal{M})^4), \quad T \text{ in } L^2(0, t_1; H^2(\mathcal{M})), \tag{4.45}$$

$$(\partial_t U, \partial_t T) \text{ in } \; L^2(0, t_1; L^2(\mathcal{M})^4). \tag{4.46}$$

We are now in position to apply a similar analysis to the approximate solutions $U_k(t)$, $T_k(t)$ defined in the next subsection by the fractional step method. We derive hereafter similar estimates on $U_k(t)$, $T_k(t)$ with minor modifications on the smallness conditions and a different choice of the auxiliary parameter ϵ arising by application of the ϵ-Young inequality used to control coupling terms between the oceanic and SST components as in (4.25) and the key quantity $Y(t)$ defined in (4.29).

4.2 A priori estimates on the approximate solutions

In this section, we start building approximate solutions which satisfies the a priori estimates given in (4.11) under the smallness conditions (4.8)–(4.9). To build the approximate solutions, we first divide the time interval $[0,t_1]$ into N intervals of length $k = \frac{t_1}{N}$. For each k, we successively define $U_k^n(t)$, $T_k^n(t)$, $1 \le n \le N$, which are meant to be an approximation of $U|_{[(n-1)k,nk)}$, $T|_{[(n-1)k,nk)}$:

$$\begin{cases} \partial_t U_k^n + \mathscr{A} U_k^n + B_1 U_k^n + B_2 U_k^n = F_U(T_k^{n-1}), & (4.47a) \\ U_k^n((n-1)k^+) = U_k^{n-1}((n-1)k^-) & (4.47b) \end{cases}$$

and

$$\begin{cases} \quad \partial_t T_k^n - \epsilon_T \Delta T_k^n + (G_0 + G_1(U_k^n) + G_2(T_k^n))(\partial_x T_k^n + \partial_y T_k^n) + (G_0 + G_2(T_k^n))T_k^n \\ = F_T + F_T G_1(U_k^n) + F_T G_2(T_k^n), & (4.48a) \\ \quad T_k^n((n-1)k^+) = T_k^{n-1}((n-1)k^-). & (4.48b) \end{cases}$$

The initial and boundary conditions on the continuous approximate functions $U_k(t)$, $T_k(t)$ given by

$$U_k(t) = U_k^n(t), \quad T_k(t) = T_k^n(t) \quad \text{for } t \in [(n-1)k, nk),\ 1 \le n \le N,$$

are the same as those for the exact functions U and T (see (4.3)–(4.4)).

By analogy with what was done to the exact equations, we want to show that the approximate solutions $U_k(t)$, $T_k(t)$ satisfy the a priori estimates (4.11) under the smallness conditions given in (4.8)–(4.9). To begin with, we define $Y_k(t)$ like in Section 4.1,

$$Y_k(t) := \frac{4c_2^2\epsilon}{c_1^2}\|U_k\|^2 + c_1\epsilon\|U_{k,x}\|^2 + c_1\epsilon\|U_{k,y}\|^2 + \|A_T^{\frac{1}{2}} T_k\|^2$$

for a new ϵ to be determined later.

Inspired by (4.35), we assume that at $n = 0$, when $T_k^0 = T_0$, $U_k^0 = U_0$,

$$Y_k^0 = Y(0) := \frac{4c_2^2\epsilon}{c_1^2}\|U_0\|^2 + c_1\epsilon\|U_{0,x}\|^2 + c_1\epsilon\|U_{0,y}\|^2 + \|A_T^{\frac{1}{2}} T_0\|^2$$

satisfies

$$Y_k^0 \le \min\Big\{\frac{2C_* c_2^2 c_3 \epsilon^2}{c_1^2}, \frac{1}{64C_*^2}\Big\}. \tag{4.49}$$

We proceed then by induction to prove that under the smallness condition given in (4.8)–(4.9):

$$Y_k^n(t) \le \overline{M} =: \min\Big\{\frac{8C_* c_2^2 c_3 \epsilon^2}{c_1^2}, \frac{1}{16C_*^2}\Big\}, \quad 1 \le n \le N. \tag{4.50}$$

At $n = 0$, the inequality (4.50) is satisfied because of the assumption on Y_k^0 (see (4.49)). Suppose that for $0 \le l \le n-1$,

$$Y_k^l \le \overline{M} = \min\Big\{\frac{8C_* c_2^2 c_3 \epsilon^2}{c_1^2}, \frac{1}{16C_*^2}\Big\}. \tag{4.51}$$

At $n = l$, repeating the calculations we did in Section 4.1 for (4.22), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2}{c_1^2}\|U_k^n\|^2 + c_1\|U_{k,x}^n\|^2 + c_1\|U_{k,y}^n\|^2\Big) + c_3\|U_k^n\|_{H^1}^2 \le C_U(\|G_0\|_{H^1}^2 + \|A_T^{\frac{1}{2}} T_k^{n-1}\|^2), \tag{4.52}$$

where c_1, c_2, c_3 are the same as in Section 4.1 and C_U here is again a constant that only depends on $\mathcal{M}$ and ϵ_0, δ, $\mathcal{S}_0$ while is independent of k.

For the SST equation, after multiplying (4.48a) by $A_T T_k^n$ and integrating on $\mathcal{M}$, we can derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|A_T^{\frac{1}{2}}T_k^n\|^2 + \|A_T T_k^n\|^2 \\
&\le C_T\|F_T\|^2 + \frac{c_3\epsilon}{2}\|U_k^n\|_{H^1}^2 + \frac{C_T}{\epsilon}\|F_T\|_{L^4}^4\|U\|^2 \\
&\quad + C_T(\|G_0\|_{L^4}^4 + \|F_T\|_{L^4}^2)\|A_T^{\frac{1}{2}}T_k^n\|^2 \\
&\quad + C_T\|U_k^n\|^2\|A_T^{\frac{1}{2}}T_k^n\|^2\|U_k^n\|_{H^1}^2 + C_T\|A_T^{\frac{1}{2}}T_k^n\|\|A_T T_k^n\|^2.
\end{aligned} \tag{4.53}$$

The parameter ϵ here is the same as that arising in the expression of $Y_k(t)$ and the constant C_T again only depends on $\mathcal{M}$ and ϵ_0, δ, $\mathcal{S}_0$ but is independent of k.

Setting $C_* = \max(C_U, C_T)$, multiplying (4.52) by ϵ and adding it to (4.53), we deduce the following inequality after we application of the Poincaré inequality on the RHS:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2\epsilon}{c_1^2}\|U_k^n\|^2 + c_1\epsilon\|U_{k,x}^n\|^2 + c_1\epsilon\|U_{k,y}^n\|^2 + \|A_T^{\frac{1}{2}}T_k^n\|^2\Big) \\
&\quad + \Big(\frac{c_3\epsilon}{2} - \frac{C_*}{\epsilon}\|F_T\|_{L^4}^4 - C_*\|U_k^n\|^2\|A_T^{\frac{1}{2}}T_k^n\|^2\Big)\|U_k^n\|_{H^1}^2 \\
&\quad + (1 - C_*c_p(\|G_0\|_{L^4}^4 + \|F_T\|_{L^4}^2) - C_*\|A_T^{\frac{1}{2}}T_k^n\|)\|A_T T_k^n\|^2 \\
&\le C_*(\|F_T\|^2 + \epsilon\|G_0\|_{H^1}^2) + C_*\epsilon\|A_T^{\frac{1}{2}}T_k^{n-1}\|^2.
\end{aligned} \tag{4.54}$$

As before, we want the coefficients of $\|U_k^n\|_{H^1}^2$ and $\|A_T T_k^n\|^2$ in the LHS of (4.54) to stay positive, so we required the smallness conditions (4.8) in Theorem 4.1 so that

$$\sup_{t\in[0,t_1]}\frac{C_*}{\epsilon}\|F_T(t)\|_{L^4}^4 \le \frac{1}{4}\cdot\frac{c_3\epsilon}{2}, \quad C_*c_p(\|G_0\|_{L^4}^4 + \sup_{t\in[0,t_1]}\|F_T(t)\|_{L^4}^2) \le \frac{1}{4}. \tag{4.55}$$

Because of the assumption (4.51) on Y_k^{l-1} and (4.47b)–(4.48b), we have

$$\|U_k^n\|^2\|A_T^{\frac{1}{2}}T_k^n\|^2 \le \frac{1}{4}\cdot\frac{c_3\epsilon}{2C_*}, \quad \|A_T^{\frac{1}{2}}T_k^n\| \le \frac{1}{4C_*} \tag{4.56}$$

at $t = (n-1)k$. Then as long as (4.56) holds, the inequality (4.54) implies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{4c_2^2\epsilon}{c_1^2}\|U_k^n\|^2 + c_1\epsilon\|U_{k,x}^n\|^2 + c_1\epsilon\|U_{k,y}^n\|^2 + \|A_T^{\frac{1}{2}}T_k^n\|^2\Big) \\
&\quad + \frac{c_3\epsilon}{4}\|U_k^n\|_{H^1}^2 + \frac{1}{2}\|A_T T_k^n\|^2 \\
&\le C_*(\|F_T\|^2 + \epsilon\|G_0\|_{H^1}^2) + C_*\epsilon\|A_T^{\frac{1}{2}}T_k^{n-1}\|^2
\end{aligned} \tag{4.57}$$

with

$$c_4 := \min\Big\{\frac{c_3}{4\max\big\{\frac{4c_2^2}{c_1^2}, c_1\big\}}, \frac{1}{2c_p}\Big\},$$

as in Section 4.1.

We deduce from (4.57) that

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_k^n(t) + c_4 Y_k^n(t) \le C_*(\|F_T\|^2 + \epsilon\|G_0\|_{H^1}^2) + C_*\epsilon\|A_T^{\frac{1}{2}}T_k^{n-1}\|^2 \quad \text{for } t \in [(n-1)k, nk). \tag{4.58}$$

Actually (4.58) holds for all Y_k^l, where $1 \le l \le n$, so we rewrite (4.58) in terms of $Y_k(t)$ and $T_k(t-k)$ for $t \in [0, nk)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y_k(t) + c_4 Y_k(t) &\le C_*(\|F_T\|^2 + \epsilon\|G_0\|_{H^1}^2) + C_*\epsilon\|A_T^{\frac{1}{2}}T_k(t-k)\|^2 \\ &= \mathrm{I}_1 + \mathrm{I}_2.\end{aligned} \tag{4.59}$$

Integrating (4.59), we get

$$Y_k(t) \le \exp(-c_4 t)Y(0) + \frac{\mathrm{I}_1}{c_4} + \frac{\mathrm{I}_2}{c_4} \quad \text{for } t \in [0, nk). \tag{4.60}$$

Using the smallness condition (4.9), the first two terms in the RHS of (4.60) can be bounded by $\frac{1}{4}\overline{M}$, where the expression of $\overline{M}$ is given in (4.50). For the third term in the RHS of (4.60), we use the assumption on $Y_k^l, 1 \le l \le n-1$. Namely, by (4.51) we have

$$\|A_T^{\frac{1}{2}}T_k(t-k)\|^2 \le \overline{M} = \min\Big\{\frac{8C_* c_2^2 c_3 \epsilon^2}{c_1^2}, \frac{1}{16C_*^2}\Big\} \quad \text{for } t \in [0, nk). \tag{4.61}$$

Now we set $\epsilon = \overline{\epsilon}_* = \frac{c_4}{4C_*}$ so that $\frac{C_*\epsilon}{c_4} = \frac{1}{4}$. Then

$$\frac{\mathrm{I}_2}{c_4} \le \frac{1}{4}\overline{M} = \min\Big\{\frac{2C_* c_2^2 c_3 \overline{\epsilon}_*^2}{c_1^2}, \frac{1}{64C_*^2}\Big\},$$

and therefore

$$Y_k(t) \le \min\Big\{\frac{8C_* c_2^2 c_3 \overline{\epsilon}_*^2}{c_1^2}, \frac{1}{16C_*^2}\Big\}, \quad \forall t \in [0, nk).$$

Restricting $Y_k(t)$ to the interval $[(n-1)k, nk)$ we obtain

$$Y_k^n(t) \le \min\Big\{\frac{8C_* c_2^2 c_3 \overline{\epsilon}_*^2}{c_1^2}, \frac{1}{16C_*^2}\Big\}, \tag{4.62}$$

which completes the induction. We have thus shown that

$$Y_k^n(t) \le \min\Big(\frac{8C_* c_2^2 c_3 \overline{\epsilon}_*}{c_1^2}, \frac{1}{16C_*^2}\Big), \quad 1 \le n \le N, \tag{4.63}$$

and (4.56), namely that for $\forall t \in [0, t_1]$, it holds:

$$\|U_k^n\|^2\|A_T^{\frac{1}{2}}T_k^n\|^2 \le \frac{c_3\overline{\epsilon}_*}{8C_*} = \frac{c_3 c_4}{32C_*^2}, \quad \|A_T^{\frac{1}{2}}T_k^n\| \le \frac{1}{4C_*}.$$

By noting that the bounds in (4.56) and (4.63) are independent of k, we have thus proved that the sequence $(U_k(t), T_k(t))$ is bounded in $L^\infty(0, t_1; H^1(\mathcal{M})^4)$.

Now we integrate (4.57) from $t = (n-1)k$ to $t = nk$ and sum the resulting inequalities from $n = 1$ to N, to arrive at

$$\int_0^{t_1} \|A_T T_k\|^2 \mathrm{d}t \le \mathcal{Q}(Y(0), t_1), \tag{4.64}$$

where $\mathcal{Q}(Y(0), t_1)$ again is a generic constant that depends on $Y(0)$ and t_1 but not on k .

Finally, by using equations (4.47)–(4.48), on one hand and the estimates (4.63) and (4.64), one the other, we deduce that

$$\|\partial_t U_k\|_{L^2(0,t_1;L^2(\mathcal{M})^3)} + \|\partial_t T_k\|_{L^2(0,t_1;L^2(\mathcal{M}))} \leq \mathcal{Q}(Y(0), t_1). \tag{4.65}$$

In other words, the sequence $(\partial_t U_k(t),\, \partial_t T_k(t))$ belongs to a bounded set of $L^2(0, t_1; L^2(\mathcal{M})^4)$.

Remark 4.1 The difference between the bound we have on $Y(t)$ in Section 4.1 and the bound we have on $Y_k(t)$ as in (4.63) lies in the different choice of ϵ. In Section 4.1, $\epsilon = \epsilon_* = \frac{1}{4C_*} \cdot \frac{1}{2c_p}$, while in $Y_k(t)$, ϵ is set to be $\overline{\epsilon}_* = \frac{c_4}{4C_*}$. Recall that

$$c_4 := \min\Big\{\frac{c_3}{4\max\big\{\frac{4c_2^2}{c_1^2}, c_1\big\}}, \frac{1}{2c_p}\Big\}.$$

As a consequence, the bounds obtained on the approximate solutions U_k, T_k are not larger than those obtained for the exact solutions U, T. The smallness conditions we need to bound U_k, T_k are nevertheless smaller as $\overline{\epsilon}_* \leq \epsilon_*$.

By now we have shown that the approximate solutions $(U_k(t),\, T_k(t))$ satisfies the a priori estimates (4.11)–(4.12), and we are in position to apply the Aubin-Lions compactness lemma (see [2, 37]) to extract convergent subsequences from $U_k(t)$ and $T_k(t)$. But in order pass to the limit $k \to 0$, we need one more estimate on the difference between $T_k(t)$ and $T_k(t-k)$. This is the purpose of the next lemma.

Lemma 4.1 *The following estimate holds:*

$$\lim_{k\to 0} \int_k^{t_1} \|T_k(t) - T_k(t-k)\|^2 \mathrm{d}t = 0. \tag{4.66}$$

To prove Lemma 4.1, we rewrite (4.47)–(4.48) in terms of U_k, T_k:

$$\begin{cases} \partial_t U_k + \mathscr{A} U_k + B_1 U_k + B_2 U_k = F_U(T_k(t-k)), & (4.67\text{a}) \\ U_k(0) = U_0, & (4.67\text{b}) \end{cases}$$

$$\begin{cases} \quad \partial_t T_k - \epsilon_T \Delta T_k + (G_0 + G_1(U_k) \\ \quad +G_2(T_k))(\partial_x T_k + \partial_y T_k) + (G_0 + G_2(T_k))T_k \\ = F_T + F_T G_1(U_k) + F_T G_2(T_k), & (4.68\text{a}) \\ \quad T_k(0) = T_0. & (4.68\text{b}) \end{cases}$$

We multiply (4.68a) by $T^b \in H^1(\mathcal{M})$ and integrate on $\mathcal{M}$ to deduce

$$\begin{aligned} \langle \partial_t T_k, T^b\rangle \leq |&(\epsilon_T \Delta T_K, T^b) - ((G_0 + G_1(U_k) + G_2(T_k))(\partial_x T_k + \partial_y T_k), T^b) \\ &- ((G_0 + G_2(T_k))T_k, T^b) + (F_T, T^b) + (F_T G_1(U_k), T^b) + (F_T G_2(T), T^b)|. \end{aligned} \tag{4.69}$$

We see that

$$\text{RHS of (4.69)} \leq [\|A_T T_k\|^2 + (\|G_0\|_{L^4} + \|U_k\|_{H^1} + \|T_k\|_{H^1})\|T_k\|_{H^1}$$

$$
\begin{aligned}
&+ (\|G_0\|_{L^4} + \|T_k\|_{H^1})\|T_k\|_{H^1} + \|F_T\| \\
&+ \|F_T\|_{L^4}\|U_k\|_{H^1} + \|F_T\|_{L^4}\|T_k\|_{H^1}]\|T^b\|_{H^1} \\
\le\, & \mathcal{Q}(Y(0), t_1)\|T^b\|_{H^1}. \qquad (4.70)
\end{aligned}
$$

As before, here $\mathcal{Q}(Y(0), t_1)$ is some generic constant depending only on $Y(0)$ and t_1.

Now for arbitrary fixed t in $[k,\, t_1]$, we integrate both sides of (4.69) from $t-k$ to t

$$
\begin{aligned}
&\langle T_k(t) - T_k(t-k), T^b\rangle \\
\le\, & \int_{t-k}^{t} [\|A_T T_k(s)\|^2 + (\|G_0\|_{L^4} + \|U_k(s)\|_{H^1} + \|T_k(s)\|_{H^1})\|T_k(s)\|_{H^1} \\
&+ (\|G_0\|_{L^4} + \|T_k(s)\|_{H^1})\|T_k(s)\|_{H^1} + \|F_T\| \\
&+ \|F_T\|_{L^4}\|U_k(s)\|_{H^1} + \|F_T\|_{L^4}\|T_k(s)\|_{H^1}]\|T^b\|_{H^1}\mathrm{d}s \\
\le\, & \Big(\int_{t-k}^{t} 1\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_{t-k}^{t} \mathcal{Q}(Y(0), t_1)^2\mathrm{d}s\Big)^{\frac{1}{2}}\|T^b\|_{H^1} \\
\le\, & \sqrt{k}\mathcal{Q}(Y(0), t_1)\|T^b\|_{H^1}. \qquad (4.71)
\end{aligned}
$$

Now in (4.71) we substitute $T^b = T_k(t) - T_k(t-k)$ and integrate on both sides with respect to t from $t = k$ to $t = t_1$ and use the fact that $T_k(t)$ is uniformly bounded in time by some constant $\mathcal{Q}(Y(0), t_1)$, we infer that

$$
\begin{aligned}
&\int_{k}^{t_1} \|T_k(t) - T_k(t-k)\|^2\mathrm{d}t \\
\le\, & \int_{k}^{t_1} \sqrt{k}\mathcal{Q}(Y(0), t_1)\|T_k(t) - T_k(t-k)\|\mathrm{d}t \\
\le\, & \sqrt{k}\Big(\int_{t-k}^{t} 1\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_{t-k}^{t} (\mathcal{Q}(Y(0), t_1)\|T_k(t) - T_k(t-k)\|)^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
\le\, & \sqrt{k}\sqrt{t_1}\mathcal{Q}(Y(0), t_1). \qquad (4.72)
\end{aligned}
$$

Passing to the limit $k \to 0$ in (4.72) gives the desired estimate (4.66) and Lemma 4.1 is proved.

4.3 Passage to the limit $k \to 0$ and existence of solutions to the original problem

Since the estimates (4.63)–(4.65) are independent of k, we infer the existence of a couple (U, T) such that

$$
U \in L^\infty(0, t_1; H^1(\mathcal{M})^3), \qquad \partial_t U \in L^2(0, t_1; L^2(\mathcal{M})^3), \qquad (4.73)
$$

$$
T \in L^\infty(0, t_1; H^1(\mathcal{M})) \cap L^2(0, t_1; H^2(\mathcal{M})), \quad \partial_t T \in L^2(0, t_1; L^2(\mathcal{M})) \qquad (4.74)
$$

for which the following convergences up to a subsequence (not relabeled), hold:

(i) $U_k \overset{*}{\rightharpoonup} U$ weakly-$*$ in $L^\infty(0, t_1; H^1(\mathcal{M})^3)$ and $\partial_t U_k \rightharpoonup \partial_t U$ weakly in $L^2(0, t_1; L^2(\mathcal{M})^3)$, as a consequence (see e.g. [37]), $U_k \to U$ strongly in $L^2(0, t_1; H^{\frac{1}{2}}(\mathcal{M})^3)$.

(ii) $T_k \overset{*}{\rightharpoonup} T$ weakly-$*$ in $L^\infty(0, t_1; H^1(\mathcal{M}))$ and weakly in $L^2(0, t_1; H^2(\mathcal{M}))$, and $\partial_t T_k \rightharpoonup \partial_t T$ weakly in $L^2(0, t_1; L^2(\mathcal{M}))$. Therefore, $T_k \to T$ strongly in $L^2(0, t_1; H^1(\mathcal{M}))$.

Besides the convergence on $T_k(t)$, we denote $T_k(t-k)$ by $T_k^1(t)$. For $T_k^1(t)$, there exists some function T^1 in the same space (4.74) as T, for which an analogue of convergence (ii) holds with T_k (resp. T) being replaced by T_k^1 (resp. T^1). Thanks to the Lemma 4.1, we see that in fact $T = T^1$ and T_k^1 converges to the same limit as T_k.

By interpolation (see e.g. [38]), we also have

$$U \in \mathcal{C}([0,t_1]; L^2(\mathcal{M})^3), \quad T \in \mathcal{C}([0,t_1]; H^1(\mathcal{M}))$$

and hence U and T satisfy the initial conditions (4.3).

By the linearity of G_1 and G_2, we have

$$\begin{aligned} G_1(U_k) &\to G_1(U) \quad \text{strongly in } L^2(0,t_1;L^2(\mathcal{M})),\\ G_2(T_k) &\to G_2(T) \quad \text{strongly in } L^2(0,t_1;H^1(\mathcal{M})). \end{aligned}$$

Let $\widetilde{U}$ be in $\mathcal{D}(\mathcal{M})^3$, $\widetilde{T}$ be in $\mathcal{D}(\mathcal{M})$ and ψ, φ be in $\mathcal{D}(0,t_1)$. We then take the L^2-inner product of (4.67a) and (4.68a) with $\widetilde{U}\psi$ and $\widetilde{T}\varphi$ respectively over $\mathcal{M} \times (0,t_1)$, which gives

$$\begin{aligned} &\int_0^{t_1} (\langle \partial_t U_k, \widetilde{U}\rangle_{L^2} + \langle \mathscr{A} U_k, \widetilde{U}\rangle_{L^2} + \langle B_1 U_k + B_2 U_k, \widetilde{U}\rangle_{L^2})\psi(t)\mathrm{d}t\\ &= \int_0^{t_1} \langle (G_0 + G_2(T_k^1), 0, 0)^t, \widetilde{U}\rangle_{L^2}\psi(t)\mathrm{d}t \end{aligned} \tag{4.75}$$

and

$$\begin{aligned} &\int_0^{t_1} (\langle \partial_t T_k, \widetilde{T}\rangle_{L^2} - \epsilon_T \langle \Delta T_k, \widetilde{T}\rangle_{L^2} + \langle G_0(\partial_x T_k + \partial_y T_k), \widetilde{T}\rangle_{L^2}\\ &+ \langle (G_1(U_k) + G_2(T_k))(\partial_x T_k + \partial_y T_k), \widetilde{T}\rangle_{L^2} + \langle (G_0 + G_2(T_k))T_k, \widetilde{T}\rangle_{L^2})\varphi(t)\mathrm{d}t\\ &= \int_0^{t_1} (\langle F_T, \widetilde{T}\rangle_{L^2} + \langle F_T G_1(U_k), \widetilde{T}\rangle_{L^2} + \langle F_T G_2(T_k), \widetilde{T}\rangle_{L^2})\varphi(t)\mathrm{d}t. \end{aligned} \tag{4.76}$$

We now pass to the limit as $k \to 0$ in (4.75)–(4.76). The linear terms in (4.75)–(4.76) converge to their corresponding limits in a straightforward fashion due to the aforementioned convergences. Regarding the nonlinear terms, we first deal with the term

$$\int_0^{t_1} \langle G_1(U_k)\partial_x T_k, \widetilde{T}\rangle_{L^2}\psi(t)\mathrm{d}t.$$

By noting that this term can be rewritten as

$$\int_0^{t_1} \langle (G_1(U_k) - G_1(U))\partial_x T_k, \widetilde{T}\rangle_{L^2}\psi(t)\mathrm{d}t + \int_0^{t_1} \langle G_1(U)\partial_x T_k, \widetilde{T}\rangle_{L^2}\psi(t)\mathrm{d}t, \tag{4.77}$$

we obtain from what precedes that

$$\int_0^{t_1} \langle G_1(U_k)\partial_x T_k, \widetilde{T}\rangle_{L^2}\psi(t)\mathrm{d}t \to \int_0^{t_1} \langle G_1(U)\partial_x T, \widetilde{T}\rangle_{L^2}\psi(t)\mathrm{d}t, \quad \text{as } k \to 0. \tag{4.78}$$

Treating the other nonlinear terms in a similar way, we are able to conclude that (4.1) holds (at least) in the sense of distributions, for the corresponding limits. Since (U_k, T_k) belongs to $H^1(\mathcal{M}) \times H^2(\mathcal{M})$, as time evolves, the boundary conditions in (4.4) are well-defined and by passing to the limit, $k \to 0$, we infer that U and T satisfy the boundary conditions in (4.4). We have thus proved the existence of solutions of the coupled system (4.1a)–(4.4) under the assumptions (4.5)–(4.9). We turn next to the study of the uniqueness.

4.4 The uniqueness

Let (U_1, T_1) and (U_2, T_2) be two solutions of the coupled SWE-SST equations (4.1a)–(4.4) which satisfy (4.11), then the differences $U = U_1 - U_2$ and $T = T_1 - T_2$ satisfy the equations

$$\begin{cases} U_t + \mathscr{A}U + B_1 U + B_2 U = F_U, \\ \partial_t T - \epsilon_T \Delta T + G_0(\partial_x T + \partial_y T) + (G_1(U_2) + G_2(T_2))(\partial_x T + \partial_y T) \\ +(G_1(U) + G_2(T))(\partial_x T_1 + \partial_y T_1) + G_2(T)T_1 + G_2(T_2)T + G_0 T \\ = F_T G_1(U) + F_T G_2(T), \end{cases} \tag{4.79}$$

where we have used the linear dependence of G_1 and G_2. Note that in (4.79), $F_U = (G_2(T), 0, 0)^{\text{tr}}$, and that the corresponding initial data are zero, i.e.,

$$U(0, x, y) = 0, \quad T(0, x, y) = 0. \tag{4.80}$$

We take the L^2-inner product of (4.79) with (U, T) and estimate term by term as follows. By integration by parts and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\langle \mathscr{A}U, U\rangle \geq 0, \quad \langle B_1 U, U\rangle \geq c_1 \|U\|^2, \quad \langle B_2 U, U\rangle = 0, \\ &\langle F_U, U\rangle_{L^2} \leq \|F_U\|_{L^2} \|U\|_{L^2} \leq C\|T\|^2 + \|U\|^2. \end{aligned} \tag{4.81}$$

The use of the Ladyzhenskaya's and Young's inequalities give

$$\begin{aligned} &\langle G_0(\partial_x T + \partial_y T), T\rangle_{L^2} \\ &\leq \|G_0\|_{L^4} \|T\|_{L^4} \|T\|_{H^1} \\ &\leq C\|G_0\|_{L^4} \|T\|^{\frac{1}{2}} \|T\|^{\frac{3}{2}}_{H^1} \\ &\leq C\|G_0\|^4_{L^4} \|T\|^2 + \frac{\epsilon_T}{16}\|T\|^2_{H^1} \end{aligned} \tag{4.82}$$

and

$$\begin{aligned} &\int_{\mathcal{M}} (G_1(U_2) + G_2(T_2))(\partial_x T + \partial_y T) T \mathrm{d}x\mathrm{d}y \\ &\leq C(\|U_2\|_{L^4} + \|T_2\|_{L^4})\|T\|_{H^1}\|T\|_{L^4} \\ &\leq C(\|U_2\|_{H^1} + \|T_2\|_{H^1})\|T\|^{\frac{3}{2}}_{H^1}\|T\|^{\frac{1}{2}} \\ &\leq C(\|U_2\|^4_{H^1} + \|T_2\|^4_{H^1})\|T\|^2 + \frac{\epsilon_T}{16}\|T\|^2_{H^1} \end{aligned} \tag{4.83}$$

for which we have used (3.18).

By using (3.18) again and the Young inequality, we arrive at

$$\begin{aligned} &\int_{\mathcal{M}} (G_1(U) + G_2(T))(\partial_x T_1 + \partial_y T_1) T \mathrm{d}x\mathrm{d}y \\ &\leq C(\|U\| + \|T\|)\|\nabla T_1\|_{L^4}\|T\|_{L^4} \\ &\leq C(\|U\| + \|T\|)\|T_1\|_{H^2}\|T\|_{H^1} \\ &\leq C\|T_1\|^2_{H^2}(\|U\|^2 + \|T\|^2) + \frac{\epsilon_T}{16}\|T\|^2_{H^1} \end{aligned} \tag{4.84}$$

and

$$\langle G_2(T)T_1 + G_2(T_2)T, T\rangle_{L^2}$$

$$\begin{aligned}&\le C\|T_1\|_{L^4}\|T\|\|T\|_{L^4} + C\|T_2\|_{L^4}\|T\|\|T\|_{L^4}\\&\le C(\|T_1\|_{H^1}^2 + \|T_2\|_{H^1}^2)\|T\|^2 + \frac{\epsilon_T}{16}\|T\|_{H^1}^2.\end{aligned} \tag{4.85}$$

The remaining terms are estimated as follows

$$\langle G_0 T, T\rangle_{L^2} \le \|G_0\|_{L^4}\|T\|\|T\|_{L^4} \le C\|G_0\|_{L^4}^2\|T\|^2 + \frac{\epsilon_T}{16}\|T\|_{H^1}^2 \tag{4.86}$$

and similarly

$$\begin{aligned}&\langle F_T G_1(U), T\rangle_{L^2} \le C\|F_T\|_{L^4}^2\|U\|^2 + \frac{\epsilon_T}{16}\|T\|_{H^1}^2,\\&\langle F_T G_2(T), T\rangle_{L^2} \le C\|F_T\|_{L^4}^2\|T\|^2 + \frac{\epsilon_T}{16}\|T\|_{H^1}^2.\end{aligned} \tag{4.87}$$

Collecting all these estimates, we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + 2c_1\|U\|^2 \le C\|T\|^2 + \|U\|^2,\\&\frac{\mathrm{d}}{\mathrm{d}t}\|T\|^2 + \epsilon_T\|T\|_{H^1}^2 \le C(\|G_0\|_{L^4}^4 + \|T_1\|_{H^2}^2 + \|U_2\|_{H^1}^4 + \|T_2\|_{H^1}^4 + \|G_0\|_{L^4}^2\\&\qquad + \|T_1\|_{H^1}^2 + \|T_2\|_{H^1}^2 + \|F_T\|_{L^4}^2)(\|U\|^2 + \|T\|^2).\end{aligned} \tag{4.88}$$

Using (4.11), the smallness assumption (4.8), and the vanishing initial data (4.80), we then infer from (4.88) and the Gronwall's lemma that for all t in $[0, t_1]$,

$$U(t) \equiv 0, \quad T(t) \equiv 0.$$

We have thus proved the uniqueness of solutions. We are now ready to state our global well-posedness result concerning the coupled system (4.1a)–(4.4).

Theorem 4.1 *We assume that condition* (C1) (*see* (3.6)) *and assumptions* (4.5)–(4.7) *hold. There exists a positive constant* $C_* > 0$ *independent of the data* G_0, F_T, T_0, U_0 *and the time* t_1*, such that if the smallness conditions* (4.8)–(4.9) *are satisfied, then there exists a unique solution* (U, T) *to the coupled system* (4.1a)–(4.4)*, satisfying*

$$(U, T) \in L^\infty(0, t_1; H^1(\mathcal{M})^4), \quad T \in L^2(0, t_1; H^2(\mathcal{M})) \tag{4.89}$$

and

$$(\partial_t U, \partial_t T) \in L^2(0, t_1; L^2(\mathcal{M})^4). \tag{4.90}$$

Remark 4.2 Let us recall that F_T in (4.1b) involves terms like $G_0(x, y)$ (see discussion before (3.5) in Section 3.1) and thus terms of the form $\delta_s \tau_z^x(x)\tau_m^y(y)$ (see Table 3.1 and (1.6)–(1.7)). The smallness conditions (4.8)–(4.9) express thus in particular a sufficient condition (not necessary) for the key physical parameters of the JN model, δ_s and δ (see Table 1), to satisfy for the existence and uniqueness of global solutions to the JN model. In other words, the smallness conditions (4.8)–(4.9) involve in particular key physical parameters of the JN model such as those that control the travel time of the equatorial waves and the strength of feedbacks due to vertical-shear currents and upwelling (see Table 1); those are central mechanisms in ENSO dynamics.

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