Chin. Ann. Math. Ser. B
40(1), 2019, 111–116
DOI: 10.1007/s11401-018-0121-5

Chinese Annals of Mathematics, Series B

Deformations on the Twisted Heisenberg-Virasoro Algebra*

Dong LIU[1] Yufeng PEI[2]

Abstract With the cohomology results on the Virasoro algebra, the authors determine the second cohomology group on the twisted Heisenberg-Virasoro algebra, which gives all deformations on the twisted Heisenberg-Virasoro algebra.

Keywords Cohomology, Deformation, Virasoro algebra, Heisenberg algebra
2000 MR Subject Classification 17B56, 17B68

1 Introduction

Deformation of a Lie algebra L means that one has a family of Lie brackets on $L[[t]] := L\otimes_k k[[t]]$, denoted by $[\ ,\ ]_t$, inducing a Lie algebra structure on the extended Lie algebra $L[[t]]$. As well-known, the second cohomology group $H^2(L,L)$ of L with coefficients in the adjoint representation classifies the infinitesimal deformations.

The twisted Heisenberg-Virasoro algebra was first studied by Arbarello et al. in [1], where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. Moreover, the twisted Heisenberg-Virasoro algebra has some relations with the full-toroidal Lie algebras and the $N=2$ super conformal algebra, which is one of the most important algebraic objects in superstring theory.

By definition, as a vector space over $\mathbb{C}$, the twisted Heisenerg-Virasoro algebra $\mathcal{T}$ has a basis $\{L_m, I_m, C, C_1, C_2 \mid m\in\mathbb{Z}\}$, subject to the following relations:

$$\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(m^3-m)C,\\
&[I_m, I_n] = \delta_{m+n,0}nC_1,\\
&[L_m, I_n] = nI_{m+n} + \delta_{m+n,0}(m^2-m)C_2.
\end{aligned}\tag{1.1}$$

Clearly the Heisenberg algebra $H=\mathbb{C}\{I_m, C_1 \mid m\in\mathbb{Z}\}$ and the Virasoro algebra $\mathcal{V}=\mathbb{C}\{L_m, C \mid m\in\mathbb{Z}\}$ are subalgebras of $\mathcal{T}$.

The structure and representations for the twisted Heisenberg-Virasoro algebra was studied in [1–2, 13–15, 21], etc. However, the deformations on the twisted Heisenberg-Virasoro algebra have not yet been considered.

Manuscript received January 27, 2016. Revised February 27, 2017.
[1]Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China.
E-mail: liudong@zjhu.edu.cn
[2]Corresponding author. Department of Mathematics, Shanghai Normal University, Shanghai 200234, China. E-mail: pei@shnu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11871249, 11371134), the Natural Science Foundation of Zhejiang Province (No. LZ14A010001) and the Shanghai Natural Science Foundation (No. 16ZR1425000).

The twisted Heisenberg-Virasoro algebra is the universal central extension of the following Lie algebra $\mathfrak{g}$, which is the Lie algebra of differential operators of order at most one. By definition, the Lie algebra $\mathfrak{g} := \mathbb{C}\{L_m, I_m \mid m \in \mathbb{Z}\}$, subject to the following relations:

$$\begin{aligned}&[L_m, L_n] = (n-m)L_{m+n};\\&[I_m, I_n] = 0;\\&[L_m, I_n] = nI_{m+n}.\end{aligned} \tag{1.2}$$

Clearly, $\mathfrak{g} = \bigoplus\limits_{m\in\mathbb{Z}} \mathfrak{g}_m$, where $\mathfrak{g}_m = \mathbb{C}\{L_m, I_m\}$, is a $\mathbb{Z}$-graded Lie algebra. Moreover, let $\mathcal{W} = \mathbb{C}\{L_m \mid m \in \mathbb{Z}\}$ be the Witt algebra and $\mathfrak{h} = \mathbb{C}\{I_p \mid p \in \mathbb{Z}\}$ be a $\mathcal{W}$-module with actions $L_m \cdot I_p = pI_{m+p}$ for any $m, p \in \mathbb{Z}$, then $\mathfrak{g}$ is just the Lie algebra $\mathcal{W} \ltimes \mathfrak{h}$.

In this paper, we shall naturally begin with the computation of $H^2(\mathfrak{g}, \mathfrak{g})$ for the Lie algebra $\mathfrak{g}$ and then give some deformations of $\mathfrak{g}$. As a by-product of the above computations, we also determine the first cohomology group and universal central extension of one deformation of $\mathfrak{g}$.

2 The Witt Algebra and Its Cohomology

In this section we recall some cohomology results about the Witt algebra.

Kaplansky-Santharoubane [9] gave a classification of $\mathcal{W}$-modules of the intermediate series.

(i) $\mathcal{A}_{a,\,b} = \sum\limits_{i\in\mathbb{Z}} \mathbb{C}v_i$: $L_m v_i = (a+i+bm)v_{m+i}$.

(ii) $\mathcal{A}(a) = \sum\limits_{i\in\mathbb{Z}} \mathbb{C}v_i$: $L_m v_i = (i+m)v_{m+i}$ if $i \neq 0$, $L_m v_0 = m(m+a)v_m$.

(iii) $\mathcal{B}(a) = \sum\limits_{i\in\mathbb{Z}} \mathbb{C}v_i$: $L_m v_i = iv_{m+i}$ if $i \neq -m$, $L_m v_{-m} = -m(m+a)v_m$, for some $a, b \in \mathbb{C}$.

If $a \in \mathbb{Z}$, then $\mathcal{A}_{a,\,b} \cong \mathcal{A}_{0,\,b}$. So we always suppose that $a \notin \mathbb{Z}$ or $a = 0$ in $\mathcal{A}_{a,b}$. As in [17], we denote by $\mathcal{F}_\lambda = \mathcal{A}_{0,\,-\lambda}$, which is also called density module of the Witt algebra. The following results mainly come from [4].

Proposition 2.1 (cf. [4, 19])

$$\dim H^0(\mathcal{W}, \mathcal{F}_b) = \begin{cases} 1, & b = 0,\\ 0, & \textit{otherwise}.\end{cases}$$

$$\dim H^1(\mathcal{W}, \mathcal{F}_b) = \begin{cases} 0, & b \notin \{0,-1,-2\},\\ 1, & b = -1,-2,\\ 2, & b = 0.\end{cases}$$

Proposition 2.2 (cf. [4, 16]) $\dim H^1(\mathcal{W}, \mathcal{W}) = 0$ *and* $\dim H^1(\mathcal{W}, \mathcal{W} \otimes \mathcal{W}) = 0$.

Proposition 2.3 (cf. [3–4]) $H^2(\mathcal{W}, \mathbb{C}) = 1$ *and* $H^2(\mathcal{W}, \mathcal{W}) = \dim H^2(\mathcal{V}, \mathcal{V}) = 0$.

Proposition 2.4 (cf. [4]) (1) $\mathrm{Inv}_W(\mathcal{F}_\lambda \otimes \mathcal{F}_\mu) = 0$ *unless* $\mu = -1-\lambda$ *and* $\mathcal{F}_\mu = \mathcal{F}_\lambda^*$; *then* $\mathrm{Inv}_W(\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda^*)$ *is one-dimensional, generated by the identity mapping.*

(2) $H^i(\mathcal{W}, \mathcal{F}_\lambda \otimes \mathcal{F}_\mu) \equiv 0$ *if* $\lambda \neq 1-\mu$ *and* λ *or* μ *are not integers.*

(3) $H^1(\mathcal{W}, \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\lambda))$ *is two-dimensional, generated by the cocycles*

$$c_1(L_m, I_n) = mI_{m+n}, \quad c_2(L_m, I_n) = I_{m+n}.$$

(4) *Invariant antisymmetric bilinear operators* $\mathcal{F}_\lambda \times \mathcal{F}_\mu \to \mathcal{F}_\nu$ *between densities have been determined by Grozman* [6]. *They are of the following type.*

(a) *The Poisson bracket for* $\nu = \lambda + \mu - 1$, *defined by*

$$\{f\mathrm{d}x^{-\lambda}, g\mathrm{d}x^{-\mu}\} = (\lambda f g' - \mu f' g)\mathrm{d}x^{-(\lambda+\mu-1)}.$$

(b) *The following three exceptional brackets*:

$$\mathcal{F}_{\frac{1}{2}} \times \mathcal{F}_{\frac{1}{2}} \to \mathcal{F}_{-1} \quad \textit{given by } (f\partial^{\frac{1}{2}}, g\partial^{\frac{1}{2}}) \to \frac{1}{2}(fg'' - gf'')\mathrm{d}x;$$
$$\mathcal{F}_0 \times \mathcal{F}_0 \to \mathcal{F}_{-3} \quad \textit{given by } (f, g) \to (f''g' - g''f')\mathrm{d}x^3;$$

and an operator $\mathcal{F}_{\frac{2}{3}} \times \mathcal{F}_{\frac{2}{3}} \to \mathcal{F}_{-\frac{5}{3}}$ *called the Grozman bracket.*

3 Deformation Theory of the Lie Algebra $\mathfrak{g}$

In this section, we shall be interested in the classification of all formal deformations of $\mathfrak{g}$. So we shall naturally begin with the computation of $H^2(\mathfrak{g}, \mathfrak{g})$ (as usual, we shall consider only local cochains, equivalently given by differential operators, or polynomial in the modes).

We shall use standard techniques in Lie algebra cohomology; the proof will be rather technical, but without specific difficulties. Let us fix the notations: set $\mathfrak{g} = \mathcal{W} \ltimes \mathfrak{h}$ as in Section 1. One can consider the exact sequence

$$0 \to \mathfrak{h} \to \mathcal{W} \ltimes \mathfrak{h} \to \mathcal{W} \to 0 \tag{3.1}$$

as a short exact sequence of $\mathfrak{g}$-modules, thus inducing a long exact sequence in cohomology:

$$\cdots \to H^1(\mathfrak{g}, \mathcal{W}) \to H^2(\mathfrak{g}, \mathfrak{h}) \to H^2(\mathfrak{g}, \mathfrak{g}) \to H^2(\mathfrak{g}, \mathcal{W}) \to H^3(\mathfrak{g}, \mathfrak{h}) \to \cdots \tag{3.2}$$

So we shall consider $H^*(\mathfrak{g}, \mathcal{W})$ and $H^*(\mathfrak{g}, \mathfrak{h})$ separately.

Lemma 3.1 $H^*(\mathfrak{g}, \mathcal{W}) = 0$ *for* $* = 0, 1, 2$.

Proof One uses the Hochschild-Serre spectral sequence associated with the exact sequence (3.1).

Let us recall that, as a module on itself, $\mathcal{W} = \mathcal{F}_1$. One gets: $E_2^{p,0} = H^p(\mathcal{W}, \mathcal{W}) = 0$ as well-known (see [4]).

$$E_2^{1,1} = H^1(\mathcal{W}, H^1(\mathfrak{h}, \mathcal{W})) = H^1(\mathcal{W}, \mathfrak{h}^* \otimes \mathcal{W}) = H^1(\mathcal{W}, \mathcal{F}_{-1} \otimes \mathcal{F}_1) = 0.$$

Moreover $E_2^{0,2} = \mathrm{Inv}_{\mathcal{W}}(H^2(\mathfrak{h}) \otimes \mathcal{W}) = \mathrm{Inv}_{\mathcal{W}}(\wedge^2\mathfrak{h}^* \otimes \mathcal{W}) = 0$. The same argument shows that $E_2^{0,1} = 0$, which ends the proof of the lemma.

From the long exact sequence (3.2) and Lemma 3.1 one has now: $H^*(\mathfrak{g}, \mathfrak{g}) = H^*(\mathfrak{g}, \mathfrak{h})$ for $* = 0, 1, 2$.

We shall compute $H^*(\mathfrak{g}, \mathfrak{h})$ by using the Hochschild-Serre spectral sequence and there are three terms to compute.

(1) $E_2^{2,0} = H^2(\mathcal{V}, H^0(\mathfrak{h}, \mathfrak{h}))$, but $H^0(\mathfrak{h}, \mathfrak{h}) = Z(\mathfrak{h}) = \mathfrak{h} = \mathcal{F}_0$ as $\mathcal{V}$-module. So $E_2^{2,0} = H^2(\mathcal{V}, \mathcal{F}_0)$. It is one dimensional, generated by the cocycle c_0 defined by $c_0(L_m, L_n) = (n - m)I_{m+n}$.

(2) $E_2^{0,2} = H^0(\mathcal{W}, H^2(\mathfrak{h}, \mathfrak{h}))$, but $H^2(\mathfrak{h}, \mathfrak{h}) = \mathrm{Hom}(\mathfrak{h} \wedge \mathfrak{h}, \mathfrak{h})$ as $\mathcal{W}$-module. So $E_2^{0,2} = 0$.

(3) $E_2^{1,1} = H^1(\mathcal{W}, H^1(\mathfrak{h}, \mathfrak{h})) = H^1(\mathcal{W}, \mathrm{Hom}(\mathfrak{h}, \mathfrak{h})) = H^1(\mathcal{W}, \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_0))$.

From [4], one gets $H^1(\mathcal{W}, H^1(\mathfrak{h}, \mathfrak{h}))$ is two-dimensional, generated by the cocycles c_1 and c_2 defined by $c_1(L_m, I_n) = mI_{m+n}$ and $c_2(L_m, I_n) = I_{m+n}$.

We have to check now that these cohomology classes shall not disappear in the spectral sequence; the only potentially non-vanishing differentials are $E_2^{0,1} \to E_2^{2,0}$ and $E_2^{1,1} \to E_2^{3,0}$. Clearly $E_2^{3,0} = H^3(\mathcal{V}, \mathfrak{h}) = H^3(\mathcal{V}, \mathcal{F}_0) = 0$ (here we consider only local cohomology). Now

$E_2^{0,1}$ is one-dimensional determined by the constant multiplication (see Proposition 2.4(1)) and direct verification shows that $E_2^{0,1} \to E_2^{2,0}$ vanishes. So we have just proved that the cocycles $c_i, i = 0, 1, 2$ defined in the theorem represent genuinely non-trivial cohomology classes in $H^2(\mathfrak{g}, \mathfrak{g})$.

Theorem 3.1 *For the Lie algebra $\mathfrak{g}$, one has $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 3$. A set of generators is provided by the cohomology classes of the cocycles $c_i, i = 0, 1, 2$, defined as follows in terms of modes (the missing components of the cocycles are meant to vanish):*

$$\begin{aligned}&c_1(L_m, I_n) = mI_{m+n}, \quad c_2(L_m, I_n) = I_{m+n},\\ &c_0(L_m, L_n) = (n - m)I_{m+n}.\end{aligned}$$

Remark 3.1 (1) The cocycle c_0 appeared in [4] and had been used in a different context in [7], and was also considered and used in [18].

(2) The cocycle c_1 gives rise to the family of Lie algebras in [2].

Theorem 3.2 implies that there are some independent infinitesimal deformations of $W(0)$, defined by the cocycles c_0, c_1 or c_2, so the most general infinitesimal deformation of $W(0)$ is of the following form:

$$[\,,\,]_{\lambda,\mu,\alpha} = [\,,\,] + \mu c_0 + \lambda c_1 + \alpha c_2.$$

So one has the following result.

Theorem 3.2 *The bracket $[\,,\,]_{\lambda,\mu,\alpha} = [\,,\,] + \lambda c_1 + \mu c_0 + \alpha c_2$, where $[\,,\,]$ is the Lie bracket on $W(0)$ and c_i, $i = 0, 1, 2$ the cocycles given in Theorem 3.2, defines a three-parameter family of Lie algebra brackets on $W(0)$.*

$$\begin{aligned}&[L_m, L_n] = (n - m)L_{n+m} + \mu(m - n)I_{n+m},\\ &[I_m, I_n] = 0,\\ &[L_m, I_n] = (\alpha + n + \lambda m)I_{n+m},\end{aligned} \tag{3.3}$$

all other terms are vanishing.

Remark 3.2 (1) Set $W(\lambda) := W(0)_{-\lambda,0,0}$, then the Lie algebra $W(\lambda)$ can be realized as $\mathcal{W} \ltimes \mathfrak{h}$, where $\mathfrak{h} = \mathbb{C}\{I_p \mid p \in \mathbb{Z}\}$ is a $\mathcal{W}$-module with actions $L_m \cdot I_p = (p - m\lambda)I_{m+p}$ for any $m, p \in \mathbb{Z}$.

(2) The Lie algebra $\mathfrak{g}$ is just $W(0)$ and the universal central extension of $W(1)$ is called $W(2, 2)$ (cf. [22]), which can be also realized from the so-called loop-Virasoro algebra $\mathcal{G}$ (cf. [8]). Let $\mathbb{C}[t, t^{-1}]$ be the Laurents polynomial ring over $\mathbb{C}$ and $\mathcal{G} := \mathrm{Vir} \otimes \mathbb{C}[t, t^{-1}]$, then the $W(2, 2) = \mathcal{G}/(t^2)$. The $W(2, 2)$ and its highest weight modules enter the picture naturally during the discussion on $L\big(\frac{1}{2}, 0\big) \otimes L\big(\frac{1}{2}, 0\big)$. Its highest weight modules produce a new class of vertex operator algebras. Several papers studied its representation theory (cf. [8, 12, 22], etc.).

(3) Schrödinger-Virasoro algebras (cf. [20]) are more connected to $W(\lambda)$. In fact, $W(0)$ $\big($resp. $W\big(\frac{1}{2}\big)\big)$ is a subalgebra (resp. quotient algebra) of the original or twisted Schrödinger-Virasoro algebra. Their representation theories were studied in [10–11, 20], etc.

(4) The structure and representations for the Lie algebra $W(\lambda)$ were studied in [5, 11], etc.

4 Computation of Some Cohomology Groups for $W(\lambda)$

Now, as a by-product of the above computations, we shall determine explicitly $H^1(\mathcal{L}, \mathcal{L})$ and $H^2(\mathcal{L}, \mathbb{C})$ for $\mathcal{L} = W(\lambda)$, although the following results were obtained in [5] by direct calculations.

Theorem 4.1 *For $\mathcal{L} = W(\lambda)$ with $\lambda \notin \{0, -1, -2\}$, $H^1(\mathcal{L}, \mathcal{L})$ is one-dimensional, generated by the cocycle c_1; for $\mathcal{L} = W(\lambda)$ with $\lambda = -1$ (resp. -2), $H^1(\mathcal{L}, \mathcal{L})$ is three-dimensional, generated by the cocycle c_{-1} (resp. c_{-2}); for $\mathcal{L} = W(0)$, $H^1(\mathcal{L}, \mathcal{L})$ is three-dimensional, generated by the cocycles c_1, c_2 and l, where c_1, c_2, l, c_{-1} and c_{-2} are given by*

$$c_1(L_n) = nI_n, \quad c_2(L_n) = I_n, \quad l(I_n) = I_n, \quad c_{-1}(L_n) = n^2 I_n, \quad c_{-2}(L_n) = n^3 I_n.$$

Proof By Lemma 3.1, one has $H^1(\mathcal{L}, \mathcal{L}) = H^1(\mathcal{L}, \mathfrak{h})$. The space $H^1(\mathcal{L}, \mathfrak{h})$ is made from two parts $H^1(\mathcal{W}, \mathfrak{h})$ and $H^1(\mathfrak{h}, \mathfrak{h})$ satisfying the compatibility condition:

$$c([X, \alpha]) - [X, c(\alpha)] = -[\alpha, c(X)] \tag{3.7}$$

for $X \in \mathcal{W}$ and $\alpha \in \mathfrak{h}$.

Clearly, one has $[\alpha, c(X)] = 0$ for $X \in \mathcal{W}$ and $\alpha \in \mathfrak{h}$. Hence the compatibility condition reduces to $c([X, \alpha]) = [X, c(\alpha)]$ and thus $c \in \mathrm{Inv}_{\mathcal{W}} H^1(\mathfrak{h}, \mathfrak{h})$. Clearly $\mathrm{Inv}_{\mathcal{W}} H^1(\mathfrak{h}, \mathfrak{h})$ is one-dimensional, generated by l (see Proposition 2.4 (1)).

The result is then easily deduced from the previous computations : $H^1(\mathcal{W}, \mathfrak{h}) = H^1(\mathcal{W}, \mathcal{F}_\lambda) = 0$ if $\lambda \notin \{0, -1, -2\}$, in the case of $\lambda = -1$(resp. -2), $H^1(\mathcal{W}, \mathfrak{h}) = H^1(\mathcal{W}, \mathcal{F}_\lambda)$ is generated by the cocycle c_{-1} (resp. c_{-2}), and in the case of $\lambda = 0$, $H^1(\mathcal{W}, \mathfrak{h}) = H^1(\mathcal{W}, \mathcal{F}_0)$ is generated by the two cocycles c_1 and c_2.

We shall again make use of the exact sequence decomposition $0 \to \mathfrak{h} \to \mathcal{L} \xrightarrow{\pi} \mathcal{W} \to 0$, and classify the cocycles with respect to their "type" along this decomposition; trivial coefficients will make computations much easier than in the above case.

First of all, $0 \to H^2(\mathcal{W}, \mathbb{C}) \xrightarrow{\pi^*} H^2(\mathcal{L}, \mathbb{C})$ is an injection. So the Virasoro class $c \in H^2(\mathcal{W}, \mathbb{C})$ always survives in $H^2(\mathcal{L}, \mathbb{C})$.

For $\mathfrak{h} = \mathcal{F}_\lambda$, one has: $H^1(\mathcal{W}, H^1(\mathfrak{h})) = H^1(\mathcal{W}, \mathfrak{h}^*) = H^1(\mathcal{W}, \mathcal{F}_{-1-\lambda})$. By Proposition 2.1, we know that

$$\dim H^1(\mathcal{W}, \mathcal{F}_{-1-\lambda}) = \begin{cases} 0, & \lambda \notin \{0, 1, -1\}, \\ 1, & \lambda = 0, 1, \\ 2, & \lambda = -1. \end{cases}$$

The last term $E_2^{0,2} = \mathrm{Inv}_{\mathcal{W}} H^2(\mathcal{F}_\lambda)$, but $H^2(\mathcal{F}_\lambda) = \mathrm{Hom}(\mathcal{F}_\lambda \wedge \mathcal{F}_\lambda, \mathbb{C})$ as $\mathcal{W}$-module. So $E_2^{0,2} = 0$ if $\lambda \neq 0$ and $E_2^{0,2} = \mathrm{Inv}_{\mathcal{W}} H^2(\mathcal{F}_0)$ is one dimensional generated by $c_{0,2}$ as $c_{0,2}(I_n, I_m) = n\delta_{n+m,0}$.

Let us summarize our results.

Theorem 4.2 *For $\mathcal{L} = W(\lambda)$, we have*

(1) *$H^2(W(0), \mathbb{C}) \simeq \mathbb{C}^3$ is generated by the Virasoro cocycle and the two independent cocycles $c_{0,1}$ and $c_{0,2}$ defined by (all other components vanishing)*

$$c_{0,1}(L_m, I_n) = n^2 \delta_{n+m,0},$$
$$c_{0,2}(I_m, I_n) = n\delta_{n+m,0}.$$

(2) *$H^2(W(-1), \mathbb{C}) \simeq \mathbb{C}^3$ is generated by the Virasoro cocycle and the two independent cocycles $c_{-1,1}$ and $c_{-1,2}$ defined by*

$$c_{-1,1}(L_m, I_n) = n\delta_{n+m,0},$$
$$c_{-1,2}(L_m, I_n) = \delta_{n+m,0}.$$

(3) $H^2(W(1), \mathbb{C}) \simeq \mathbb{C}^2$ *is generated by the Virasoro cocycle and the cocycle* $c_{1,1}$ *defined by*

$$c_{1,1}(L_m, I_n) = n^3 \delta_{n+m,0}.$$

(4) $H^2(W(\lambda), \mathbb{C}) \simeq \mathbb{C}$ *is generated by the Virasoro cocycle.*

References

[1] Arbarello, E., De Concini, C., Kac, V. G. and Procesi, C., Moduli spaces of curves and representation theory, *Comm. Math. Phys.*, **117**, 1988, 1–36.

[2] Billig, Y., Representations of the twisted Heisenberg-Virasoro algebra at level zero, *Canad. Math. Bull.*, **46**(4), 2003, 529–533.

[3] Fialowski, A., Formal rigidity of the Witt and Virasoro algebra, *J. Math. Phys.,* **53**, 2012, 073501.

[4] Fuks, D. B., Cohomology of Infinite-Dimensional Lie Algebras, Nauka, Moscow, 1984. (in Russian)

[5] Gao, S., Jiang, C. and Pei, Y., Low dimensional cohomology groups of the Lie algebras $W(a, b)$, *Commun. Alg.,* **39**(2), 2011, 397–423.

[6] Grozman, P. Ja., Classification of bilinear invariant operators on tensor fields, *Functional Anal. Appl,* **14**(2), 1980, 127–128.

[7] Guieu, L. and Roger, C., Algebra and the Virasoro Group, Geometric and Algebraic Aspects, Generalizations, Les Publications CRM, Montreal, QC, 2007.

[8] Guo, X., Lv, R. and Zhao, K., Simple Harish-Chandra modules, intermediate series modules, and Verma modules over the loop-Virasoro algebra, *Forum Math.,* **23**, 2011, 1029–1052.

[9] Kaplansky, I. and Santharoubane, L. J., Harish-Chandra modules over the Virasoro algebra, Math. Sci. Res. Inst. Publ., Vol. 4, Springer-Verlag, New York, 1985.

[10] Li, J. and Su, Y., Representations of the Schrödinger-Virasoro algebras, *J. Math. Phys.,* **49**, 2008, 053512.

[11] Liu, D., Classification of Harish-Chandra modules over some Lie algebras related to the Virasoro algebra, *J. Algebra*, **447**, 2016, 548–559.

[12] Liu, D., Gao, S. and Zhu, L., Classification of irreducible weight modules over W-algebra $W(2, 2)$, *J. Math. Phys.,* **49**(11), 2008, 113503.

[13] Liu, D. and Jiang, C., Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra, *J. Math. Phys.,* **49**(1), 2008, 012901.

[14] Liu, D., Pei, Y. and Zhu, L., Lie bialgebra structures on the twisted Heisenberg-Virasoro algebra, *J. Algebra,* **359**, 2012, 35–48.

[15] Lv, R. and Zhao, K., Classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra, *Commun. Contemp. Math.* **12**(2), 2010, 183–205

[16] Ng, S. H. and Taft, E. J., Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Alg.,* **151**, 2000, 67–88.

[17] Ovsienko, V. Yu. and Rozhe, K., Extension of Virasoro group and Virasoro algebra by modules of tensor densities on S^1, *Functional Anal. Appl.*, **30**, 1996, 290–291.

[18] Ovsienko, V. and Roger, C., Generalizations of the Virasoro group and Virasoro algebra through extensions by modules of tensor densities on S^1, *Indag. Math.(N.S.),* **9**(2), 1998, 277–288.

[19] Qiu, S., The cohomology of the Virasoro algebra with coefficients in a basic Harish-Chandra modules, *Northeastern Math. J.,* **5**(1), 1989, 75–83.

[20] Roger, C. and Unterberger, J., The Schrödinger-Virasoro Lie group and algebra: From geometry to representation theory, *Annales Henri Poincaré*, **7**, 2006, 1477–1529.

[21] Shen, R. and Jiang, C., The derivation algebra and automorphism group of the twisted Heisenberg-Virasoro algebra, *Commun. Alg.*, **34**, 2006, 2547–2558.

[22] Zhang, W. and Dong, C., W-algebra $W(2, 2)$ and the vertex operator algebra $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right)$, *Commun. Math. Phys.,* **285**(3), 2009, 991–1004.