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On Characterizations of Special Elements in Rings with Involution*

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Abstract Let R be a ring with involution. It is well-known that an EP element in R is a core invertible element, but the question when a core invertible element is an EP element, the authors answer in this paper. Several new characterizations of star-core, normal and Hermitian elements in R are also presented.

Keywords Moore-Penrose inverse, Core inverse, EP element, Star-core element, Normal element, Hermitian element
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1 Introduction

The core inverse and the dual core inverse for a complex matrix were introduced by Baksalary and Trenkler in [1]. Recently, Rakić et al. in [12] generalized core inverse of a complex matrix to the case of an element in a ring. For $a, x \in R$, if

$$axa = a, \quad xR = aR, \quad Rx = Ra^*,$$

then x is called a core inverse of a and if such an element x exists, then it is unique and denoted by $a^{\#}$. The set of all core invertible elements of R will be denoted by $R^{\#}$.

In [12], Rakić et al. gave several equivalent conditions for a core invertible element in a ring with involution to be an EP element. In Section 3, we will present more equivalent conditions which ensure that a core invertible element in a ring with involution is an EP element. Star-dagger element was introduced by Hartwig and Spindelböck [5]. In [8–9], Mosić and Djordjević investigated various equivalent conditions for an element to be star-dagger element, normal element, Hermitian element and partial isometry in the setting of rings. Motivated by [8–9], we give the definition of a star-core element. The results in [8–9] are under the hypothesis $a \in R^{\dagger}$. In Section 4, we will give several equivalent conditions which ensure that an element a of a ring

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with involution R is a star-core, normal, Hermitian element or $a^* = a^{\#}$, under the hypothesis $a \in R^{\#}$.

2 Definitions and Notations

Let R be a ring with involution, that is a ring with unity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$.

An element $a \in R$ is called normal if $aa^* = a^*a$. An element $a \in R$ is called Hermitian if $a^* = a$. An element $a \in R$ (R is not necessary to be a ring with involution) is said to be group invertible if there exists $b \in R$ such that the following equations equations hold:

$$aba = a, \quad bab = b, \quad ab = ba.$$

The element b which satisfies the above equations is called a group inverse of a. If such an element b exists, then it is unique and denoted by $a^{\#}$. The set of all group invertible elements of R will be denoted by $R^{\#}$. We say that $b \in R$ is the Moore-Penrose inverse of $a \in R$, if the following equations hold:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

There is at most one b such that above four equations hold. If such an element b exists, then it is denoted by $a^{\dagger}$. The set of all Moore-Penrose invertible elements of R will be denoted by $R^{\dagger}$. An element $a \in R$ is said to be an EP element if $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} = a^{\#}$. The set of all EP elements of R will be denoted by R^{EP}.

An element $b \in R$ is an inner inverse of $a \in R$ if $aba = a$ holds. The set of all inner inverses of a will be denoted by $a\{1\}$. Set $a\{1,2\} = \{b \in R : aba = a \text{ and } bab = b\}$. An element $\widetilde{a} \in R$ is called a $\{1,3\}$-inverse of a if we have $a\widetilde{a}a = a$, $(a\widetilde{a})^* = a\widetilde{a}$. Let $a\{1,3\} = \{\widetilde{a} \in R : a\widetilde{a}a = a \text{ and } (a\widetilde{a})^* = a\widetilde{a}\}$. The set of all $\{1,3\}$-invertible elements of R will be denoted by $R^{\{1,3\}}$. Similarly, an element $\widehat{a} \in R$ is called a $\{1,4\}$-inverse of a if $a\widehat{a}a = a$, $(\widehat{a}a)^* = \widehat{a}a$. The set of all $\{1,4\}$-invertible elements of R will be denoted by $R^{\{1,4\}}$. Also, denote by $a\{1,4\} = \{\widehat{a} \in R : a\widehat{a}a = a \text{ and } (\widehat{a}a)^* = \widehat{a}a\}$.

We will also use the following notations: $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, ${}^{\circ}a = \{x \in R \mid xa = 0\}$, $a^{\circ} = \{x \in R \mid ax = 0\}$ and $[a,b] = ab - ba$.

3 When a Core Invertible Element is an EP Element

It is well-known that an EP element in R is a core invertible element, but the question when a core invertible element is an EP element, we answer in this section. Let us begin this section with three lemmas which will be used in the rest of the paper.

Lemma 3.1 (cf. [7, Theorem 7.3], [12, Theorem 3.1]) *Let $a \in R$. Then the following statements are equivalent*:

(i) $a \in R^{\mathrm{EP}}$;

(ii) $a \in R^{\#}$ *and* $[a, a^{\#}] = 0$;

(iii) $a \in R^{\dagger} \cap R^{\#}$ *and* $a^{\dagger} = a^{\oplus}$;

(iv) $a \in R^{\oplus}$ *and* $a^{\#} = a^{\oplus}$;

(v) $a \in R^{\#}$ *and* $a^{\#}a$ *is Hermitian (cf. [7, Theorem 7.3]).*

Lemma 3.2 (cf. [13, Theorem 2.6, Theorem 3.1]) *Let* $a \in R$. *Then the following conditions are equivalent*:

(i) $a \in R^{\oplus}$;

(ii) $a \in R^{\#} \cap R^{\{1,3\}}$;

(iii) *There exists* $x \in R$ *such that* $(ax)^* = ax$, $xa^2 = a$ *and* $ax^2 = x$.

In this case, $x = a^{\oplus} = a^{\#}aa^{(1,3)}$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$.

In the following theorem, we present 24 necessary and sufficient conditions for an element a of a ring with involution to be EP in the case that $a \in R^{\oplus}$. By Lemma 3.2, $a \in R^{\oplus}$ if and only if $a \in R^{\#} \cap R^{\{1,3\}}$. Thus, the next characterizations of EP elements involve the assumption $a \in R^{\{1,3\}}$ instead of stronger condition $a \in R^{\dagger}$ which appears in characterizations proved in [10–11].

Theorem 3.1 *Let* $m, n \in N$. *An element* $a \in R$ *is EP if and only if* $a \in R^{\oplus}$ *and one of the following equivalent conditions holds*:

(i) $a^{\oplus}a$ *is Hermitian*;

(ii) $(a^{\#})^{n+m-1} = (a^{\#})^{n-1}(a^{\oplus})^m$;

(iii) $a(a^{\#})^n(a^{\oplus})^m = a^{\oplus}a(a^{\#})^{n+m-1}$;

(iv) $(a^*)^n a^{\oplus}a = (a^*)^n$;

(v) $a^{\oplus}a(a^*)^n = (a^*)^n$;

(vi) $a^{\oplus}a(a^*)^n = (a^*)^n a^{\oplus}a$;

(vii) $(a^{\oplus})^2(a^{\#})^n = a^{\oplus}(a^{\#})^n a^{\oplus}$;

(viii) $(a^{\#})^{n+1}a^{(1,3)} = a^{\#}a^{(1,3)}(a^{\#})^n$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$;

(ix) $(a^{\#})^n = (a^{\oplus})^n$;

(x) $a^*(a^{\oplus})^n = a^*(a^{\#})^n$;

(xi) $a^{\oplus}(a^{\#})^n = (a^{\#})^n a^{\oplus}$;

(xii) $(a^{\oplus})^{n+1} = a^{\oplus}(a^{\#})^n$;

(xiii) $aa^{(1,3)}(a^*)^n = (a^*)^n$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$;

(xiv) $aa^{(1,3)}(a^*)^n a^m = (a^*)^n a^m aa^{(1,3)}$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$;

(xv) $aa^{(1,3)}(a^m(a^*)^n - (a^*)^n a^m) = (a^m(a^*)^n - (a^*)^n a^m)aa^{(1,3)}$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$;

(xvi) $(a^*)^n a^{\#}a + aa^{\#}(a^*)^n = 2(a^*)^n$;

(xvii) $a^{\oplus}(a^{\#})^n a + aa^{\#}(a^{\oplus})^n = 2(a^{\oplus})^n$;

(xviii) $a^n aa^{\oplus} + a^{\oplus}aa^n = 2a^n$;

(xix) $a^n aa^{(1,3)} + (a^n aa^{(1,3)})^* = a^n + (a^*)^n$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$;

(xx) $a^n = a^n aa^{(1,3)}$ *for arbitrary* $a^{(1,3)} \in a\{1,3\}$;

(xxi) $a^n a^{\oplus} = a^{\oplus}a^n$;

(xxii) $[(a^{\oplus})^*]^n = [(a^{\oplus})^*]^n a^{\oplus}a$;

(xxiii) $a \in Ra^{\oplus}$;

(xxiv) $a \in R^{-1}a^{\oplus}$.

Proof If a is EP, then $a^{\oplus} = a^{\#} = a^{\dagger}$ by Lemma 3.1 and $aa^{\dagger} = aa^{(1,3)}$ for arbitrary $a^{(1,3)} \in a\{1,3\}$. It is not difficult to verify that conditions (i)–(xxiii) hold. Also, we have that $a = (a^2 + 1 - a^{\oplus}a)a^{\oplus}$, where $a^2 + 1 - a^{\oplus}a \in R^{-1}$ and $a^2 + 1 - a^{\oplus}a = ((a^{\oplus})^2 + 1 - a^{\oplus}a))^{-1}$. So, (xxiv) is satisfied.

Conversely, we suppose that $a \in R^{\#} \cap R^{\{1,3\}}$. To conclude that a is EP, we show that one of the conditions of Lemma 3.1 is satisfied, or that the element a is subject to one of the preceding already established conditions of this theorem.

(i) Because $a^{\oplus}a$ is Hermitian, then $aa^{\#} = a^{\oplus}a^2a^{\#} = a^{\oplus}a$ is Hermitian. By Lemma 3.1, we deduce that a is EP.

(ii) From $(a^{\#})^{n+m-1} = (a^{\#})^{n-1}(a^{\oplus})^m$, we get

$$\begin{aligned} aa^{\#} &= a^{n+m-1}(a^{\#})^{n+m-1} = a^{n+m-1}(a^{\#})^{n-1}(a^{\oplus})^m \\ &= a^{n+m-1}(a^{\#})^{n-1}(a^{\oplus})^m aa^{\oplus} = a^{n+m-1}(a^{\#})^{n+m-1}aa^{\oplus} = aa^{\oplus}. \end{aligned}$$

Since $aa^{\oplus}$ is Hermitian, we have that $aa^{\#}$ is Hermitian too.

(iii) Multiplying the equality $a(a^{\#})^n(a^{\oplus})^m = a^{\oplus}a(a^{\#})^{n+m-1}$ by $a^{\#}a$ from the left side, we obtain that (ii) holds.

(iv) As in part (i), $a^{\oplus}a = aa^{\#} = a^n(a^{\#})^n$. By the hypothesis $(a^*)^n a^{\oplus}a = (a^*)^n$, we obtain $(a^{\oplus}a)^* = [(a^{\#})^n]^*(a^*)^n = [(a^{\#})^n]^*(a^*)^n a^{\oplus}a = (a^{\oplus}a)^*a^{\oplus}a$. Since $(a^{\oplus}a)^*a^{\oplus}a$ is Hermitian, we conclude that $a^{\oplus}a$ is Hermitian too, i.e., (i) is satisfied.

(v) Similarly as (iv).

(vi) The hypothesis $a^{\oplus}a(a^*)^n = (a^*)^n a^{\oplus}a$ implies

$$(a^*)^n = (a^*)^n aa^{\oplus} = ((a^*)^n a^{\oplus}a)aa^{\oplus} = a^{\oplus}a(a^*)^n aa^{\oplus} = a^{\oplus}a(a^*)^n.$$

So, the condition (v) is satisfied.

(vii) Applying $(a^{\oplus})^2(a^{\#})^n = a^{\oplus}(a^{\#})^n a^{\oplus}$, we get

$$\begin{aligned} a^{\oplus}(a^{\#})^n a^{\oplus} &= (a^{\oplus})^2(a^{\#})^n = ((a^{\oplus})^2(a^{\#})^n)aa^{\#} \\ &= a^{\oplus}(a^{\#})^n aa^{\#} = a^{\oplus}(a^{\#})^{n+1}aa^{\oplus}aa^{\#} \\ &= a^{\oplus}(a^{\#})^{n+1} \end{aligned}$$

implying

$$\begin{aligned} aa^{\oplus} &= a^{n+1}aa^{\oplus}a(a^{\#})^{n+1}a^{\oplus} = a^{n+2}(a^{\oplus}(a^{\#})^n a^{\oplus}) \\ &= a^{n+2}a^{\oplus}(a^{\#})^{n+1} = a^{n+1}aa^{\oplus}a(a^{\#})^{n+2} = aa^{\#}. \end{aligned}$$

Hence, the element $aa^{\#}$ is Hermitian and, by Lemma 3.1, a is EP.

(viii) Multiplying $(a^{\#})^{n+1}a^{(1,3)} = a^{\#}a^{(1,3)}(a^{\#})^n$ from the left side by a^{n+2}, we get $aa^{(1,3)} = a^{n+1}a^{(1,3)}a(a^{\#})^{n+1} = aa^{\#}$, that is, $aa^{\#}$ is Hermitian.

(ix) Assume that $(a^{\#})^n = (a^{\oplus})^n$. Then $aa^{\#} = a^n(a^{\#})^n = a^n(a^{\oplus})^n = aa^{\oplus}$ is Hermitian and so a is EP.

(x) Using $a^*(a^{\oplus})^n = a^*(a^{\#})^n$, we have that condition (ix) is satisfied:

$$(a^{\oplus})^n = aa^{\oplus}(a^{\oplus})^n = (a^{\oplus})^*a^*(a^{\oplus})^n = (a^{\oplus})^*a^*(a^{\#})^n$$
$$= aa^{\oplus}a(a^{\#})^{n+1} = (a^{\#})^n.$$

(xi) The equality $a^{\oplus}(a^{\#})^n = (a^{\#})^n a^{\oplus}$ gives

$$aa^{\#} = a^n aa^{\oplus}a(a^{\#})^{n+1} = a^{n+1}(a^{\oplus}(a^{\#})^n) = a^{n+1}(a^{\#})^n a^{\oplus} = aa^{\oplus},$$

i.e., $aa^{\#}$ is Hermitian.

(xii) Multiplying $(a^{\oplus})^{n+1} = a^{\oplus}(a^{\#})^n$ by a^*a from the left side, we obtain (x).

(xiii) Applying involution to $aa^{(1,3)}(a^*)^n = (a^*)^n$, we have $a^n aa^{(1,3)} = a^n$. Multiplying the last equality by $(a^{\#})^n$ from the left side, we get $aa^{(1,3)} = a^{\#}a$ and $a^{\#}a$ is Hermitian.

(xiv) Multiplying $aa^{(1,3)}(a^*)^n a^m = (a^*)^n a^m aa^{(1,3)}$ by $a(a^{\#})^m$ from the right side, we get $aa^{(1,3)}(a^*)^n a = (a^*)^n a$. If we multiply the last equality by $a^{(1,3)}$ from the right side, the condition (xiii) $aa^{(1,3)}(a^*)^n = (a^*)^n$ is satisfied.

(xv) From $aa^{(1,3)}(a^m(a^*)^n - (a^*)^n a^m) = (a^m(a^*)^n - (a^*)^n a^m)aa^{(1,3)}$, we observe that

$$a^m(a^*)^n - aa^{(1,3)}(a^*)^n a^m = a^m(a^*)^n - (a^*)^n a^m aa^{(1,3)},$$

that is, $aa^{(1,3)}(a^*)^n a^m = (a^*)^n a^m aa^{(1,3)}$. So, the equality (xiv) holds.

(xvi) Multiplying the hypothesis $(a^*)^n a^{\#}a + aa^{\#}(a^*)^n = 2(a^*)^n$ by $aa^{\oplus}$ from the right side, we get $(a^*)^n = aa^{\#}(a^*)^n = a^{\oplus}a(a^*)^n$, i.e., (v) is satisfied.

(xvii) If we multiply $a^{\oplus}(a^{\#})^n a + aa^{\#}(a^{\oplus})^n = 2(a^{\oplus})^n$ by $a^{\#}a$ from the left side, we get $(a^{\oplus})^n = a^{\oplus}a(a^{\#})^n = aa^{\#}(a^{\#})^n = (a^{\#})^n$. Thus, the condition (ix) holds.

(xviii) Multiplying $a^n aa^{\oplus} + a^{\oplus}aa^n = 2a^n$ by $(a^{\#})^n$ from the left side, we have that $a^{\#}a = aa^{\oplus}$ is Hermitian.

(xix) When we multiply $a^n aa^{(1,3)} + (a^n aa^{(1,3)})^* = a^n + (a^*)^n$ by $aa^{(1,3)}$ from the right side, we obtain $aa^{(1,3)}(a^*)^n = (a^*)^n$, that is, condition (xiii) holds.

(xx) Applying involution to $a^n = a^n aa^{(1,3)}$, we get (xiii).

(xxi) The assumption $a^n a^{\oplus} = a^{\oplus}a^n$ yields

$$a^{\#}a = (a^{\#})^n a^n = (a^{\#})^n a(a^{\oplus}a^n) = (a^{\#})^n aa^n a^{\oplus} = aa^{\oplus}.$$

So, $a^{\#}a$ is Hermitian.

(xxii) Applying involution to $[(a^{\oplus})^*]^n = [(a^{\oplus})^*]^n a^{\oplus}a$, note that $(a^{\oplus})^n = (a^{\oplus}a)^*(a^{\oplus})^n$ which gives $a^{\oplus}a = (a^{\oplus})^n a^n = (a^{\oplus}a)^*(a^{\oplus})^n a^n = (a^{\oplus}a)^*a^{\oplus}a$. Since $(a^{\oplus}a)^*a^{\oplus}a$ is Hermitian and so is $a^{\oplus}a$.

(xxiii) Since $a \in Ra^{\oplus},\ a = aaa^{\oplus}$ gives that $a^{\#}a = aa^{\oplus}$ is Hermitian.

(xxiv) The conditions $a \in R^{-1}a^{\oplus}$ implies (xxiii).

Recall that, for $a \in R$, if there exists $x \in R$ such that $axa = a$, $xR = a^*R$ and $Rx = Ra$, then the element x is the dual core inverse of a (cf. [12]). The dual core inverse of a is unique, if it exists and will be denoted by $a_{\oplus}$. If we denote by $R_{\oplus}$ the set of all dual core invertible elements of R, observe that $a \in R_{\oplus}$ if and only if $a \in R^{\#} \cap R^{\{1,4\}}$ (cf. [13]). Similarly as in the proof of Theorem 3.1, we can get new equivalent conditions for an element $a \in R^{\#} \cap R^{\{1,4\}}$ to be EP.

4 Star-Core, Normal and Hermitian Elements in Rings

In this section, some fundamental properties and characterizations of star-core, normal and Hermitian elements in rings are obtained. Let us start this section with an example.

Example 4.1 Let R be the ring of all 2×2 matrices over the real field with involution as transpose. Taking $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \in R$, then $M^{\oplus} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $M^{\oplus}M^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M^*M^{\oplus} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Thus $M^{\oplus}M^* \neq M^*M^{\oplus}$.

Since the equality $a^{\oplus}a^* = a^*a^{\oplus}$ is not true in general, we introduce a star-core element in the following definition.

Definition 4.1 *Let $a \in R^{\oplus}$. Then a is called a star-core (SC for short) element if $a^{\oplus}a^* = a^*a^{\oplus}$.*

Lemma 4.1 (cf. [2, Corollary 3.4]) *Let $a \in R^{\oplus}$ and $b \in R$ such that $[a, b] = 0$ and $[a^*, b] = 0$. Then $[a^{\oplus}, b] = 0$.*

From [2, Corollary 3.4, Theorem 3.5], we have the following proposition. For the convenience to readers, here we give the proof.

Proposition 4.1 *Let $a, b \in R^{\oplus}$ such that $[a, b] = 0$ and $[a^*, b] = 0$. Then $[a^{\oplus}, b] = [a^{\oplus}, b^*] = [a, b^{\oplus}] = [a^*, b^{\oplus}] = [a^{\oplus}, b^{\oplus}] = 0$.*

Proof Suppose that $a, b \in R^{\oplus}$ satisfy $[a, b] = 0$ and $[a^*, b] = 0$. Then, by Lemma 4.1, we deduce that $[a^{\oplus}, b] = 0$. Taking involution on $[a, b] = 0$ and $[a^*, b] = 0$, we get $[a^*, b^*] = 0$ and $[a, b^*] = 0$, respectively. From $[a, b] = 0$, $[a, b^*] = 0$ and Lemma 4.1, we obtain $[a, b^{\oplus}] = 0$. By $[a^*, b^*] = 0$, $[a, b^*] = 0$ and Lemma 4.1, notice that $[a^{\oplus}, b^*] = 0$. So, by $[a^*, b^*] = 0$, $[a^*, b] = 0$ and Lemma 4.1, we have $[a^*, b^{\oplus}] = 0$.

Under assumption $a \in R^{\oplus}$, we now characterize normal elements of rings with involution.

Theorem 4.1 *Let $a \in R^{\oplus}$. Then the following conditions are equivalent:*

(i) *a is normal;*

(ii) *$a \in R^{\mathrm{EP}}$ and a is SC;*

(iii) *$a \in R^{\{1,4\}}$ and $[aa^*, (a^{\oplus})^2 a] = 0$;*

(iv) *$a \in R^{\{1,4\}}$ and $[a^*, (a^{\oplus})^2 a] = 0$.*

Proof (i) $\Rightarrow$ (ii). If a is normal, then $[a, a] = [a^*, a] = 0$. Thus, by Proposition 4.1, $[a, a^{\oplus}] = 0$, notice that $a \in R^{\mathrm{EP}}$, by Lemma 3.1. Taking involution on $[a, a] = [a^*, a] = 0$, we get $[a^*, a^*] = [a^*, a] = 0$. By Proposition 4.1, we have $[a^*, a^{\oplus}] = 0$, that is, a is SC.

(ii) $\Rightarrow$ (iii). Suppose that $a \in R^{\mathrm{EP}}$ and a is SC. Then, by Lemma 3.1, we observe that $a^{\dagger} = a^{\#} = a^{\oplus}$. Thus

$$aa^*(a^{\oplus})^2 a = aa^*(a^{\#})^2 a = aa^*a^{\#} = aa^*a^{\oplus} = aa^{\oplus}a^* = a^*,$$
$$a^* = (aa^{\dagger}a)^* = a^{\dagger}aa^* = a^{\oplus}aa^* = (a^{\oplus})^2 a^2 a^* = ((a^{\oplus})^2 a)aa^*.$$

Hence, $a \in R^{\{1,4\}}$ and $[aa^*, (a^{\oplus})^2 a] = 0$.

(iii) $\Rightarrow$ (iv). The assumptions $a \in R^{\{1,4\}}$ and $[aa^*, (a^{\oplus})^2 a] = 0$ yield $aa^*(a^{\oplus})^2 a = (a^{\oplus})^2 a^2 a^* = a^{\oplus} aa^*$ and for $a^{(1,4)} \in a\{1,4\}$,

$$\begin{aligned} a^{(1,4)}a^2a^* &= a^{(1,4)}a^2(aa^{\oplus}a)^* = a^{(1,4)}a^2a^*aa^{\oplus} = a^{(1,4)}a^2a^*aa^{\oplus}aa^{\oplus} \\ &= a^{(1,4)}a^2a^*a(a^{\oplus})^2a^2a^{\oplus} = a^{(1,4)}a^2a^*(a^{\oplus})^2a^3a^{\oplus} \\ &= a^{(1,4)}a(aa^*(a^{\oplus})^2a)a^2a^{\oplus} = a^{(1,4)}a(a^{\oplus}aa^*)a^2a^{\oplus} \\ &= a^{(1,4)}aa^*a^2a^{\oplus} = a^*a^2a^{\oplus}, \end{aligned}$$

that is, $a^{(1,4)}a^2a^* = a^*a^2a^{\oplus}$. Then

$$\begin{aligned} a^*aa^{\oplus} &= a^* = a^{(1,4)}aa^* = a^{(1,4)}aa^{\oplus}aa^* = a^{(1,4)}a(a^{\oplus}aa^*) \\ &= a^{(1,4)}a(aa^*(a^{\oplus})^2a) = (a^{(1,4)}a^2a^*)(a^{\oplus})^2a \\ &= a^*a^2a^{\oplus}(a^{\oplus})^2a = a^*a(a^{\oplus})^2a = a^*a^{\oplus}a. \end{aligned}$$

So, $a^*aa^{\oplus} = a^*a^{\oplus}a = a^*a(a^{\oplus})^2a$, i.e., $a^*a(a^{\oplus} - (a^{\oplus})^2a) = 0$ implying

$$aa^{\oplus} - a^{\oplus}a = aa^{\oplus}a(a^{\oplus} - (a^{\oplus})^2a) = (a^{\oplus})^*a^*a(a^{\oplus} - (a^{\oplus})^2a) = 0.$$

Hence, $aa^{\oplus} = a^{\oplus}a$ and, by Lemma 3.1, $a \in R^{\mathrm{EP}}$ and $a^{\#} = a^{\oplus}$. Now

$$a^*a = aa^{\oplus}a^*a = a^{\oplus}aa^*a = (aa^*(a^{\oplus})^2a)a = aa^*a^{\oplus}a = aa^*aa^{\oplus} = aa^*$$

gives $a^*(a^{\oplus})^2a = a^*(a^{\#})^2a = a^*a^{\#} = a^{\#}a^* = (a^{\#})^2aa^* = (a^{\oplus})^2aa^*$, that is, $[a^*, (a^{\oplus})^2a] = 0$.

(iv) $\Rightarrow$ (i). From $a \in R^{\{1,4\}}$ and $[a^*, (a^{\oplus})^2a] = 0$, we have $a^*(a^{\oplus})^2a = (a^{\oplus})^2aa^*$. Premultiplication of previous equation by a now yields

$$aa^*(a^{\oplus})^2a = a(a^{\oplus})^2aa^* = a^{\oplus}aa^* = (a^{\oplus})^2a^2a^* = (a^{\oplus})^2a(aa^*),$$

i.e., $[aa^*, (a^{\oplus})^2a] = 0$. As in part (iii) $\Rightarrow$ (iv), we get $a^*a = aa^*$ and so a is normal.

Some characterizations of Hermitian elements are proved in the next results in the cases that $a \in R^{\{1,3\}}$ or $a \in R^{\oplus}$.

Lemma 4.2 *Suppose that $a \in R^{\{1,3\}}$. Then a is Hermitian if and only if $aaa^{(1,3)} = a^*$ for arbitrary $a^{(1,3)} \in a\{1,3\}$.*

Proof If $a = a^*$ and $a^{(1,3)} \in R^{\{1,3\}}$, then $aaa^{(1,3)} = a^*aa^{(1,3)} = a^*$. Conversely, applying involution to $aaa^{(1,3)} = a^*$ for $a^{(1,3)} \in a\{1,3\}$, we obtain $a = aa^{(1,3)}a^* = aa^{(1,3)}aaa^{(1,3)} = aaa^{(1,3)} = a^*$.

Theorem 4.2 *Let $a \in R^{\#} \cap R^{\{1,3\}}$ and $m \in N$. Then a is Hermitian if and only if one of the following equivalent conditions holds*:

(i) $aa^m = a^*a^m$;

(ii) $a(a^{\oplus})^m = a^*(a^{\oplus})^m$;

(iii) $a^*(a^{\#})^m = a(a^{\oplus})^m$;

(iv) $a^*(a^{\oplus})^{m+1} = (a^{\oplus})^m$;

(v) $a(a^{\#})^m = a^*(a^{\#})^m$;

(vi) $a^*a^m a^{\oplus} = a^m$;
(vii) $a^*(a^{\#})^{m+1} = (a^{\#})^m$;
(viii) $a^*a^*a^{\#} = a^*$;
(ix) $a^*a^{\oplus}a^{\oplus} = a^{\#}$;
(x) $a^*a^{\oplus}a^{\#} = a^{\oplus}$;
(xi) $a^*a^{\oplus}a^{\#} = a^{\#}$;
(xii) $a^*a^{\#}a^{\#} = a^{\#}$;
(xiii) $a^{\#}a^*a^{\#} = a^{\oplus}$;
(xiv) $aa^*a^{\oplus} = a$;
(xv) $a^2a^{\oplus} = a^*$;
(xvi) $a^{\oplus}a^* = a^{\oplus}a$.

Proof If a is Hermitian, then it commutes with its core inverse by Lemma 4.1 and $a^{\oplus} = a^{\#} = a^{\dagger}$. It is not difficult to verify that conditions (i)–(xvi) hold.

Conversely, we show that a satisfies the equality $a = a^*$ or the conditions of Lemma 4.2, or one of the preceding, already established conditions of this theorem.

(i) The assumption $aa^m = a^*a^m$ implies

$$aaa^{(1,3)} = (aa^m)(a^{\#})^{m-1}a^{(1,3)} = a^*a^m(a^{\#})^{m-1}a^{(1,3)} = a^*aa^{(1,3)} = a^*$$

for arbitrary $a^{(1,3)} \in a\{1,3\}$. Thus, a is Hermitian by Lemma 4.2.

(ii) Applying $a(a^{\oplus})^m = a^*(a^{\oplus})^m$, we get (i):

$$aa^m = aa^{\oplus}aa^m = (a(a^{\oplus})^m)a^{2m} = a^*(a^{\oplus})^m a^{2m} = a^*a^m.$$

(iii)–(v) Similarly as (ii).

(iv) If we multiply $a^*(a^{\oplus})^{m+1} = (a^{\oplus})^m$ by a^{m+1} from the right side, we obtain $a^*a^{\oplus}a = a^{\oplus}a^2 = a$. Multiplying the last equality by $(a^{\oplus})^m$, we have that (ii) holds.

(vi) Multiplying the equality $a^*a^m a^{\oplus} = a^m$ by a from the right side, we obtain that condition (i) is satisfied.

(vii) In the same way as (iv).

(viii) Using the hypothesis $a^*a^*a^{\#} = a^*$, we observe that

$$aa^{\oplus} = (a^{\oplus})^*a^* = (a^{\oplus})^*a^*a^*a^{\#} = aa^{\oplus}a^*a^{\#} = aa^{\oplus}a^*a^{\#}aa^{\#} = aa^{\oplus}aa^{\#} = aa^{\#},$$

that is, $aa^{\#}$ is Hermitian. By Lemma 3.2, a is EP and

$$a = (aa^{\oplus})a = (aa^{\oplus})a^*(a^{\#}a) = a^{\dagger}aa^*aa^{\dagger} = a^*.$$

(ix) From $a^*a^{\oplus}a^{\oplus} = a^{\#}$, we conclude that

$$aa^{\#} = aa^*a^{\oplus}a^{\oplus} = a(a^*a^{\oplus}a^{\oplus})aa^{\oplus} = aa^{\#}aa^{\oplus} = aa^{\oplus}$$

is Hermitian and a is EP. Now $a = a^{\#}a^2 = a^*a^{\oplus}a^{\oplus}a^2 = a^*a^{\oplus}a = a^*aa^{\oplus} = a^*$.

(x) Multiplying $a^*a^{\oplus}a^{\#} = a^{\oplus}$ by a^{m+2} from the right side, we see that (i) holds.

(xi)–(xii) Similarly as (x).

(xiii) Assume that $a^{\#}a^*a^{\#} = a^{\oplus}$. Then $aa^{\#} = aa^{\oplus}aa^{\#} = aa^{\#}a^*a^{\#}aa^{\#} = a(a^{\#}a^*a^{\#}) = aa^{\oplus}$ is Hermitian, which implies that a is EP. Therefore

$$a = aa^{\oplus}a = aa^{\#}a^*a^{\#}a = a^{\dagger}aa^*aa^{\dagger} = a^*.$$

(xiv) The hypothesis $aa^*a^{\oplus} = a$ gives $a^{\#}a = a^{\#}aa^*a^{\oplus} = a^{\#}(aa^*a^{\oplus})aa^{\oplus} = a^{\#}aaa^{\oplus} = aa^{\oplus}$, i.e., a is EP. Thus, $a = a^{\#}a^2 = a^{\dagger}aa^*a^{\oplus}a = a^*$.

(xv) By $a^2a^{\oplus} = a^*$, we get $a = (a^*)^* = (a^2a^{\oplus})^* = aa^{\oplus}a^* = aa^{\oplus}a^2a^{\oplus} = a^2a^{\oplus} = a^*$.

(xvi) Since $a^{\oplus}a^* = a^{\oplus}a$, (xv) holds: $a^* = (aa^{\oplus}a)^* = (aa^{\oplus}a^*)^* = a^2a^{\oplus}$.

Necessary and sufficient conditions for an element $a \in R^{\oplus}$ to satisfy the equality $a^* = a^{\oplus}$ are given in the following result.

Theorem 4.3 *Suppose that $a \in R^{\#} \cap R^{\{1,3\}}$ and let $n \in N$. Then $a^* = a^{\oplus}$ if and only if one of the following equivalent conditions holds*:

(i) $a^*a^n = a^{\oplus}a^n$;

(ii) $a^*(a^{\#})^n = a^{\oplus}(a^{\#})^n$;

(iii) *a is EP and a partial isometry;*

(iv) $a^*a^n = a^na^{\oplus}$;

(v) $a^*a^n = a^na^{\#}$;

(vi) $a^*(a^{\oplus})^n = a^{\oplus}(a^{\#})^n$;

(vii) $a^*(a^{\oplus})^n = (a^{\#})^na^{\oplus}$;

(viii) $a^*(a^{\#})^n = (a^{\#})^na^{\oplus}$;

(ix) $a^*(a^{\oplus})^n = (a^{\#})^{n+1}$;

(x) $a^*(a^{\#})^n = (a^{\oplus})^{n+1}$;

(xi) $a^*(a^{\#})^n = (a^{\#})^{n+1}$;

(xii) $a^*a^{n+1} = a^n$;

(xiii) $a^{n+1}a^* = a^n$;

(xiv) $a^*(a^{\oplus})^na = (a^{\#})^n$;

(xv) $a(a^{\oplus})^na^* = (a^{\#})^n$;

(xvi) $a^*(a^{\oplus})^n = (a^{\oplus})^{n+1}$;

(xvii) $a^{\oplus}a^*a = a^{\oplus}$.

Proof If $a^* = a^{\oplus}$, it is not difficult to check that conditions (i)–(ii) hold. Since $a^{\oplus}a = a^*a$ is Hermitian, a is EP, by Theorem 3.1, and $a^* = a^{\oplus} = a^{\#} = a^{\dagger}$. Thus, a is a partial isometry and (iii)–(xvii) are satisfied.

(i) Multiplying $a^*a^n = a^{\oplus}a^n$ by $(a^{\oplus})^n$ from the right side, we obtain $a^*aa^{\oplus} = a^{\oplus}aa^{\oplus}$, i.e., $a^* = a^{\oplus}$.

(ii) If we multiply the assumption $a^*(a^{\#})^n = a^{\oplus}(a^{\#})^n$ by $a^{n+1}a^{\oplus}$ from the right side, we get $a^*aa^{\oplus} = a^{\oplus}aa^{\oplus}$, that is, $a^* = a^{\oplus}$.

(iii) Obviously, $a^* = a^{\dagger} = a^{\oplus}$.

(iv) Using $a^*a^n = a^na^{\oplus}$, we obtain

$$a^* = a^*aa^{\oplus} = (a^*a^n)(a^{\oplus})^n = a^na^{\oplus}(a^{\oplus})^n = a^n(a^{\oplus})^na^{\oplus} = a(a^{\oplus})^2 = a^{\oplus}.$$

(v) The equality $a^*a^n = a^n a^\#$ gives

$$a^* = a^*aa^{\oplus} = (a^*a^n)(a^{\oplus})^n = a^n a^\# a(a^{\oplus})^{n+1} = a^n(a^{\oplus})^{n+1} = a^{\oplus}.$$

(vi) Multiplying $a^*(a^{\oplus})^n = a^{\oplus}(a^\#)^n$ by a^{n+1} from the right side, we observe that $a^*a^{\oplus}a^2 = a^{\oplus}a$ which yields $a^* = a^*aa^{\oplus} = (a^*a^{\oplus}a^2)a^{\oplus} = a^{\oplus}aa^{\oplus} = a^{\oplus}$.

(vii) From the hypothesis $a^*(a^{\oplus})^n = (a^\#)^n a^{\oplus}$, we get

$$a^* = (a^*(a^{\oplus})^n)a^{n+1}a^{\oplus} = (a^\#)^n a^{\oplus}a^{n+1}a^{\oplus} = a^\# aa^{\oplus} = a^{\oplus}aa^{\oplus} = a^{\oplus}.$$

(viii) Applying $a^*(a^\#)^n = (a^\#)^n a^{\oplus}$, notice that (vii) is satisfied:

$$\begin{aligned}(a^\#)^n a^{\oplus} &= ((a^\#)^n a^{\oplus})aa^{\oplus} = a^*(a^\#)^n aa^{\oplus} = a^*(a^\#)^n a^n(a^{\oplus})^n \\ &= a^*a^\# a(a^{\oplus})^n = a^*a^{\oplus}a(a^{\oplus})^n = a^*(a^{\oplus})^n.\end{aligned}$$

(ix) Multiplying $a^*(a^{\oplus})^n = (a^\#)^{n+1}$ by $aa^{\oplus}$ from the right side, we see that (vii) holds.

(x) If we multiply $a^*(a^\#)^n = (a^{\oplus})^{n+1}$ by $a^{n+1}a^{\oplus}$ from the right side, we obtain $a^* = a^{\oplus}$.

(xi) Multiplying $a^*(a^\#)^n = (a^\#)^{n+1}$ by a^{2n} from the right side, we have that (v) is satisfied.

(xii) Multiplying $a^*a^{n+1} = a^n$ by $a^\#$ from the right side, we obtain (v).

(xiv) If we multiply $a^*(a^{\oplus})^n a = (a^\#)^n$ by $a^{\oplus}$ from the right side, we have that (vii) holds.

(xiii) and (xv) Similarly as (xii) and (xiv), respectively.

(xvi) The assumption $a^*(a^{\oplus})^n = (a^{\oplus})^{n+1}$ implies that (i) holds:

$$a^*a^n = a^*a^{\oplus}a^{n+1} = a^*(a^{\oplus})^n a^{2n} = (a^{\oplus})^{n+1}a^{2n} = a^{\oplus}a^n.$$

(xvii) Using $a^{\oplus}a^*a = a^{\oplus}$, we obtain $a^* = (aa^{\oplus}a)^* = a^*aa^{\oplus} = a^*aa^{\oplus}a^*a = a^*a^*a$, which gives $a = (a^*)^* = (a^*a^*a)^* = a^*a^2$. Therefore, $a^{\oplus} = a(a^{\oplus})^2 = a^*a^2(a^{\oplus})^2 = a^*$.

Example 4.2 Neither the implication a is normal implies $a \in R^{\mathrm{EP}}$ nor $a \in R^{\mathrm{EP}}$ implies a is normal is valid in general rings.

(I) In $\mathbb{Z}_{12}$, taking the identity map as involution, then $a = 2$ is normal and $a = 2$ is not an EP element.

(II) Let $M_2(\mathcal{F})$ be the ring of all 2×2 matrices over the real field $\mathcal{F}$. Taking conjugate transpose as involution, if we consider the following matrix $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $M^\dagger = M^\# = M^{-1}$. Thus M is an EP matrix. Yet, we have $MM^* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = M^*M$.

In the next result, for $a \in R^{\oplus}$ which satisfies $a^* = a^{\oplus}$, notice that a is EP if and only if a is normal.

Corollary 4.1 *Let $a \in R^{\oplus}$ such that $a^* = a^{\oplus}$. Then the following conditions are equivalent*:

(i) $a \in R^{\mathrm{EP}}$;

(ii) *a is normal*;

(iii) $a^* = (a^{\oplus})^2 a$.

Proof (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Suppose that $a^* = a^{\oplus}$ and $a \in R^{\mathrm{EP}}$. Then by Lemma 3.1, we have $a^* = a^\# = a^{\oplus}$. Thus, (ii) and (iii) hold.

(ii) $\Rightarrow$ (i). It is easy to see by Theorem 4.1.

(iii) $\Rightarrow$ (i). The hypothesis $a^* = (a^{\oplus})^2 a$ yields

$$aa^* = a(a^{\oplus})^2 a = a^{\oplus} a = (a^{\oplus})^2 a^2 = ((a^{\oplus})^2 a)a = a^* a.$$

Several sufficient conditions for an element a to be star-core are considered now.

Proposition 4.2 *Let $a \in R^{\oplus}$. Then each of the following conditions is sufficient for a to be SC*:

(i) $a^* = a$;

(ii) $a^* = a^* a^{\oplus}$;

(iii) $a^{\oplus} = (a^{\oplus})^2$;

(iv) $a^* = (a^{\oplus})^2$;

(v) $a^{\oplus} = (a^*)^2$.

Proof (i) If $a^* = a$, we deduce that a is normal and, by Theorem 4.1, a is a SC.

(ii) Assume that $a^* = a^* a^{\oplus}$. Then $a^* a^{\oplus} = a^* = (aa^{\oplus} a)^* = a^* aa^{\oplus}$ implies

$$a^* aa^{\oplus} = a^* a^{\oplus} = (aa^{\oplus} a)^* a^{\oplus} = a^* aa^{\oplus} a^{\oplus}.$$

Since $a \in R^{\oplus}$, a is $*$-cancellable. Therefore, $aa^{\oplus} = a(a^{\oplus})^2 = a$ gives that a is Hermitian and the condition (i) is satisfied.

(iii) By $a^{\oplus} = (a^{\oplus})^2$, we conclude that (ii) holds:

$$a^*(1 - a^{\oplus}) = (aa^{\oplus} a)^*(1 - a^{\oplus}) = a^* aa^{\oplus}(1 - a^{\oplus}) = a^* a(a^{\oplus} - (a^{\oplus})^2) = 0.$$

(iv) The equality $a^* = (a^{\oplus})^2$ yields $a^* a^{\oplus} = (a^{\oplus})^3 = a^{\oplus}(a^{\oplus})^2 = a^{\oplus} a^*$.

(v) From $a^{\oplus} = (a^*)^2$, we get $a^* a^{\oplus} = (a^*)^3 = (a^*)^2 a^* = a^{\oplus} a^*$.

Definition 4.2 (cf. [5]) *An element $a \in R^{\dagger}$ satisfying $[a^*, a^{\dagger}] = 0$ is called star-dagger (SD for short).*

An infinite matrix M is said to be bi-finite if it is both row-finite and column-finite.

Example 4.3 If a is SD, then a need not to be SC in general rings. Let R be the ring of all bi-finite matrices over field $\mathcal{F}$ with transpose as involution and $e_{i,j}$ be the matrix in R with 1 in the (i, j) position and 0 elsewhere. Consider the following matrices A and B over R: $A = \sum\limits_{i=1}^{\infty} e_{i,i+1}$ and $B = A^*$, then $BA = \sum\limits_{i=2}^{\infty} e_{i,i}$ and $AB = I$. So, $ABA = A$, $BAB = B$ $(AB)^* = AB$ and $(BA)^* = BA$. Thus, $A^* A^{\dagger} = B^2 = A^{\dagger} A^*$, that is, A is SD. Since A is not group invertible, A is not core invertible, which yields A is not SC.

Lemma 4.3 (cf. [14, Theorem 2.16]) *Let $a \in R$. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$;

(ii) $a \in aa^* aR$;

(iii) $a \in Raa^* a$.

In this case, $a^{\dagger} = (ar)^* ara^* = a^* sa(sa)^*$, where $a = aa^* ar = saa^* a$ for some $r, s \in R$.

Lemma 4.4 *Let $a \in R$. Then we have $a \in R^{\oplus}$ if and only if $a^* \in R_{\oplus}$. In this case, $(a^*)_{\oplus} = (a^{\oplus})^*$.*

Proof Let $a \in R^{\oplus}$ with core inverse x, then

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a, \quad ax^2 = x. \tag{4.1}$$

Taking involution on the equality (4.1), we have $a^*x^*a^* = a^*$, $x^*a^*x^* = x^*$, $x^*a^* = ax = (x^*a^*)^*$, $a^* = (a^*)^2x^*$, $x^* = (x^*)^2a^*$, hence $a_{\oplus} = x^*$.

By related characterizations and equivalent conditions of Moore-Penrose inverse, group inverse, (dual) core inverse and EP element in [2–6, 9, 11] etc. Some of the equivalences in the following lemma were proved for matrices, operators and elements of rings. We collect these results here.

Lemma 4.5 *Let $a \in R$. If $a^* = a$, then the following conditions are equivalent*:
(i) $a \in R^{\mathrm{EP}}$;
(ii) $a \in R^{\dagger}$;
(iii) $a \in R^{\#}$;
(iv) $a \in R^{\oplus}$;
(v) $a \in R_{\oplus}$;
(vi) $[a, a^{(1,2)}] = 0$ *for some* $a^{(1,2)} \in a\{1,2\}$;
(vii) $[a, a^{(1)}] = 0$ *for some* $a^{(1)} \in a\{1\}$;
(viii) $Ra \subseteq Ra^n$ *for all choices* $n \geqslant 2$;
(ix) $aR \subseteq a^nR$ *for all choices* $n \geqslant 2$;
(x) $a^{2n} \in R^{\dagger}$ *for all choices* $n \geqslant 1$;
(xi) $a^{2n} \in R^{\#}$ *for all choices* $n \geqslant 1$;
(xii) $a^{2n} \in R^{\oplus}$ *for all choices* $n \geqslant 1$;
(xiii) $a^{2n} \in R_{\oplus}$ *for all choices* $n \geqslant 1$;
(xiv) $a \in R^{\{1,3\}}$;
(xv) $a \in R^{\{1,4\}}$.

Corollary 4.2 *Let $a \in R$ and $n \geqslant 2$. Then the following conditions are equivalent*:
(i) $a \in R^{\dagger}$;
(ii) $aa^* \in R^{\oplus}$ *and a is right $*$-cancellable*;
(iii) $aa^* \in R^{\oplus}$ *and* $a = aa^*(aa^*)^{\oplus}a$;
(iv) $aa^* \in R^{\oplus}$ *and* $a \in R^{\{1,4\}}$;
(v) $a^*a \in R^{\oplus}$ *and a is left $*$-cancellable*;
(vi) $a^*a \in R^{\oplus}$ *and* $a = a(a^*a)^{\oplus}a^*a$;
(vii) $a^*a \in R^{\oplus}$ *and* $a \in R^{\{1,3\}}$;
(viii) $a^*a \in R(a^*a)^n$ *and a is left $*$-cancellable*;
(ix) $a^*a \in (a^*a)^nR$ *and a is left $*$-cancellable*;
(x) $aa^* \in R(aa^*)^n$ *and a is left $*$-cancellable*;
(xi) $aa^* \in (aa^*)^nR$ *and a is left $*$-cancellable*;

(xii) $a^*a \in R(a^*a)^n$ *and* a *is right* $*$*-cancellable*;

(xiii) $a^*a \in (a^*a)^n R$ *and* a *is right* $*$*-cancellable*;

(xiv) $aa^* \in R(aa^*)^n$ *and* a *is right* $*$*-cancellable*;

(xv) $aa^* \in (aa^*)^n R$ *and* a *is right* $*$*-cancellable.*

In this case, $a^\dagger = a^*(aa^*)^{\oplus} = (aa^*)^{\oplus} a^*$.

Proof (i) $\Rightarrow$ (ii)–(vii). If $a \in R^\dagger$, then $aa^*, a^*a \in R^{\mathrm{EP}}$ and $(aa^*)^\dagger = (aa^*)^{\oplus}$, $(a^*a)^\dagger = (a^*a)^{\oplus}$ by [12, Theorem 3.1]. By [7, Theorem 5.4], we deduce that (i) $\Rightarrow$ (ii)–(vii).

(ii)–(vii) $\Rightarrow$ (i). Since $(aa^*)^* = aa^*$, we have $aa^*, a^*a \in R^{\mathrm{EP}}$ by Lemma 4.5 and $(aa^*)^\dagger = (aa^*)^{\oplus}$, $(a^*a)^\dagger = (a^*a)^{\oplus}$ by [12, Theorem 3.1]. Therefore, (ii)–(vii) $\Rightarrow$ (i) by [7, Theorem 5.4].

By Lemma 4.5 and the conditions (ii) and (xv), the equivalences between (viii)–(xv) and (ii) are obvious.

In the ring of square complex matrices, since every complex matrix has a Moore-Penrose inverse and therefore a $\{1,3\}$-inverse, we only need to assume that a is a group invertible and we obtain that results of this paper are valid. Also, for Hilbert space operators and elements of C^*-algebras, only regular operators and elements of C^*-algebras possess the Moore-Penrose inverse. Thus, anything with a group inverse automatically has a Moore-Penrose inverse and $\{1,3\}$-inverse. So, the results presented in this paper hold in a C^*-algebra A with the conditions that $a \in A^{\#}$ instead of $a \in A^{\oplus}$. In rings with involution the regularity is not enough to ensure the existence of a Moore-Penrose inverse.

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