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Triangulated Structures Induced by Triangle Functors*

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Abstract Given a triangle functor $F\colon \mathcal{A}\to\mathcal{B}$, the authors introduce the half image hImF, which is an additive category closely related to F. If F is full or faithful, then hImF admits a natural triangulated structure. However, in general, one can not expect that hImF has a natural triangulated structure. The aim of this paper is to prove that hImF admits a natural triangulated structure if and only if F satisfies the condition (SM). If this is the case, hImF is triangle-equivalent to the Verdier quotient $\mathcal{A}/\mathrm{Ker}F$.

Keywords Triangulated category, Triangle functor, Half image, Verdier quotient
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1 Introduction and Preliminaries

Comparing with general additive functors between additive categories, or exact functors between abelian categories, triangle functors have special properties. One of the most fundamental results on triangle functors is that in an adjoint pair (F,G) of additive functors, F is a triangle functor if and only if so is G (see [2, 6.7] or [3, p.179]); also, a sincere full triangle functor is faithful (see [4, p.446]). For more information on triangle functors we refer to [5].

Let $F\colon \mathcal{A}\to\mathcal{B}$ be an additive functor between additive categories. We consider the category hImF, called the half image of F, whose objects are the same as ones of $\mathcal{A}$, and the set of morphisms from X to Y is

$$\mathrm{Hom}_{\mathrm{hIm}F}(X,Y) := F\mathrm{Hom}_{\mathcal{A}}(X,Y).$$

Thus $\mathrm{Hom}_{\mathrm{hIm}F}(X,Y)=\{F(f)\mid f\in\mathrm{Hom}_{\mathcal{A}}(X,Y)\}$, but we emphasize that in general

$$\mathrm{Hom}_{\mathrm{hIm}F}(X,Y)\subsetneqq \mathrm{Hom}_{\mathcal{B}}(FX,FY).$$

It is clear that hImF is an additive category. Here, “h” in hImF refers to “half ”.

Suppose that $F\colon \mathcal{A}\to\mathcal{B}$ is a triangle functor between triangulated categories. Then the translation functor $[1]_{\mathcal{A}}$ of $\mathcal{A}$ induces naturally an automorphism functor of hImF, denoted by

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[1]. To be precise, $X[1]\colon = X[1]_{\mathcal{A}}$ for any object X of hImF, and $(F(f))[1]\colon = F(f[1]_{\mathcal{A}})$ for any morphism $F(f)\colon X \to Y$ of hImF. Our main aim is to investigate when the category hImF together with the automorphism [1] admits a natural triangulated structure (see Theorem 2.1). In fact, this is the case provided that F is full or faithful. However, in general, one can not expect that hImF has a natural triangulated structure. See Examples 3.2–3.3 for details.

In this paper, we show that the category hImF has a natural triangulated structure if and only if the triangle functor F satisfies the condition (SM) (see Theorem 2.1). If this is the case, hImF is exactly triangle-equivalent to the Verdier quotient $\mathcal{A}/\mathrm{Ker}F$. Note that the condition (SM) is weaker than that F is full or faithful. As an application, we describe when the Verdier quotient functor is full.

All the functors in this paper are covariant and additive between additive categories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. Recall from [5, 6, Appendix] that F is objective, provided that any morphism $f : X \to Y$ in $\mathcal{A}$ with $F(f) = 0$ factors through an object K with $F(K) = 0$. We consider the following conditions (I) and (SM).

(I) For each morphism $u\colon X \to Y$ in $\mathcal{A}$ such that $F(u)$ is an isomorphism in $\mathcal{B}$, there exists a morphism $u'\colon Y \to X$ such that $(F(u))^{-1} = F(u')$.

(SM) For each morphism $u\colon X \to Y$ in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism, there exists a morphism $u'\colon Y \to X$ such that $F(u')F(u) = \mathrm{id}_{F(X)}$.

Lemma 1.1 (see [5]) *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. If F is full or faithful, then F satisfies the condition* (SM).

Let $(\mathcal{A}, [1])$ be a triangulated category and $\mathcal{K}$ a triangulated subcategory of $\mathcal{A}$. Recall that the objects of the Verider quotient $\mathcal{A}/\mathcal{K}$ are the same as ones of $\mathcal{A}$, and a morphism of $\mathcal{A}/\mathcal{K}$ from X to Y is a right fraction a/s, where $a\colon Z \to Y$ and $s\colon Z \Rightarrow_{\mathcal{K}} X$ are morphisms in $\mathcal{A}$ and $s\colon Z \Rightarrow_{\mathcal{K}} X$ means that there exists a distinguish triangle $Z \xrightarrow{s} X \to K \to Z[1]$ in $\mathcal{A}$ with $K \in \mathcal{K}$.

Denote by $V_{\mathcal{K}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ the localization functor. That is, $V_{\mathcal{K}}(X) = X$ for any object X of $\mathcal{A}$, and $V_{\mathcal{K}}(f) = f/\mathrm{id}_X$ for $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Then $(V_{\mathcal{K}}, \mathrm{id})\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ is a dense triangle functor, which is called the Verider functor. It has the universal property with respect to $V_{\mathcal{K}}(\mathcal{K}) = 0$. To be precise, if $F\colon \mathcal{A} \to \mathcal{B}$ is a triangle functor with $F(\mathcal{K}) = 0$, then there exists a unique triangle functor $\widetilde{F}\colon \mathcal{A}/\mathcal{K} \to \mathcal{B}$ such that $F = \widetilde{F}V_{\mathcal{K}}$ (see [3, 7]).

Lemma 1.2 (see [5]) *Let $\mathcal{A}$ be a triangulated category and $\mathcal{K}$ a triangulated subcategory of $\mathcal{A}$. Then the Verdier functor $V_{\mathcal{K}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ is objective.*

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. We denote by Ker$F$ the full subcategory of $\mathcal{A}$ consisting of all objects X of $\mathcal{A}$ satisfying $F(X) = 0$. For simplicity, we denote by V_F the Verdier functor $\mathcal{A} \to \mathcal{A}/\mathrm{Ker}F$. Since $F(\mathrm{Ker}F) = 0$, it follows from the universal property that there exists a unique triangle functor $\widetilde{F}\colon \mathcal{A}/\mathrm{Ker}F \to \mathcal{B}$ such that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "V_F"'] & & \mathcal{B} \\ & \mathcal{A}/\mathrm{Ker}F \arrow[ur, "\widetilde{F}"'] & \end{tikzcd} \tag{1.1}$$

commutes. Thus $\widetilde{F}(X) = F(X)$ for any object X of $\mathcal{A}/\mathrm{Ker}F$, and

$$\widetilde{F}(a/s) = F(a)(F(s))^{-1}. \tag{1.2}$$

We need the following results.

Lemma 1.3 (see [5]) *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then F is objective if and only if the induced functor $\widetilde{F}\colon \mathcal{A}/\mathrm{Ker}F \to \mathcal{B}$ is faithful.*

Lemma 1.4 (see [5]) *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then the following are equivalent*:

(i) *F satisfies the condition* (SM);

(ii) *F is objective and V_F is full*;

(iii) *There is a factorization $F = F_2F_1$, where F_1 is a full triangle functor and F_2 is a faithful triangle functor.*

2 Main Results

Let $\mathrm{hIm}F$ be the half image of additive functor $F\colon \mathcal{A} \to \mathcal{B}$. Observe that if FX is the zero object of $\mathcal{B}$, then X is the zero object of $\mathrm{hIm}F$ (note that X may not be the zero object of $\mathcal{A}$); and that the coproduct in $\mathrm{hIm}F$ coincides with the one in $\mathcal{A}$, and therefore $\mathrm{hIm}F$ is an additive category.

Proposition 2.1 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Then there is a canonical factorization of functors*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "F'"'] & & \mathcal{B} \\ & \mathrm{hIm}F, \arrow[ur, "\sigma"'] & \end{tikzcd} \tag{2.1}$$

where F' is dense and full, and σ is faithful.

Moreover, F is full if and only if σ is fully faithful, and F is faithful if and only if F' is an equivalence.

Proof For $X \in \mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, define $F'X := X$ and $F'(f) := F(f)$; and for $X \in \mathrm{hIm}F$ and $F(f) \in \mathrm{Hom}_{\mathrm{hIm}F}(X, Y)$, define $\sigma X := FX$ and $\sigma(F(f)) := F(f)$. Clearly,

$F' : \mathcal{A} \to \mathrm{hIm}F$ is a dense and full functor and $\sigma : \mathrm{hIm}F \to \mathcal{B}$ is a faithful functor, with $F = \sigma F'$.

Thus, if F is full, then $\mathrm{hIm}F$ can be viewed as a full subcategory of $\mathcal{B}$, and if F is faithful, then $\mathrm{hIm}F$ is equivalent to $\mathcal{A}$.

For a triangle functor $F\colon \mathcal{A} \to \mathcal{B}$, we have already an additive category $\mathrm{hIm}F$ with an automorphism functor [1]. Denote by Ω the class of all the triangles in $\mathrm{hIm}F$ of the form

$$X \xrightarrow{F(u)} X' \xrightarrow{F(u')} X'' \xrightarrow{F(u'')} X[1],$$

where $X \xrightarrow{u} X' \xrightarrow{u'} X'' \xrightarrow{u''} X[1]_{\mathcal{A}}$ is a distinguished triangle in $\mathcal{A}$. Suppose that $\mathcal{E}$ is the class of all the triangles $Y \xrightarrow{F(v)} Y' \xrightarrow{F(v')} Y'' \xrightarrow{F(v'')} Y[1]$ in $\mathrm{hIm}F$, such that there is an isomorphism of triangles in $\mathrm{hIm}F$

$$\begin{tikzcd}
Y \arrow[r, "F(v)"] \arrow[d, "F(f)"'] & Y' \arrow[r, "F(v')"] \arrow[d, "F(g)"'] & Y'' \arrow[r, "F(v'')"] \arrow[d, "F(h)"'] & Y[1] \arrow[d, "F(f)[1]"] \\
X \arrow[r, "F(u)"] & X' \arrow[r, "F(u')"] & X'' \arrow[r, "F(u'')"] & X[1],
\end{tikzcd}$$

where the triangle in the second row belongs to Ω.

We say that $\mathrm{hIm}F$ has a natural triangulated structure, provided that $(\mathrm{hIm}F, [1], \mathcal{E})$ is a triangulated category.

The main result of this paper is follows.

Theorem 2.1 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then the following are equivalent*:

(i) $\mathrm{hIm}F$ *has a natural triangulated structure;*

(ii) $\mathrm{hIm}F$ *has a natural triangulated structure, and there is a unique triangle-equivalence* $(\widetilde{F'}, \mathrm{id})\colon \mathcal{A}/\mathrm{Ker}F \to \mathrm{hIm}F$, *such that the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F'"] \arrow[dr, "V_F"'] & \mathrm{hIm}F \arrow[r, "\sigma"] & \mathcal{B} \\
& A/\mathrm{Ker}F \arrow[u, "\widetilde{F'}"'] \arrow[ur, "\widetilde{F}"'] &
\end{tikzcd} \tag{2.2}$$

commutes, where V_F and $\widetilde{F}$ are the same as in (1.1), *F' and σ are the same as in* (2.1);

(iii) *there is an equivalence $\widetilde{F'}\colon A/\mathrm{Ker}F \to \mathrm{hIm}F$ as categories, such that $F' = \widetilde{F'}V_F$;*

(iv) *F is objective and the Verdier functor V_F is full;*

(v) *F satisfies the condition* (SM).

In order to prove Theorem 2.1, we need some preparations. The following fact is well known and the proof is straightforward.

Lemma 2.1 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Suppose that $(\mathcal{A}, [1]_{\mathcal{A}})$ is a triangulated category and $\mathcal{B}$ has an automorphism, denoted by $[1]_{\mathcal{B}}$, such that there is a* *natural isomorphism* $\xi\colon F\circ[1]_{\mathcal{A}}\to[1]_{\mathcal{B}}\circ F$*. If* F *is an equivalence of categories, then* $(\mathcal{B},[1]_{\mathcal{B}})$ *admits a unique triangulated structure such that* $(F,\xi)\colon\mathcal{A}\to\mathcal{B}$ *is a triangle-equivalence.*

Lemma 2.2 *Let* $F\colon\mathcal{A}\to\mathcal{B}$ *be a triangle functor between triangulated categories. Then* F *satisfies the condition* (I) *if and only if the functor* $\widetilde{F}\colon\mathcal{A}/\mathrm{Ker}F\to\mathcal{B}$ *in* (1.1) *induces a dense and full functor* $\widetilde{F'}\colon\mathcal{A}/\mathrm{Ker}F\to\mathrm{hIm}F$ *such that the diagram* (2.2) *commutes.*

Proof Assume that F satisfies the condition (I). For any $a/s\colon X\to Y$ in $\mathcal{A}/\mathrm{Ker}F$ with $a\colon Z\to Y$ and $s\colon Z\Rightarrow_{\mathrm{Ker}F}X$, $F(s)$ is an isomorphism since there exists a distinguished triangle $Z\xrightarrow{s}X\to K\to Z[1]$ with $F(K)=0$ in $\mathcal{B}$ (see [1, Lemma 1.7]). By the condition (I), there exists $s'\colon X\to Z$ such that $(F(s))^{-1}=F(s')$. By (1.2) we have $\widetilde{F}(a/s)=F(a)(F(s))^{-1}=F(as')$. Define $\widetilde{F'}\colon\mathcal{A}/\mathrm{Ker}F\to\mathrm{hIm}F$ as follows: $\widetilde{F'}(X)=X$ for any object X of $A/\mathrm{Ker}F$, and $\widetilde{F'}(a/s)=F(as')$ for any morphism $a/s\colon X\to Y$ of $\mathcal{A}/\mathrm{Ker}F$. Since $\widetilde{F}\colon\mathcal{A}/\mathrm{Ker}F\to\mathcal{B}$ is well-defined, it follows that $\widetilde{F'}$ is also a well-defined functor, and it is dense and full. By a direct calculation we have $\sigma\widetilde{F'}=\widetilde{F}$ and $\widetilde{F'}V_F=F'$, i.e., the diagram (2.2) commutes.

Conversely, suppose that $\widetilde{F}\colon\mathcal{A}/\mathrm{Ker}F\to\mathcal{B}$ induces a dense and full functor $\widetilde{F'}\colon\mathcal{A}/\mathrm{Ker}F\to\mathrm{hIm}F$ such that the diagram (2.2) commutes. Let $s\colon Z\to X$ be a morphism in $\mathcal{A}$ such that $F(s)$ is an isomorphism in $\mathcal{B}$. Since by construction σ fixes morphisms (see the proof of Proposition 2.1), by (1.2) we have

$$\widetilde{F'}(\mathrm{id}_Z/s)=\sigma\widetilde{F'}(\mathrm{id}_Z/s)=\widetilde{F}(\mathrm{id}_Z/s)=(F(s))^{-1}\colon F(X)\to F(Z).$$

By Proposition 2.1 we know that F' is full. It follows that there exists $s'\colon X\to Z$ such that $F(s')=F'(s')=(F(s))^{-1}$, and hence F satisfies the condition (I).

Proof of Theorem 2.1 (ii) $\Rightarrow$ (iii) is trivial, and (iv)$\Leftrightarrow$ (v) follows from Lemma 1.4.

(i) $\Rightarrow$ (ii) Assume that $\mathrm{hIm}F$ has a natural triangulated structure. Then by construction $F'\colon\mathcal{A}\to\mathrm{hIm}F$ is a full triangle functor and $\sigma\colon\mathrm{hIm}F\to\mathcal{B}$ is a faithful triangle functor. It follows from Lemma 1.4 that $F=\sigma F'$ is objective. Hence $\widetilde{F}\colon\mathcal{A}/\mathrm{Ker}F\to\mathcal{B}$ is faithful by Lemma 1.3.

On the other hand, if $X\in\mathrm{Ker}F$, then X is a zero object of $\mathrm{hIm}F$ and hence $F'X:=X=0$ in $\mathrm{hIm}F$. That is $F'(\mathrm{Ker}F)=0$. By the universal property of the Verdier quotient, there exists a unique triangle functor $\widetilde{F'}\colon\mathcal{A}/\mathrm{Ker}F\to\mathrm{hIm}F$ such that the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "F'"] \arrow[dr, "V_F"'] & & \mathrm{hIm}F \\
& \mathcal{A}/\mathrm{Ker}F \arrow[ur, "\widetilde{F'}"'] &
\end{tikzcd}$$

commutes. Thus by (1.1) and (2.1) we have

$$\widetilde{F}V_F=F=\sigma F'=(\sigma\widetilde{F'})V_F,$$

and by the uniqueness we have $\widetilde{F}=\sigma\widetilde{F'}$. So we get the commutative diagram (2.2).

Since $\widetilde{F}$ is faithful and $\widetilde{F} = \sigma\widetilde{F'}$, it follows that $\widetilde{F'}$ is faithful. Similarly, since F' is full and $F' = \widetilde{F'}V_F$, it follows that $\widetilde{F'}$ is full. Clearly $\widetilde{F'}$ is dense and thus $\widetilde{F'}\colon \mathcal{A}/\mathrm{Ker}F \to \mathrm{hIm}F$ is a triangle-equivalence.

(iii) $\Rightarrow$ (i) Assume that there is an equivalence $\widetilde{F'}\colon \mathcal{A}/\mathrm{Ker}F \to \mathrm{hIm}F$ as categories such that $F' = \widetilde{F'}V_F$. Note that an equivalence between additive categories is always additive. By the definitions of $\mathcal{A}/\mathrm{Ker}F$ and $\mathrm{hIm}F$, we have $\widetilde{F'}(X[1]_{\mathcal{A}}) = \widetilde{F'}(X)[1]$ for any object X of $\mathcal{A}/\mathrm{Ker}F$. It follows from Lemma 2.1 that $(\mathrm{hIm}F, [1])$ has a triangulated structure such that $(\widetilde{F'}, \mathrm{id})\colon \mathcal{A}/\mathrm{Ker}F \to \mathrm{hIm}F$ is a triangle functor. This means that the triangulated structure of $\mathrm{hIm}F$ is exactly the natural one.

(ii) $\Rightarrow$ (iv) By the commutative diagram (2.2), we have $F' = \widetilde{F'}V_F$, where $\widetilde{F'}\colon \mathcal{A}/\mathrm{Ker}F \to \mathrm{hIm}F$ is a triangle-equivalence. By construction $F'\colon \mathcal{A} \to \mathrm{hIm}F$ is a full triangle functor, since $\mathrm{hIm}F$ has a natural triangulated structure. It follows that V_F is also full. On the other hand, as in the proof of (i) $\Rightarrow$ (ii) we have known that F is objective.

(v) $\Rightarrow$ (iii) Assume that F satisfies the condition (SM). So F satisfies the condition (I). By Lemma 2.2, the functor $\widetilde{F}\colon \mathcal{A}/\mathrm{Ker}F \to \mathcal{B}$ induces a dense and full functor $\widetilde{F'}\colon \mathcal{A}/\mathrm{Ker}F \to \mathrm{hIm}F$ such that the diagram (2.2) commutes.

On the other hand, by Lemma 1.4, F is objective. It follows from Lemma 1.3 that $\widetilde{F}$ is faithful. Since $\widetilde{F} = \sigma\widetilde{F'}$, $\widetilde{F'}$ is also faithful, and hence $\widetilde{F'}$ is an equivalence.

This completes the proof.

3 Consequences and Examples

Recall that a functor F is sincere, provided that F sends non-zero objects to non-zero objects. Clearly, a functor is faithful if and only if it is objective and sincere. By Theorem 2.1, we immediately get the follwing result.

Corollary 3.1 *Let $(F, \xi)\colon \mathcal{A} \to \mathcal{B}$ be a sincere triangle functor. Then the following are equivalent*:

(i) $\mathrm{hIm}F$ *has a natural triangulated structure*;

(ii) F *satisfies the condition* (SM);

(iii) F *is faithful.*

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. If F is full or faithful, then F satisfies the condition (SM) (see Lemma 1.1), and hence $\mathrm{hIm}F$ admits the natural triangulated structure.

The following examples show that the condition (SM) is weaker than that F is full (resp. faithful), and hence Theorem 2.1 really provides category $\mathrm{hIm}F$, which has a natural triangulated structure, but F is not full (resp. faithful).

Example 3.1 Let $\mathcal{T} \times \mathcal{T}$ be the product of a triangulate category $\mathcal{T}$ with itself. Then

$\mathcal{T}\times\mathcal{T}$ is again triangulated. Let $F\colon \mathcal{T}\to\mathcal{T}\times\mathcal{T}$ be the triangle functor given by $F(X)=(X,X)$ and $F(f)=(f,f)$, for any X in $\mathcal{T}$ and for any $f\in \mathrm{Hom}_{\mathcal{T}}(X,Y)$. Then (F,id) is a faithful triangle functor but not full. It follows from Lemma 1.1 that F satisfies the condition (SM).

Let $G\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$ be the triangle functor given by $G(X,Y)=X$ and $G(f,g)=f$. Then (G,id) is a full triangle functor but not faithful. It follows from Lemma 1.1 that G satisfies the condition (SM).

Consider the triangle functor $F\circ G\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}\times\mathcal{T}$. By Lemma 1.4 we know that $F\circ G$ satisfies the condition (SM), since F is faithful and G is full. But $F\circ G$ is neither full nor faithful.

The functor in the following example does not satisfy the conditions (SM), and therefore $\mathrm{hIm}F$ has no the natural triangulated structure.

Example 3.2 (see [5]) Let A be the path algebra of the quiver $(b\circ\to\circ a)$ over a field k and B the semisimple algebra given by the quiver with the two vertices a,b and no arrow. Then B is a subalgebra of A and we consider the forgetful functor $F_0\colon A$-mod$\to B$-mod, which is induced by the inclusion $B\hookrightarrow A$.

The functor F_0 sends $S_A(x)$ to $S_B(x)$ for $x=a,b$, and it sends $P_A(b)$ to $S_B(a)\oplus S_B(b)$, where $S_A(x)$ and $S_B(x)$ are the simple A-module and B-module corresponding to vertex x, respectively, and $P_A(x)$ is the indecomposable projective A-module corresponding to vertex x. Then F_0 is an exact and faithful functor.

The induced functor $F: D^b(A)\to D^b(B)$ sends $S_A(x)[i]$ to $S_B(x)[i]$ for $x=a,b$, and sends $P_A(b)[i]$ to $S_B(a)[i]\oplus S_B(b)[i]$ for all $i\in\mathbb{Z}$. Since $S_A(a)$ is a submodule of $P_A(b)$, we can consider the inclusion map $u\colon S_A(a)\to P_A(b)$. Applying the functor F, we obtain the inclusion map

$$F(u): S_B(a)\hookrightarrow S_B(a)\oplus S_B(b),$$

which is a splitting monomorphism. In fact, we have a projection map

$$u': S_B(a)\oplus S_B(b)\to S_B(a)$$

such that $u'F(u)=1_{S_B(a)}$. Since there is no non-zero map $P_A(b)\to S_A(a)$, such a map u' is not in the image of F. It follows that the functor F does not satisfy the condition (SM).

Let $(\mathcal{A},[1])$ be a triangulated category, $\mathcal{K}$ a triangulated subcategory of $\mathcal{A}$, and $\overline{\mathcal{K}}$ the full subcategory of $\mathcal{A}$ consisting of all the direct summands of objects of $\mathcal{K}$. Then $\overline{\mathcal{K}}$ is the smallest thick subcategory of $\mathcal{A}$ containing $\mathcal{K}$, which is called the thick closure of $\mathcal{K}$. Then

$$\mathrm{Ker}V_{\mathcal{K}}=\overline{\mathcal{K}}=\mathrm{Ker}V_{\overline{\mathcal{K}}} \tag{3.1}$$

and there is a triangle-equivalence

$$\varphi:\mathcal{A}/\mathcal{K}\cong\mathcal{A}/\overline{\mathcal{K}}, \tag{3.2}$$

such that

$$V_{\overline{\mathcal{K}}} = \varphi V_{\mathcal{K}}. \tag{3.3}$$

Thus $\mathrm{Ker}V_{\mathcal{K}} = \mathcal{K}$ if and on if $\mathcal{K}$ is a thick subcategory.

Applying Theorem 2.1 to the Verdier functor $V_{\mathcal{K}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$, by Lemma 1.2 together with Lemma 1.4 and (3.3) we know the following result.

Corollary 3.2 *Let $\mathcal{K}$ be a triangulated subcategory of triangulated category $\mathcal{A}$, and $V_{\mathcal{K}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ the Verdier functor. Then* $\mathrm{hIm}V_{\mathcal{K}}$ *has a natural triangulated structure if and only if $V_{\mathcal{K}}$ is full.*

Example 3.3 Let A be an Artin algebra, A-mod the category of finitely generated left A-modules, $K^-(A)$ the homotopy category of the upper bounded complexes over A-mod, $K^-_{ac}(A)$ the full subcategory of $K^-(A)$ consisting of the upper bounded acyclic complexes, and $D^-(A) := K^-(A)/K^-_{ac}(A)$ the derived category of the upper bounded complexes over A-mod. We have the Verdier quotient functor $V\colon K^-(A) \to D^-(A)$. It is well known that V is full if and only if A is semisimple.

For convenience we include a justification. If A is semi-simple, then each A-module is projective, and hence V is an equivalence. Conversely, assume that A is not semi-simple. Let M be a simple A-module which is not projective, and $\pi\colon P \to M$ with

$$P : \cdots \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \to 0$$

be a minimal projective resolution. Then $\pi\colon P \to M$ is a quasi-isomorphism. We claim that $\mathrm{id}_P/\pi \in \mathrm{Hom}_{D^-(A)}(M,P)$ is not of the form a/id_M with $a \in \mathrm{Hom}_{K^-(A)}(M,P)$, and hence V is not full. In fact, if otherwise, by the calculation of the right fractions there exists morphism $t\colon Z \to P$ such that $\pi t\colon Z \to M$ is a quasi-isomorphism, and $a\pi t = t$ in $K^-(A)$. Then t and hence a is a quasi-isomorphism. But $a\colon M \to P$ is a quasi-isomorphism means that a is a splitting epimorphism, i.e., there is $g\colon P \to M$ such that ag is homotopic to id_P. Since $\mathrm{Im}d^1 \subset \mathrm{rad}P^0$, it follows that $a\colon M \to P^0$ is surjective, which contradicts the assumption of M.

Now, let A be an Artin algebra which is not semisimple. Then the Verdier functor $V\colon K^-(A) \to D^-(A)$ is not full. By Corollary 3.2, $\mathrm{hIm}V$ has no natural triangulated structure.

Finally, we include a direct proof for the implication (v) $\Rightarrow$ (i) in Theorem 2.1. That is, if a triangle functor F satisfies the condition (SM), then $\mathrm{hIm}F$ has the natural triangulated structure.

Let Ω and $\mathcal{E}$ be defined as in Section 2. In order to say that $(\mathrm{hIm}F, [1], \mathcal{E})$ is a triangulated category, it suffices to show that Ω and $\mathcal{E}$ satisfy (tr1), (tr2), (tr3) and (tr4) in Lemma 1.4.4 in [8].

Note that Ω and $\mathcal{E}$ trivially satisfy (tr1), (tr2) and (tr4) even without the assumption that F satisfies the condition (SM). We only need to show that Ω satisfies (tr3).

Given two triangles $X \xrightarrow{F(u)} Y \xrightarrow{F(v)} Z \xrightarrow{F(w)} X[1]$ and $X' \xrightarrow{F(u')} Y' \xrightarrow{F(v')} Z' \xrightarrow{F(w')} X'[1]$ in Ω, and given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F(u)"] \arrow[d, "F(f)"'] & Y \arrow[d, "F(g)"] \\ X' \arrow[r, "F(u')"] & Y' \end{tikzcd}$$

in hImF, since F satisfies the condition (SM), by Lemma 1.4, F is objective, and hence $\widetilde{F}\colon \mathcal{A}/\mathrm{Ker}F \to \mathcal{B}$ is faithful by Lemma 1.3. From $0 = F(gu - u'f) = \widetilde{F}V_F(gu - u'f)$ we know that $V_F(gu - u'f) = (gu - u'f)/\mathrm{id}_X = 0$ in $\mathcal{A}/\mathrm{Ker}F$. This means that there exists a morphism $t\colon W \Rightarrow_{\mathrm{Ker}F} X$ in $\mathcal{A}$ such that

$$(gu - u'f)t = 0, \tag{3.4}$$

where $t\colon W \Rightarrow_{\mathrm{Ker}F} X$ means that there exists a distinguished triangle $W \xrightarrow{t} X \to K \to W[1]$ with $K \in \mathrm{Ker}F$. Therefore, $F(t)$ is an isomorphism in $\mathcal{B}$ by [8, Lemma 1.3.7] or [1, Lemma 1.7].

Since $X \xrightarrow{F(u)} Y \xrightarrow{F(v)} Z \xrightarrow{F(w)} X[1]$ is in Ω, by the definition of Ω we know that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is a distinguished triangle in $\mathcal{A}$. Embedding ut into a distinguished triangle $W \xrightarrow{ut} Y \to \widetilde{Z} \to W[1]$ in $\mathcal{A}$, by using (TR3) for $\mathcal{A}$, we get the following morphism of distinguished triangles in $\mathcal{A}$:

$$\begin{tikzcd} W \arrow[r, "ut"] \arrow[d, "t"'] & Y \arrow[r] \arrow[d, equal] & \widetilde{Z} \arrow[r] \arrow[d, "a"'] & W[1] \arrow[d, "t[1]_{\mathcal{A}}"] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]. \end{tikzcd} \tag{3.5}$$

Applying the functor F, we get $F(a)\colon F(\widetilde{Z}) \to F(Z)$ is an isomorphism since $F(t)$ is an isomorphism in $\mathcal{B}$. By assumption, F satisfies the condition (SM) and hence satisfies the condition (I). So there exist $t'\colon X \to W$ and $a'\colon Z \to \widetilde{Z}$ such that $F(t)^{-1} = F(t')$ and $F(a)^{-1} = F(a')$, respectively.

Similarly, we get the following morphism of distinguished triangles by (3.4):

$$\begin{tikzcd} W \arrow[r, "ut"] \arrow[d, "ft"'] & Y \arrow[r] \arrow[d, "g"'] & \widetilde{Z} \arrow[r] \arrow[d, "b"'] & W[1] \arrow[d, "(ft)[1]_{\mathcal{A}}"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]. \end{tikzcd} \tag{3.6}$$

Combining (3.5) and (3.6), we get the following morphism of distinguished triangles in $\mathrm{hIm}F$:

$$\begin{array}{ccccccc}
X & \xrightarrow{F(u)} & Y & \xrightarrow{F(v)} & Z & \xrightarrow{F(w)} & X[1] \\
{\scriptstyle F(t')}\downarrow & & \| & & {\scriptstyle F(a')}\downarrow & & \downarrow{\scriptstyle F(t'[1]_{\mathcal{A}})=F(t')[1]} \\
W & \xrightarrow{F(ut)} & Y & \longrightarrow & \widetilde{Z} & \longrightarrow & W[1] \\
{\scriptstyle F(ft)}\downarrow & & {\scriptstyle F(g)}\downarrow & & {\scriptstyle F(b)}\downarrow & & \downarrow{\scriptstyle F(ft[1]_{\mathcal{A}})=F(ft)[1]} \\
X' & \xrightarrow{F(u')} & Y' & \xrightarrow{F(v')} & Z' & \xrightarrow{F(w')} & X'[1].
\end{array}$$

Finally, we get the following morphism of distinguished triangles in $\mathrm{hIm}F$:

$$\begin{array}{ccccccc}
X & \xrightarrow{F(u)} & Y & \xrightarrow{F(v)} & Z & \xrightarrow{F(w)} & X[1] \\
{\scriptstyle F(f)}\downarrow & & {\scriptstyle F(g)}\downarrow & & {\scriptstyle F(ba')}\downarrow & & \downarrow{\scriptstyle F(f)[1]} \\
X' & \xrightarrow{F(u')} & Y' & \xrightarrow{F(v')} & Z' & \xrightarrow{F(w')} & X'[1].
\end{array}$$

This completes the proof.

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