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Some Properties of Tracially Quasidiagonal Extensions*

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Abstract Suppose that $0 \to I \to A \to A/I \to 0$ is a tracially quasidiagonal extension of C^*-algebras. In this paper, the authors give two descriptions of the K_0, K_1 index maps which are induced by the above extension and show that for any $\epsilon > 0$, any τ in the tracial state space of A/I and any projection $\overline{p} \in A/I$ (any unitary $\overline{u} \in A/I$), there exists a projection $p \in A$ (a unitary $u \in A$) such that $|\tau(\overline{p}) - \tau(\pi(p))| < \epsilon$ $(|\tau(\overline{u}) - \tau(\pi(u))| < \epsilon)$.

Keywords Tracially topological rank, Quasidiagonal extension, Tracially quasidiagonal extension
2000 MR Subject Classification 46L05, 46L35

1 Introduction

Let A be a unital C^*-algebra, I be a two-sided closed ideal of A and π: $A \to A/I$ be the quotient map. Then we have the short exact sequence

$$0 \to I \to A \xrightarrow{\pi} A/I \to 0. \qquad (\star)$$

We denote the extension $(\star)$ by the pair (A, I). It is well known that the extensions of C^*-algebra were first studied by Busby [2] and the attension was not attracted until the development of BDF theory (see [1]). The extension theory becomes more and more important since it describes how complicated C^*-algebras can be constructed. One of the applications is using associated homological invariants to distinguish between C^*-algebras.

As the extension theory is concerned, there is a special case called the quasidiagonal extension (see Definition 2.1) and many results have been obtained up to now. The tracial topological rank of a C^*-algebra (see [7]) and the tracially quasidiagonal extension (see [9] and Definition 2.2) were first raised and studied by Lin (see [10–12]). Thus, we can find more C^*-algebras by extensions and make it possible that new C^*-algebras can be classified with K-theory. It is easy to know that the tracially quasidiagonal extensions are more general than the quasidiagonal ones. Here we only mention some facts related to this paper. Suppose that the extension (A, I) is tracially quasidiagonal. If A/I and I have tracial topological rank zero, then A has tracial topological rank zero (see [6]). In [4], suppose that I has tracial topological rank no more than one and A/I is a TAI algebra. It was showed that A has tracial topological rank no more than

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one if the extension is quasidiagonal, and A has the property (P_1) if the extension is tracially quasidiagonal. In [5], assume that I has tracial topological rank no more than one and A/I is in a class of certain C^*-algebras. Then A has tracial topological rank no more than one if the extension is quasidiagonal, and A has the property (P_1) if the extension is tracially quasidiagonal. Recently in [13], suppose that I and A/I both have tracial topological rank no more than one. It was shown that A has tracial topological rank no more than one if the extension (A, I) is quasidiagonal, and A has the property (P_1) if the extension is tracially quasidiagonal. From these facts, we know that the tracially quasidiagonal extension of C^*-algebras is meaningful and applied widely.

Let $0 \to I \to A \to A/I \to 0$ be a quasidiagonal extension of C^*-algebras. Suppose that any $\overline{a} \in A/I$ is a projection (a unitary). Then $\overline{a}$ has a projection lift (a unitary lift) in A. Therefore, it means that the index maps K_0, K_1 induced by the extension are both zero. Inspired by the results above, it might become a very interesting question that what will happen if the extension is tracially quasidiagonal.

2 Preliminaries

In this paper, we assume that A is a C^*-algebra. The following conventions will be used:

$\mathbb{N}$ is the set of natural numbers;

A_{sa} is the set of all self-adjoint elements of A;

A_+ is the set of all positive elements of A;

$\mathrm{P}(A)$ is the set of all projections of A;

$\mathrm{U}(A)$ is the set of all unitary elements of A;

$\mathrm{PI}(A)$ is the set of all partial isometries of A;

$\mathrm{GL}(A)$ is the set of all invertible elements of A.

For $a, b \in A$, we write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$.

For a C^*-subalgebra C of A, we write $a \in_\epsilon C$ if there is $b \in C$ such that $\|a - b\| < \epsilon$.

For $a \in A$, we denote by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra of A generated by a.

Let a, b be two positive elements of a C^*-algebra A. We write $[a] \leq [b]$ if there is $x \in A$ such that $x^*x = a$, $xx^* \in \mathrm{Her}(b)$. We write $n[a] \leq [b]$ if there are $x_1, \cdots, x_n \in A$ such that ${x_i}^*x_i = a$, $x_i{x_i}^* \in \mathrm{Her}(b)$ and $x_1{x_1}^*, \cdots, x_n{x_n}^*$ are mutually orthogonal.

Let $0 < \sigma_2 < \sigma_1 < 1$. Define $f_{\sigma_2}^{\sigma_1}$ by

$$f_{\sigma_2}^{\sigma_1}(t) = \begin{cases} 1, & t \geq \sigma_1, \\ \dfrac{t - \sigma_2}{\sigma_1 - \sigma_2}, & \sigma_2 < t < \sigma_1, \\ 0, & 0 \leq t \leq \sigma_2. \end{cases}$$

Definition 2.1 (see [1]) *Let A and I be as in ($\star$). Then the extension (A, I) is called quasidiagonal if there is a quasicenter approximate unit $\{r_n\}_{n=1}^\infty$ of I consisting of projections such that*

$$\lim_{n\to\infty} \|r_n x - x r_n\| = 0, \quad \forall x \in A.$$

Theorem 2.1 (see [3]) *Let I and A be as in ($\star$). Suppose that (A, I) is quasidiagonal. If $\overline{a}_i, \cdots, \overline{a}_j$ are any finite elements in A/I, then for any quasicentral approximate unit $\{r_n\}_{n=1}^\infty$ of I consisting of projections, we can choose a subsequence $\{r_{n_k}\}_{k=1}^\infty$ of $\{r_n\}_{n=1}^\infty$, which is clearly* *again a quasicentral approximate unit of I, and $a_1, \cdots, a_k$ in A such that*

$$\pi(a_i) = \overline{a}_i, \quad r_{n_k} a_i = a_i r_{n_k}, \quad k \geq 1,\ i = 1, \cdots, j.$$

Furthermore,

(1) *if $\overline{a}_i \in A/I_+$ for some i, then we can require that $a_i \in A_+$;*

(2) *if $\overline{a}_i \in \mathrm{P}(A/I)$ for some i, then we can require that $a_i \in \mathrm{P}(A)$;*

(3) *if $\overline{a}_i \in \mathrm{U}(A/I)$ for some i, then we can require that $a_i \in \mathrm{U}(A)$;*

(4) *if $\overline{a}_i \in \mathrm{PI}(A/I)$ for some i, then we can require that $a_i \in \mathrm{PI}(A)$.*

Definition 2.2 (see [7]) *Let A and I be as in $(\star)$. Then the extension (A, I) is called tracially quasidiagonal if for any $\epsilon > 0$, $n \in \mathbb{N}$, $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any finite subset $F \subset A$ containing a nonzero positive element a, there exists $p \in \mathrm{P}(A)$ and a C^*-subalgebra $C \subset A$ with $1_C = p$ such that*

(1) $\|xp - px\| < \epsilon,\ \forall x \in F$;

(2) $pxp \in_\epsilon C,\ \forall x \in F$;

(3) $n[1-p] \leq [p]$ *and* $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$;

(4) $C \cap I = pIp$ *and the extension (C, pIp) is quasidiagonal.*

Lemma 2.1 (see [8]) *If $a \in A_{sa}$ such that $\|a - a^2\| < \frac{1}{4}$, then there exists a projection p in the C^*-subalgebra generated by a such that*

$$\|a - p\| < 2\|a - a^2\|.$$

Lemma 2.2 (see [8]) *Let A be a unital Banach algebra and a be an element of A such that $\|1 - a\| < 1$. Then $a \in \mathrm{GL}(A)$ and*

$$a^{-1} = \sum_{n=0}^{\infty} (1-a)^n.$$

Moreover, $\|a^{-1}\| \leq \frac{1}{1 - \|1-a\|}$ and $\|1 - a^{-1}\| \leq \frac{\|1-a\|}{1-\|1-a\|}$.

3 Main Results and Proofs

Now we begin to give our main results and their proofs.

Theorem 3.1 *Suppose that the extension (A, I) is tracially quasidiagonal. Let $\overline{p} \in \mathrm{P}(A/I)$ and $G_0 \subset K_0(A/I)$ be any finitely generated subgroup. Then for any $\epsilon > 0$, $n \in \mathbb{N}$, $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any finite subset $F \subset A$ containing a nonzero positive element a, there is a projection $r \in \mathrm{P}(A)$ such that*

(1) $\|rx - xr\| < \epsilon$, $\forall x \in F$;

(2) $n[r] \leq [1-r]$ *and* $n[f_{\sigma_2}^{\sigma_1}(rar)] \leq [f_{\sigma_4}^{\sigma_3}((1-r)a(1-r))]$;

(3) *there exists a projection $p \in A$ and $\overline{p}'' \in \pi(r)(A/I)\pi(r)$ such that*

$$\|\overline{p} - \pi(p) - \overline{p}''\| < \frac{\epsilon}{2};$$

(4) $\delta_0(G_0) \subset \mathrm{Im}(\iota_{r*1} \circ \delta_0^r)$, *where $\delta_0 : K_0(A/I) \to K_1(I)$, $\delta_0^r : K_0(\pi(rAr)) \to K_1(rIr)$ are the index maps and $\iota_{r*1} : K_1(rIr) \to K_1(I)$ is induced by $rIr \hookrightarrow I$.*

Proof Since $\overline{p} \in \mathrm{P}(A/I)$, we can find $b \in A_+$ such that

$$\|b\| \leq 1, \quad \pi(b) = \overline{p}.$$

Let G_0 be a finitely generated subgroup of $K_0(A/I)$. Then we can find $\overline{p}_i, \overline{q}_i \in M_{n_i}(A/I)$, $i = 1, \cdots, j$, such that $\{[\overline{p}_i]_0 - [\overline{q}_i]_0 | i = 1, \cdots, j\}$ generates G_0. Let $m = \max\{n_1, \cdots, n_i\}$. If $\overline{p}_i, \overline{q}_i \notin M_m(A/I)$, then we can replace $\overline{p}_i$ and $\overline{q}_i$ by $\overline{p}'_i = \mathrm{diag}(\overline{p}_i, \underbrace{0, \cdots, 0}_{m-n_i})$ and $\overline{q}'_i = \mathrm{diag}(\overline{q}_i, \underbrace{0, \cdots, 0}_{m-n_i})$ respectively. It still holds that $[\overline{p}_i]_0 = [\overline{p}'_i]_0$ and $[\overline{q}_i]_0 = [\overline{q}'_i]_0$. Without loss of generality, we assume that $\overline{p}_1, \cdots, \overline{p}_j, \overline{q}_1, \cdots, \overline{q}_j \in M_m(A/I)$ for some integer $m > 0$.

For $i = 1, \cdots, j$, there exist self-adjoint elements $a_1, \cdots, a_j, b_1, \cdots, b_j$ in $M_m(A)$ such that

$$\pi(a_i) = \overline{p}_i \quad \text{and} \quad \pi(b_i) = \overline{q}_i.$$

Given $\epsilon > 0$, $n \in \mathbb{N}$, $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, for any finite subset $F \subset A$ containing a positive element a, let $F' = \{a_{lt} \mid (a_{lt})_{m \times m} \in \{a_i, b_i \mid i = 1, \cdots, j\}\} \cup F \cup \{b\}$ and $0 < \delta < \min\left\{1, \frac{\epsilon}{2}\right\}$, where δ will be decided later. Since the extension (A, I) is tracially quasidiagonal, there exists a C^*-subalgebra $C \subset A$ and $s \in \mathrm{P}(A)$ with $1_C = s$ such that

(1′) $\|sx - xs\| < \delta$, $\forall x \in F'$;

(2′) $sxs \in_\delta C$, $\forall x \in F'$;

(3′) $n[1-s] \leq [s]$ and $n[f_{\sigma_2}^{\sigma_1}((1-s)a(1-s))] \leq [f_{\sigma_4}^{\sigma_3}(sas)]$;

(4′) $C \cap I = sIs$ and the extension (C, sIs) is quasidiagonal.

Let $r = 1 - s$. Directly from (1′) and (2′), it follows that

(1) $\|rx - xr\| < \epsilon$, $\forall x \in F$;

(2) $n[r] \leq [1-r]$ and $n[f_{\sigma_2}^{\sigma_1}(rar)] \leq [f_{\sigma_4}^{\sigma_3}((1-r)a(1-r))]$.

By (2′), we can find $c \in C_{sa}$ such that

$$\|sbs - c\| < \delta.$$

Without loss of generality, we may assume that $\|c\| \leq 1$. Then

$$\|\pi(s)\overline{p}\pi(s) - \pi(c)\| = \|\pi(s)\pi(b)\pi(s) - \pi(c)\| < \delta.$$

It follows $\pi(c)^2 - \pi(c) \approx_\delta \pi(s)\overline{p}\pi(s)\pi(c) - \pi(c) \approx_\delta \pi(s)\overline{p}\pi(s)\pi(c) - \pi(s)\overline{p}\pi(s) \approx_\delta \pi(s)\overline{p}\pi(s)\overline{p}\pi(s) - \pi(s)\overline{p}\pi(s) \approx_\delta \pi(s)\overline{p}\pi(s) - \pi(s)\overline{p}\pi(s) = 0$ (by (1′)).

By Lemma 2.1, there exists a projection $\overline{p}' \in \pi(C)$ such that

$$\|\pi(c) - \overline{p}'\| < 8\delta.$$

Since

$$\begin{aligned} \|\overline{p} - \pi(sbs) - \pi(rbr)\| &= \|\pi(sbr) + \pi(rbs)\| \leq \max\{\|\pi(sbr)\|, \|\pi(rbs)\|\}, \\ \overline{p} &\approx_\delta \pi(sbs) + \pi(rbr)(\|\pi(sbr)\|, \|\pi(rbs)\| \approx_\delta 0 \,\text{from}\, (1')) \\ &\approx_\delta \pi(c) + \pi(rbr) \\ &\approx_{8\delta} \overline{p}' + \pi(rbr), \end{aligned}$$

we have

$$\|\overline{p} - \overline{p}' - \pi(rbr)\| < 10\delta.$$

It follows from (4′) that the extension (C, sIs) is quasidiagonal. By Theorem 2.1, there exists a projection $p \in C$ such that $\pi(p) = \overline{p}'$. We have

$$\|\overline{p} - \pi(p) - \pi(rbr)\| < 10\delta.$$

Similarly, since

$$\begin{aligned}[\pi(rbr)]^2 - \pi(rbr) &= \pi(rbrbr) - \pi(rbr)\\ &\approx_\delta \pi(rb^2r) - \pi(rbr)\\ &= \pi(r)\overline{p}\pi(r) - \pi(r)\overline{p}\pi(r)\\ &= 0,\end{aligned}$$

again by Lemma 2.1, there exists a projection $\overline{p}'' \in \pi(r)(A/I)\pi(r)$ such that

$$\|\pi(rbr) - \overline{p}''\| < 2\delta.$$

Thus

$$\|\overline{p} - \pi(p) - \overline{p}''\| < 12\delta.$$

Let δ be sufficiently small $\left(< \frac{\epsilon}{24}\right)$. Then the statement (3) follows, that is,

(3) $\|\overline{p} - \pi(p) - \overline{p}''\| < \frac{\epsilon}{2}$.

Let $s_m = \mathrm{diag}(\underbrace{s, s, \cdots, s}_{m})$. By (1′), for $1 \le i \le j$, we have

$$a_i \approx_\delta s_m a_i s_m + (1 - s_m) a_i (1 - s_m)$$

and

$$b_i \approx_\delta s_m b_i s_m + (1 - s_m) b_i (1 - s_m).$$

Since

$$\begin{aligned}[\pi(s_m a_i s_m)]^2 - \pi(s_m a_i s_m) &= \pi(s_m a_i s_m a_i s_m) - \pi(s_m a_i s_m)\\ &\approx_\delta \pi(s_m a_i^2 s_m) - \pi(s_m a_i s_m)\\ &= \pi(s_m)\overline{p}_i\pi(s_m) - \pi(s_m)\overline{p}_i\pi(s_m)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}[\pi(s_m b_i s_m)]^2 - \pi(s_m b_i s_m) &= \pi(s_m b_i s_m b_i s_m) - \pi(s_m b_i s_m)\\ &\approx_\delta \pi(s_m b_i^2 s_m) - \pi(s_m b_i s_m)\\ &= \pi(s_m)\overline{q}_i\pi(s_m) - \pi(s_m)\overline{q}_i\pi(s_m)\\ &= 0,\end{aligned}$$

by Lemma 2.1, there exist $\overline{x}_i, \overline{y}_i \in \mathrm{P}(\pi(M_m(C)))$ such that

$$\pi(s_m a_i s_m) \approx_{2\delta} \overline{x}_i, \quad \pi(s_m b_i s_m) \approx_{2\delta} \overline{y}_i.$$

Since the extension $0 \to sIs \to C \to \pi(C) \to 0$ is quasidiagonal, the extension $0 \to M_m(sIs) \to M_m(C) \to M_m(\pi(C)) \to 0$ is also quasidiagonal and there exist $x_i, y_i \in \mathrm{P}(M_m(C))$ such that

$$\pi(x_i) = \overline{x}_i, \quad \pi(y_i) = \overline{y}_i.$$

Moreover, we can find self-adjoint elements $c_i, d_i \in M_m(I)$ such that

$$\|s_m a_i s_m - c_i - x_i\| < 2\delta, \quad \|s_m b_i s_m - d_i - y_i\| < 2\delta.$$

Similarily, for $i = 1, \cdots, j$, since

$$\begin{aligned}
&[\pi((1-s_m)a_i(1-s_m))]^2 - \pi((1-s_m)a_i(1-s_m)) \\
&= \pi((1-s_m)a_i(1-s_m)a_i(1-s_m)) - \pi((1-s_m)a_i(1-s_m)) \\
&\approx_\delta \pi((1-s_m)a_i^2(1-s_m)) - \pi((1-s_m)a_i(1-s_m)) \\
&= \pi(1-s_m)\overline{p}_i\pi(1-s_m) - \pi(1-s_m)\overline{p}_i\pi(1-s_m) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&[\pi((1-s_m)b_i(1-s_m))]^2 - \pi((1-s_m)b_i(1-s_m)) \\
&= \pi((1-s_m)b_i(1-s_m)b_i(1-s_m)) - \pi((1-s_m)b_i(1-s_m)) \\
&\approx_\delta \pi((1-s_m)b_i^2(1-s_m)) - \pi((1-s_m)b_i(1-s_m)) \\
&= \pi(1-s_m)\overline{q}_i\pi(1-s_m) - \pi(1-s_m)\overline{q}_i\pi(1-s_m) \\
&= 0,
\end{aligned}$$

there exist elements $\overline{x}'_i, \overline{y}'_i \in \mathrm{P}(\pi((1-s_m)M_m(A)(1-s_m)))$ and self-adjoint elements $x'_i, y'_i \in (1-s_m)M_m(A)(1-s_m)$ such that

$$\begin{aligned}
&\pi((1-s_m)a_i(1-s_m)) \approx_{2\delta} \overline{x}'_i, \quad \pi((1-s_m)b_i(1-s_m)) \approx_{2\delta} \overline{y}'_i; \\
&\pi(x'_i) = \overline{x}'_i, \quad \pi(y'_i) = \overline{y}'_i.
\end{aligned}$$

Therefore, we can find self-adjoint elements $e_i, f_i \in M_m(I)$ such that

$$\|x'_i + e_i - (1-s_m)a_i(1-s_m)\| < 2\delta, \quad \|y'_i + f_i - (1-s_m)b_i(1-s_m)\| < 2\delta.$$

From the discussions above, it can be concluded that

$$\begin{aligned}
&a_i - c_i - e_i \approx_{5\delta} x_i + x'_i, \quad \pi(a_i) = \pi(a_i - c_i - e_i) = \overline{p}_i \approx_{5\delta} \overline{x}_i + \overline{x}'_i; \\
&b_i - d_i - f_i \approx_{5\delta} y_i + y'_i, \quad \pi(b_i) = \pi(b_i - d_i - f_i) = \overline{q}_i \approx_{5\delta} \overline{y}_i + \overline{y}'_i.
\end{aligned}$$

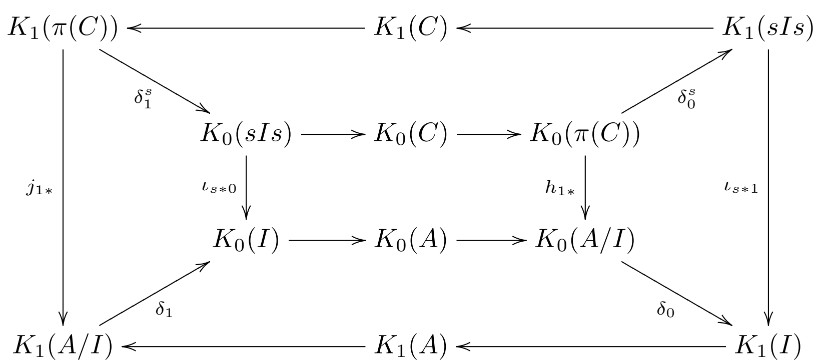
Since

$$\overline{p}_i \approx_{5\delta} \overline{x}_i + \overline{x}'_i, \quad \overline{q}_i \approx_{5\delta} \overline{y}_i + \overline{y}'_i$$

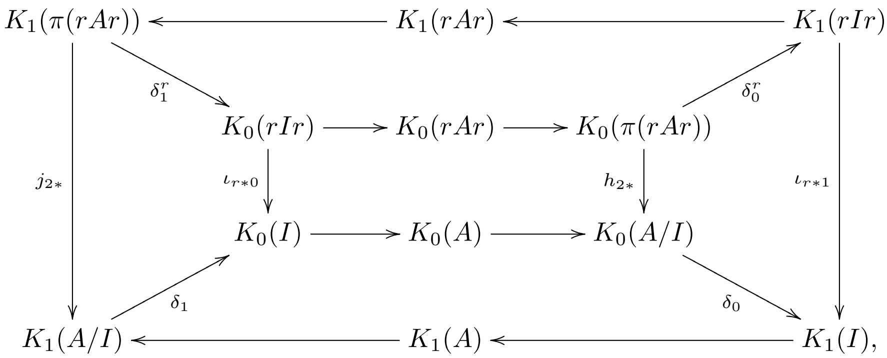
with δ sufficiently small, we have

$$[\overline{x}_i + \overline{x}'_i]_0 = [\overline{p}_i]_0, \quad [\overline{y}_i + \overline{y}'_i]_0 = [\overline{q}_i]_0.$$

By the following two commutative diagrams



and



for $1 \leq i \leq j$, we have

$$\begin{aligned}
&\iota_{s*1}(\delta_0^s([\overline{x}_i]_0)) + \iota_{r*1}(\delta_0^r([\overline{x}'_i]_0)) \\
= {}& \delta_0 \circ h_{1*}([\overline{x}_i]_0) + \delta_0 \circ h_{2*}([\overline{x}'_i]_0) \\
= {}& \delta_0([\overline{x}_i]_0 + [\overline{x}'_i]_0) \\
= {}& \delta_0([\overline{x}_i + \overline{x}'_i]_0) \\
= {}& \delta_0([\overline{p}_i]_0)
\end{aligned}$$

and

$$\begin{aligned}
&\iota_{s*1} \circ \delta_0^s([\overline{y}_i]_0) + \iota_{r*1} \circ \delta_0^r([\overline{y}'_i]_0) \\
= {}& \delta_0 \circ h_{1*}([\overline{y}_i]_0) + \delta_0 \circ h_{2*}([\overline{y}'_i]_0) \\
= {}& \delta_0([\overline{y}_i]_0 + [\overline{y}'_i]_0) \\
= {}& \delta_0([\overline{y}_i + \overline{y}'_i]_0) \\
= {}& \delta_0([\overline{q}_i]_0).
\end{aligned}$$

Since the extension $0 \to sIs \to C \to \pi(C) \to 0$ is quasidiagonal, the index map δ_0^s is zero. We have

$$\delta_0([\overline{p}_i]_0 - [\overline{q}_i]_0) = \iota_{r*1} \circ \delta_0^r([\overline{x}'_i]_0) - \iota_{r*1} \circ \delta_0^r([\overline{y}'_i]_0) = \iota_{r*1} \circ \delta_0^r([\overline{x}'_o]_0 - [\overline{y}'_i]_0).$$

It shows that

(4) $\delta_0(G_0) \subset \mathrm{Im}(\iota_{r*1} \circ \delta_0^r)$.

Corollary 3.1 *Suppose that the extension (A, I) is tracially quasidiagonal. Let $\mathrm{T}(A/I)$ be the tracial state space of A/I. Then for any $\epsilon > 0$, $\overline{p} \in \mathrm{P}(A/I)$ and $\tau \in \mathrm{T}(A/I)$, there exists $p \in \mathrm{P}(A)$ such that*

$$|\tau(\overline{p}) - \tau(\pi(p))| < \epsilon.$$

Proof Fix $\epsilon > 0$. For $\frac{\epsilon}{2}$, $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{\epsilon}{2}$, $\tau \in \mathrm{T}(A/I)$ and $\overline{p} \in \mathrm{P}(A/I)$, then from Theorem 3.1 (2)–(3), we can find $r, p \in \mathrm{P}(A)$ and $\overline{p}'' \in \pi(r)(A/I)\pi(r)$ such that

$$n[r] \leq [1 - r], \quad \|\overline{p} - \pi(p) - \overline{p}''\| < \frac{\epsilon}{2}.$$

Since $n[r] \leq [1 - r]$, there exist $v_1, \cdots, v_n$ such that $v_i v_i^* = r$ and $v_i^* v_i \in \mathrm{Her}((1 - r)A(1 - r))$ are mutually orthogonal. Thus,

$$|\tau(\overline{p}) - \tau(\pi(p))| \leq |\tau(\overline{p}) - \tau(\pi(p)) - \tau(\overline{p}'')| + \tau(\overline{p}'')$$

$$
\begin{aligned}
&< \frac{\epsilon}{2} + \tau(\overline{p}'') \\
&\le \frac{\epsilon}{2} + \tau(\pi(r)) \\
&= \frac{\epsilon}{2} + \tau(\pi(v_i^* v_i)) \\
&= \frac{\epsilon}{2} + \frac{1}{n}\Big(\tau\Big(\pi\Big(\sum_{i=1}^{n} v_i^* v_i\Big)\Big)\Big) \\
&\le \frac{\epsilon}{2} + \frac{1}{n}\tau(\pi(1-r)) \\
&\le \frac{\epsilon}{2} + \frac{1}{n} \\
&< \epsilon.
\end{aligned}
$$

It shows that

$$|\tau(\overline{p}) - \tau(\pi(p))| < \epsilon.$$

Theorem 3.2 *Suppose that the extension (A, I) is tracially quasidiagonal. Let $\overline{u} \in \mathrm{U}(A/I)$ and $G_1 \subset K_1(A/I)$ be any finitely generated subgroup. Then for any $\epsilon > 0$, $n \in \mathbb{N}$, $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any finite subset $F \subset A$ containing a positive element a, there is a projection $r \in \mathrm{P}(A)$ such that*

(1) $\|rx - xr\| < \epsilon$, $\forall x \in F$;

(2) $n[r] \le [1-r]$ *and* $n[f_{\sigma_2}^{\sigma_1}(rar)] \le [f_{\sigma_4}^{\sigma_3}((1-r)a(1-r))]$;

(3) *there exists a partial unitary v with $v^*v = vv^* = 1-r$ and $\overline{u}'$ which is in $\mathrm{U}(\pi(r)(A/I)\pi(r))$ such that*

$$\|\overline{u} - \pi(v) - \overline{u}'\| < \frac{\epsilon}{2};$$

(4) $\delta_1(G_1) \subset \mathrm{Im}(\iota_{r*0} \circ \delta_1^r)$, *where $\delta_1 : K_1(A/I) \to K_0(I)$ and $\delta_1^r : K_1(\pi(rAr)) \to K_0(rIr)$ are the index maps, and $\iota_{r*0} : K_0(rIr) \to K_0(I)$ is induced by $rIr \hookrightarrow I$.*

Proof Without loss of generality, we can choose $b \in A$ with $\|b\| \le 1$ such that

$$\pi(b) = \overline{u}.$$

Let $\overline{u}_1 \in M_{n_1}(A/I), \cdots, \overline{u}_j \in M_{n_j}(A/I)$ be such that $[\overline{u}_1]_1, \cdots, [\overline{u}_j]_1$ are the generators of G_1. Let $m = \max\{n_1, \cdots, n_i\}$. If $\overline{u}_i \notin M_m(A/I)$, then we replace $\overline{u}_i$ by

$$\overline{u}_i' = \begin{pmatrix} \overline{u}_i & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{m\times m}$$

and obtain $[\overline{u}_i]_1 = [\overline{u}_i']_1$. Thus, it can be assumed that $\overline{u}_1, \cdots, \overline{u}_j \in M_m(A/I)$ for some integer $m > 0$.

For any $1 \le i \le j$, let v_i be

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{U}(M_{2m}(A)),$$

where $a_i, b_i, c_i, d_i \in M_m(A)$, such that

$$\pi(v_i) = \begin{pmatrix} \overline{u}_i & 0 \\ 0 & \overline{u}_i^* \end{pmatrix}.$$

It follows that there exist $b_i, c_i, e_i \in M_m(I)$ such that

$$d_i = a_i^* + e_i$$

and we obtain

$$v_i = \begin{pmatrix} a_i & b_i \\ c_i & a_i^* + e_i \end{pmatrix}.$$

Let $F' = \{a_{lt} \mid (a_{lt})_{m\times m} \in \{a_i, b_i, c_i, a_i^* + e_i \mid i = 1, \cdots, j\}\} \cup F \cup \{b\}$ and let $\epsilon > 0$, $n \in \mathbb{N}$, $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$. Given $0 < \delta < \min\{\frac{\epsilon}{16}, \frac{1}{3}\}$, since the extension (A, I) is tracially quasidiagonal, there exists $s \in \mathrm{P}(A)$ and a C^*-subalgebra $C \subset A$ with $1_C = s$ such that

(1′) $\|sx - xs\| < \delta$, $\forall x \in F'$;

(2′) $sxs \in_\delta C$, $\forall x \in F'$;

(3′) $n[1 - s] \leq [s]$ and $n[f_{\sigma_2}^{\sigma_1}((1 - s)b(1 - s))] \leq [f_{\sigma_4}^{\sigma_3}(sbs)]$;

(4′) $C \cap I = sIs$, the extension (C, sIs) is quasidiagonal.

Write $r = 1 - s$. From (1′)–(2′), it follows that

(1) $\|rx - xr\| < \epsilon$, $\forall x \in F$;

(2) $n[r] \leq [1 - r]$ and $n[f_{\sigma_2}^{\sigma_1}(rar)] \leq [f_{\sigma_4}^{\sigma_3}((1 - r)a(1 - r))]$.

From (1′), since $sb - bs \approx_\delta 0$, we have

$$b \approx_\delta sbs + rbr$$

and

$$\overline{u} \approx_\delta \pi(s)\overline{u}\pi(s) + \pi(r)\overline{u}\pi(r).$$

By (2′), we can find $c \in C$ with $\|c\| \leq 1$ such that

$$\|sbs - c\| < \delta.$$

Therefore

$$\|\pi(s)\overline{u}\pi(s) - \pi(c)\| \leq \|sbs - c\| < \delta.$$

Since

$$\begin{aligned} \pi(c)^*\pi(c) &\approx_\delta \pi(s)\overline{u}^*\pi(s)\pi(c) \\ &\approx_\delta \pi(s)\overline{u}^*\pi(s)\pi(s)\overline{u}\pi(s) \\ &= \pi(s)\overline{u}^*\pi(s)\overline{u}\pi(s) \\ &\approx_\delta \pi(s) \end{aligned}$$

and

$$\begin{aligned} \pi(c)\pi(c)^* &\approx_\delta \pi(c)\pi(s)\overline{u}^*\pi(s) \\ &\approx_\delta \pi(s)\overline{u}\pi(s)\pi(s)\overline{u}^*\pi(s) \\ &= \pi(s)\overline{u}\pi(s)\overline{u}^*\pi(s) \\ &\approx_\delta \pi(s), \end{aligned}$$

we have

$$\|\pi(c)^*\pi(c) - \pi(s)\| < 3\delta, \quad \|\pi(c)\pi(c)^* - \pi(s)\| < 3\delta.$$

Since $\delta < \frac{1}{3}$,

$$\|\pi(c)^*\pi(c) - \pi(s)\| < 3\delta < 1, \quad \|\pi(c)\pi(c)^* - \pi(s)\| < 3\delta < 1.$$

It is known that $\pi(c)^*\pi(c)$ and $\pi(c)\pi(c)^*$ are invertible in C. Let $\overline{w} = \pi(c)|\pi(c)|^{-1}$, then $\overline{w} \in \mathrm{U}(\pi(C))$. Since

$$0 \leq \pi(s) - |\pi(c)| \leq \pi(s) - \pi(c)^*\pi(c) < 3\delta,$$

we have $\|\pi(s) - |\pi(c)|\| \leq \|\pi(s) - \pi(c)^*\pi(c)\| < 1$ and

$$\begin{aligned}
\|\overline{w} - \pi(c)\| &= \|\pi(c)|\pi(c)|^{-1} - \pi(c)\| \\
&\leq \|\pi(s) - |\pi(c)|^{-1}\| \\
&\leq \frac{\|\pi(s) - |\pi(c)|\|}{1 - \|\pi(s) - |\pi(c)|\|} \quad \text{(by Lemma 2.2)} \\
&\leq \frac{\|\pi(s) - \pi(c)^*\pi(c)\|}{1 - \|\pi(s) - \pi(c)^*\pi(c)\|} \quad \Big(f(x) = \frac{x}{1-x} \text{ is increasing on } [0,1)\Big) \\
&< \frac{3\delta}{1-3\delta}.
\end{aligned}$$

That is

$$\|\overline{w} - \pi(c)\| < \frac{3\delta}{1-3\delta}.$$

On the other side, since

$$\begin{aligned}
[\pi(r)\overline{u}\pi(r)]^*[\pi(r)\overline{u}\pi(r)] &\approx_\delta \pi(r)\overline{u}^*\overline{u}\pi(r) \\
&= \pi(r)
\end{aligned}$$

and

$$\begin{aligned}
[\pi(r)\overline{u}\pi(r)][\pi(r)\overline{u}\pi(r)]^* &\approx_\delta \pi(r)\overline{u}\overline{u}^*\pi(r) \\
&= \pi(r),
\end{aligned}$$

similarly, by the above discussions, there is a $\overline{u}' \in \mathrm{U}(\pi(r)(A/I)\pi(r))$ such that

$$\|\pi(r)\overline{u}\pi(r) - \overline{u}'\| < \frac{\delta}{1-\delta}.$$

It is from $(4')$ that the extension (C, sIs) is quasidiagonal. By Theorem 2.1, there can be found $v \in \mathrm{U}(C)$ such that $\pi(v) = \overline{w}$. Then

$$\begin{aligned}
\overline{u} &\approx_\delta \pi(s)\overline{u}\pi(s) + \pi(r)\overline{u}\pi(r) \\
&\approx_\delta \pi(c) + \pi(r)\overline{u}\pi(r) \\
&\approx_{\frac{3\delta}{1-3\delta}} \pi(v) + \pi(r)\overline{u}\pi(r) \\
&\approx_{\frac{\delta}{1-\delta}} \pi(v) + \overline{u}'.
\end{aligned}$$

Therefore

$$\|\overline{u} - \pi(v) - \overline{u}'\| < 2\delta + \frac{3\delta}{1-3\delta} + \frac{\delta}{1-\delta} < \frac{8\delta}{1-8\delta} < \frac{\epsilon}{2}.$$

It follows that

(3) $\|\overline{u} - \pi(v) - \overline{u}'\| < \frac{\epsilon}{2}$.

For $i = 1, \cdots, j$, let

$$s_m = \begin{pmatrix} s & & \\ & \ddots & \\ & & s \end{pmatrix}_{m \times m},$$

then

$$\begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} \begin{pmatrix} a_i & b_i \\ c_i & a_i^* + e_i \end{pmatrix} \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} = \begin{pmatrix} s_m a_i s_m & s_m b_i s_m \\ s_m c_i s_m & s_m a_i^* s_m + s_m e_i s_m \end{pmatrix}.$$

From $(1')$–$(2')$, it is known that

$$\|s_m a_i - a_i s_m\| < \delta, \quad \|s_m b_i - b_i s_m\| < \delta,$$

$$\|s_m c_i - c_i s_m\| < \delta, \quad \|s_m d_i - d_i s_m\| < \delta.$$

We can find $x_i \in M_m(C)$ such that $\|s_m a_i s_m - x_i\| < \delta$. Let

$$w_i' = \begin{pmatrix} x_i & s_m b_i s_m \\ s_m c_i s_m & s_m x_i^* s_m + s_m e_i s_m \end{pmatrix},$$

then $\|w_i' - \mathrm{diag}(s_m, s_m) v_i \mathrm{diag}(s_m, s_m)\| < \delta$ and

$$\begin{aligned} (w_i')^* w_i' &\approx_{2\delta} \mathrm{diag}(s_m, s_m) v_i^* \mathrm{diag}(s_m, s_m) v_i \mathrm{diag}(s_m, s_m) \\ &\approx_{\delta} \mathrm{diag}(s_m, s_m) v_i^* v_i \mathrm{diag}(s_m, s_m) \\ &= \mathrm{diag}(s_m, s_m). \end{aligned}$$

It implies that

$$\|(w_i')^* w_i' - \mathrm{diag}(s_m, s_m)\| < 3\delta.$$

On the other side, we have

$$\|w_i'(w_i')^* - \mathrm{diag}(s_m, s_m)\| < 3\delta.$$

It is easy to check that w_i' is invertible in $M_{2m}(C)$. Let $y_i = w_i'|w_i'|^{-1}$, then $y_i \in \mathrm{U}(M_{2m}(C))$ and $\pi(y_i) = \mathrm{diag}(\omega_i, \omega_i^*)$ for some $\omega_i \in \mathrm{U}(\pi(M_m(C)))$. Since

$$0 \le \mathrm{diag}(s_m, s_m) - |w_i'| \le \mathrm{diag}(s_m, s_m) - (w_i')^*(w_i') < 3\delta,$$

we have $\|y_i - w_i'\| < \frac{3\delta}{1-3\delta}$. Then

$$\|y_i - \mathrm{diag}(s_m, s_m) v_i \mathrm{diag}(s_m, s_m)\| < \delta + \frac{3\delta}{1-3\delta}.$$

Let $w_i'' = (1 - \mathrm{diag}(s_m, s_m)) v_i (1 - \mathrm{diag}(s_m, s_m))$. By calculation, we have

$$\|(w_i'')^* w_i'' - (1 - \mathrm{diag}(s_m, s_m))\| < 3\delta$$

and

$$\|w_i''(w_i'')^* - (1 - \mathrm{diag}(s_m, s_m))\| < 3\delta.$$

It follows that w_i'' is invertible in $M_{2m}((1-s)A(1-s))$ and we put $z_i = w_i''|w_i''|^{-1}$. Since

$$0 \le (1 - \mathrm{diag}(s_m, s_m)) - |w_i''| \le (1 - \mathrm{diag}(s_m, s_m)) - (w_i'')^*(w_i'') < 3\delta,$$

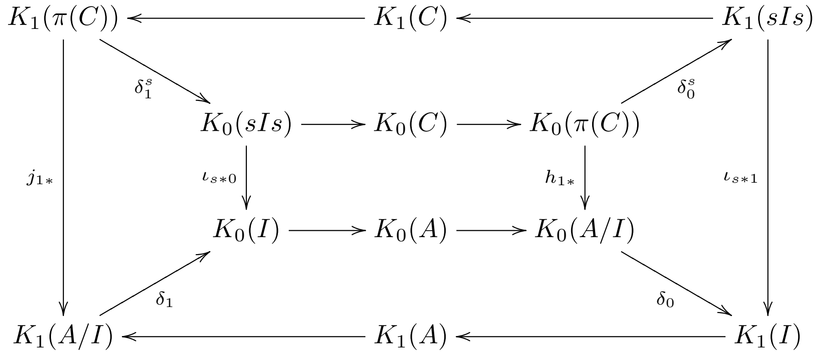
we have $\|z_i - w_i''\| < \frac{3\delta}{1-3\delta}$ and $\pi(z_i) = \mathrm{diag}(\omega_i', \omega_i'^*)$ for some $\omega_i' \in \mathrm{U}(\pi(M_m((1-s)A(1-s))))$. Thus, we obtain

$$\begin{aligned} v_i &\approx_{\delta} \mathrm{diag}(s_m, s_m) v_i \mathrm{diag}(s_m, s_m) + (1 - \mathrm{diag}(s_m, s_m)) v_i (1 - \mathrm{diag}(s_m, s_m)) \\ &\approx_{\delta + \frac{3\delta}{1-3\delta}} y_i + (1 - \mathrm{diag}(s_m, s_m)) v_i (1 - \mathrm{diag}(s_m, s_m)) \\ &\approx_{\frac{3\delta}{1-3\delta}} y_i + z_i, \end{aligned}$$

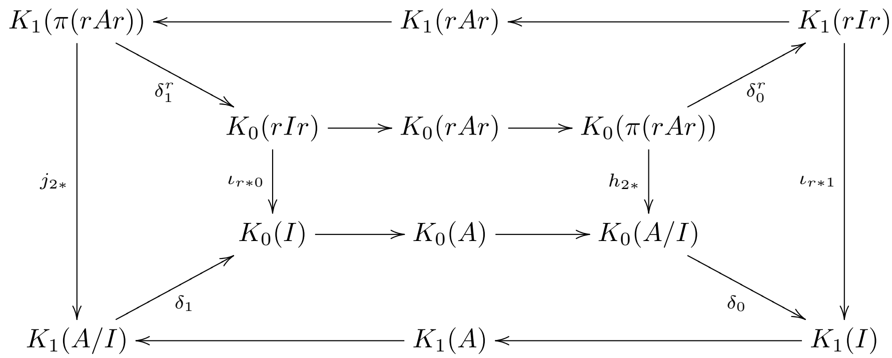
which shows that

$$\|v_i - (y_i + z_i)\| < 2\delta + \frac{3\delta}{1-3\delta} + \frac{3\delta}{1-3\delta} < \frac{8\delta}{1-3\delta}.$$

Applying the following two commutative diagrams of K-group



and



for $i = 1, \cdots, j$, we have

$$\begin{aligned}
&\iota_{s*0}(\delta_1^s([\omega_i]_1)) + \iota_{r*0}(\delta_1^r([\omega_i']_1)) \\
&= \delta_1 \circ j_{1*}([\omega_i]_1) + \delta_1 \circ j_{2*}([\omega_i']_1) \\
&= \delta_1([\omega_i + 1 - s_m]_1) + \delta_1([\omega_i' + s_m]_1) \\
&= \delta_1([\mathrm{diag}(\omega_i + 1 - s_m, \omega_i' + s_m)]_1) \\
&= \delta_1([\mathrm{diag}(\omega_i + \omega_i', 1)]_1) \\
&= \delta_1([\omega_i + \omega_i']_1).
\end{aligned}$$

Since $\|v_i - (y_i + z_i)\| < \frac{8\delta}{1-3\delta}$, we have

$$\|\pi(v_i) - \pi(y_i + z_i)\| < \frac{8\delta}{1 - 3\delta},$$

that is,

$$\|\mathrm{diag}(\overline{u}_i, \overline{u}_i^*) - \mathrm{diag}(\omega_i, \omega_i^*) - \mathrm{diag}(\omega_i', \omega_i'^*)\| < \frac{8\delta}{1 - 3\delta}.$$

Let δ be sufficiently small such that

$$[\overline{u}_i]_1 = [\omega_i + \omega_i']_1.$$

Therefore

$$\delta_1([\overline{u}_i]_1) = \delta_1([\omega_i + \omega_i']_1) = \iota_{s*0} \circ \delta_1^s([\omega_i]_1) + \iota_{r*0} \circ ([\omega_i']_1).$$

By (4′), since the extension $0 \to sIs \to C \to \pi(C) \to 0$ is quasidiagonal, we see that the index map δ_1^s is zero and we have

$$\delta_1([\overline{u}_i]_1) = \iota_{r*0} \circ ([\omega_i']_1), \quad i = 1, \cdots, j.$$

It shows that

(4) $\delta_1(G_1) \subset \mathrm{Im}(\iota_{r*0} \circ \delta_1^r)$.

Corollary 3.2 *Suppose that the extension (A, I) is tracially quasidiagonal. Let $\mathrm{T}(A/I)$ be the tracial state space of A/I. Then for any $\epsilon > 0$, any $\overline{u} \in \mathrm{U}(A/I)$ and $\tau \in \mathrm{T}(A/I)$, there exists $u \in \mathrm{U}(A)$ such that*

$$\|\tau(\overline{u}) - \tau(\pi(u))\| < \epsilon.$$

Proof Let $\epsilon > 0$. Given $\frac{\epsilon}{2}$ and n with $n^{\frac{1}{2}} < \frac{\epsilon}{2}$, by Theorem 3.2 (2)–(3), there exists a projection r, a partial unitary $v \in A$ with $v^*v = vv^* = 1 - r$ and a unitary $\overline{u}' \in \pi(r)(A/I)\pi(r)$ such that

$$n[r] \le [1 - r], \quad \|\overline{u} - \pi(v) - \overline{u}'\| < \frac{\epsilon}{2}.$$

Write $u = v + r$. Then $uu^* = u^*u = 1$ and u is a unitary in A. Since

$$\begin{aligned}
\tau(\overline{u}) - \tau(\pi(u)) &= \tau(\overline{u}) - \tau(\pi(v + r)) \\
&= \tau(\overline{u}) - \tau(\pi(v)) - \tau(\pi(r)) \\
&= \tau(\overline{u}) - \tau(\pi(v)) - \tau(\overline{u}') + \tau(\overline{u}') - \tau(\pi(r)) \\
&\approx_{\frac{\epsilon}{2}} \tau(\pi(r)\overline{u}'\pi(r)) - \tau(\pi(r)),
\end{aligned}$$

we have

$$\begin{aligned}
|\tau(\overline{u}) - \tau(u)| &< \frac{\epsilon}{2} + |\tau(\pi(r)\overline{u}'\pi(r)) - \tau(\pi(r))| \\
&= \frac{\epsilon}{2} + |\tau(\pi(r)(\overline{u}' - 1)\pi(r))| \\
&\le \frac{\epsilon}{2} + |\tau(\pi(r)(\overline{u}' - 1)^*\pi(r)(\overline{u}' - 1)\pi(r))|^{\frac{1}{2}} \\
&\le \frac{\epsilon}{2} + 2\tau(\pi(r))^{\frac{1}{2}}.
\end{aligned}$$

Since $n[r] \le [1 - r]$, there are $v_1, \cdots, v_n$ such that $v_iv_i^* = r$ and $v_i^*v_i (\le 1 - r)$ are mutually orthogonal. Therefore

$$\begin{aligned}
|\tau(\overline{u}) - \tau(u)| &< \frac{\epsilon}{2} + 2\tau(\pi(r))^{\frac{1}{2}} \\
&= \frac{\epsilon}{2} + 2\tau(\pi(v_iv_i^*))^{\frac{1}{2}} \\
&= \frac{\epsilon}{2} + 2\Big(\frac{1}{n}\Big(\tau\Big(\pi\Big(\sum_{i=1}^{n} v_i^*v_i\Big)\Big)\Big)\Big)^{\frac{1}{2}} \\
&\le \frac{\epsilon}{2} + \frac{2}{n^{\frac{1}{2}}} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

That is, $|\tau(\overline{u}) - \tau(u)| < \epsilon$.

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