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Convergences of Random Variables Under Sublinear Expectations*

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Abstract In this note, the authors survey the existing convergence results for random variables under sublinear expectations, and prove some new results. Concretely, under the assumption that the sublinear expectation has the monotone continuity property, the authors prove that convergence in capacity is stronger than convergence in distribution, and give some equivalent characterizations of convergence in distribution. In addition, they give a dominated convergence theorem under sublinear expectations, which may have its own interest.

Keywords Sublinear expectation, Capacity, The dominated convergence theorem
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1 Introduction

It is well known that limit theory plays an important role in probability theory and statistics. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X, X_n, n \geq 1\}$ be a sequence of random variables. Then we have the following convergences:

(1) $\{X_n, n \geq 1\}$ is said to almost surely converge to X, if there exists a set $N \in \mathcal{F}$ such that $P(N) = 0$ and $\forall \omega \in \Omega \backslash N$, $\lim\limits_{n\to\infty} X_n(\omega) = X(\omega)$, which is denoted by $X_n \xrightarrow{\text{a.s.}} X$ or $X_n \to X$ a.s.

(2) $\{X_n, n \geq 1\}$ is said to converge to X in probability, if for any $\varepsilon > 0$, $\lim\limits_{n\to\infty} P(\{|X_n - X| \geq \varepsilon\}) = 0$, which is denoted by $X_n \xrightarrow{P} X$.

(3) $\{X_n, n \geq 1\}$ is said to L^p converge to X $(p > 0)$, if $\lim\limits_{n\to\infty} E[|X_n - X|^p] = 0$, which is denoted by $X_n \xrightarrow{L^p} X$.

(4) $\{X_n, n \geq 1\}$ is said to converge to X in distribution, if for any bounded continuous function f, $\lim\limits_{n\to\infty} E[f(X_n)] = E[f(X)]$, which is denoted by $X_n \xrightarrow{d} X$.

(5) $\{X_n, n \geq 1\}$ is said to completely converge to X, if for any $\varepsilon > 0$, $\sum\limits_{n=1}^{\infty} P(\{|X_n - X| \geq \varepsilon\}) < \infty$, which is denoted by $X_n \xrightarrow{\text{c.c.}} X$ (see [4]).

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(6) $\{X_n, n \geq 1\}$ is said to s-L^r converge to X ($r > 0$), if $\sum_{n=1}^{\infty} E[|X_n - X|^r] < \infty$, which is denoted by $X_n \xrightarrow{s\text{-}L^r} X$ (see [5, Definition 1.4]).

Then we have

$$X_n \xrightarrow{s\text{-}L^r} X \quad \Rightarrow \quad X_n \xrightarrow{\text{c.c.}} X \quad \Rightarrow \quad X_n \xrightarrow{\text{a.s.}} X \quad \Rightarrow \quad X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{d} X,$$

$$\Uparrow$$

$$X_n \xrightarrow{L^p} X$$

and

(a) if $X_n \xrightarrow{P} X$, then there exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$ as $k \to \infty$;

(b) if $X_n \xrightarrow{d} C$, where C is a constant, then $X_n \xrightarrow{P} C$;

(c) if $X_n \xrightarrow{d} X$, then by Skorokhod's theorem, there exists a sequence of random variables $\{Y, Y_n, n \geq 1\}$ such that for any $n \geq 1$, X_n and Y_n have the same distribution, X and Y have the same distribution, and $Y_n \xrightarrow{\text{a.s.}} Y$.

Recently, motivated by the risk measures, superhedge pricing and modeling uncertainty in finance, Peng [6–12] initiated the notion of independent and identically distributed (IID for short) random variables under sublinear expectations, proved the weak law of large numbers and the central limit theorems, defined the G-expectations, G-Brownian motions and built Itô's type stochastic calculus.

In this note, we will survey the existing convergence results for random variables under sublinear expectations, and prove some new results. Concretely, under the assumption that the sublinear expectation has the monotone continuity property, we will prove that convergence in capacity is stronger than convergence in distribution, and give some equivalent characterizations of convergence in distribution. In addition, a dominated convergence theorem under sublinear expectations is given, which may have its own interest.

2 Sublinear Expectations

In this section, we present some basic settings about sublinear expectations. Please refer to Peng [6–12] for more details.

Let Ω be a given set and $\mathcal{H}$ be a linear space of real functions defined on Ω such that for any constant number $c, c \in \mathcal{H}$; if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$; if $X_1, \cdots, X_n \in \mathcal{H}$, then for any $\varphi \in C_{l,\text{Lip}}(\mathbb{R}^n)$, $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$, where $C_{l,\text{Lip}}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x,\ y \in \mathbb{R}^n$$

for some $C > 0, m \in \mathbb{N}$ depending on φ.

Definition 2.1 *A sublinear expectation $\widehat{E}$ on $\mathcal{H}$ is a functional $\widehat{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties*:

(a) *Monotonicity*: $\widehat{E}[X] \geq \widehat{E}[Y]$, *if* $X \geq Y$.

(b) *Constant preserving*: $\widehat{E}[c] = c, \forall c \in \mathbb{R}$.

(c) *Sub-additivity*: $\widehat{E}[X+Y] \le \widehat{E}[X] + \widehat{E}[Y]$.

(d) *Positive homogeneity*: $\widehat{E}[\lambda X] = \lambda \widehat{E}[X]$, $\forall \lambda \ge 0$.

The triple $(\Omega, \mathcal{H}, \widehat{E})$ *is called a sublinear expectation space.*

For simplicity, we assume that Ω is a complete separable metric space, and use $\mathcal{B}(\Omega)$ to denote the Borel σ-algebra of Ω. Further, we assume that there exists a family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\widehat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \forall X \in \mathcal{H}.$$

Suppose that for any $A \in \mathcal{B}(\Omega)$, $I_A \in \mathcal{H}$. Define

$$V(A) := \widehat{E}[I_A] = \sup_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{B}(\Omega).$$

Obviously, $V(\emptyset) = 0, V(\Omega) = 1$.

Theorem 2.1 (see [3, Theorem 1] or [12, Theorem VI.1.1]) *The set function* $V(\cdot)$ *is a Choquet capacity, i.e.,*

(1) $0 \le V(A) \le 1, \forall A \in \mathcal{B}(\Omega)$.

(2) *If* $A \subset B$, *then* $V(A) \le V(B)$.

(3) *If* $(A_n)_{n=1}^{\infty}$ *is a sequence in* $\mathcal{B}(\Omega)$, *then* $V(\cup A_n) \le \sum_n V(A_n)$.

(4) *If* $(A_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathcal{B}(\Omega)$: $A_n \uparrow A = \cup A_n$, *then* $V(A) = \lim_{n\to\infty} V(A_n)$.

Definition 2.2 (see [12, Definition VI.1.3]) *We use the standard capacity-related vocabulary: a set* A *is polar if* $V(A) = 0$ *and a property holds "quasi-surely" (q.s.) if it holds outside a polar set.*

Definition 2.3 (see [2, Definition 3.1]) *For* $p \in [1, \infty)$, *the map*

$$\|\cdot\|_p : X \mapsto (\widehat{E}[|X|^p])^{\frac{1}{p}}$$

forms a seminorm on $\mathcal{H}$. *Define the space* $\mathcal{L}^p(\mathcal{F})$ *as the completion under* $\|\cdot\|_p$ *of the set*

$$\{X \in \mathcal{H} : \|X\|_p < \infty\}$$

and then $L^p(\mathcal{F})$ *as the equivalent classes of* $\mathcal{L}^p$ *modulo equality in* $\|\cdot\|_p$.

Definition 2.4 (see [2, Definition 3.2]) *Consider* $K \subset L^1$. K *is said to be uniformly integrable if* $\widehat{E}[I_{\{|X| \ge c\}}|X|]$ *converges to* 0 *uniformly in* $X \in K$ *as* $c \to \infty$.

Definition 2.5 (see [2, Definition 3.3]) *Let* L_b^p *be the completion of the set of bounded functions* $X \in \mathcal{H}$, *under the norm* $\|\cdot\|_p$. *Note that* $L_b^p \subset L^p$.

Lemma 2.1 (see [3, Proposition 15] or [2, Lemma 3.4]) *For each* $p \ge 1$,

$$L_b^p = \{X \in L^p : \lim_{n\to\infty} \widehat{E}[|X|^p I_{\{|X|>n\}}] = 0\}.$$

Theorem 2.2 (Monotone Convergence Theorem) (see [2, Theorem 2.2]) *Let* $\{X_n, n \ge 1\}$ *be a sequence in* $\mathcal{H}$ *and lower bounded. If* $X_n \uparrow X \in \mathcal{H}$, *then* $\widehat{E}[X_n] \uparrow \widehat{E}[X]$.

3 Convergences Under Sublinear Expectations

Let $(\Omega, \mathcal{H}, \widehat{E})$ be a sublinear expectation space introduced in Section 2. We further suppose that for any $X \in \mathcal{H}$ and any bounded continuous function f, $f(X) \in \mathcal{H}$. In this section, we consider the convergences of random variables under sublinear expectations. Let $\{X, X_n, n \geq 1\}$ be a sequence of random variables in $\mathcal{H}$. We have the following convergences:

(1) $\{X_n, n \geq 1\}$ is said to quasi-surely converge to X, if there exists a set $N \subset \Omega$ such that $\widehat{E}[I_N] = 0$ and $\forall \omega \in \Omega \backslash N$, $\lim_{n\to\infty} X_n(\omega) = X(\omega)$, which is denoted by $X_n \xrightarrow{\text{q.s.}} X$ or $X_n \to X$ q.s.

(2) $\{X_n, n \geq 1\}$ is said to converge to X in capacity, if for any $\varepsilon > 0$, $\lim_{n\to\infty} V(\{|X_n - X| \geq \varepsilon\}) = 0$, which is denoted by $X_n \xrightarrow{V} X$.

(3) $\{X_n, n \geq 1\}$ is said to L^p converge to X $(p > 0)$, if $\lim_{n\to\infty} \widehat{E}[|X_n - X|^p] = 0$, which is denoted by $X_n \xrightarrow{L^p} X$.

(4) $\{X_n, n \geq 1\}$ is said to converge to X in distribution, if for any bounded continuous function φ, $\lim_{n\to\infty} \widehat{E}[\varphi(X_n)] = \widehat{E}[\varphi(X)]$, which is denoted by $X_n \xrightarrow{d} X$.

(5) $\{X_n, n \geq 1\}$ is said to completely converge to X, if for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} V(\{|X_n - X| \geq \varepsilon\}) < \infty$, which is denoted by $X_n \xrightarrow{\text{c.c.}} X$.

(6) $\{X_n, n \geq 1\}$ is said to s-L^r converge to X $(r > 0)$, if $\sum_{n=1}^{\infty} \widehat{E}[|X_n - X|^r] < \infty$, which is denoted by $X_n \xrightarrow{s\text{-}L^r} X$.

3.1 Without the monotone continuity property

(a) By Markov's inequality (see [3, Lemma 13] or [12, Lemma VI.1.13]), we get

$$X_n \xrightarrow{s\text{-}L^r} X \Rightarrow X_n \xrightarrow{\text{c.c.}} X.$$

(b) By the Borel–Cantelli lemma (see [12, Lemma VI.1.5]), we get

$$X_n \xrightarrow{\text{c.c.}} X \Rightarrow X_n \xrightarrow{\text{q.s.}} X.$$

(c) By Markov's inequality (see [3, Lemma 13] or [12, Lemma VI.1.13]), we get

$$X_n \xrightarrow{L^p} X \Rightarrow X_n \xrightarrow{V} X.$$

(d) By [12, Proposition VI.1.17], we know that if $X_n \xrightarrow{L^p} X$, then there exists a subsequence X_{n_k} such that $X_{n_k} \xrightarrow{\text{q.s.}} X$.

In general, we do not have "$X_n \xrightarrow{\text{q.s.}} X \Rightarrow X_n \xrightarrow{V} X$" and the dominated convergence theorem as in the classic probability space.

By using the idea in the sufficiency proof of [2, Theorem 3.2], we give the following dominated convergence theorem.

Theorem 3.1 *Suppose that $\{X_n\}$ is a sequence in L^1, and $|X_n| \leq Y, \forall n \geq 1$ with $Y \in L^1_b$. Further suppose that $\{X_n\}$ converges to X in capacity and $X \in L^1_b$. Then $\{X_n\}$ converges to X in L^1 norm.*

Proof The idea comes from the sufficiency proof of [2, Theorem 3.2]. For the reader's convenience, we spell out the details.

By Definition 2.4 and Lemma 2.1, and the assumption that $|X_n| \le Y$ for all $n \ge 1$ and $Y \in L_b^1$, we get that $X_n \in L_b^1$ for all $n \ge 1$ and $\{X_n\}$ is uniformly integrable.

For any $c > 0$, we have

$$\begin{aligned}\widehat{E}[|X_n - X|] &= \widehat{E}[|X_n(I_{\{|X_n|\le c\}} + I_{\{|X_n|>c\}}) - X(I_{\{|X|\le c\}} + I_{\{|X|>c\}})|] \\ &\le \widehat{E}[|X_n I_{\{|X_n|\le c\}} - X I_{\{|X|\le c\}}|] + \widehat{E}[|X_n| I_{\{|X_n|>c\}}] + \widehat{E}[|X| I_{\{|X|>c\}}] \\ &\le \widehat{E}[|X_n - X| I_{\{|X_n|\le c, |X|\le c\}}] + \widehat{E}[|X_n| I_{\{|X_n|\le c, |X|>c\}}] \\ &\quad + \widehat{E}[|X| I_{\{|X_n|>c, |X|\le c\}}] + \widehat{E}[|X_n| I_{\{|X_n|>c\}}] + \widehat{E}[|X| I_{\{|X|>c\}}] \\ &\le \widehat{E}[|X_n - X| I_{\{|X_n|\le c, |X|\le c\}}] + \widehat{E}[c I_{\{|X|>c\}}] + \widehat{E}[c I_{\{|X_n|>c\}}] \\ &\quad + \widehat{E}[|X_n| I_{\{|X_n|>c\}}] + \widehat{E}[|X| I_{\{|X|>c\}}] \\ &\le \widehat{E}[|X_n - X| I_{\{|X_n|\le c, |X|\le c\}}] + 2\widehat{E}[|X_n| I_{\{|X_n|>c\}}] + 2\widehat{E}[|X| I_{\{|X|>c\}}]. \end{aligned} \tag{3.1}$$

For any fixed $\varepsilon > 0$, as $\{X_n\}$ is uniformly integrable and $X \in L_b^1$, there exists $c > 0$ such that

$$\sup_{n\ge 1} \widehat{E}[|X_n| I_{\{|X_n|>c\}}] < \frac{\varepsilon}{8}, \quad \widehat{E}[|X| I_{\{|X|>c\}}] < \frac{\varepsilon}{8}. \tag{3.2}$$

Since $\{X_n\}$ converges to X in capacity, there exists N such that for any $n \ge N$, $\widehat{E}[I_{\{|X_n - X| > \frac{\varepsilon}{4}\}}] < \frac{\varepsilon}{8c}$. Obviously, $|X_n - X| I_{\{|X_n|\le c, |X|\le c\}} \le 2c$. Hence, for any $n \ge N$, we have

$$\begin{aligned}\widehat{E}[|X_n - X| I_{\{|X_n|\le c, |X|\le c\}}] &\le \widehat{E}\Big[(|X_n - X| I_{\{|X_n|\le c, |X|\le c\}}) \vee \frac{\varepsilon}{4}\Big] \\ &\le \frac{\varepsilon}{4} + 2c\widehat{E}[I_{\{|X_n - X| > \frac{\varepsilon}{4}\}}] \le \frac{\varepsilon}{2}. \end{aligned} \tag{3.3}$$

By (3.1)–(3.3), we get that for any $n \ge N$, $\widehat{E}[|X_n - X|] < \varepsilon$. Hence $\{X_n\}$ converges to X in L^1 norm.

Remark 3.1 If there exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \to X$ q.s. and $\{X_{n_k}\}$ is uniformly integrable, then $X \in L_b^1$. In fact, by Theorem 2.2 and the monotonicity of $\widehat{E}$, we get that for any $A \in \mathcal{F}$,

$$\begin{aligned}\widehat{E}[I_A|X|] &= \widehat{E}[\lim_{k\to\infty} I_A |X_{n_k}|] = \widehat{E}\Big[\liminf_{k\to\infty} I_A |X_{n_k}|\Big] \\ &= \widehat{E}[\lim_{n\to\infty} \inf_{k\ge n} I_A |X_{n_k}|] \\ &= \lim_{n\to\infty} \widehat{E}[\inf_{k\ge n} I_A |X_{n_k}|] \\ &\le \lim_{n\to\infty} \inf_{k\ge n} \widehat{E}[I_A |X_{n_k}|]. \end{aligned} \tag{3.4}$$

By (3.4) (setting $A = \Omega$), the fact that $\{X_{n_k}\}$ is uniformly integrable and [2, Theorem 3.1], we get that $\widehat{E}[|X|] < \infty$. Then by the Markov's inequality, we get that

$$\lim_{m\to\infty} \widehat{E}[I_{\{|X|>m\}}] \le \lim_{m\to\infty} \frac{\widehat{E}[|X|]}{m} = 0. \tag{3.5}$$

By (3.4)–(3.5), the fact that $\{X_{n_k}\}$ is uniformly integrable and [2, Theorem 3.1], we get that

$$\lim_{m\to\infty} \widehat{E}[|X|I_{\{|X|>m\}}] = 0,$$

which together with Lemma 2.1 implies that $X \in L_b^1$.

3.2 With the monotone continuity property

Throughout the rest of the paper, we assume that $\widehat{E}$ has the monotone continuity property ([2, Definition 2.2(vii)]), i.e., for any $X_n \downarrow 0$ on Ω, we have $\widehat{E}[X_n] \downarrow 0$. Then we have the following result.

Remark 3.2 $V(\cdot)$ is a continuous capacity, i.e., if $(A_n)_{n=1}^{\infty}$ is a decreasing sequence in $\mathcal{B}(\Omega)$: $A_n \downarrow A = \cap A_n$, then $V(A) = \lim_{n\to\infty} V(A_n)$. In fact, we have $I_{A_n} - I_A = I_{A_n \setminus A} \downarrow 0$. Thus for any $n \geq 1$, we have

$$0 \leq V(A_n) - V(A) = \widehat{E}[I_{A_n}] - \widehat{E}[I_A] \leq \widehat{E}[I_{A_n} - I_A] \downarrow 0,$$

which implies that $V(A) = \lim_{n\to\infty} V(A_n)$.

By [2, Lemma 3.7], we know that $X_n \xrightarrow{\text{q.s.}} X \Rightarrow X_n \xrightarrow{V} X$, and if $X_n \xrightarrow{V} X$, then there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \xrightarrow{\text{q.s.}} X$. Thus, by Remark 3.1 we can rewrite Theorem 3.1 in this case as follows.

Theorem 3.2 (see [1, Theorem 3.11]) *Suppose that $\{X_n\}$ is a sequence in L^1, and $|X_n| \leq Y, \forall n \geq 1$ with $Y \in L_b^1$. If $X_n \to X$ q.s. or $\{X_n\}$ converges to X in capacity, then $\{X_n\}$ converges to X in L^1 norm.*

3.2.1 Convergence in capacity

In this subsection, we give some discussions of convergence in capacity.

Proposition 3.1 $X_n \xrightarrow{V} X$ *if and only if for any subsequence $\{X_{n'}\}$ of $\{X_n\}$, there exists a subsequence $\{X_{n'_k}\}$ of $\{X_{n'}\}$ such that $X_{n'_k} \xrightarrow{\text{q.s.}} X$.*

Proof Necessity. Suppose that $\{X_{n'}\}$ is a subsequence of $\{X_n\}$. Then $X'_n \xrightarrow{V} X$, and thus by [2, Lemma 3.7], there exists a subsequence $\{X_{n'_k}\}$ of $\{X_{n'}\}$ such that $X_{n'_k} \xrightarrow{\text{q.s.}} X$.

Sufficiency. We use proof by contradiction. We assume that $\{X_n\}$ does not converge to X in capacity. Then by the definition, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\limsup_{n\to\infty} V(\{|X_n - X| \geq \varepsilon\}) > \delta.$$

It follows that there exists a subsequence $\{X_{n_k}\}$ such that for any $k \geq 1$, $V(\{|X_{n_k} - X| \geq \varepsilon\}) > \delta$. By the assumption, the exists a subsequence $\{X_{n_{k_l}}\}$ of $\{X_{n_k}\}$ such that $X_{n_{k_l}} \xrightarrow{\text{q.s.}} X$, which implies that

$$V\Big(\bigcap_{m=1}^{\infty}\bigcup_{l=m}^{\infty}\{|X_{n_{k_l}} - X| \geq \varepsilon\}\Big) = 0,$$

i.e.,

$$\lim_{m\to\infty} V\Big(\bigcup_{l=m}^{\infty}\{|X_{n_{k_l}}-X|\geq\varepsilon\}\Big)=0. \tag{3.6}$$

But for any $j\geq m$, we have

$$V\Big(\bigcup_{l=m}^{\infty}\{|X_{n_{k_l}}-X|\geq\varepsilon\}\Big)\geq V(\{|X_{n_{k_j}}-X|\geq\varepsilon\})>\delta>0.$$

It contradicts with (3.6). Hence we obtain that $\{X_n\}$ converges to X in capacity.

Corollary 3.1 *Suppose that* $X_n \xrightarrow{V} X$*, and* f *is a continuous function such that* $\{f(X), f(X_n), n\geq 1\}\subset\mathcal{H}$*. Then* $f(X_n)\xrightarrow{V} f(X)$*.*

Proof Suppose that $\{f(X_{n_k})\}$ is a subsequence of $\{f(X_n)\}$. Then by the assumption and Proposition 3.1, there exists a subsequence $\{X_{n_{k_l}}\}$ of $\{X_{n_k}\}$ such that $X_{n_{k_l}}\xrightarrow{\text{q.s.}} X$. Since f is a continuous function, we get that $f(X_{n_{k_l}})\xrightarrow{\text{q.s.}} f(X)$. By Proposition 3.1 again, we obtain that $f(X_n)\xrightarrow{V} f(X)$.

3.2.2 Convergence in distribution

In this subsection, we discuss convergence in distribution. At first, we prove that convergence in capacity is stronger than convergence in distribution.

Proposition 3.2 $X_n\xrightarrow{V} X\Rightarrow X_n\xrightarrow{d} X$.

Proof Suppose that $X_n\xrightarrow{V} X$ and f is a bounded continuous function. By Corollary 3.1, we get that $f(X_n)\xrightarrow{V} f(X)$. Then by the dominated convergence theorem (see Theorem 3.2 above), we obtain that

$$\lim_{n\to\infty}\widehat{E}[|f(X_n)-f(X)|]=0,$$

which implies

$$\lim_{n\to\infty}\widehat{E}[f(X_n)]=\widehat{E}[f(X)].$$

Hence $X_n\xrightarrow{d} X$.

Now we have the following relations under sublinear expectations as in the classic setting:

$$\begin{array}{ccccccccc}
X_n\xrightarrow{s\text{-}L^r} X & \Rightarrow & X_n\xrightarrow{\text{c.c.}} X & \Rightarrow & X_n\xrightarrow{\text{q.s.}} X & \Rightarrow & X_n\xrightarrow{V} X & \Rightarrow & X_n\xrightarrow{d} X.\\
 & & & & & & \Uparrow & & \\
 & & & & & & X_n\xrightarrow{L^p} X & &
\end{array}$$

In order to state the next result, we introduce one notion and one assumption.

Definition 3.1 *Let* X *be a random variable in* $\mathcal{H}$*. Define a set function* C_X *on* $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ *as follows:*

$$C_X(A)=\widehat{E}[I_{X^{-1}(A)}],\quad \forall A\in\mathcal{B}(\mathbb{R}).$$

Then C_X *is a continuous capacity, and we call it the distribution capacity of* X*.*

For simplicity, we define the following Assumption (A).

Assumption (A) The set $\{x \in \mathbb{R} : \widehat{E}[I_{\{X=x\}}] > 0\}$ is at most countable.

Remark 3.3 Let X be a random variable in $\mathcal{H}$ and C_X be its distribution capacity. Suppose that C_X is 2-monotone, i.e.,

$$C_X(A \cup B) + C_X(A \cap B) \geq C_X(A) + C_X(B).$$

It follows that if $\{A_k, k = 1, 2, \cdots, n\}$ satisfies that $A_i \cap A_j = \emptyset, \forall i \neq j$, then

$$C_X\Big(\bigcup_{k=1}^{n} A_k\Big) \geq \sum_{k=1}^{n} C_X(A_k).$$

It follows that X satisfies Assumption (A) in this case.

Theorem 3.3 *Suppose that* $\{X, X_n, n \geq 1\}$ *is a sequence of random variables in the sublinear expectation space* $(\Omega, \mathcal{H}, \widehat{E})$. *Define the following six claims*:

(1) $X_n \xrightarrow{d} X$.

(2) *For any bounded lower semi-continuous function* $f : \mathbb{R} \to \mathbb{R}, \liminf_{n\to\infty} \widehat{E}[f(X_n)] \geq \widehat{E}[f(X)]$, *and for any bounded upper semi-continuous function* $g : \mathbb{R} \to \mathbb{R}$, $\limsup_{n\to\infty} \widehat{E}[g(X_n)] \leq \widehat{E}[g(X)]$.

(3) *For any bounded* X*-q.s. continuous function* $f : \mathbb{R} \to \mathbb{R}$ (*i.e., if* $D_f := \{x \in \mathbb{R} \mid f \text{ is not continuous at } x\}$, *then* $\{\omega \in \Omega \mid X(\omega) \in D_f\}$ *is polar*), $\lim_{n\to\infty} \widehat{E}[f(X_n)] = \widehat{E}[f(X)]$.

(4) *For any finite open subsets* $\{A_1, \cdots, A_m\}$ *of* $\mathbb{R}$ *and any finite positive constants* $\{t_1, \cdots, t_m\}$, *we have*

$$\liminf_{n\to\infty} \widehat{E}\Big[\sum_{i=1}^{m} t_i I_{A_i}(X_n)\Big] \geq \widehat{E}\Big[\sum_{i=1}^{m} t_i I_{A_i}(X)\Big],$$

$$\limsup_{n\to\infty} \widehat{E}\Big[-\sum_{i=1}^{m} t_i I_{A_i}(X_n)\Big] \leq \widehat{E}\Big[-\sum_{i=1}^{m} t_i I_{A_i}(X)\Big].$$

(5) *For any finite closed subsets* $\{F_1, \cdots, F_m\}$ *of* $\mathbb{R}$ *and any finite positive constants* $\{t_1, \cdots, t_m\}$, *we have*

$$\limsup_{n\to\infty} \widehat{E}\Big[\sum_{i=1}^{m} t_i I_{F_i}(X_n)\Big] \leq \widehat{E}\Big[\sum_{i=1}^{m} t_i I_{F_i}(X)\Big],$$

$$\liminf_{n\to\infty} \widehat{E}\Big[-\sum_{i=1}^{m} t_i I_{F_i}(X_n)\Big] \geq \widehat{E}\Big[-\sum_{i=1}^{m} t_i I_{F_i}(X)\Big].$$

(6) *For any finite* X*-q.s. continuous sets* $\{A_1, \cdots, A_m\}$ (*i.e.,* $\{\omega \in \Omega \mid X(\omega) \in \partial A_i\}$ *is polar for any* $i = 1, \cdots, m$) *and any finite positive constants* $\{t_1, \cdots, t_m\}$, *we have* $\widehat{E}\big[\sum_{i=1}^{m} t_i I_{A_i}(X_n)\big] \to \widehat{E}\big[\sum_{i=1}^{m} t_i I_{A_i}(X)\big]$ *and* $\widehat{E}\big[-\sum_{i=1}^{m} t_i I_{A_i}(X_n)\big] \to \widehat{E}\big[-\sum_{i=1}^{m} t_i I_{A_i}(X)\big]$.

Then we have

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5) \Rightarrow (6).$$

Further, if Assumption (A) *holds, then* (6) $\Rightarrow$ (1). *Hence, in this case,* (1)–(6) *are equivalent.*

Proof (1) ⇒ (2) If $f' \le f$ and f' is a bounded continuous function, then by (1), we have

$$\liminf_{n\to\infty} \widehat{E}[f(X_n)] \ge \liminf_{n\to\infty} \widehat{E}[f'(X_n)] = \widehat{E}[f'(X)]. \tag{3.7}$$

Suppose that f is a bounded lower semi-continuous function, then there exists a sequence $\{f_n\}$ of increasing bounded continuous functions such that $f_n \uparrow f$. By (3.7) and the monotone convergence theorem (see Theorem 2.2 above), we get that

$$\liminf_{n\to\infty} \widehat{E}[f(X_n)] \ge \lim_{m\to\infty} \widehat{E}[f_m(X)] = \widehat{E}[f(X)].$$

If $g' \ge g$ and g' is a bounded continuous function, then by (1), we have

$$\limsup_{n\to\infty} \widehat{E}[g(X_n)] \le \limsup_{n\to\infty} \widehat{E}[g'(X_n)] = \widehat{E}[g'(X)]. \tag{3.8}$$

Suppose that g is a bounded upper semi-continuous function, then there exists a sequence $\{g_n\}$ of decreasing bounded continuous functions such that $g_n \downarrow g$. By (3.8), we get that

$$\limsup_{n\to\infty} \widehat{E}[g(X_n)] \le \lim_{m\to\infty} \widehat{E}[g_m(X)]. \tag{3.9}$$

Since $g_m \downarrow g$, we have $g_m(X) - g(X) \downarrow 0$, and thus

$$0 \le \widehat{E}[g_m(X)] - \widehat{E}[g(X)] \le \widehat{E}[g_m(X) - g(X)] \downarrow 0.$$

It follows that

$$\lim_{m\to\infty} \widehat{E}[g_m(X)] = \widehat{E}[g(X)],$$

which together with (3.9) implies that

$$\limsup_{n\to\infty} \widehat{E}[g(X_n)] \le \widehat{E}[g(X)].$$

Hence (2) holds.

(2) ⇒ (3) Suppose that f is a function in (3). Define

$$\underline{f}(x) = \liminf_{y\to x} f(y) = \sup_{n\ge 1} \inf_{y\in B\left(x,\frac{1}{n}\right)} f(y),$$
$$\overline{f}(x) = \limsup_{y\to x} f(y) = \inf_{n\ge 1} \sup_{y\in B\left(x,\frac{1}{n}\right)} f(y),$$

where $B\big(x, \frac{1}{n}\big)$ stands for the open ball $\big\{y \in \mathbb{R} : |y-x| < \frac{1}{n}\big\}$. Then $\underline{f}$ is a bounded lower semi-continuous function, $\overline{f}$ is a bounded upper semi-continuous function, $\underline{f}(x) \le f(x) \le \overline{f}(x), \forall x$, and if x is a continuous point of f, $\underline{f}(x) = f(x) = \overline{f}(x)$. By the assumption on f, we have

$$\widehat{E}[\underline{f}(X)] = \widehat{E}[f(X)] = \widehat{E}[\overline{f}(X)],$$

which together with (2) implies that

$$\widehat{E}[f(X)] = \widehat{E}[\underline{f}(X)] \le \liminf_{n\to\infty} \widehat{E}[\underline{f}(X_n)]$$

$$
\begin{aligned}
&\le \liminf_{n\to\infty} \widehat{E}[f(X_n)] \le \limsup_{n\to\infty} \widehat{E}[f(X_n)] \\
&\le \limsup_{n\to\infty} \widehat{E}[\overline{f}(X_n)] \\
&\le \widehat{E}[\overline{f}(X)] = \widehat{E}[f(X)].
\end{aligned}
$$

Hence (3) holds.

(3) $\Rightarrow$ (1) It is obvious.

(3) $\Rightarrow$ (4) Suppose that (3) holds. Then (2) holds. If A_i is an open set, then I_{A_i} is a bounded lower semi-continuous function, and $-I_{A_i}$ is a bounded upper semi-continuous function. It follows that $\sum\limits_{i=1}^m t_i I_{A_i}$ is a bounded lower semi-continuous function and $-\sum\limits_{i=1}^m t_i I_{A_i}$ is a bounded upper semi-continuous function. Then by (2), we obtain (4).

(4) $\Leftrightarrow$ (5) Suppose that (4) holds and F_i is a closed set for $i = 1, \cdots, m$. Then $I_{F_i} = 1 - I_{F_i^c}$, where $F_i^c = \mathbb{R} - F_i$ is an open set. Further, we have

$$
\begin{aligned}
\widehat{E}\Big[\sum_{i=1}^m t_i I_{F_i}(X_n)\Big] &= \widehat{E}\Big[\sum_{i=1}^m t_i(1 - I_{F_i^c})(X_n)\Big] = \sum_{i=1}^m t_i + \widehat{E}\Big[-\sum_{i=1}^m t_i I_{F_i^c}(X_n)\Big], \\
\widehat{E}\Big[\sum_{i=1}^m t_i I_{F_i}(X)\Big] &= \widehat{E}\Big[\sum_{i=1}^m t_i(1 - I_{F_i^c})(X)\Big] = \sum_{i=1}^m t_i + \widehat{E}\Big[-\sum_{i=1}^m t_i I_{F_i^c}(X)\Big], \\
\widehat{E}\Big[-\sum_{i=1}^m t_i I_{F_i}(X_n)\Big] &= \widehat{E}\Big[\sum_{i=1}^m t_i(I_{F_i^c} - 1)(X_n)\Big] = \widehat{E}\Big[\sum_{i=1}^m t_i I_{F_i^c}(X_n)\Big] - \sum_{i=1}^m t_i, \\
\widehat{E}\Big[-\sum_{i=1}^m t_i I_{F_i}(X)\Big] &= \widehat{E}\Big[\sum_{i=1}^m t_i(I_{F_i^c} - 1)(X)\Big] = \widehat{E}\Big[\sum_{i=1}^m t_i I_{F_i^c}(X)\Big] - \sum_{i=1}^m t_i.
\end{aligned}
$$

It follows that (5) holds. Similarly, we can prove (5) $\Rightarrow$ (4).

(5) $\Rightarrow$ (6) Suppose that (5) holds. Then (4) holds also. For any $i = 1, \cdots, m$, denote by A_i^o the interior of A_i, by $\overline{A}_i$ the closure of A_i. Then $A_i^o \subset A_i \subset \overline{A}_i$, A_i^o is an open set and $\overline{A}_i$ is a closed set. By the assumption on A_i, we have that for any $i = 1, \cdots, m$,

$$
V(\{\omega \in \Omega : X(\omega) \in \partial A_i\}) = 0,
$$

which together with Theorem 2.1(3) implies that $V\big(\{\omega \in \Omega : X(\omega) \in \bigcup\limits_{i=1}^m \partial A_i\}\big) = 0$. Thus

$$
\widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i^o}(X)\Big] = \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i}(X)\Big] = \widehat{E}\Big[\sum_{i=1}^m t_i I_{\overline{A}_i}(X)\Big],
$$

which together with (4) and (5) implies that

$$
\limsup_{n\to\infty} \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i}(X_n)\Big] \le \limsup_{n\to\infty} \widehat{E}\Big[\sum_{i=1}^m t_i I_{\overline{A}_i}(X_n)\Big] \le \widehat{E}\Big[\sum_{i=1}^m t_i I_{\overline{A}_i}(X)\Big] = \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i}(X)\Big]
$$

and

$$
\liminf_{n\to\infty} \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i}(X_n)\Big] \ge \limsup_{n\to\infty} \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i^o}(X_n)\Big] \ge \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i^o}(X)\Big] = \widehat{E}\Big[\sum_{i=1}^m t_i I_{A_i}(X)\Big].
$$

Hence $\widehat{E}\big[\sum_{i=1}^{m} t_i I_{A_i}(X_n)\big] \to \widehat{E}\big[\sum_{i=1}^{m} t_i I_{A_i}(X)\big]$. Similarly, we can get $\widehat{E}\big[-\sum_{i=1}^{m} t_i I_{A_i}(X_n)\big] \to \widehat{E}\big[-\sum_{i=1}^{m} t_i I_{A_i}(X)\big]$. So (6) holds.

(6) $\Rightarrow$ (1) Under Assumption (A): Let $f \in C_b(\mathbb{R})$. Then there exists $M > 0$ such that $|f| < M$. By $\widehat{E}[f(X_n)] = \widehat{E}[(f + M - M)(X_n)] = \widehat{E}[(f + M)(X_n)] - M$ and $\widehat{E}[f(X)] = \widehat{E}[(f+M)(X)]-M$, we can assume that there exist two positive constants M_1 and M_2 such that $M_1 < f(x) < M_2$ for any $x \in \mathbb{R}$. Define $B = \{c \in \mathbb{R} \mid C_X(f^{-1}(c)) \neq 0\}$. By Assumption (A), we get that B is at most countable. For any $\varepsilon > 0$, we construct a partition $\{t_i\}_{i=1}^{j}$ of $[M_1, M_2]$ satisfying that $M_1 = t_0 < t_1 < \cdots < t_j = M_2, t_i \notin B$, and $\sup_i(t_i - t_{i-1}) < \varepsilon, i = 1, 2, \cdots, j$. Define $B_i = \{x : t_i \le f(x) < t_{i+1}\}, i = 0, 1, \cdots, j-1$, then B_i is an X-q.s. continuous set, and for any $Y \in \mathcal{H}$, $\big|f(Y) - \sum_{i=0}^{j-1} t_i I_{B_i}(Y)\big| < \varepsilon$.

By (6), we know that

$$\widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X_n)\Big] \to \widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X)\Big]. \tag{3.10}$$

By the triangle inequality and the sublinear property, we have

$$\begin{aligned}
|\widehat{E}[f(X_n)] - \widehat{E}[f(X)]| &\le \Big|\widehat{E}[f(X_n)] - \widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X_n)\Big]\Big| \\
&\quad + \Big|\widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X_n)\Big] - \widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X)\Big]\Big| \\
&\quad + \Big|\widehat{E}[f(X)] - \widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X)\Big]\Big| \\
&\le \widehat{E}\Big[\Big|\Big(f - \sum_{i=0}^{j-1} t_i I_{B_i}\Big)(X_n)\Big|\Big] \\
&\quad + \Big|\widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X_n)\Big] - \widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X)\Big]\Big| \\
&\quad + \widehat{E}\Big[\Big|\Big(f - \sum_{i=0}^{j-1} t_i I_{B_i}\Big)(X)\Big|\Big] \\
&\le 2\varepsilon + \Big|\widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X_n)\Big] - \widehat{E}\Big[\sum_{i=0}^{j-1} t_i I_{B_i}(X)\Big]\Big|,
\end{aligned}$$

which together with (3.10) implies that

$$\limsup_{n\to\infty} |\widehat{E}[f(X_n)] - \widehat{E}[f(X)]| \le 2\varepsilon.$$

By the arbitrariness of ε, we get

$$\lim_{n\to\infty} |\widehat{E}[f(X_n)] - \widehat{E}[f(X)]| = 0.$$

Hence (1) holds.

Definition 3.2 *Let X be a random variable in the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{E})$. For any $x \in \mathbb{R}$, define*

$$\overline{F}(x) = \widehat{E}[I_{\{X \le x\}}], \quad \underline{F}(x) = -\widehat{E}[-I_{\{X \le x\}}],$$

where $\{X \le x\} := \{\omega \in \Omega \mid X(\omega) \le x\}$. We call $(\overline{F}, \underline{F})$ the distribution function pair of X.

Proposition 3.3 *Let $(\overline{F}, \underline{F})$ be the distribution function pair of X. Then*

(i) $0 \le \underline{F}(x) \le \overline{F}(x) \le 1, \ \forall x \in \mathbb{R}$.

(ii) *Both $\overline{F}$ and $\underline{F}$ are increasing functions and right continuous.*

(iii) $\lim_{x \to -\infty} \overline{F}(x) = 0, \ \lim_{x \to +\infty} \overline{F}(x) = 1; \ \lim_{x \to -\infty} \underline{F}(x) = 0, \ \lim_{x \to +\infty} \underline{F}(x) = 1.$

(iv) *If $\widehat{E}[I_{\{X=x\}}] = 0$, then x is a continuous point of $\overline{F}$; if $\widehat{E}[-I_{\{X=x\}}] = 0$, then x is a continuous point of $\underline{F}$.*

Proof (i) For any x, we have $0 \le I_{\{X \le x\}} \le 1$. Then we have

$$0 \le -\widehat{E}[-I_{\{X \le x\}}] \le \widehat{E}[I_{\{X \le x\}}] \le 1,$$

i.e., (i) holds.

In the following, we only prove the results for $\overline{F}$ and the proofs for $\underline{F}$ are similar.

(ii) Obviously, $\overline{F}$ is an increasing function. When $y \downarrow x$, we have

$$0 \le I_{\{X \le y\}} - I_{\{X \le x\}} \downarrow 0,$$

which together with the monotone continuity property of $\widehat{E}$ implies that

$$0 \le \widehat{E}[I_{\{X \le y\}}] - \widehat{E}[I_{\{X \le x\}}] \le \widehat{E}[I_{\{X \le y\}} - I_{\{X \le x\}}] \downarrow 0,$$

i.e., $\lim_{y \downarrow x} \overline{F}(y) = \overline{F}(x)$. Hence $\overline{F}$ is right continuous.

(iii) When $x \to -\infty$, we have $I_{\{X \le x\}} \downarrow 0$. Then by the monotone continuity property of $\widehat{E}$, we get

$$\lim_{x \to -\infty} \widehat{E}[I_{\{X \le x\}}] = 0,$$

i.e., $\lim_{x \to -\infty} \overline{F}(x) = 0$. When $x \to +\infty$, we have $I_{\{X \le x\}} \uparrow 1$. Then by the monotone convergence theorem (see Theorem 2.2), we get

$$\lim_{x \to +\infty} \widehat{E}[I_{\{X \le x\}}] = 1,$$

i.e., $\lim_{x \to +\infty} \overline{F}(x) = 1$.

(iv) If $\widehat{E}[I_{\{X=x\}}] = 0$, then

$$0 \le \widehat{E}[I_{\{X \le x\}}] - \widehat{E}[I_{\{X < x\}}] \le \widehat{E}[I_{\{X \le x\}} - I_{\{X < x\}}] = \widehat{E}[I_{\{X=x\}}] = 0.$$

It follows that $\widehat{E}[I_{\{X \le x\}}] = \widehat{E}[I_{\{X < x\}}]$, i.e., $\overline{F}(x) = \overline{F}(x-)$. Hence x is a continuous point of $\overline{F}$.

Theorem 3.4 *Suppose that* $\{X, X_n, n \geq 1\}$ *is a sequence of random variables in the sublinear expectation space* $(\Omega, \mathcal{H}, \widehat{E})$ *and the distribution function pairs are* $(\overline{F}, \underline{F}), (\overline{F}_n, \underline{F}_n), n \geq 1$, *respectively. Define the following three claims*:

(1) $X_n \xrightarrow{d} X$.

(2) *For any positive integer* m *and* m *sequences* $\{a^i_k, b^i_k, k = 1, 2, \cdots, l_i\}$ *satisfying* $a^i_1 < b^i_1 < a^i_2 < b^i_2 < \cdots < a^i_{l_i} < b^i_{l_i}$ *and* $\widehat{E}[I_{\{X=a^i_k\}}] = \widehat{E}[I_{\{X=b^i_k\}}] = 0, \forall k = 1, 2, \cdots, l_i, \forall i = 1, \cdots, m$, *and* m *positive constants* $\{t_1, \cdots, t_m\}$, *it holds that* $\lim\limits_{n\to\infty} \widehat{E}\big[\sum\limits_{i=1}^m t_i I_{\bigcup\limits_{k=1}^{l_i}(a^i_k, b^i_k]}(X_n)\big] = \widehat{E}\big[\sum\limits_{i=1}^m t_i I_{\bigcup\limits_{k=1}^{l_i}(a^i_k, b^i_k]}(X)\big]$, $\lim\limits_{n\to\infty} \widehat{E}\big[-\sum\limits_{i=1}^m t_i I_{\bigcup\limits_{k=1}^{l_i}(a^i_k, b_k]}(X_n)\big] = \widehat{E}\big[-\sum\limits_{i=1}^m t_i I_{\bigcup\limits_{k=1}^{l_i}(a_k, b_k]}(X)\big]$.

(3) *For any point* x *with* $\widehat{E}[I_{\{X=x\}}] = 0$, *it holds that* $\lim\limits_{n\to\infty} \overline{F}_n(x) = \overline{F}(x)$ *and* $\lim\limits_{n\to\infty} \underline{F}_n(x) = \underline{F}(x)$.

Then we have

$$(1) \Rightarrow (2) \quad \textit{and} \quad (1) \Rightarrow (3).$$

If X *satisfies Assumption* (A), *then* (2) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) For the sequence $\{a^i_k, b^i_k, k = 1, 2, \cdots, l_i\}$ satisfying the conditions in (2), the set $\bigcup\limits_{k=1}^{l_i}(a^i_k, b^i_k]$ is an X-q.s. continuous set. Then by Theorem 3.3, we get (2).

(1) $\Rightarrow$ (3) For any point x with $\widehat{E}[I_{\{X=x\}}] = 0$, the set $(-\infty, x]$ is an X-q.s. continuous set. Then by Theorem 3.3, we get (3).

(2) $\Rightarrow$ (1) under Assumption (A): Suppose that (2) and Assumption (A) hold. Suppose that m is a positive integer and $\{A_1, \cdots, A_m\}$ is a sequence of open subset of $\mathbb{R}$. In the following, we will prove that Theorem 3.3(4) holds and thus by Theorem 3.3, we obtain that (1) holds. For simplicity, in the following, we only give the proof for the case that $m = 1$. The general proof is similar. Let A_1 be an open subset of $\mathbb{R}$ and t_1 be a positive constant. Then A_1 can be expressed to be the disjoint union of open intervals $\{I_k, k \in I\}$. Without loss of generality, we assume that $I = \mathbb{N}$.

(a) For any $m \in \mathbb{N}$, we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \widehat{E}[t_1 I_{A_1}(X_n)] &= \liminf_{n\to\infty} \widehat{E}[t_1 I_{\bigcup\limits_{k=1}^{\infty} I_k}(X_n)] \\ &\geq \liminf_{n\to\infty} \widehat{E}[t_1 I_{\bigcup\limits_{k=1}^{m} I_k}(X_n)] \\ &= \liminf_{n\to\infty} \widehat{E}\Big[t_1 \sum_{k=1}^m I_{I_k}(X_n)\Big]. \end{aligned} \tag{3.11}$$

Given $\varepsilon > 0$. For any k, by Assumption (A) and the monotone continuity property of the sublinear expectation, there exists a sub-interval $(a_k, b_k]$ of I_k such that $\widehat{E}[I_{\{X=a_k\}}] = \widehat{E}[I_{\{X=b_k\}}] = 0$ and

$$\widehat{E}[I_{I_k}(X) - I_{(a_k,b_k]}(X)] < \frac{\varepsilon}{2^k}. \tag{3.12}$$

It follows that for any $m \geq 1$, we have

$$\begin{aligned} 0 &\leq \widehat{E}[t_1 I_{\bigcup_{k=1}^m I_k}(X)] - \widehat{E}[t_1 I_{\bigcup_{k=1}^m (a_k,b_k]}(X)] \\ &\leq \widehat{E}[t_1 I_{\bigcup_{k=1}^m I_k}(X) - t_1 I_{\bigcup_{k=1}^m (a_k,b_k]}(X)] \\ &\leq \sum_{k=1}^m t_1 \widehat{E}[I_{I_k}(X) - I_{(a_k,b_k]}(X)] \\ &\leq t_1 \sum_{k=1}^m \frac{\varepsilon}{2^k}. \end{aligned} \tag{3.13}$$

By (3.11), the monotone property of $\widehat{E}$, the condition (2), (3.13) and the monotone convergence theorem (Theorem 2.2), we obtain that

$$\begin{aligned} \liminf_{n\to\infty} \widehat{E}[t_1 I_{A_1}(X_n)] &\geq \lim_{m\to\infty} \liminf_{n\to\infty} \widehat{E}\Big[t_1 \sum_{k=1}^m I_{I_k}(X_n)\Big] \\ &\geq \lim_{m\to\infty} \liminf_{n\to\infty} \widehat{E}\Big[t_1 \sum_{k=1}^m I_{(a_k,b_k]}(X_n)\Big] \\ &= \lim_{m\to\infty} \widehat{E}\Big[t_1 \sum_{k=1}^m I_{(a_k,b_k]}(X)\Big] \\ &\geq \lim_{m\to\infty} \widehat{E}\Big[t_1 \sum_{k=1}^m I_{I_k}(X)\Big] - t_1\varepsilon \\ &= \widehat{E}[t_1 I_{A_1}(X)] - t_1\varepsilon. \end{aligned}$$

By the arbitrariness of ε, we get that

$$\liminf_{n\to\infty} \widehat{E}[t_1 I_{A_1}(X_n)] \geq \widehat{E}[t_1 I_{A_1}(X)]. \tag{3.14}$$

(b) Given $\varepsilon > 0$. For any k, take the interval $(a_k, b_k]$ as in (a). For any m, we have

$$\begin{aligned} &\limsup_{n\to\infty} \widehat{E}[-t_1 I_{A_1}(X_n)] \\ &\leq \limsup_{n\to\infty} \widehat{E}[-t_1 I_{\bigcup_{k=1}^m I_k}(X_n)] \leq \limsup_{n\to\infty} \widehat{E}[-t_1 I_{\bigcup_{k=1}^m (a_k,b_k]}(X_n)] \end{aligned} \tag{3.15}$$

and by (3.12), we have

$$\begin{aligned} 0 &\leq \widehat{E}[-t_1 I_{\bigcup_{k=1}^m (a_k,b_k]}(X)] - \widehat{E}[-t_1 I_{\bigcup_{k=1}^m I_k}(X)] \\ &\leq \widehat{E}[-t_1 I_{\bigcup_{k=1}^m (a_k,b_k]}(X) - (-t_1 I_{\bigcup_{k=1}^m I_k}(X))] \\ &\leq \sum_{k=1}^m t_1 \widehat{E}[I_{I_k}(X) - I_{(a_k,b_k]}(X)] \\ &\leq t_1 \sum_{k=1}^m \frac{\varepsilon}{2^k}. \end{aligned} \tag{3.16}$$

By the monotone continuity property of the sublinear expectation, we have

$$
\begin{aligned}
0 &\le \widehat{E}[-t_1 I_{\bigcup\limits_{k=1}^{m} I_k}(X)] - \widehat{E}[-t_1 I_{A_1}(X)] \\
&\le \widehat{E}[-t_1 I_{\bigcup\limits_{k=1}^{m} I_k}(X) - (-t_1 I_{A_1}(X))] \\
&= t_1 \widehat{E}[I_{\bigcup\limits_{k=m+1}^{\infty} I_k}(X)] \downarrow 0 \quad \text{as } m \to \infty. \qquad (3.17)
\end{aligned}
$$

By (3.15), the condition (2) and (3.16)–(3.17), we get

$$
\begin{aligned}
\limsup_{n\to\infty} \widehat{E}[-t_1 I_A(X_n)] &\le \lim_{m\to\infty} \limsup_{n\to\infty} \widehat{E}[-t_1 I_{\bigcup\limits_{k=1}^{m} (a_k,b_k]}(X_n)] \\
&\le \lim_{m\to\infty} \widehat{E}[-t_1 I_{\bigcup\limits_{k=1}^{m} (a_k,b_k]}(X)] \\
&\le \lim_{m\to\infty} \widehat{E}[-t_1 I_{\bigcup\limits_{k=1}^{m} I_k}(X)] + t_1\varepsilon \\
&= \widehat{E}[-t_1 I_{A_1}(X)] + t_1\varepsilon.
\end{aligned}
$$

By the arbitrariness of ε, we get that

$$
\limsup_{n\to\infty} \widehat{E}[-t_1 I_{A_1}(X_n)] \le \widehat{E}[-t_1 I_{A_1}(X)]. \qquad (3.18)
$$

Proposition 3.4 *Suppose that $\{X_n, n \ge 1\}$ is a sequence of random variables in the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{E})$ and $X_n \xrightarrow{d} C$, where C is a constant. Then $X_n \xrightarrow{V} C$.*

Proof Define $X := C$. Denote by $(\overline{F}, \underline{F})$ the distribution function pair of X, and by $(\overline{F}_n, \underline{F}_n)$ the distribution function pair of X_n for any $n \ge 1$. We have

$$
\overline{F}(x) = \underline{F}(x) = \begin{cases} 1, & \text{when } x \ge c, \\ 0, & \text{otherwise,} \end{cases}
$$

and for any $x \ne c$, we have $\widehat{E}[I_{\{X=x\}}] = 0$. Then by Theorem 3.4, we get that for any $x \ne c$,

$$
\lim_{n\to\infty} \overline{F}_n(x) = \lim_{n\to\infty} \underline{F}_n(x) = \begin{cases} 1, & \text{when } x > c, \\ 0, & \text{when } x < c. \end{cases} \qquad (3.19)
$$

For any $\varepsilon > 0$, we have

$$
\begin{aligned}
\widehat{E}[I_{\{|X_n - C| \ge \varepsilon\}}] &= \widehat{E}[I_{\{X_n \ge C+\varepsilon\} \bigcup \{X_n \le C-\varepsilon\}}] \\
&\le \widehat{E}[I_{\{X_n \ge C+\varepsilon\}}] + \widehat{E}[I_{\{X_n \le C-\varepsilon\}}] \\
&= \widehat{E}[1 - I_{\{X_n < C+\varepsilon\}}] + \widehat{E}[I_{\{X_n \le C-\varepsilon\}}] \\
&\le 1 + \widehat{E}[-I_{\{X_n < C+\varepsilon\}}] + \widehat{E}[I_{\{X_n \le C-\varepsilon\}}] \\
&\le 1 + \widehat{E}[-I_{\{X_n \le C+\frac{\varepsilon}{2}\}}] + \widehat{E}[I_{\{X_n \le C-\varepsilon\}}] \\
&= 1 - \underline{F}_n\Big(C + \frac{\varepsilon}{2}\Big) + \overline{F}_n(C - \varepsilon),
\end{aligned}
$$

which together with (3.19) implies that

$$\lim_{n\to\infty} \widehat{E}[I_{\{|X_n - C|\geq \varepsilon\}}] = 0,$$

i.e., $X_n \xrightarrow{V} C$.

3.3 Remarks

Remark 3.4 An anonymous referee pointed out that quasi sure convergence does not imply convergence in capacity in G-expectation setting. Thus our assumption that $\widehat{E}$ has the monotone continuity property is strong in some sense. We would explore the corresponding problems under weaker conditions in future.

Remark 3.5 As to Slutsky's theorem under sublinear expectations, refer to [13, Lemma 4.2].

Remark 3.6 A natural question arise: Can we give a Skorokhod type theorem under sublinear expectation? We tried, but we have not found the right way yet.

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