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Local Exact Boundary Synchronization for a Kind of First Order Quasilinear Hyperbolic Systems

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Abstract In this paper, the synchronization for a kind of first order quasilinear hyperbolic system is taken into account. In this system, all the equations share the same positive wave speed. To realize the synchronization, a uniform constructive method is adopted, rather than an iteration process usually used in dealing with nonlinear systems. Furthermore, similar results on the exact boundary synchronization by groups can be obtained for a kind of first order quasilinear hyperbolic system of equations with different positive wave speeds by groups.

Keywords Exact boundary synchronization, Quasilinear hyperbolic system, Exact boundary synchronization by groups
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1 Introduction

Synchronization is a widespread natural phenomenon that was first observed by Huygens [3] in 1665. The theoretical research on synchronization phenomenon in the mathematical perspective dates back to Wiener [17] in the 1950s. While almost all the previous works focused on systems described by ODEs, for systems governed by PDEs, the study of synchronization was initiated by Li and Rao, who proposed the concept of exact boundary synchronization, the aim of which is to achieve synchronization in a limited time period by means of boundary controls, such that, after switching off all the controls, the state of synchronization remains. They considered the exact boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls in any given space dimensions in the framework of weak solutions (see [9, 11]), and also acquired related results for the same system with all kinds of boundary controls in one space dimension in the framework of classical solutions (see [2, 13]). Moreover, there are some results on the exact boundary synchronization by groups (see [10, 12]). These results are all restricted to coupled systems of linear wave equations. While, Hu, Li and Qu [1] derived the local exact boundary synchronization for 1-D coupled system of quasilinear wave equations with all kinds of boundary controls by an iteration process based on a uniform constructive method, and the fixed point theory. In this paper, the synchronization for a kind of first order linear and quasilinear hyperbolic system is taken into account. Noting that all the equations in the system share the same positive wave speed, we can directly get the local exact boundary synchronization by a constructive method rather than an iteration process.

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Furthermore, similar results on the exact boundary synchronization by groups can be obtained for a kind of first order quasilinear hyperbolic system of equations with different positive wave speeds.

Consider the following 1-D first order quasilinear hyperbolic system:

$$U_t + a(U)U_x = A(U)U, \tag{1.1}$$

where $U = (u_1, \cdots, u_n)^{\mathrm{T}}$ is an unknown vector function of (t, x), $a(U)$ is a C^1 function of U, $A(U) = (a_{ij}(U))$ is an $n \times n$ C^1 matrix of U. Obviously, $U = 0$ is an equilibrium for system (1.1).

(1.1) can be also written as

$$u_{it} + a(U)u_{ix} = \sum_{j=1}^{n} a_{ij}(U)u_j, \quad i = 1, \cdots, n, \tag{1.2}$$

which can be regarded as a first order coupled system of quasilinear hyperbolic equations with the same propagation speed $a(U)$. Assume that on the domain under consideration, we have

$$a(U) > 0. \tag{1.3}$$

Boundary controls are added on one end $x = 0$ as

$$x = 0: \quad U = H(t), \tag{1.4}$$

where

$$H(t) = (h_1(t), \cdots, h_n(t))^{\mathrm{T}},$$

and h_i $(i = 1, \cdots, n)$ are C^1 functions of t.

For the forward mixed initial-boundary value problem (1.1), (1.4) and the initial condition

$$t = 0: \quad U = \Phi(x), \quad 0 \le x \le L, \tag{1.5}$$

where

$$\Phi(x) = (\varphi_1(x), \cdots, \varphi_n(x))^{\mathrm{T}}$$

is a C^1 vector function of x, the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ are

$$\varphi_i(0) = h_i(0), \quad i = 1, \cdots, n \tag{1.6}$$

and

$$\sum_{j=1}^{n} a_{ij}(\Phi(0))\varphi_j(0) - a(\Phi(0))\varphi_i'(0) = \dot{h}_i(0), \quad i = 1, \cdots, n. \tag{1.7}$$

Lemma 1.1 *Under the assumptions mentioned above, assume that the conditions of C^1 compatibility (1.6)–(1.7) at the point $(t, x) = (0, 0)$ are satisfied, and that the forward mixed initial-boundary value problem (1.1), (1.4) and (1.5) admits a unique global C^1 solution $U = U(t, x)$ on the domain $D = \{(t, x) \mid t \ge 0,\ 0 \le x \le L\}$, which satisfies*

$$\|U\|_{C^1[D]} \le C(\|\Phi\|_{C^1[0,L]} + \|H\|_{C^1[0,+\infty)}), \tag{1.8}$$

where C is a positive constant depending only on L, provided that the C^1 norms $\|\Phi\|_{C^1[0,L]}$ and $\|H\|_{C^1[0,+\infty)}$ are small.

Proof Noting (1.3), by changing the status of t and x, the system (1.1) can be rewritten as

$$U_x + \frac{1}{a(U)}U_t = \frac{1}{a(U)}A(U)U. \tag{1.9}$$

Correspondingly, we solve the rightward mixed initial-boundary value problem (1.9) with the initial data

$$x = 0: \quad U = H(t), \quad t \geq 0 \tag{1.10}$$

and the boundary condition (1.5) on the domain $D = \{(t, x) \mid t \geq 0,\ 0 \leq x \leq L\}$. According to the theory on the semi-global C^1 solution (see [4–5]), as long as $\|\Phi\|_{C^1[0,L]}$ and $\|H\|_{C^1[0,+\infty)}$ are small (depending on L), we can get a unique rightward semi-global C^1 solution $U = U(t, x)$ on D. This solution is exactly the global C^1 solution to the mixed initial-boundary value problem (1.1), (1.4) and (1.5), which satisfies (1.8).

By the theory on the local exact boundary controllability for quasilinear hyperbolic system (see [4]), the system (1.1) and (1.4) possesses the local exact boundary controllability under the assumption that the number of boundary controls equals that of unknown variables. Generally speaking, the exact boundary controllability can not be realized with lack of controls. However, we can study the local exact boundary synchronization for the system (1.1) and (1.4).

Definition 1.1 *If there exists $T > 0$ such that for any given initial data $\Phi(x)$ with small $C^1[0, L]$ norm, we can find a part of control functions in $H(t)$ with compact support on $[0, T]$ in essence, the $C^1[0, +\infty)$ norm of which is also small, such that the corresponding mixed initial-boundary value problem* (1.1), (1.4) *and* (1.5) *admits a unique C^1 solution $U = U(t, x)$ with small C^1 norm on the domain $D = \{(t, x) \mid t \geq 0,\ 0 \leq x \leq L\}$, and for $t \geq T$, this solution satisfies*

$$u_1(t, x) \equiv \cdots \equiv u_n(t, x) \overset{\text{def.}}{=} \widetilde{\widetilde{u}}(t, x), \quad 0 \leq x \leq L, \tag{1.11}$$

where $\widetilde{\widetilde{u}}(t, x)$ is a priori unknown, then, the coupled system (1.1) *and* (1.4) *is called to possess the exact boundary synchronization for $t \geq T$, while $\widetilde{\widetilde{u}}(t, x)$ is called the state of exact boundary synchronization.*

Especially, if

$$\widetilde{\widetilde{u}}(t, x) \equiv 0, \tag{1.12}$$

then, the coupled system (1.1) *and* (1.4) *is called to possess the exact boundary null controllability for $t \geq T$.*

In [4, 6–8], the two-sided exact boundary controllability, the one-sided exact boundary controllability, the two-sided exact boundary controllability with fewer controls, and the exact boundary null controllability are realized, respectively, by a constructive method. For the case considered in this paper, we have specially the following result.

Lemma 1.2 *Let*

$$T > \frac{L}{a(0)}. \tag{1.13}$$

For any given initial data $\Phi(x)$ with small $C^1[0,L]$ norm, there exists a C^1 boundary control $H(t)$ on $x=0$ with compact support on $[0,T]$, satisfying the conditions of C^1 compatibility at the point $(t,x)=(0,0)$ and

$$\|H\|_{C^1[0,+\infty)} \leq C(T)\|\Phi\|_{C^1[0,L]}, \tag{1.14}$$

where $C(T)$ is a positive constant depending only on T, such that the system (1.1) *and* (1.4) *is exact boundary null controllable for $t \geq T$.*

The aim of this paper is to realize the local exact boundary synchronization for the corresponding system with fewer boundary controls by properly specifying the constructive method mentioned before.

Remark 1.1 In order to study the synchronization, we may fix the boundary function $h_1(t)$ beforehand and take $h_i(t)$ $(i=2,\cdots,n)$ as the boundary controls, and assume that $h_i(t)-h_1(t)$ $(i=2,\cdots,n)$ have compact supports on $[0,T]$. Assuming that the system (1.1) and (1.4) is exactly boundary synchronizable at the time $t=T$ for $t \geq T$, although all the boundary controls are not necessarily equal to zero, but $h_i(t) \equiv h_1(t)$ $(i=1,\cdots,n)$, thus the state of synchronization remains.

Remark 1.2 If the system (1.1) and (1.4) is exactly boundary synchronizable, but not exactly boundary null controllable, then the given boundary function $h_1(t) \not\equiv 0$ for $t \geq T$.

2 Synchronization for Linear System

First, we consider the linear system

$$U_t + aU_x = AU, \tag{2.1}$$

where $a>0$ is a constant, $A=(a_{ij})$ is an $n\times n$ matrix with constant elements. (2.1) can be written as

$$u_{it} + au_{ix} = \sum_{j=1}^{n} a_{ij}u_j, \quad i=1,\cdots,n. \tag{2.2}$$

Theorem 2.1 *If there exists $T>0$, such that the linear system* (2.1) *and* (1.4) *possesses the exact boundary synchronization for $t \geq T$, that is, for any given C^1 initial data $\Phi(x)$, we can find a C^1 boundary control $H(t)$ such that the C^1 solution $U=U(t,x)$ to the corresponding mixed initial-boundary value problem* (2.1), (1.4) *and* (1.5) *satisfies* (1.11) *for $t \geq T$, but it is not exact boundary null controllable, then the coupling matrix $A=(a_{ij})$ should satisfy the following condition of compatibility* (*the sum of elements in every row is equal to each other*), *called the row sum condition:*

$$\sum_{j=1}^{n} a_{ij} \overset{\text{def.}}{=} \widetilde{\widetilde{a}}, \quad i=1,\cdots,n, \tag{2.3}$$

where $\widetilde{\widetilde{a}}$ is a constant independent of $i=1,\cdots,n$.

Proof If the system (2.1) and (1.4) is exactly boundary synchronizable for $t \geq T$, then we have

$$\widetilde{\widetilde{u}}_t + a\widetilde{\widetilde{u}}_x = \Big(\sum_{j=1}^n a_{ij}\Big)\widetilde{\widetilde{u}}, \quad i = 1, \cdots, n \tag{2.4}$$

for $t \geq T$. Therefore, for $i, k = 1, \cdots, n$, we have

$$\Big(\sum_{j=1}^n a_{kj} - \sum_{j=1}^n a_{ij}\Big)\widetilde{\widetilde{u}} = 0 \tag{2.5}$$

for $t \geq T$. By assumption, this system is not exactly boundary null controllable, namely, at least for some C^1 initial data $\Phi(x)$, the corresponding $\widetilde{\widetilde{u}} \not\equiv 0$, thus we get

$$\sum_{j=1}^n a_{kj} = \sum_{j=1}^n a_{ij}, \quad i, k = 1, \cdots, n, \tag{2.6}$$

which is the condition of compatibility (2.3).

Let

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}_{(n-1)\times n} \tag{2.7}$$

denote the matrix of synchronization. C_1 is a matrix with full row-rank.

Lemma 2.1 (see [14]) *For any given $N \times N$ matrix A and any given full row-rank $M \times N$ (where $M < N$) matrix C, there exists a unique $M \times M$ matrix $\overline{A}$, such that*

$$CA = \overline{A}C \tag{2.8}$$

if and only if $\mathrm{Ker}(C)$ *is an invariant subspace of A :*

$$A\mathrm{Ker}(C) \subseteq \mathrm{Ker}(C). \tag{2.9}$$

The elements of $\overline{A}$ in (2.8) *is given by*

$$\overline{A} = CAC^+, \tag{2.10}$$

where C^+ denotes the Moore-Penrose generalized inverse of C,

$$C^+ = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}. \tag{2.11}$$

Corollary 2.1 *The following facts are equivalent*:

(1) *Condition of compatibility* (2.3) *holds.*

(2) *There exists a unique $(n-1)$ matrix $\overline{A}$ such that*

$$C_1A = \overline{A}C_1. \tag{2.12}$$

$\overline{A} = (\overline{a}_{ij})$ is called the reduced matrix of A, the elements of which can be precisely given by

$$\overline{a}_{ij} = \sum_{p=j+1}^n (a_{i+1,p} - a_{ip}) = \sum_{p=1}^j (a_{ip} - a_{i+1,p}), \quad i, j = 1, \cdots, n-1. \tag{2.13}$$

Theorem 2.2 *Under the condition of compatibility* (2.3), *let*

$$T > \frac{L}{a}. \tag{2.14}$$

For any given C^1 *initial data* $\Phi(x)$, *without loss of generality, suppose that the* C^1 *boundary function* $h_1(t)$ *is fixed beforehand on the end* $x = 0$, *which satisfies the corresponding conditions of* C^1 *compatibility at the point* $(t, x) = (0, 0)$. *Then there exist* $(n-1)$ C^1 *boundary controls* $h_2(t), \cdots, h_n(t)$, *the differences of which with* $h_1(t)$ *have compact supports on* $[0, T]$, *and*

$$\|h_i(t) - h_1(t)\|_{C^1[0,+\infty)} \le C(T)\|\Phi\|_{C^1[0,L]}, \quad i = 2, \cdots, n, \tag{2.15}$$

and thus

$$\|h_i\|_{C^1[0,+\infty)} \le C(T)(\|\Phi\|_{C^1[0,L]} + \|h_1\|_{C^1[0,+\infty)}), \quad i = 2, \cdots, n, \tag{2.16}$$

such that the system (2.1) *and* (1.4) *is exactly boundary synchronizable for* $t \ge T$. *Here and hereafter,* $C(T)$ *denotes a positive constant depending only on* T.

Proof Under the condition of compatibility (2.3), let

$$W = C_1 U.$$

By Corollary 2.1, the original system (2.1), (1.4) and (1.5) for the variable U can be reduced to the following self-closed problem for the variable W:

$$\begin{cases} W_t + aW_x = \overline{A}W, \\ x = 0: \ W = \overline{H}, \\ t = 0: \ W = W_0, \ 0 \le x \le L, \end{cases} \tag{2.17}$$

where $\overline{H} = C_1 H$, $W_0 = C_1\Phi$, and $\overline{A}$ is given by (2.12). In this way, the exact boundary synchronization of the original system (2.1) and (1.4) for U and $t \ge T$ is equivalent to the exact boundary null controllability of the reduced system (2.17) for W and $t \ge T$. Hence, by Lemma 1.2, applying the same constructive method to this linear system, we get that for any given initial data W_0, the system (2.17) can realize the exact boundary null controllability by properly choosing the C^1 boundary control $\overline{H} = (\overline{h}_1(t), \cdots, \overline{h}_{n-1}(t))^{\mathrm{T}}$ with compact support on $[0, T]$. Moreover,

$$\|\overline{H}\|_{C^1[0,T]} \le C(T)\|\Phi\|_{C^1[0,L]}. \tag{2.18}$$

Noting that C_1 has full row-rank, when fixing $h_1(t)$ beforehand, H can be solved by $\overline{H} = C_1 H$. This H is exactly the boundary control which can realize the exact boundary synchronization for the system (2.1) and (1.4). Furthermore, (2.15) follows from (2.18), and $h_i(t) - h_1(t)$ $(i = 2, \cdots, n)$ have compact supports on $[0, T]$.

3 Exact Boundary Synchronization for Quasilinear Hyperbolic System

According to the discussion above, the condition of compatibility (2.3) is a necessary and sufficient condition of exact boundary synchronization for the linear hyperbolic system (2.1) and (1.4). For the corresponding quasilinear system, we naturally assume that such condition still holds, that is, on the domain under consideration (namely, in a neighborhood of $U = 0$), we have

$$\sum_{j=1}^{n} a_{ij}(U) \overset{\text{def.}}{=} \widetilde{\widetilde{a}}(U), \quad i = 1, \cdots, n, \tag{3.1}$$

where $\widetilde{\widetilde{a}}(U)$ is independent of $i = 1, \cdots, n$. By Corollary 2.1, there exists a unique matrix $\overline{A}(U)$ of order $(n-1)$, such that

$$C_1 A(U) = \overline{A}(U) C_1, \tag{3.2}$$

where $\overline{A}(U) = (\overline{a}_{ij}(U))$ is the reduced matrix of $A(U)$, the elements of which can be explicitly given by

$$\overline{a}_{ij}(U) = \sum_{p=j+1}^{n} (a_{i+1,p}(U) - a_{ip}(U)) = \sum_{p=1}^{j} (a_{ip}(U) - a_{i+1,p}(U)), \quad i, j = 1, \cdots, n-1. \tag{3.3}$$

Theorem 3.1 *Under the condition of compatibility* (3.1), *let*

$$T > \frac{L}{a(0)}. \tag{3.4}$$

For any given initial data $\Phi(x)$ with small $C^1[0, L]$ norm, without loss of generality, assume that the C^1 boundary function $h_1(t)$ with small $C^1[0, +\infty)$ norm is fixed beforehand on the end $x = 0$, which satisfies the corresponding conditions of C^1 compatibility at the point $(t, x) = (0, 0)$. There exist $(n-1)$ C^1 boundary controls $h_2(t), \cdots, h_n(t)$, the differences of which with $h_1(t)$ have compact supports on $[0, T]$, and

$$\|h_i - h_1\|_{C^1[0,+\infty)} \le C(T) \|\Phi\|_{C^1[0,L]}, \quad i = 2, \cdots, n, \tag{3.5}$$

and thus

$$\|h_i\|_{C^1[0,+\infty)} \le C(T)(\|\Phi\|_{C^1[0,L]} + \|h_1\|_{C^1[0,+\infty)}), \quad i = 2, \cdots, n, \tag{3.6}$$

such that the problem (1.1), (1.4) *and* (1.5) *admits a unique global C^1 solution $U = U(t, x)$ with small C^1 norm on the domain $D = \{(t, x) \mid t \ge 0,\ 0 \le x \le L\}$, which possesses the exact boundary synchronization shown by* (1.11) *for $t \ge T$.*

Proof Noting (3.4), there exists $\varepsilon_0 > 0$ so small that

$$T > T_1, \tag{3.7}$$

where

$$T_1 = \sup_{|U| \le \varepsilon_0} \frac{L}{a(U)}. \tag{3.8}$$

Let

$$W = C_1 U. \tag{3.9}$$

The mixed problem (1.1), (1.4) and (1.5) for U can be reduced to the following mixed problem for W:

$$\begin{cases} W_t + a(U)W_x = \overline{A}(U)W, \\ x = 0: \ W = \overline{H}, \\ t = 0: \ W = W_0, \quad 0 \le x \le L, \end{cases} \tag{3.10}$$

where $\overline{A}(U) = (\overline{a}_{ij}(U))$ is given by (3.2), while

$$W_0 = C_1\Phi, \quad \overline{H} = C_1 H = (\overline{h}_1, \cdots, \overline{h}_{n-1})^{\mathrm{T}}. \tag{3.11}$$

Thus, noting that if the solution to the problem (3.10) satisfies

$$W \equiv 0, \quad t \ge T \tag{3.12}$$

(namely, the system (3.10) is exactly boundary null controllable for $t \ge T$), then the system (1.1) and (1.4) should be exactly boundary synchronizable for $t \ge T$. Hence, in order to get Theorem 3.1, we need only to prove the following things.

(1) For any given C^1 function $U = U(t,x)$ with $|U(t,x)| \le \varepsilon_0$, we can properly construct a boundary control $\overline{H}$ independent of $U(t,x)$, such that the system (3.10) is exactly boundary null controllable for $t \ge T$ and any given initial data W_0 with small $C^1[0,L]$ norm.

Consider the corresponding rightward mixed initial-boundary value problem, by the uniqueness of the C^1 solution to the Cauchy problem (see [15]), if we can find a C^1 control function $\overline{H} = \overline{H}(t)$ independent of U on $x = 0$, such that

$$\overline{H}(t) \equiv 0, \quad t \ge T_0$$

and the $C^1[0,T_0]$ norm of $\overline{H}(t)$ is small, in which

$$T_0 = T - T_1 > 0.$$

Besides, the conditions of C^1 compatibility at the point $(t,x) = (0,0)$ are satisfied, then it is easy to get (3.12).

In a similar way to (1.6)–(1.7), for the mixed problem (3.10), the conditions of C^1 compatibility at the point $(t,x) = (0,0)$

$$\begin{cases} \overline{H}(0) = C_1\Phi(0), \\ \overline{H}'(0) = \overline{A}(\Phi(0))C_1\Phi(0) - a(\Phi(0))C_1\Phi'(0) \end{cases} \tag{3.13}$$

depend only on $\Phi(0)$ and $\Phi'(0)$, and are independent of $U = U(t,x)$. By the value of $\overline{H}(0)$ and $\overline{H}'(0)$ given by (3.13) and the fact that $\overline{H}(T_0) = \overline{H}'(T_0) = 0$ at the time $t = T_0$, we can use Hermite interpolation to construct a C^1 function $\overline{H}(t)$ on $[0,T_0]$ as follows:

$$\overline{h}_i(t) = b_i g_1(t) + c_i g_2(t), \quad 0 \le t \le T_0, \ i = 1, \cdots, n-1, \tag{3.14}$$

where

$$g_1(t) = \Big(1 + \frac{2t}{T_0}\Big)\Big(\frac{t - T_0}{T_0}\Big)^2,$$
$$g_2(t) = t\Big(\frac{t - T_0}{T_0}\Big)^2$$

and

$$b_i = \varphi_i(0) - \varphi_{i+1}(0), \quad i = 1, \cdots, n-1,$$
$$c_i = \sum_{j=1}^{n-1} \overline{a}_{ij}(\Phi(0))(\varphi_j(0) - \varphi_{j+1}(0)) - a(\Phi(0))(\varphi_i'(0) - \varphi_{i+1}'(0)), \quad i = 1, \cdots, n-1.$$

By (3.14), it is easy to see that

$$\|\overline{h}_i\|_{C^1[0,T]} \le C(T)\|\Phi\|_{C^1[0,L]}, \quad i = 1, \cdots, n-1. \tag{3.15}$$

(2) The corresponding value of $H(t)$ with small $C^1[0,T]$ norm can be determined by $\overline{H}(t)$ acquired above.

Noting (3.11), the value of C^1 boundary controls $h_i(t)$ $(i = 2, \cdots, n)$ on $x = 0$ can be solved by

$$h_{i+1}(t) = h_i(t) - \overline{h}_i(t), \quad i = 1, \cdots, n-1, \tag{3.16}$$

where $h_1(t)$ is a beforehand fixed boundary function with small $C^1[0, +\infty)$ norm, satisfying the corresponding conditions of C^1 compatibility at the point $(t, x) = (0, 0)$.

By (3.3) and (3.15), it is easy to see that $h_i(t)$ $(i = 2, \cdots, n)$ and $h_1(t)$ satisfy (3.5) and the conditions of C^1 compatibility (1.6)–(1.7) at the point $(t, x) = (0, 0)$.

(3) The mixed initial-boundary value problem (1.1), (1.4) and (1.5) admits a unique global C^1 solution $U = U(t, x)$ on the domain $D = \{(t, x) \mid t \ge 0,\ 0 \le x \le L\}$, satisfying

$$|U| \le \varepsilon_0. \tag{3.17}$$

By the theory of semi-global C^1 solution to the rightward mixed problem and the corresponding estimation on its C^1 norm (see [4–5]), as long as the C^1 norm of Φ and h_1 are small, we have

$$\|U\|_{C^1[D]} \le C(T)(\|\Phi\|_{C^1[0,L]} + \|h_1\|_{C^1[0,+\infty)}). \tag{3.18}$$

Then we get (3.17).

4 Synchronization by Groups

In the case of further losing boundary controls, we can consider the exact boundary synchronization by p-groups ($p \ge 1$; when $p = 1$, it goes back to the exact boundary synchronization). This means that the components of U are divided into p groups:

$$(u_1, \cdots, u_{m_1}), (u_{m_1+1}, \cdots, u_{m_2}), \cdots, (u_{m_{p-1}+1}, \cdots, u_{m_p}), \tag{4.1}$$

where $0 = m_0 < m_1 < m_2 < \cdots < m_p = n$. The synchronization should be required for every group of elements, respectively, and the states of synchronization by groups are mutually independent.

Besides, in the case of synchronization by groups, all the components are not necessarily required to share the same propagation speed, but those who correspond to the same state of synchronization should have the same propagation speed. In other words, we may consider the exact boundary synchronization by p-groups for the following quasilinear hyperbolic system:

$$U_t + \Lambda(U)U_x = A(U)U, \tag{4.2}$$

where

$$\Lambda(U) = \mathrm{diag}\,\{\lambda_1(U)I_{m_1}, \lambda_2(U)I_{m_2-m_1}, \cdots, \lambda_p(U)I_{n-m_{p-1}}\} \tag{4.3}$$

is a diagonal matrix of order n, and

$$\lambda_s(U) > 0, \quad 1 \le s \le p \tag{4.4}$$

are C^1 functions of U, while I_i is the identity matrix of order i.

In a similar way to Lemma 1.1, we have the following result.

Lemma 4.1 *Under the assumptions mentioned above, assume that the conditions of C^1 compatibility are satisfied at the point $(t,x) = (0,0)$. As long as the C^1 norms $\|\Phi\|_{C^1[0,L]}$ and $\|H\|_{C^1[0,+\infty)}$ are small, there exists a unique global C^1 solution $U = U(t,x)$ to the forward mixed initial-boundary value problem* (4.2), (1.4) *and* (1.5) *on the domain $D = \{(t,x) \mid t \ge 0,\ 0 \le x \le L\}$, which satisfies*

$$\|U\|_{C^1[D]} \le C(\|\Phi\|_{C^1[0,L]} + \|H\|_{C^1[0,+\infty)}), \tag{4.5}$$

where C is a positive constant depending only on L.

Definition 4.1 *If there exists $T > 0$ such that for any given initial data $\Phi(x)$ with small $C^1[0,L]$ norm, we can find a part of boundary controls in $H(t)$ with small $C^1[0,+\infty)$ norm, such that the corresponding mixed initial-boundary value problem* (4.2), (1.4) *and* (1.5) *admits a unique C^1 solution $U = U(t,x)$ with small C^1 norm on the domain $\{(t,x) \mid t \ge 0,\ 0 \le x \le L\}$, which satisfies*

$$u_k \equiv u_l \stackrel{\text{def.}}{=} \tilde{\tilde{u}}_s, \quad m_{s-1}+1 \le k,\ l \le m_s,\ 1 \le s \le p \tag{4.6}$$

for $t \ge T$, where $\tilde{\tilde{u}}_s$ $(1 \le s \le p)$ are priori unknown, then the system (4.2) *and* (1.4) *is called to possess the exact boundary synchronization by p-groups for $t \ge T$, and $(\tilde{\tilde{u}}_1, \cdots, \tilde{\tilde{u}}_p)^{\mathrm{T}}$ is called the state of exact boundary synchronization by p-groups.*

Let C_p denote the following $(N-p) \times N$ matrix of synchronization by p-groups

$$C_p = \begin{pmatrix} C_{1,1} & & & \\ & C_{1,2} & & \\ & & \ddots & \\ & & & C_{1,p} \end{pmatrix}, \tag{4.7}$$

where $C_{1,s}$ is an $(m_s - m_{s-1} - 1) \times (m_s - m_{s-1})$ matrix with full row-rank

$$C_{1,s} = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \le s \le p. \tag{4.8}$$

Denoting

$$(e_s)_j = \begin{cases} 1, & m_{s-1}+1 \le j \le m_s, \\ 0, & \text{others}, \end{cases} \qquad 1 \le s \le p, \tag{4.9}$$

it is easy to see that

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\} \tag{4.10}$$

and (4.6) can be written as

$$t \ge T: \ U = \sum_{s=1}^{p} \widetilde{\widetilde{u}}_s e_s. \tag{4.11}$$

By [4], we have the following result.

Lemma 4.2 *Let*

$$T > L \max_{1 \le s \le p} \frac{1}{\lambda_s(0)}. \tag{4.12}$$

For any given initial data $\Phi(x)$ *with small* $C^1[0,L]$ *norm, there exists a* C^1 *boundary control* $H(t)$ *on* $x=0$ *with compact support on* $[0,T]$, *which satisfies the conditions of* C^1 *compatibility at the point* $(t,x)=(0,0)$ *and*

$$\|H\|_{C^1[0,T]} \le C(T)\|\Phi\|_{C^1[0,L]}, \tag{4.13}$$

such that the system (4.2) *and* (1.4) *is exactly boundary null controllable for* $t \ge T$.

4.1 Exact boundary synchronization by *p*-groups for linear system

First, we consider the linear system

$$U_t + \Lambda U_x = AU, \tag{4.14}$$

where

$$\Lambda = \mathrm{diag}\ \{\lambda_1 I_{m_1}, \lambda_2 I_{m_2-m_1}, \cdots, \lambda_p I_{n-m_{p-1}}\}$$

is a diagonal matrix of order n, and

$$\lambda_s > 0, \quad 1 \le s \le p$$

are constants.

Theorem 4.1 *Assume that the linear system* (4.14) *and* (1.4) *is exactly boundary synchronizable by p-groups, but not exactly boundary synchronizable by* $(p-1)$*-groups under any invertible linear transformation of state variables* $u_1, \cdots, u_n$. *Then the coupling matrix* $A = (a_{ij})$ *should satisfy the following condition of compatibility:*

$$A\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p), \tag{4.15}$$

or equivalently, there exists a unique matrix $\overline{A}_p$ *of order* $(N-p)$, *such that*

$$C_p A = \overline{A}_p C_p. \tag{4.16}$$

Proof If the system (4.14) and (1.4) is exactly boundary synchronizable by p-groups for $t \geq T$, then, for $t \geq T$, multiplying (4.14) by C_p, and noting (4.10), we get

$$\sum_{r=1}^{p} \widetilde{\widetilde{u}}_r C_p A e_r = 0. \tag{4.17}$$

If for some given initial data Φ, the corresponding $\widetilde{\widetilde{u}}_1, \widetilde{\widetilde{u}}_2, \cdots, \widetilde{\widetilde{u}}_p$ are linear independent, we have

$$C_p A e_r = 0, \quad 1 \leq r \leq p. \tag{4.18}$$

Then, (4.15) follows from (4.10). On the other hand, if for any given initial data Φ, $\widetilde{\widetilde{u}}_1, \cdots, \widetilde{\widetilde{u}}_p$ are linear dependent, then the linear dependence could be shown by (4.18), which contradicts the fact that the system (4.14) and (1.4) is not exact boundary synchronizable by $(p-1)$-groups under any invertible linear transformation of state variables. Thus (4.15) is proved.

Remark 4.1 By the condition of compatibility (4.15) or (4.16), there exist constants $\widetilde{\widetilde{a}}_{sr}$ $(1 \leq s, r \leq p)$ such that

$$Ae_s = \sum_{r=1}^{p} \widetilde{\widetilde{a}}_{sr} e_r, \quad 1 \leq s \leq p, \tag{4.19}$$

or the following row sum conditions in the block sense hold:

$$\sum_{j=m_{s-1}+1}^{m_s} a_{ij} = \widetilde{\widetilde{a}}_{sr}, \quad m_{r-1}+1 \leq i \leq m_r, \ 1 \leq s, r \leq p. \tag{4.20}$$

In particular, when $p = 1$, the condition of compatibility (4.20) is exactly (2.3).

Theorem 4.2 *Under the condition of compatibility* (4.15), *let*

$$T > L \max_{1 \leq s \leq p} \frac{1}{\lambda_s}. \tag{4.21}$$

For any given C^1 initial data $\Phi(x)$, without loss of generality, assume that the C^1 boundary functions $h_{m_{s-1}+1}(t)$ $(1 \leq s \leq p)$ on $x = 0$ are given beforehand, which satisfy the corresponding conditions of C^1 compatibility at the point $(t, x) = (0, 0)$. Then there exist $(n-p)$ C^1 boundary functions $h_i(t)$ $(m_{s-1}+2 \leq i \leq m_s, \ 1 \leq s \leq p)$, the differences of which with the corresponding $h_{m_{s-1}+1}(t)$ $(1 \leq s \leq p)$ have compact supports on $[0, T]$, and

$$\|h_i - h_{m_{s-1}+1}\|_{C^1[0,+\infty)} \leq C(T)\|\Phi\|_{C^1[0,L]}, \quad m_{s-1}+2 \leq i \leq m_s, \ 1 \leq s \leq p, \tag{4.22}$$

and thus

$$\begin{aligned} \|h_i\|_{C^1[0,+\infty)} \leq C(T)(\|\Phi\|_{C^1[0,L]} + \|h_{m_{s-1}+1}\|_{C^1[0,+\infty)}), \\ m_{s-1}+2 \leq i \leq m_s, \ 1 \leq s \leq p, \end{aligned} \tag{4.23}$$

such that the corresponding mixed initial-boundary value problem (4.14), (1.4) *and* (1.5) *admits a unique global C^1 solution $U = U(t, x)$ on the domain $\{(t, x) \mid t \geq 0, \ 0 \leq x \leq L\}$, which possesses the exact boundary synchronization by p-groups defined by* (4.6).

Proof Under the condition of compatibility (4.15), let

$$W = C_p U.$$

Noting

$$C_p \Lambda = \overline{\Lambda} C_p, \tag{4.24}$$

where

$$\overline{\Lambda} = \text{diag}\,\{\lambda_1 I_{m_1-1},\ \lambda_2 I_{m_2-m_1-1}, \cdots,\ \lambda_p I_{n-m_{p-1}-1}\} \tag{4.25}$$

is a diagonal matrix of order $(n-p)$, the original mixed problem (4.14), (1.4) and (1.5) for U can be reduced to the following mixed problem for W:

$$\begin{cases} W_t + \overline{\Lambda} W_x = \overline{A}_p W, \\ x = 0: \ W = \overline{H}, \\ t = 0: \ W = W_0, \quad 0 \le x \le L, \end{cases} \tag{4.26}$$

where $\overline{A}_p$ is the reduced matrix of A defined by (4.16), and

$$W_0 = C_p \Phi, \quad \overline{H} = C_p H = (\overline{h}_1, \cdots, \overline{h}_{n-p})^{\mathrm{T}}. \tag{4.27}$$

The rest of the proof is similar to that of Theorem 2.2.

4.2 Exact boundary synchronization by p-groups for quasilinear system

For the quasilinear system (4.2) and (1.4), in order to consider its exact boundary synchronization by p-groups, similar to (4.15), assume that the coupling matrix $A(U)$ satisfies the following condition of compatibility:

$$A(U)\text{Ker}(C_p) \subseteq \text{Ker}(C_p) \tag{4.28}$$

on the domain under consideration (namely, in a neighborhood of $U = 0$), where C_p is defined by (4.7).

Remark 4.2 By Lemma 2.1, the condition of compatibility (4.28) is equivalent to the fact that there exists a unique C^1 matrix $\overline{A}_p(U)$ of order $(N-p)$, such that

$$C_p A(U) = \overline{A}_p(U) C_p. \tag{4.29}$$

Or equivalently, the row sum conditions in the block sense are satisfied

$$\sum_{j=m_{s-1}+1}^{m_s} a_{ij}(U) = \widetilde{\widetilde{a}}_{sr}(U), \quad m_{r-1}+1 \le i \le m_r,\ 1 \le s, r \le p. \tag{4.30}$$

In particular, when $p = 1$, the condition of compatibility (4.30) is exactly (3.1).

Similarly, we can equivalently transform the exact boundary synchronization by groups for the system (4.2) and (1.4) to the exact boundary null controllability for its reduced system.

Theorem 4.3 *Under the condition of compatibility* (4.28)*, let*

$$T > L \max_{1 \le s \le p} \frac{1}{\lambda_s(0)}. \tag{4.31}$$

For any given initial data $\Phi(x)$ *with small* $C^1[0,L]$ *norm, without loss of generality, assume that the* C^1 *boundary functions* $h_{m_{s-1}+1}(t)$ $(1 \le s \le p)$ *with small* $C^1[0,+\infty)$ *norm on* $x=0$ *are given beforehand, which satisfy the corresponding conditions of* C^1 *compatibility at the point* $(t,x)=(0,0)$*. Then there exist* $(n-p)$ C^1 *boundary functions* $h_i(t)$ $(m_{s-1}+2 \le i \le m_s,\ 1 \le s \le p)$*, the differences of which with the corresponding* $h_{m_{s-1}+1}(t)$ $(1 \le s \le p)$ *have compact supports on* $[0,T]$*, and*

$$\|h_i - h_{m_{s-1}+1}\|_{C^1[0,+\infty)} \le C(T)\|\Phi\|_{C^1[0,L]}, \quad m_{s-1}+2 \le i \le m_s,\ 1 \le s \le p, \tag{4.32}$$

and thus

$$\begin{aligned} \|h_i\|_{C^1[0,+\infty)} \le C(T)(\|\Phi\|_{C^1[0,L]} + \|h_{m_{s-1}+1}\|_{C^1[0,+\infty)}), \\ m_{s-1}+2 \le i \le m_s,\ 1 \le s \le p, \end{aligned} \tag{4.33}$$

such that the mixed initial-boundary value problem (4.2), (1.4) *and* (1.5) *admits a unique global* C^1 *solution* $U=U(t,x)$ *with small* C^1 *norm on the domain* $\{(t,x) \mid t \ge 0,\ 0 \le x \le L\}$*, which satisfies the exact boundary synchronization by p-groups defined by* (4.6).

Proof Noting (4.31), there exists $\varepsilon_0 > 0$ so small that

$$T > T_2, \tag{4.34}$$

where

$$T_2 = \sup_{|U| \le \varepsilon_0} \max_{1 \le s \le p} \frac{L}{\lambda_s(U)}. \tag{4.35}$$

Under the condition of compatibility (4.28), let

$$W = C_p U.$$

Noting that

$$C_p \Lambda(U) = \overline{\Lambda}(U) C_p, \tag{4.36}$$

where

$$\overline{\Lambda}(U) = \text{diag}\ \{\lambda_1(U) I_{m_1-1},\ \lambda_2(U) I_{m_2-m_1-1}, \cdots,\ \lambda_p(U) I_{n-m_{p-1}-1}\} \tag{4.37}$$

is a diagonal matrix of order $(n-p)$, the mixed problem (4.2), (1.4) and (1.5) for U can be reduced to the following mixed problem for W:

$$\begin{cases} W_t + \overline{\Lambda}(U) W_x = \overline{A}_p(U) W, \\ x=0: \ W = \overline{H}, \\ t=0: \ W = W_0, \quad 0 \le x \le L, \end{cases} \tag{4.38}$$

where $\overline{A}_p(U)$ is the reduced matrix of $A(U)$ (see Remark 4.2), and

$$W_0 = C_p\Phi, \quad \overline{H} = C_pH = (\overline{h}_1, \cdots, \overline{h}_{n-p})^{\mathrm{T}}. \tag{4.39}$$

Here, the corresponding conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ are

$$\begin{cases} \overline{H}(0) = C_p\Phi(0), \\ \overline{H}'(0) = \overline{A}(\Phi(0))C_p\Phi(0) - \overline{\Lambda}(\Phi(0))C_p\Phi'(0). \end{cases} \tag{4.40}$$

In a similar way to Theorem 3.1, we can construct a boundary control $\overline{H}(t)$ as in (3.14), such that

$$\overline{H}(t) \equiv 0, \quad t \geq T_0$$

and $\overline{H}(t)$ has a small $C^1[0, T_0]$ norm, where

$$T_0 = T - T_2 > 0,$$

and the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ are satisfied, hence the system (4.38) possesses the exact boundary null controllability for $t \geq T_0$, the rest of the proof is similar to that of Theorem 3.1.

5 Stability of Synchronization Solution

By Lemma 1.1, the mixed initial-boundary value problem (1.1), (1.4) and (1.5) admits a unique global classical solution on the domain $\{(t, x) \mid t \geq 0,\ 0 \leq x \leq L\}$, therefore we can consider the limit behavior of the solution as $t \to +\infty$.

Assume that the system possesses the synchronization state for $t \geq T$,

$$U = \overline{U}(t, x) \stackrel{\text{def.}}{=} \widetilde{\widetilde{u}}(t, x)e_1, \tag{5.1}$$

where $e_1 = (1, \cdots, 1)^{\mathrm{T}}$. Then the synchronization solution $\overline{U}(t, x)$ satisfies

$$\begin{cases} \overline{U}_t + a(\overline{U})\overline{U}_x = A(\overline{U})\overline{U} = \widetilde{\widetilde{a}}(\overline{U})\overline{U}, \\ x = 0: \ \overline{U} = h_1(t)e_1, \\ t = T: \ \overline{U} = u_1(x)e_1, \quad 0 \leq x \leq L, \end{cases} \tag{5.2}$$

where $u_1(x)$ is the initial data of the synchronization solution at $t = T$: $u_1(x) = \widetilde{\widetilde{u}}(T, x)$.

Taking some small perturbations $a(x)$ and $b(t)$ to the initial data and the boundary data of the synchronization solution $\overline{U}$, respectively, we get a new solution $U = U(t, x)$, satisfying

$$\begin{cases} U_t + a(U)U_x = A(U)U, \\ x = 0: \ U = h_1(t)e_1 + b(t), \\ t = T: \ U = u_1(x)e_1 + a(x), \quad 0 \leq x \leq L. \end{cases} \tag{5.3}$$

Lemma 5.1 (see [16]) *Under the assumptions of Lemma* 1.1*, assume that there are two different initial data* $\Phi^{(1)}(x), \Phi^{(2)}(x) \in C^1[0, L]$ *and two different boundary data* $H^{(1)}(t), H^{(2)}(t) \in C^1[0, +\infty)$*, both of which satisfy the conditions of* C^1 *compatibility at the point* $(t, x) = (0, 0)$*, respectively. The corresponding solutions to the mixed initial-boundary value problem* (1.1)*,* *(1.4) and (1.5) on the domain $\{(t,x) \mid t \geq 0,\ 0 \leq x \leq L\}$ are denoted by $U^{(1)}(t,x)$ and $U^{(2)}(t,x)$, respectively. Then for any given $\delta > 0$, there exists $\eta > 0$, such that when*

$$\|\Phi^{(1)} - \Phi^{(2)}\|_{C^1[0,L]} + \|H^{(1)} - H^{(2)}\|_{C^1[0,+\infty)} \leq \eta, \tag{5.4}$$

we have

$$\|U^{(1)}(t,\cdot) - U^{(2)}(t,\cdot)\|_{C^1[0,L]} \leq \delta, \quad \forall\, t \in [0,+\infty). \tag{5.5}$$

Theorem 5.1 *Assume that the mixed initial-boundary value problem (1.1), (1.4) and (1.5) possesses the exact boundary synchronization for $t \geq T$, and that the corresponding synchronization solution $\overline{U}$ satisfies problem (5.2). Taking some small perturbations $a(x)$ and $b(t)$ to the initial data and the boundary data, respectively, the corresponding perturbed solution to the mixed problem (5.3) is denoted by U. Then for any given $\delta > 0$, there exists $\eta > 0$, such that when*

$$\|a\|_{C^1[0,L]} + \|b\|_{C^1[T,+\infty)} \leq \eta, \tag{5.6}$$

we have

$$\|U(t,\cdot) - \overline{U}(t,\cdot)\|_{C^1[0,L]} \leq \delta, \quad \forall\, t \in [T,+\infty). \tag{5.7}$$

Furthermore, for any given $\delta > 0$, there exists $\eta > 0$, such that when

$$\|b\|_{C^1[T,+\infty)} \leq \eta, \tag{5.8}$$

we have

$$\|U(t,\cdot) - \overline{U}(t,\cdot)\|_{C^1[0,L]} \leq \delta, \tag{5.9}$$

when the time t is large enough $\big(t \geq T + \frac{L}{a(0)}\big)$.

Proof The former conclusion results from Lemma 5.1 directly.

Furthermore, consider the corresponding rightward mixed problem, noting the fact that all the characteristics travel from left to right, the behavior of the solution essentially depends on the boundary condition on $x = 0$ for time large enough, and has nothing to do with the perturbation on the initial data at $t = 0$. Therefore, we need only to consider the solution to the following rightward Cauchy problem:

$$\begin{cases} \overline{U}_t + a(\overline{U})\overline{U}_x = A(\overline{U})\overline{U} \equiv \widetilde{\widetilde{a}}(\overline{U})\overline{U}, \\ x = 0:\ \overline{U} = h_1(t)e_1, \end{cases} \tag{5.10}$$

where

$$\overline{U} = \widetilde{\widetilde{u}} e_1$$

and

$$\begin{cases} U_t + a(U)U_x = A(U)U, \\ x = 0:\ U = h_1(t)e_1 + b(t). \end{cases} \tag{5.11}$$

Thus, the latter conclusion can be proved by Lemma 5.1.

Remark 5.1 In fact, for any given global solution $\overline{U} = (u_1, \cdots, u_n)^{\mathrm{T}}$ (not necessarily a synchronization solution), when time is large enough, the difference between its perturbed solution and the solution itself depends only on the perturbation on the boundary condition. If $b(t) \to 0$ as $t \to +\infty$, then the perturbed solution $U(t,x)$ approaches to $\overline{U}(t,x)$ itself at last.

The result mentioned above can be extended to the case of synchronization by groups, namely, we have the following theorem.

Theorem 5.2 *Under the assumptions of Theorem* 4.3*, assuming that the mixed initial-boundary value problem* (4.2), (1.4) *and* (1.5) *possesses the exact boundary synchronization by p-groups for $t \geq T$, and for the corresponding synchronization solution $\overline{U}$ satisfying* (4.6)*, we have*

$$\begin{cases} \overline{U}_t + \Lambda(\overline{U})\overline{U}_x = A(\overline{U})\overline{U}, \\ x = 0: \ \overline{U} = \sum\limits_{s=1}^{p} h_{m_{s-1}+1}(t)e_s, \\ t = T: \ \overline{U} = \sum\limits_{s=1}^{p} \overline{u}_s(x)e_s, \quad 0 \leq x \leq L, \end{cases} \tag{5.12}$$

where $\overline{u}_s(x)$ is the initial data of the synchronization solution by p-groups at $t = T$: $\overline{u}_s(x) = \widetilde{\widetilde{u}}_s(T,x)$ $(1 \leq s \leq p)$. Taking some small perturbations $a(x)$ and $b(t)$ to the initial data and the boundary data of the synchronization solution $\overline{U}$ by p-groups, respectively, the corresponding perturbed solution $U = U(t,x)$ satisfies

$$\begin{cases} U_t + \Lambda(U)U_x = A(U)U, \\ x = 0: \ U = \sum\limits_{s=1}^{p} h_{m_{s-1}+1}(t)e_s + b(t), \\ t = T: \ U = \sum\limits_{s=1}^{p} \overline{u}_s(x)e_s + a(x), \quad 0 \leq x \leq L. \end{cases} \tag{5.13}$$

Then for any given $\delta > 0$, there exists $\eta > 0$, such that when

$$\|a\|_{C^1[0,L]} + \|b\|_{C^1[T,+\infty)} \leq \eta, \tag{5.14}$$

we have

$$\|U(t,\cdot) - \overline{U}(t,\cdot)\|_{C^1[0,L]} \leq \delta, \quad \forall\, t \in [T,+\infty). \tag{5.15}$$

Furthermore, for any given $\delta > 0$, there exists $\eta > 0$, such that when

$$\|b\|_{C^1[T,+\infty)} \leq \eta, \tag{5.16}$$

we have

$$\|U(t,\cdot) - \overline{U}(t,\cdot)\|_{C^1[0,L]} \leq \delta \tag{5.17}$$

for time t large enough $\big(t \geq T + L \max\limits_{1 \leq s \leq p} \frac{1}{\lambda_s(0)}\big)$.

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