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Rosenthal's Inequalities for Asymptotically Almost Negatively Associated Random Variables Under Upper Expectations*

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Abstract In this paper, the authors generalize the concept of asymptotically almost negatively associated random variables from the classic probability space to the upper expectation space. Within the framework, the authors prove some different types of Rosenthal's inequalities for sub-additive expectations. Finally, the authors prove a strong law of large numbers as the application of Rosenthal's inequalities.

Keywords Upper expectations, Asymptotically almost negatively associated, Rosenthal's inequalities, Strong law of large numbers
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1 Introduction

In recent decades, nonlinear probabilities and nonlinear expectations play crucial roles in the study of statistical uncertainties, risk measuring, and nonlinear stochastic calculus. Peng [9–13] extended the classical linear expectation and introduced the general sublinear expectation by replacing the linear property with the sub-additivity and positive homogeneity. Peng also defined the independence of random variables in the setting of sublinear expectations. Several definitions of dependence in the classical framework can also be extended to nonlinear cases.

Joag-Dev and Proschan [7] and Block et al. [1] brought up the concept of negative association (NA for short) in 1982 and it led to numerous applications in reliability theory, percolation theory and multivariate statistical analysis. Chandra and Ghosal [2] extended this concept and introduced asymptotically almost negative association (AANA, for short) by noticing the fact that maximal inequality for the NA random variables in Matula [8] can also hold when small negative correlations are considered. Then some convergence theorems for NA and AANA random variables were achieved by scholars, such as Zhang [18]. The concepts of NA and AANA can be well defined in the setting of nonlinear expectations by replacing linear expectations with nonlinear ones.

Within the framework of nonlinear expectations, numerous results of the limit theory were gained. Recently, Chen and Hu [3] investigated a law of the iterated logarithm for bounded IID

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random variables according to Peng's definition under upper expectations. Zhang [20] obtained a law of the iterated logarithm for NA random variables. Inspecting the proofs of the two results, we can find that the estimate of moments and maximum for partial sums is the key step in the proofs. There are various elegant results of estimations for NA and AANA random variables in both linear and nonlinear settings. Matula [8] first derived the Kolmogorov type inequality and Shao [14] obtained Rosenthal's inequalities for NA random variables and a comparison theorem on moment inequalities between negatively associated and independent random variables (see [15]) in classical setting. Zhang [19–20] achieved Kolmogorovs exponential inequalities and Rosenthal's inequalities for NA random variables under sublinear expectations. For AANA random variables, Yuan and An [17] deduced Rosenthal's inequalities and investigated their applications within linear framework.

However, we find that Rosenthal's inequalities for AANA random variables under the nonlinear expectation is still not available, which is the main motivation of this paper. Our purpose of this paper is to fill in this blank. The paper is organized as below: In Section 2, we introduce some preliminaries about upper expectations and ANNA random variables. Then the main results of Rosenthal's inequalities for ANNA random variables are derived in Section 3. In Section 4, we will prove a strong law of large numbers as the application of our Rosenthal's inequality.

2 Preliminary

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{P}$ be a set of probability measures on Ω. We define a pair $(\mathbb{V}, v)$ of upper-lower probabilities by

$$\mathbb{V}(A) := \sup_{P\in\mathcal{P}} P(A), \quad v(A) := \inf_{P\in\mathcal{P}} P(A), \quad \forall A \in \mathcal{F}.$$

It is easy to check that $\mathbb{V}(A) + v(A^c) = 1$ and $\mathbb{V}$ satisfies the following properties:

(i) $\mathbb{V}(\emptyset) = 0$, $\mathbb{V}(\Omega) = 1$;

(ii) $\mathbb{V}(A) \le \mathbb{V}(B)$, whenever $A \subset B$ and $A, B \in \mathcal{F}$;

(iii) $\mathbb{V}(A_n) \uparrow \mathbb{V}(A)$, if $A_n \uparrow A$, where $A_n, A \in \mathcal{F}$.

Now we define the upper expectation $\mathbb{E}[\cdot]$ and the lower expectation $\mathcal{E}[\cdot]$ on $(\Omega, \mathcal{F})$ generated by $\mathcal{P}$. For each random variable X such that $E_P[X]$ exists for each $P \in \mathcal{P}$, we define

$$\mathbb{E}[X] := \sup_{P\in\mathcal{P}} E_P[X], \quad \mathcal{E}[X] := \inf_{P\in\mathcal{P}} E_P[X].$$

$(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$ is called an upper expectation space. It is obvious that $\mathcal{E}[X] := -\mathbb{E}[-X]$ and $\mathbb{E}$ is a sublinear expectation. In other words, $\mathbb{E}$ have the following properties.

Proposition 2.1 *For random variables X and Y on the upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$, we have*

(1) *monotonicity*: $X \ge Y$ *implies* $\mathbb{E}[X] \ge \mathbb{E}[Y]$;

(2) *constant preserving*: $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$;

(3) *sub-additivity*: $\mathbb{E}[X + Y] \le \mathbb{E}[X] + \mathbb{E}[Y]$;

(4) *positive homogeneity*: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \le 0$.

In addition, by the definition of $\mathbb{E}$ and $\mathcal{E}$, we get the following proposition.

Proposition 2.2 *For random variables X and Y on the upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$, we have*

(1) $\mathbb{E}[X] \geq \mathcal{E}[X]$;

(2) $\mathbb{E}[aX] = a^+\mathbb{E}[X] + a^-\mathbb{E}[-X]$, $\forall a \in \mathbb{R}$;

(3) $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$;

(4) $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, $\forall c \in \mathbb{R}$.

Remark 2.1 Here and the sequel, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$, x can be real numbers or random variables.

Next, we will list some inequalities under upper expectations, which can be regarded as an extension of inequalities of classic probability theory.

Proposition 2.3 *Let X, Y be two random variables on the upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$. Then*

(1) *Hölder's inequality. For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} \cdot (\mathbb{E}[|Y|^q])^{\frac{1}{q}}.$$

(2) *Jensen's inequality. Let $f(\cdot)$ be a convex function on $\mathbb{R}$. Suppose that $\mathbb{E}[X]$ and $\mathbb{E}[f(X)]$ exist. Then*

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).$$

(3) *Minkowski's inequality. For $p > 1$, we have*

$$(\mathbb{E}[|X + Y|^p])^{\frac{1}{p}} \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} + (\mathbb{E}[|Y|^p])^{\frac{1}{p}}.$$

(4) *C_r inequality. For any $r \geq 1$, we have*

$$\mathbb{E}[|X + Y|^r] \leq 2^{r-1}(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r]).$$

The proofs of these inequalities can be found in [5] and [13].

Proposition 2.4 *Let X, Y be two random variables on the upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$. Then*

(1) $(\mathbb{E}[X])^2 \leq \mathbb{E}[X^2]$;

(2) *for any $1 \leq p \leq p'$, there is* $(\mathbb{E}[|X|^p])^{\frac{1}{p}} \leq (\mathbb{E}[|X|^{p'}])^{\frac{1}{p'}}$.

Proof The proof of this proposition also can be found in [13]. We outline the proof for the convenience of readers.

Since $f(x) = x^2$ is a convex function, (1) is directly deduced by Proposition 2.3 (Jensen's inequality).

Set $Y = Z^p$, $X = 1$, $q = \frac{p'}{p}$. By Proposition 2.3 (Hölder's inequality), we have

$$\mathbb{E}[|Z|^p] \leq (\mathbb{E}[|Z|^{p \cdot \frac{p'}{p}}])^{\frac{p}{p'}},$$

that is

$$\mathbb{E}[|Z|^p] \leq (\mathbb{E}[|Z|^{p'}])^{\frac{p}{p'}}.$$

Since $p \geq 1$, then we have $(\mathbb{E}[|X|^p])^{\frac{1}{p}} \leq (\mathbb{E}[|X|^{p'}])^{\frac{1}{p'}}$.

We denote the norm of random variable X on the upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$ by

$$\|X\|_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}.$$

It is easy to prove the following lemma.

Lemma 2.1 *For random variables X and Y on the upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{E})$, we have*

(1) $\mathbb{E}[XY] \leq \mathbb{E}[X^+Y^+] + \mathbb{E}[X^-Y^-]$;

(2) $\mathbb{E}[(X - \mathbb{E}[X])^2] \leq 4\mathbb{E}[X^2]$.

Proof (1) By monotonicity and sub-additivity of $\mathbb{E}$, we have

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[(X^+ - X^-)(Y^+ - Y^-)] \\
&= \mathbb{E}[X^+Y^+ + X^-Y^- - X^+Y^- - X^-Y^+] \\
&\leq \mathbb{E}[X^+Y^+ + X^-Y^-] \\
&\leq \mathbb{E}[X^+Y^+] + \mathbb{E}[X^-Y^-].
\end{aligned}$$

(2) By sub-additivity of $\mathbb{E}$ and Proposition 2.4, we have

$$\begin{aligned}
\mathbb{E}[(X - \mathbb{E}[X])^2] &= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\
&\leq \mathbb{E}[X^2] + 2|\mathbb{E}[X]|\mathbb{E}[|-X|] + (\mathbb{E}[X])^2 \\
&\leq 3\mathbb{E}[X^2] + (\mathbb{E}[X])^2 \\
&\leq 4\mathbb{E}[X^2].
\end{aligned}$$

Now, we will extend the concept of asymptotically almost negatively associated sequence of random variables to the upper expectation space.

Definition 2.1 *A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables is called asymptotically almost negatively associated (AANA) under $\mathbb{E}$ if there exists a nonnegative sequence $\{\eta(n)\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} \eta(n) = 0$ and*

$$\begin{aligned}
&\mathbb{E}[f(X_n)g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})] - \mathbb{E}[f(X_n)]\mathbb{E}[g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})] \\
\leq\ &\eta(n)\{\mathbb{E}[(f(X_n) - \mathbb{E}[f(X_n)])^2]\}^{\frac{1}{2}} \\
&\cdot \{\mathbb{E}[(g(X_{n+1}, X_{n+2}, \cdots, X_{n+k}) - \mathbb{E}[g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})])^2]\}^{\frac{1}{2}}
\end{aligned}$$

for all $n, k \geq 1$ and for all coordinatewise nondecreasing or nonincreasing continuous functions f and g whenever the expectations exist. And $\{\eta(n)\}_{n=1}^{\infty}$ are called mixing coefficients.

3 Rosenthal's Inequalities

Lemma 3.1 *Let $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables with mixing coefficients $\{\eta(n)\}_{n=1}^{\infty}$. Then $\{f_n(X_n)\}_{n=1}^{\infty}$ is also an AANA sequence with the same coefficients $\{\eta(n)\}_{n=1}^{\infty}$, where $\{f_n(\cdot)\}_{n=1}^{\infty}$ are all nondecreasing or nonincreasing functions.*

Lemma 3.2 *Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables with mixing coefficients $\{\eta(n)\}_{n=1}^{\infty}$, then*

$$\begin{aligned}
&\mathbb{E}[X_n g(X_{n+1}, \cdots, X_{n+k})] - \mathbb{E}[X_n]\mathbb{E}[g(X_{n+1}, \cdots, X_{n+k})] \\
&\le 15\eta^{\frac{2}{q} \wedge \frac{2}{p}}(n)\|X_n\|_p \|g(X_{n+1}, \cdots, X_{n+k})\|_q
\end{aligned}$$

for all $n, k \ge 1$ and coordinatewise nondecreasing or coordinatewise nonincreasing functions.

Proof Case 1 Assume $1 < p \le 2$ and denote X_n by Z and $g(X_{n+1}, \cdots, X_{n+k})$ by Y for convenience. Set $Z_1 = (-C) \vee (Z \wedge C)$ and $Z_2 = Z - Z_1$, where C is a positive constant whose value will be defined later. At first, we will prove that

$$\mathbb{E}[ZY] - \mathbb{E}[Z]\mathbb{E}[Y] \le \mathbb{E}[Z_1 Y] - \mathbb{E}[Z_1]\mathbb{E}[Y] + \mathbb{E}[Z_2 Y] + \mathbb{E}[|Z_2|]\mathbb{E}[|Y|]. \tag{3.1}$$

If $\mathbb{E}[Y] \ge 0$, we get

$$\begin{aligned}
\mathbb{E}[ZY] - \mathbb{E}[Z]\mathbb{E}[Y] &= \mathbb{E}[(Z_1 + Z_2)Y] - \mathbb{E}[Z_1 - (-Z_2)]\mathbb{E}[Y] \\
&\le \mathbb{E}[Z_1 Y] + \mathbb{E}[Z_2 Y] - (\mathbb{E}[Z_1] - \mathbb{E}[-Z_2])\mathbb{E}[Y] \\
&= \mathbb{E}[Z_1 Y] + \mathbb{E}[Z_2 Y] - \mathbb{E}[Z_1]\mathbb{E}[Y] + \mathbb{E}[-Z_2]\mathbb{E}[Y] \\
&\le \mathbb{E}[Z_1 Y] - \mathbb{E}[Z_1]\mathbb{E}[Y] + \mathbb{E}[Z_2 Y] + \mathbb{E}[|Z_2|]\mathbb{E}[|Y|].
\end{aligned}$$

Otherwise, if $\mathbb{E}[Y] < 0$, we achieve

$$\begin{aligned}
\mathbb{E}[ZY] - \mathbb{E}[Z]\mathbb{E}[Y] &= \mathbb{E}[(Z_1 + Z_2)Y] - \mathbb{E}[Z_1 + Z_2]\mathbb{E}[Y] \\
&\le \mathbb{E}[Z_1 Y] + \mathbb{E}[Z_2 Y] - (\mathbb{E}[Z_1] + \mathbb{E}[Z_2])\mathbb{E}[Y] \\
&= \mathbb{E}[Z_1 Y] + \mathbb{E}[Z_2 Y] - \mathbb{E}[Z_1]\mathbb{E}[Y] - \mathbb{E}[Z_2]\mathbb{E}[Y] \\
&\le \mathbb{E}[Z_1 Y] - \mathbb{E}[Z_1]\mathbb{E}[Y] + \mathbb{E}[Z_2 Y] + \mathbb{E}[|Z_2|]\mathbb{E}[|Y|].
\end{aligned}$$

Therefore, we prove that (3.1) holds. Next, we will split the right side of (3.1) into three parts and estimate them respectively.

(i) By the monotonicity of $\mathbb{E}$ and Proposition 2.4, we have

$$\begin{aligned}
\mathbb{E}[|Z_2|]\mathbb{E}[|Y|] &\le \mathbb{E}[|Z| I_{|Z| \ge C}](\mathbb{E}[|Y|^q])^{\frac{1}{q}} \\
&\le \mathbb{E}\Big[|Z|\Big(\frac{|Z|}{C}\Big)^{p-1}\Big](\mathbb{E}[|Y|^q])^{\frac{1}{q}} \\
&\le \mathbb{E}[|Z|^p] C^{1-p}\|Y\|_q \\
&= C^{-\frac{p}{q}}\|Z\|_p^p \|Y\|_q.
\end{aligned}$$

(ii) Applying Lemma 2.1 and Lemma 3.1, since $f_n(Z) = (-C) \vee (Z \wedge C)$ is a nondecreasing continuous function, we have

$$\begin{aligned}
\mathbb{E}[Z_1 Y] - \mathbb{E}[Z_1]\mathbb{E}[Y] &\le \eta(n)(\mathbb{E}[(Z_1 - \mathbb{E}[Z_1])^2])^{\frac{1}{2}}(\mathbb{E}[(Y - \mathbb{E}[Y])^2])^{\frac{1}{2}} \\
&\le 4\eta(n)((\mathbb{E}[|Z_1|^p |Z_1|^{2-p}])^{\frac{1}{p}})^{\frac{p}{2}}(\mathbb{E}[|Y|^q])^{\frac{1}{q}} \\
&\le 4\eta(n) C^{1-\frac{p}{2}}((\mathbb{E}[|Z_1|^p])^{\frac{1}{p}})^{\frac{p}{2}}\|Y\|_q \\
&\le 4\eta(n) C^{1-\frac{p}{2}}||Z_1||_p^{\frac{p}{2}}\|Y\|_q.
\end{aligned}$$

(iii) From Lemma 2.1, we achieve $\mathbb{E}[Z_2Y] \le \mathbb{E}[Z_2^+Y^+] + \mathbb{E}[Z_2^-Y^-]$. Then applying Proposition 2.3 (Hölder′s inequality), we have

$$\begin{aligned}\mathbb{E}[Z_2^+Y^+] &= \mathbb{E}[(Z_2^+)^{\frac{p(q-2)}{q}}((Z_2^+)^{1-\frac{p(q-2)}{q}}Y^+)]\\ &\le (\mathbb{E}[|(Z_2^+)^{\frac{p(q-2)}{q}}|^{\frac{q}{q-2}}])^{\frac{q-2}{q}}(\mathbb{E}[|(Z_2^+)^{1-\frac{p(q-2)}{q}}Y^+|^{\frac{q}{2}}])^{\frac{2}{q}}\\ &= (\mathbb{E}[|Z_2^+|^p])^{1-\frac{2}{q}}(\mathbb{E}[|Z_2^+|^{\frac{p}{2}}|Y^+|^{\frac{q}{2}}])^{\frac{2}{q}}.\end{aligned}$$

Again by Lemma 3.1, we get

$$\begin{aligned}&\mathbb{E}[|Z_2^+|^{\frac{p}{2}}|Y^+|^{\frac{q}{2}}] - \mathbb{E}[(Z_2^+)^{\frac{p}{2}}]\mathbb{E}[(Y^+)^{\frac{q}{2}}]\\ &\le \eta(n)(\mathbb{E}[((Z_2^+)^{\frac{p}{2}} - \mathbb{E}[(Z_2^+)^{\frac{p}{2}}])^2])^{\frac{1}{2}}(\mathbb{E}[((Y^+)^{\frac{q}{2}} - \mathbb{E}[(Y^+)^{\frac{q}{2}}])^2])^{\frac{1}{2}}\\ &\le \eta(n)(\mathbb{E}[(2(Z_2^+)^{\frac{p}{2}})^2])^{\frac{1}{2}}(\mathbb{E}[(2(Y^+)^{\frac{q}{2}})^2])^{\frac{1}{2}}\\ &= 4\eta(n)(\mathbb{E}[(Z_2^+)^p])^{\frac{1}{2}}(\mathbb{E}[(Y^+)^q])^{\frac{1}{2}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\mathbb{E}[Z_2^+Y^+]\\ &\le (\mathbb{E}[|Z_2^+|^p])^{1-\frac{2}{q}}(\mathbb{E}[(Z_2^+)^{\frac{p}{2}}]\mathbb{E}[(Y^+)^{\frac{q}{2}}] + 4\eta(n)(\mathbb{E}[(Z_2^+)^p])^{\frac{1}{2}}(\mathbb{E}[(Y^+)^q])^{\frac{1}{2}})^{\frac{2}{q}}\\ &\le (\mathbb{E}[|Z|^p])^{1-\frac{2}{q}}(\mathbb{E}[|Z|^{\frac{p}{2}}I_{|Z|>C}]\mathbb{E}[|Y|^{\frac{q}{2}}] + 4\eta(n)(\mathbb{E}[|Z|^p])^{\frac{1}{2}}(\mathbb{E}[|Y|^q])^{\frac{1}{2}})^{\frac{2}{q}}\\ &\le (\mathbb{E}[|Z|^p])^{1-\frac{2}{q}}\Big(\mathbb{E}\Big[|Z|^{\frac{p}{2}}\Big(\frac{|Z|}{C}\Big)^{\frac{p}{2}}\Big]\mathbb{E}[|Y|^{\frac{q}{2}}] + 4\eta(n)(\mathbb{E}[|Z|^p])^{\frac{1}{2}}(\mathbb{E}[|Y|^q])^{\frac{1}{2}}\Big)^{\frac{2}{q}}\\ &\le (\mathbb{E}[|Z|^p])^{1-\frac{2}{q}}(C^{-\frac{p}{2}}\mathbb{E}[|Z|^p]\mathbb{E}[|Y|^{\frac{q}{2}}] + 4\eta(n)(\mathbb{E}[|Z|^p])^{\frac{1}{2}}(\mathbb{E}[|Y|^q])^{\frac{1}{2}})^{\frac{2}{q}}\\ &\le (\mathbb{E}[|Z|^p])^{1-\frac{2}{q}}(C^{-\frac{p}{q}}(\mathbb{E}[|Z|^p])^{\frac{2}{q}}(\mathbb{E}[|Y|^{\frac{q}{2}}])^{\frac{2}{q}} + 4^{\frac{2}{q}}\eta^{\frac{2}{q}}(n)(\mathbb{E}[|Z|^p])^{\frac{1}{q}}(\mathbb{E}[|Y|^q])^{\frac{1}{q}})\\ &= C^{-\frac{p}{q}}\|Z\|_p^p\|Y\|_{\frac{q}{2}} + 4^{\frac{2}{q}}\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q\\ &\le C^{-\frac{p}{q}}\|Z\|_p^p\|Y\|_q + 4^{\frac{2}{q}}\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q.\end{aligned}$$

The penultimate inequality is from the fact that for any $a > 0$, $b > 0$ and $0 < t < 1$, $(a+b)^t \le a^t + b^t$ always holds. Similarly, we also have

$$\mathbb{E}[Z_2^-Y^-] \le C^{-\frac{p}{q}}\|Z\|_p^p\|Y\|_q + 4^{\frac{2}{q}}\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q.$$

Consequently, we achieve that

$$\mathbb{E}[Z_2Y] \le 2C^{-\frac{p}{q}}\|Z\|_p^p\|Y\|_q + 2^{\frac{4}{q}+1}\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q.$$

Then by (i),(ii) and (iii) we have

$$\mathbb{E}[ZY] - \mathbb{E}[Z]\mathbb{E}[Y] \le 4C^{1-\frac{p}{2}}\eta(n)\|Z_1\|_p^{\frac{p}{2}}\|Y\|_q + 3C^{-\frac{p}{q}}\|Z\|_p^p\|Y\|_q + 2^{\frac{4}{q}+1}\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q.$$

Setting $C = \eta^{-\frac{2}{p}}(n)\|Z\|_p$, then we get

$$\begin{aligned}&\mathbb{E}[ZY] - \mathbb{E}[Z]\mathbb{E}[Y]\\ &\le 4\eta^{\frac{2}{q}}(n)\|Z\|_p^{1-\frac{p}{2}}\|Z_1\|_p^{\frac{p}{2}}\|Y\|_q + 3\eta^{\frac{2}{q}}(n)\|Z\|_p^{-\frac{p}{q}}\|Z\|_p^p\|Y\|_q + 2^{\frac{4}{q}+1}\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q\\ &\le 15\eta^{\frac{2}{q}}(n)\|Z\|_p\|Y\|_q.\end{aligned}$$

Case 2 When $p > 2$, denoting X_n by Y and $g(X_{n+1}, \cdots, X_{n+k})$ by Z, setting $Z_1 = (-C) \vee (Z \wedge C)$, $Z_2 = Z - Z_1$, then by the same method of Case 1, we obtain that

$$\mathbb{E}[ZY] - \mathbb{E}[Z]\mathbb{E}[Y] \le 15\eta^{\frac{2}{p}}(n)\|X_n\|_p\|g(X_{n+1}, \cdots, X_{n+k})\|_q.$$

Theorem 3.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $1<p\le 2$ and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables under $\mathbb{E}$ with $\mathbb{E}[X_n]=\mathcal{E}[X_n]=0$. And $\{\eta(n)\}_{n=1}^{\infty}$ are the corresponding mixing coefficients. Then there exists a positive constant C_p depending only on p such that for any $n\ge 1$, we have*

$$\mathbb{E}\Big[\max_{1\le i\le n}|S_i|^p\Big]\le C_p\Big\{\sum_{i=1}^{n}\|X_i\|_p^p+\Big(\sum_{i=1}^{n-1}\eta^{\frac{2}{q}}(i)\|X_i\|_p\Big)^p\Big\}. \tag{3.2}$$

In particular, if $\sum\limits_{n=1}^{\infty}\eta^2(n)<\infty$, then for any $n\ge 1$, we have

$$\mathbb{E}\Big[\max_{1\le i\le n}|S_i|^p\Big]\le C_p\sum_{i=1}^{n}\|X_i\|_p^p.$$

Proof For a fixed n, set $U_i=\max(X_i, X_i+X_{i+1},\cdots,X_i+\cdots+X_n)$, for all $1\le i\le n$. It is obvious that $U_i=X_i+U_{i+1}^+$. By elementary inequality $|x+y|^p\le 2^{2-p}|x|^p+|y|^p+px|y|^{p-1}\text{sgn}y$ for any $x,y\in\mathbb{R}$ and $1<p\le 2$, we obtain for $1\le i\le n-1$,

$$\begin{aligned}\mathbb{E}[|U_i|^p]&=\mathbb{E}[|U_i|^pI_{\{U_{i+1}\le 0\}}+|U_i|^pI_{\{U_{i+1}>0\}}]\\&=\mathbb{E}[|X_i|^pI_{\{U_{i+1}\le 0\}}+|X_i+U_{i+1}|^pI_{\{U_{i+1}>0\}}]\\&\le\mathbb{E}[|X_i|^pI_{\{U_{i+1}\le 0\}}+(2^{2-p}|X_i|^p+|U_{i+1}|^p+pX_i|U_{i+1}|^{p-1}\text{sgn}U_{i+1})I_{\{U_{i+1}>0\}}]\\&=\mathbb{E}[|X_i|^pI_{\{U_{i+1}\le 0\}}+2^{2-p}|X_i|^pI_{\{U_{i+1}>0\}}+|U_{i+1}|^p+pX_i|U_{i+1}|^{p-1}I_{\{U_{i+1}>0\}}]\\&\le 2^{2-p}\mathbb{E}[|X_i|^p]+\mathbb{E}[|U_{i+1}|^p]+p\mathbb{E}[X_i|U_{i+1}|^{p-1}I_{\{U_{i+1}>0\}}].\end{aligned}$$

It is easy to show that $g(X_{i+1},\cdots,X_n):=(U_{i+1}^+)^{p-1}$ is a coordinatewise nondecreasing function. Since $\mathbb{E}[X_i]=0$, by Lemma 3.2, we have

$$\begin{aligned}\mathbb{E}[X_i|U_{i+1}|^{p-1}I_{\{U_{i+1}>0\}}]&\le 15\eta^{\frac{2}{q}}(i)\|X_i\|_p(\mathbb{E}[(|U_{i+1}|^{p-1}I_{\{U_{i+1}>0\}})^q])^{\frac{1}{q}}\\&\le 15\eta^{\frac{2}{q}}(i)\|X_i\|_p(\mathbb{E}[|U_{i+1}|^p])^{\frac{1}{q}}\\&=15\eta^{\frac{2}{q}}(i)\|X_i\|_p\|U_{i+1}\|_p^{\frac{p}{q}}.\end{aligned}$$

Therefore, we have

$$\|U_i\|_p^p\le 2^{2-p}\|X_i\|_p^p+\|U_{i+1}\|_p^p+15p\eta^{\frac{2}{q}}(i)\|X_i\|_p\|U_{i+1}\|_p^{\frac{p}{q}}.$$

Next, we will establish a sequence of numbers by the following rules. Let

$$\xi_i^p=\begin{cases}2^{2-p}\|X_i\|_p^p+\xi_{i+1}^p+15p\eta^{\frac{2}{q}}(i)\|X_i\|_p\xi_{i+1}^{\frac{p}{q}}, & 1\le i\le n-1,\\ 2^{2-p}\|X_n\|_p^p, & i=n.\end{cases}$$

Since $U_n=X_n$, there is $\|U_n\|_p^p=\|X_n\|_p^p\le 2^{2-p}\|X_n\|_p^p=\xi_n^p$. Therefore, we achieve

$$\begin{aligned}\|U_{n-1}\|_p^p&\le 2^{2-p}\|X_{n-1}\|_p^p+\|U_n\|_p^p+15p\eta^{\frac{2}{q}}(n-1)\|X_{n-1}\|_p\|U_n\|_p^{\frac{p}{q}}\\&\le 2^{2-p}\|X_{n-1}\|_p^p+\xi_n^p+15p\eta^{\frac{2}{q}}(n-1)\|X_{n-1}\|_p\xi_n^{\frac{p}{q}}\\&=\xi_{n-1}^p.\end{aligned}$$

Then we can deduce that $\|U_i\|_p \le \xi_i$ for any $1 \le i \le n$ by analogy. And it is easy to prove that $\{\xi_i\}_{i=1}^n$ is a decreasing sequence. Therefore, for any $1 \le i \le n-1$, we have

$$\begin{cases} \xi_i^p \le 2^{2-p}\|X_i\|_p^p + \xi_{i+1}^p + 15p\eta^{\frac{2}{q}}(i)\|X_i\|_p \xi_1^{\frac{p}{q}}, & 1 \le i \le n-1, \\ \xi_n^p = 2^{2-p}\|X_n\|_p^p. \end{cases}$$

Substituting sequentially, we conclude that

$$\begin{aligned} \xi_1^p &\le 2^{2-p}\sum_{i=1}^n \|X_i\|_p^p + 15p\xi_1^{\frac{p}{q}} \sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \\ &= 2^{2-p}\sum_{i=1}^n \|X_i\|_p^p + (\xi_1^p)^{\frac{1}{q}} \Big(\Big(15p \sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \Big)^p \Big)^{\frac{1}{p}} \\ &\le 2^{2-p}\sum_{i=1}^n \|X_i\|_p^p + q^{-1}\xi_1^p + p^{-1}\Big(15p \sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \Big)^p. \end{aligned}$$

The last inequality is from an elementary inequality $a^\alpha b^\beta \le \alpha a + \beta b$ for nonnegative real numbers a, b, α, β with $\alpha + \beta = 1$. Therefore, we have

$$\xi_1^p \le 2^{2-p}p\sum_{i=1}^n \|X_i\|_p^p + \Big(15p \sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \Big)^p.$$

Since $\|U_1\|_p \le \xi_1^p$, we obtain that

$$\mathbb{E}\Big[\Big| \max_{1\le i\le n} S_i \Big|^p\Big] \le 2^{2-p}p\sum_{i=1}^n \|X_i\|_p^p + (15p)^p\Big(\sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \Big)^p.$$

By Lemma 3.1, $\{-X_n\}_{n=1}^\infty$ is also an AANA sequence of random variables with the same mixing coefficients $\{\eta(n)\}_{n=1}^\infty$. Consequently, similarly as above steps, we can achieve that

$$\mathbb{E}\Big[\Big| \max_{1\le i\le n} (-S_i) \Big|^p\Big] \le 2^{2-p}p\sum_{i=1}^n \|X_i\|_p^p + (15p)^p\Big(\sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \Big)^p.$$

Thus, by Proposition 2.3 (C_r Inequality), we have

$$\begin{aligned} \mathbb{E}\Big[\max_{1\le i\le n} |S_i|^p \Big] &= \mathbb{E}\Big[\Big(\max_{1\le i\le n} |S_i| \Big)^p\Big] \\ &\le \mathbb{E}[\{\max(0, S_1, \cdots, S_n) + \max(0, -S_1, \cdots, -S_n)\}^p] \\ &\le \mathbb{E}[2^{p-1}\{\max(0, S_1, \cdots, S_n)\}^p + 2^{p-1}\{\max(0, -S_1, \cdots, -S_n)\}^p] \\ &\le \mathbb{E}\Big[2^{p-1}\Big| \max_{1\le i\le n} S_i \Big|^p + 2^{p-1}\Big| \max_{1\le i\le n} (-S_i) \Big|^p \Big] \\ &\le 2^{p-1}\mathbb{E}\Big[\Big| \max_{1\le i\le n} S_i \Big|^p\Big] + 2^{p-1}\mathbb{E}\Big[\Big| \max_{1\le i\le n} (-S_i) \Big|^p\Big] \\ &\le 4p\sum_{i=1}^n \|X_i\|_p^p + 2^p(15p)^p\Big(\sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p \Big)^p. \end{aligned}$$

Therefore, Inequality (3.2) is proved. In addition, by classic Hölder's inequality, we get

$$\begin{aligned}\sum_{i=1}^{n-1}\eta^{\frac{2}{q}}(i)\|X_i\|_p &= \sum_{i=1}^{n-1}(\eta^2(i))^{\frac{1}{q}}(\|X_i\|_p^p)^{\frac{1}{p}} \\ &\le \Big(\sum_{i=1}^{n-1}\eta^2(i)\Big)^{\frac{1}{q}}\Big(\sum_{i=1}^{n-1}\|X_i\|_p^p\Big)^{\frac{1}{p}}.\end{aligned}$$

Combining with the assumption that $\sum\limits_{n=1}^{\infty}\eta^2(n)<\infty$, we get

$$\begin{aligned}\mathbb{E}\Big[\max_{1\le i\le n}|S_i|^p\Big] &\le 4p\sum_{i=1}^{n}\|X_i\|_p^p + 2^p(15p)^p\Big(\sum_{i=1}^{n-1}\eta^2(i)\Big)^{\frac{p}{q}}\sum_{i=1}^{n-1}\|X_i\|_p^p \\ &\le \Big(4p+2^p(15p)^p\Big(\sum_{i=1}^{n-1}\eta^2(i)\Big)^{\frac{p}{q}}\Big)\sum_{i=1}^{n}\|X_i\|_p^p \\ &= C_p\sum_{i=1}^{n}\|X_i\|_p^p,\end{aligned}$$

where $C_p=4p+2^p(15p)^p\big(\sum\limits_{i=1}^{\infty}\eta^2(i)\big)^{\frac{p}{q}}$ is a constant depending only on p. Therefore, the proof is complete.

From the above proof, it is easy to achieve the following corollary.

Corollary 3.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $1<p\le 2$ and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables under $\mathbb{E}$ with $\mathbb{E}[X_n]=0$. And $\{\eta(n)\}_{n=1}^{\infty}$ are the corresponding mixing coefficients. Then there exists a positive constant C_p depending only on p such that for any $n\ge 1$, we have*

$$\mathbb{E}\Big[\Big|\max_{1\le i\le n}S_i\Big|^p\Big]\le C_p\Big\{\sum_{i=1}^{n}\|X_i\|_p^p+\Big(\sum_{i=1}^{n}\eta^{\frac{2}{q}}(i)\|X_i\|_p\Big)^p\Big\}.$$

In particular, if $\sum\limits_{n=1}^{\infty}\eta^2(n)<\infty$, then for any $n\ge 1$, we have

$$\mathbb{E}\Big[\Big|\max_{1\le i\le n}S_i\Big|^p\Big]\le C_p\sum_{i=1}^{n}\|X_i\|_p^p.$$

Theorem 3.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p\ge 2$ and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables under $\mathbb{E}$ with $\mathbb{E}[X_n]=\mathcal{E}[X_n]=0$. And $\{\eta(n)\}_{n=1}^{\infty}$ are the corresponding mixing coefficients. Then there exist positive constants C_p and C'_p depending only on p such that for any $n\ge 1$, we have*

$$\mathbb{E}\Big[\max_{1\le i\le n}|S_i|^p\Big]\le C_p\Big\{\sum_{i=1}^{n}\|X_i\|_p^p+\Big(\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p\Big\}+C'_pn^{\frac{p}{2}-1}\sum_{i=1}^{n-1}\|X_i\|_p^p. \tag{3.3}$$

In particular, if $\sum\limits_{n=1}^{\infty}\eta^{\frac{q}{p}}(n)<\infty$, then for any $n\ge 1$, we have

$$\mathbb{E}\Big[\max_{1\le i\le n}|S_i|^p\Big]\le (C_p+C'_pn^{\frac{p}{2}-1})\sum_{i=1}^{n}\|X_i\|_p^p.$$

Proof Let U_i be defined as in the proof of Theorem 3.1, then $U_i = X_i + U_{i+1}^+$. By elementary inequality $|x+y|^p \le 2^p|x|^p + |y|^p + px|y|^{p-1}\mathrm{sgn}y + 2^p p^2 x^2|y|^{p-2}$ for any $x, y \in \mathbb{R},\ p \ge 2$ and Lemma 3.2, we obtain for $1 \le i \le n-1$,

$$\begin{aligned}\mathbb{E}[|U_i|^p] &\le \mathbb{E}[2^p|X_i| + (U_{i+1}^+)^p + pX_i(U_{i+1}^+)^{p-1} + 2^p p^2 X_i^2 (U_{i+1}^+)^{p-2}]\\ &\le 2^p\mathbb{E}[|X_i|^p] + \mathbb{E}[|U_{i+1}|^p] + p\mathbb{E}[X_i(U_{i+1}^+)^{p-1}] + 2^p p^2\mathbb{E}[X_i^2(U_{i+1}^+)^{p-2}]\\ &\le 2^p\mathbb{E}[|X_i|^p] + \mathbb{E}[|U_{i+1}|^p] + 15p\eta^{\frac{2}{p}}(i)\|X_i\|_p\|U_{i+1}\|_p^{\frac{p}{q}} + 2^p p^2\mathbb{E}[X_i^2|U_{i+1}|^{p-2}].\end{aligned}$$

Therefore, we have

$$\|U_i\|_p^p \le 2^p\|X_i\|_p^p + \|U_{i+1}\|_p^p + 15p\eta^{\frac{2}{p}}(i)\|X_i\|_p\|U_{i+1}\|_p^{\frac{p}{q}} + 2^p p^2\mathbb{E}[X_i^2|U_{i+1}|^{p-2}].$$

Next, we will establish a sequence of numbers by the following rules. Let

$$\zeta_i^p = \begin{cases} 2^p\|X_i\|_p^p + \zeta_{i+1}^p + 15p\eta^{\frac{2}{p}}(i)\|X_i\|_p\zeta_{i+1}^{\frac{p}{q}} + 2^p p^2\mathbb{E}[X_i^2|U_{i+1}|^{p-2}], & 1 \le i \le n-1,\\ 2^p\|X_n\|_p^p, & i = n.\end{cases}$$

It is clear that $\|U_i\| \le \zeta_i$ and $\zeta_{i+1} \le \zeta_i$ for any $1 \le i \le n-1$. Therefore, we have

$$\begin{cases} \zeta_i^p \le 2^p\|X_i\|_p^p + \zeta_{i+1}^p + 15p\eta^{\frac{2}{p}}(i)\|X_i\|_p\zeta_1^{\frac{p}{q}} + 2^p p^2\mathbb{E}[X_i^2|U_{i+1}|^{p-2}], & 1 \le i \le n-1,\\ \zeta_n^p = 2^p\|X_n\|_p^p, & i = n.\end{cases}$$

Similar as the proof of Theorem 3.1, we conclude that

$$\begin{aligned}\zeta_1^p &\le 2^p\sum_{i=1}^n\|X_i\|_p^p + 15p\zeta_1^{\frac{p}{q}}\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p + 2^p p^2\sum_{i=1}^{n-1}\mathbb{E}[X_i^2|U_{i+1}|^{p-2}]\\ &= 2^p\sum_{i=1}^n\|X_i\|_p^p + (\zeta_1^p)^{\frac{1}{q}}\Big(\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p\Big)^{\frac{1}{p}}\\ &\quad + 2^p p^2\sum_{i=1}^{n-1}(\mathbb{E}[(X_i^2)^{\frac{p}{2}}])^{\frac{2}{p}}(\mathbb{E}[(|U_{i+1}|^{p-2})^{\frac{p}{p-2}}])^{\frac{p-2}{p}}\\ &\le 2^p\sum_{i=1}^n\|X_i\|_p^p + q^{-1}\zeta_1^p + p^{-1}\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p\\ &\quad + 2^p p^2\sum_{i=1}^{n-1}(\mathbb{E}[(X_i^2)^{\frac{p}{2}}])^{\frac{2}{p}}(\mathbb{E}[|U_{i+1}|^p])^{1-\frac{2}{p}}.\end{aligned}$$

The last inequality is from an elementary inequality $a^\alpha b^\beta \le \alpha a + \beta b$ for nonnegative real numbers a,b,α,β with $\alpha + \beta = 1$. Therefore, we have

$$\zeta_1^p \le 2^p p\sum_{i=1}^n\|X_i\|_p^p + \Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p + 2^p p^3\sum_{i=1}^{n-1}(\mathbb{E}[(X_i^2)^{\frac{p}{2}}])^{\frac{2}{p}}(\mathbb{E}[|U_{i+1}|^p])^{1-\frac{2}{p}}.$$

Set $A_n = \max_{i\le n}\mathbb{E}[|U_i|^p]$. Then we have

$$A_n \le 2^p p\sum_{i=1}^n\|X_i\|_p^p + \Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p + \Big(2^p p^3\sum_{i=1}^{n-1}(\mathbb{E}[(X_i^2)^{\frac{p}{2}}])^{\frac{2}{p}}\Big)A_n^{1-\frac{2}{p}}.$$

Consequently

$$A_n \leq 2^{p-1}p^2\sum_{i=1}^{n}\|X_i\|_p^p + \frac{p}{2}\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p + \Big(2^p p^3\sum_{i=1}^{n-1}(\mathbb{E}[(X_i^2)^{\frac{p}{2}}])^{\frac{2}{p}}\Big)^{\frac{p}{2}}.$$

From the fact that $\mathbb{E}\big[\big|\max_{1\leq i\leq n} S_i\big|^p\big] = \mathbb{E}[|U_1|^p] \leq A_n$, we obtain

$$\mathbb{E}\Big[\Big|\max_{1\leq i\leq n} S_i\Big|^p\Big] \leq 2^{p-1}p^2\sum_{i=1}^{n}\|X_i\|_p^p + \frac{p}{2}\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p + 2^{\frac{p^2}{2}}p^{\frac{3p}{2}}\Big(\sum_{i=1}^{n-1}(\mathbb{E}[(X_i^2)^{\frac{p}{2}}])^{\frac{2}{p}}\Big)^{\frac{p}{2}}.$$

Applying classic Hölder's inequality, we have

$$\mathbb{E}\Big[\Big|\max_{1\leq i\leq n} S_i\Big|^p\Big] \leq 2^{p-1}p^2\sum_{i=1}^{n}\|X_i\|_p^p + \frac{p}{2}\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p + 2^{\frac{p^2}{2}}p^{\frac{3p}{2}}n^{\frac{p}{2}-1}\sum_{i=1}^{n-1}\|X_i\|_p^p.$$

Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\mathbb{E}\Big[\Big|\max_{1\leq i\leq n}(-S_i)\Big|^p\Big] &\leq 2^{p-1}p^2\sum_{i=1}^{n}\|X_i\|_p^p + \frac{p}{2}\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p \\ &\quad + 2^{\frac{p^2}{2}}p^{\frac{3p}{2}}n^{\frac{p}{2}-1}\sum_{i=1}^{n-1}\|X_i\|_p^p.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{E}\Big[\max_{1\leq i\leq n}|S_i|^p\Big] \\ &\leq 2^{p-1}\mathbb{E}\Big[\Big|\max_{1\leq i\leq n} S_i\Big|^p\Big] + 2^{p-1}\mathbb{E}\Big[\Big|\max_{1\leq i\leq n}(-S_i)\Big|^p\Big] \\ &\leq 2^{2p-1}p^2\sum_{i=1}^{n}\|X_i\|_p^p + 2^{p-1}p\Big(15p\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p + 2^{\frac{p^2+p}{2}}p^{\frac{3p}{2}}n^{\frac{p}{2}-1}\sum_{i=1}^{n-1}\|X_i\|_p^p.\end{aligned}$$

Therefore, Inequality (3.3) is proved. In addition, by classic Hölder's inequality and with the assumption that $\sum_{n=1}^{\infty}\eta^{\frac{q}{p}}(n) < \infty$, we get

$$\begin{aligned}&\mathbb{E}\Big[\max_{1\leq i\leq n}|S_i|^p\Big] \\ &\leq 2^{2p-1}p^2\sum_{i=1}^{n}\|X_i\|_p^p + 2^{p-1}p(15p)^p\Big(\sum_{i=1}^{n-1}\eta^{\frac{2q}{p}}(i)\Big)^{\frac{p}{q}}\sum_{i=1}^{n-1}\|X_i\|_p^p + 2^{\frac{p^2+p}{2}}p^{\frac{3p}{2}}n^{\frac{p}{2}-1}\sum_{i=1}^{n-1}\|X_i\|_p^p \\ &\leq (C_p + C_p' n^{\frac{p}{2}-1})\sum_{i=1}^{n}\|X_i\|_p^p,\end{aligned}$$

where $C_p = 2^{2p-1}p^2 + 2^{p-1}p(15p)^p\big(\sum_{i=1}^{\infty}\eta^{\frac{2q}{p}}(i)\big)^{\frac{p}{q}}$ and $C_p' = 2^{\frac{p^2+p}{2}}p^{\frac{3p}{2}}$ are constants depending only on p. Therefore, the proof is complete.

Corollary 3.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p\geq 2$ and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables under $\mathbb{E}$ with $\mathbb{E}[X_n]=0$. And $\{\eta(n)\}_{n=1}^{\infty}$ are the corresponding mixing coefficients.*

Then there exist positive constants C_p and C'_p depending only on p such that for any $n \geq 1$, we have

$$\mathbb{E}\Big[\Big|\max_{1\leq i\leq n} S_i\Big|^p\Big] \leq C_p\Big\{\sum_{i=1}^{n}\|X_i\|_p^p + \Big(\sum_{i=1}^{n-1}\eta^{\frac{2}{p}}(i)\|X_i\|_p\Big)^p\Big\} + C'_p n^{\frac{p}{2}-1}\sum_{i=1}^{n-1}\|X_i\|_p^p.$$

In particular, if $\sum_{n=1}^{\infty}\eta^{\frac{q}{p}}(n) < \infty$, then for any $n \geq 1$, we have

$$\mathbb{E}\Big[\Big|\max_{1\leq i\leq n} S_i\Big|^p\Big] \leq (C_p + C'_p n^{\frac{p}{2}-1})\sum_{i=1}^{n}\|X_i\|_p^p.$$

4 Strong Law of Large Numbers

In this section, we will come up with a version of strong law of large numbers as the application of our Rosenthal's inequality which is an extension of the classic case (see [6, 16]). Firstly, we will give the concept of 'quasi-surely' under upper expectations, which is derived from the concept of 'almost surely' in classic probability (see [4–5]).

Definition 4.1 *If a property holds on a set D, such that $\mathbb{V}(D^c) = 0$, then we called this property holds 'quasi-surely' (q.s. for short).*

Lemma 4.1 *Let $\beta_1, \cdots, \beta_n$ be a nondecreasing sequence of positive numbers and $\alpha_1, \cdots, \alpha_n$ be nonnegative numbers. Let r be a fixed positive number. Assume that for each $1 \leq m \leq n$, $\mathbb{E}\big[\max_{1\leq l\leq m}|S_l|^r\big] \leq \sum_{l=1}^{m}\alpha_l$. Then*

$$\mathbb{E}\Big[\max_{1\leq l\leq n}\Big|\frac{S_l}{\beta_l}\Big|^r\Big] \leq 4\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r}.$$

Proof Without loss of generality, assume that $\beta_1 = 1$. Consider the sets $A_i = \{k : 2^{\frac{i}{r}} \leq \beta_k < 2^{\frac{i+1}{r}}\}$, $i = 1, 2, \cdots$. Denote the index of the last nonempty A_i by I_m. Let $k(i) = \max\{k, k \in A_i\}$, if A_i is nonempty, while $k(i) = k(i-1)$ if A_i is empty, and let $k(-1) = 0$. And denote $\delta_l = \sum_{j=k(l-1)+1}^{k(l)}\alpha_j$, $l = 0, 1, 2, \cdots$, where $\delta_l = 0$ if A_l is empty. Then

$$\begin{aligned}
\mathbb{E}\Big[\max_{1\leq l\leq n}\Big|\frac{S_l}{\beta_l}\Big|^r\Big] &\leq \mathbb{E}\Big[\sum_{i=0}^{I_m}\max_{j\in A_i}\Big|\frac{S_j}{\beta_j}\Big|^r\Big] \leq \sum_{i=0}^{I_m}\mathbb{E}\Big[\max_{j\in A_i}\Big|\frac{S_j}{\beta_j}\Big|^r\Big] \leq \sum_{i=0}^{I_m}2^{-i}\mathbb{E}\Big[\max_{j\in A_i}|S_j|^r\Big] \\
&\leq \sum_{i=0}^{I_m}2^{-i}\mathbb{E}\Big[\max_{j\leq k(i)}|S_j|^r\Big] \leq \sum_{i=0}^{I_m}2^{-i}\sum_{j=1}^{k(i)}\alpha_j = \sum_{i=0}^{I_m}2^{-i}\sum_{l=0}^{i}\delta_l \\
&= \sum_{l=0}^{I_m}\delta_l\sum_{i=l}^{I_m}2^{-i} \leq \sum_{l=0}^{I_m}\delta_l\sum_{i=l}^{\infty}2^{-i} \leq \sum_{l=0}^{I_m}2^{-l+1}\delta_l \\
&= \sum_{l=0}^{I_m}2^{-l+1}\sum_{j=k(l-1)+1}^{k(l)}\alpha_j \leq 4\sum_{l=0}^{I_m}\sum_{j=k(l-1)+1}^{k(l)}\alpha_j\beta_j^{-r} = 4\sum_{i=1}^{n}\frac{\alpha_l}{\beta_l^r}.
\end{aligned}$$

Lemma 4.2 *Let $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers, $\alpha_1, \alpha_2, \cdots$ be nonnegative numbers and r be fixed positive number. For each $n \leq 1$, we have $\mathbb{E}\big[\max\limits_{1\leq l\leq n} |S_l|^r\big] \leq \sum\limits_{l=1}^{n} \alpha_l$. If $\sum\limits_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty$, then $\lim\limits_{n\to\infty} \frac{S_n}{b_n} = 0$ q.s.*

Proof If there exists a number N such that $\alpha_n = 0$ for any $n \geq N$. Then $\mathbb{E}\big[\max\limits_{n\geq 1} |S_n|^r\big] < \infty$. Then our result follows. Therefore, we only consider the case that $\alpha_n > 0$ for infinite numbers. Set $\gamma_n = \sum\limits_{k=n}^{\infty} \frac{\alpha_k}{b_k^r}$ and $\beta_n = \max\limits_{1\leq k\leq n} b_k \gamma_k^{\frac{1}{2r}}$. Obviously $0 < \gamma_n < \infty$. And it can be proved that $\sum\limits_{n=1}^{\infty} \frac{\alpha_n}{b_n^r \gamma_n^{\frac{1}{2}}} < \infty$. Therefore, the sequence β_n satisfy the following properties:

(1) $\beta_k \leq \beta_{k+1},\ k = 1, 2, \cdots$;

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{\alpha_k}{\beta_k^r} < \infty$;

(3) $\displaystyle\lim_{k\to\infty} \frac{\beta_k}{b_k} = 0$.

Then from (2) and Lemma 4.1, we have $\mathbb{E}\big[\max\limits_{1\leq l\leq n} \big|\frac{S_l}{\beta_l}\big|^r\big] \leq 4\sum\limits_{l=1}^{n} \frac{\alpha_l}{\beta_l^r} < \infty$. Consequently, $\sup\limits_{l\geq 1} \big|\frac{S_l}{\beta_l}\big| < \infty$ q.s. Therefore, we obtain

$$0 \leq \Big|\frac{S_l}{b_l}\Big| = \Big|\frac{S_l}{\beta_l}\Big|\frac{\beta_l}{b_l} \leq \Big\{\sup_{l\geq 1}\Big|\frac{S_l}{\beta_l}\Big|\Big\}\frac{\beta_l}{b_l} \to 0 \quad q.s.\ as\ l \to \infty.$$

Theorem 4.1 *Let $1 < p \leq 2$, $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables under $\mathbb{E}$ with $\mathbb{E}[X_n] = \mathcal{E}[X_n] = 0$. And $\{\eta(n)\}_{n=1}^{\infty}$ are the corresponding mixing coefficients. If $\sum\limits_{n=1}^{\infty} \eta^2(n) < \infty$ and $\sum\limits_{n=1}^{\infty} \frac{\mathbb{E}[|X_n|^p]}{b_n^p} < \infty$, then $\lim\limits_{n\to\infty} \frac{S_n}{b_n} = 0$ q.s.*

The theorem can be deduced from Theorem 3.1 and Lemma 4.2 immediately. So we omit the proof here.

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