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Sharp Threshold of Global Existence for a Nonlocal Nonlinear Schrödinger System in $\mathbb{R}^3$*

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Abstract In this paper, the authors investigate the sharp threshold of a three-dimensional nonlocal nonlinear Schrödinger system. It is a coupled system which provides the mathematical modeling of the spontaneous generation of a magnetic field in a cold plasma under the subsonic limit. The main difficulty of the proof lies in exploring the inner structure of the system due to the fact that the nonlocal effect may bring some hinderance for establishing the conservation quantities of the mass and of the energy, constructing the corresponding variational structure, and deriving the key estimates to gain the expected result. To overcome this, the authors must establish local well-posedness theory, and set up suitable variational structure depending crucially on the inner structure of the system under study, which leads to define proper functionals and a constrained variational problem. By building up two invariant manifolds and then making a priori estimates for these nonlocal terms, the authors figure out a sharp threshold of global existence for the system under consideration.

Keywords Nonlocal nonlinear Schrödinger system, Sharp threshold, Blow-up, Global existence

2000 MR Subject Classification 35A15, 35E55, 35Q55

1 Introduction

The main purpose of this work is to establish a sharp threshold for the blow-up and global existence to the Cauchy problem of the following nonlocal nonlinear Schrödinger system in $\mathbb{R}^3$:

$$\begin{aligned}&\mathrm{i}\partial_t E_1 + \Delta E_1 + (|E_1|^2 + |E_2|^2 + |E_3|^2)E_1\\&+ A_1(E_1, E_2, E_3) + A_2(E_1, E_2, E_3) = 0,\end{aligned}\tag{1.1}$$

$$\begin{aligned}&\mathrm{i}\partial_t E_2 + \Delta E_2 + (|E_1|^2 + |E_2|^2 + |E_3|^2)E_2\\&+ B_1(E_1, E_2, E_3) + B_2(E_1, E_2, E_3) = 0,\end{aligned}\tag{1.2}$$

$$\begin{aligned}&\mathrm{i}\partial_t E_3 + \Delta E_3 + (|E_1|^2 + |E_2|^2 + |E_3|^2)E_3\\&+ C_1(E_1, E_2, E_3) + C_2(E_1, E_2, E_3) = 0,\end{aligned}\tag{1.3}$$

$$E_1(0, x) = E_{10}(x), \quad E_2(0, x) = E_{20}(x), \quad E_3(0, x) = E_{30}(x).\tag{1.4}$$

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Here

$$A_1(E_1,E_2,E_3) = -E_2\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3) - (\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)+\xi_2\xi_3\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)]\Big\}, \tag{1.5}$$

$$A_2(E_1,E_2,E_3) = E_3\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2) - (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)+\xi_1\xi_2\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)]\Big\}, \tag{1.6}$$

$$B_1(E_1,E_2,E_3) = -E_3\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3) - (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)+\xi_1\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)]\Big\}, \tag{1.7}$$

$$B_2(E_1,E_2,E_3) = E_1\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3) - (\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)+\xi_2\xi_3\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)]\Big\}, \tag{1.8}$$

$$C_1(E_1,E_2,E_3) = -E_1\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2) - (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)+\xi_1\xi_2\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)]\Big\}, \tag{1.9}$$

$$C_2(E_1,E_2,E_3) = E_2\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3) - (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)+\xi_1\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)]\Big\}, \tag{1.10}$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively (see [14, 16–18]), $\eta > 0$, $\delta \le 0$, $(E_1,E_2,E_3)(t,x)$ is a pair of complex-valued functions from $\mathbb{R}^+\times\mathbb{R}^3$ into $\mathbb{C}^3$, and $\overline{E}_i$ $(i=1,2,3)$ denotes the complex conjugate of E_i. Let $\mathbf{E}=(E_1,E_2,E_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$. Due to rotational invariance of (1.1)–(1.3), one can recast the system (1.1)–(1.3) in the form of the following vector-valued nonlinear Schrödinger equations:

$$\mathrm{i}\mathbf{E}_t+\Delta\mathbf{E}+|\mathbf{E}|^2\mathbf{E}+\mathrm{i}(\mathbf{E}\wedge\mathbf{B})=0, \tag{1.11}$$

$$\mathbf{B}(\mathbf{E})=\mathcal{F}^{-1}\Big[\frac{\mathrm{i}\eta}{|\xi|^2-\delta}(\xi\wedge(\xi\wedge\mathcal{F}(\mathbf{E}\wedge\overline{\mathbf{E}})))\Big]. \tag{1.12}$$

Equations (1.11)–(1.12) provide the mathematical modeling of the spontaneous generation of a magnetic field in a cold plasma under the subsonic limit. $\mathbf{E}$ denotes a slowly varying complex amplitude of the high-frequency electric field, and $\mathbf{B}$ denotes the self-generated magnetic field (see [4, 13, 24–25]). The gauge invariance yields that for any $\omega>0$ and for $j=1,2$,

$$\begin{aligned}
A_j(\mathrm{e}^{\mathrm{i}\omega t}E_1,\mathrm{e}^{\mathrm{i}\omega t}E_2,\mathrm{e}^{\mathrm{i}\omega t}E_3)&=\mathrm{e}^{\mathrm{i}\omega t}A_j(E_1,E_2,E_3),\\
B_j(\mathrm{e}^{\mathrm{i}\omega t}E_1,\mathrm{e}^{\mathrm{i}\omega t}E_2,\mathrm{e}^{\mathrm{i}\omega t}E_3)&=\mathrm{e}^{\mathrm{i}\omega t}B_j(E_1,E_2,E_3),\\
C_j(\mathrm{e}^{\mathrm{i}\omega t}E_1,\mathrm{e}^{\mathrm{i}\omega t}E_2,\mathrm{e}^{\mathrm{i}\omega t}E_3)&=\mathrm{e}^{\mathrm{i}\omega t}C_j(E_1,E_2,E_3).
\end{aligned}$$

Hence, it is natural to consider the solutions of the system (1.1)–(1.3) in the form $E_i(t,x)=\mathrm{e}^{\mathrm{i}\omega t}u_i(x)$ $(i=1,2,3)$ with the initial condition (1.4); $\omega>0$ is a constant called frequency, and (u_1,u_2,u_3) is a pair of complex-valued functions with respect to $x\in\mathbb{R}^3$ which solves the following nonlinear elliptic system:

$$-\omega u_1+\Delta u_1+(|u_1|^2+|u_2|^2+|u_3|^2)u_1-u_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\Big]$$

$$
\begin{aligned}
&+ u_2\mathcal{F}^{-1}\Big[\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-u_2\overline{u}_1)\Big] - u_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\Big]\\
&+ u_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)\Big] + u_3\mathcal{F}^{-1}\Big[\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)\Big]\\
&+ u_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\Big] = 0, \qquad (1.13)\\
&-\omega u_2+\Delta u_2+(|u_1|^2+|u_2|^2+|u_3|^2)u_2 - u_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\Big]\\
&+ u_3\mathcal{F}^{-1}\Big[\frac{\eta(\xi_3^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\Big] - u_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)\Big]\\
&+ u_1\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\Big] + u_1\mathcal{F}^{-1}\Big[\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\Big]\\
&+ u_1\mathcal{F}^{-1}\Big[\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\Big] = 0, \qquad (1.14)\\
&-\omega u_3+\Delta u_3+(|u_1|^2+|u_2|^2+|u_3|^2)u_3 - u_1\mathcal{F}^{-1}\Big[\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)\Big]\\
&+ u_1\mathcal{F}^{-1}\Big[\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\Big] - u_1\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\Big]\\
&+ u_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\Big] + u_2\mathcal{F}^{-1}\Big[\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\Big]\\
&+ u_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)\Big] = 0. \qquad (1.15)
\end{aligned}
$$

For the nonlinear Schrödinger equations without nonlocal terms, there have been many works on the blow-up and global existence for their solutions. Ginibre and Velo [6] established the local existence of the Cauchy problem in energy space $H^1(\mathbb{R}^N)$. Glassey [8], Ogawa and Tsutsumi [20–21] proved that for some initial data, especially for a class of sufficiently large data, the solutions to the Cauchy problem blow-up in finite time. On the other hand, the authors proved that, for sufficiently small initial data, the solutions to the Cauchy problem exist globally for all time (see [5, 7, 9–10, 12]). Thus in the course of nature, the topic to explore sharp threshold for global existence and blow-up to the related Cauchy problem arises, and it was mentioned by Strauss and Cazenave in their monographs [3, 22]. Furthermore, for the nonlinear Schrödinger equations without nonlocal terms, the sharp threshold of global existence and blow-up was studied extensively in [2, 15, 22, 26–27].

For the system (1.1)–(1.3), however, the sharp threshold of global existence and blow-up seems to be elusive. In this paper, we will investigate the sharp threshold of global existence and blow-up of solutions to the Cauchy problem (1.1)–(1.4). The main difficulty of the proof lies in exploring the inner structure of the system (1.1)–(1.3) due to the fact that the nonlocal effect may bring some hinderance for establishing the conservation quantities of the mass and of the energy, constructing the corresponding variational structure, and deriving the key estimates to gain the expected result. To overcome this, we must establish local well-posedness theory, and set up a suitable variational structure depending crucially on the inner structure of the system under study, which requires us to define proper functionals and a constrained variational problem. Fortunately, by building up two invariant manifolds and then making a priori estimates for these nonlocal terms, we finally figure out a sharp threshold of global existence and blow-up for the system under consideration.

This paper is arranged as follows. In Section 2, we establish some basic facts including local well-posedness, conservation laws of the mass and of the energy as well as variational structure. In Section 3, we show the sharp threshold for global existence and blow-up of solutions to the Cauchy problem (1.1)–(1.4). We will give the proof of Lemma 2.2 in Section 4.

For simplicity, we denote any positive constant by C throughout this paper.

2 Preliminaries

In this section, we establish some basic facts including local well-posedness, conservation laws of the mass and of the energy for the Cauchy problem (1.1)–(1.4), as well as the variational structure for (1.5)–(1.7).

2.1 Local well-posedness

The Cauchy problem (1.1)–(1.4) can be recast in the integral equation below:

$$E_i(t) = U(t)E_{i0} + \mathrm{i}\int_0^t U(t-t')[(|E_1|^2+|E_2|^2+|E_3|^2)E_i + K_i(E_1,E_2,E_3)](t')\mathrm{d}t', \qquad (2.1)$$

where $i=1,2,3$, $U(t)=\mathrm{e}^{\mathrm{i}t\triangle}$ is the unitary group generated by the free Schrödinger equation $\mathrm{i}E_t+\Delta E=0$ in $H^s(\mathbb{R}^3)$ $(s\in\mathbb{R})$, and $K_i(E_1,E_2,E_3)$ has the following form:

$$\begin{aligned}
&K_1(E_1,E_2,E_3)\\
&= -E_2\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\\
&\quad+\xi_2\xi_3\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)]\Big\}+E_3\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\\
&\quad-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)+\xi_1\xi_2\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)]\Big\}, \qquad (K\text{-}1)\\
&K_2(E_1,E_2,E_3)\\
&= -E_3\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)-(\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\\
&\quad+\xi_1\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)]\Big\}+E_1\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\\
&\quad-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)+\xi_2\xi_3\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)]\Big\}, \qquad (K\text{-}2)\\
&K_3(E_1,E_2,E_3)\\
&= -E_1\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\\
&\quad+\xi_1\xi_2\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)]\Big\}+E_2\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\\
&\quad-(\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)+\xi_1\xi_3\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)]\Big\}. \qquad (K\text{-}3)
\end{aligned}$$

Let (E_1^j,E_2^j,E_3^j) $(j=1,2)$ solve the Cauchy problem (1.1)–(1.4). We then need to make a priori estimates for the term $K_i(E_1^1,E_2^1,E_3^1)-K_i(E_1^2,E_2^2,E_3^2)$ $(i=1,2,3)$. Direct calculation yields

$$K_1(E_1^1,E_2^1,E_3^1)-K_1(E_1^2,E_2^2,E_3^2)$$

$$
\begin{aligned}
&= -E_2^1\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1^1\overline{E_2^1}-\overline{E_1^1}E_2^1)\\
&\quad+\xi_2\xi_3\mathcal{F}(\overline{E_1^1}E_3^1-E_1^1\overline{E_3^1})]\Big\}+E_3^1\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(E_1^1\overline{E_2^1}-\overline{E_1^1}E_2^1)\\
&\quad-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1^1}E_3^1-E_1^1\overline{E_3^1})+\xi_1\xi_2\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)]\Big\}\\
&\quad-\Big\{-E_2^2\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(E_2^2\overline{E_3^2}-\overline{E_2^2}E_3^2)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1^2\overline{E_2^2}-\overline{E_1^2}E_2^2)\\
&\quad+\xi_2\xi_3\mathcal{F}(\overline{E_1^2}E_3^2-E_1^2\overline{E_3^2})]\Big\}+E_3^2\mathcal{F}^{-1}\Big\{\frac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(E_1^2\overline{E_2^2}-\overline{E_1^2}E_2^2)\\
&\quad-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1^2}E_3^2-E_1^2\overline{E_3^2})+\xi_1\xi_2\mathcal{F}(E_2^2\overline{E_3^2}-\overline{E_2^2}E_3^2)]\Big\}\Big\}. \qquad (2.2)
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
&-E_2^1\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)\Big]-\Big\{-E_2^2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2^2\overline{E_3^2}-\overline{E_2^2}E_3^2)\Big]\Big\}\\
&=-(E_2^1-E_2^2)\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)\Big]\\
&\quad+E_2^2\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}[(E_2^2-E_2^1)\overline{E_3^2}+E_2^1(\overline{E_3^2}-\overline{E_3^1})\\
&\quad+(\overline{E_2^1}-\overline{E_2^2})E_3^1+\overline{E_2^2}(E_3^1-E_3^2)]\Big\}. \qquad (2.3)
\end{aligned}
$$

Making similar estimates to (2.3), we can calculate the other terms of $K_i(E_1^1,E_2^1,E_3^1)-K_i(E_1^2,E_2^2,E_3^2)$ $(i=1,2,3)$.

Note that $\big|\frac{\eta\xi_i\xi_k}{|\xi|^2-\delta}\big|\le\eta$ by $\eta>0$ and $\delta\le 0$. We obtain by applying the contraction mapping principle that the solutions to the integral equation (2.1) is local well-posedness in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ (see [6, 11, 16–19, 22]). That is, we claim the following result.

Proposition 2.1 *For $\eta>0,\ \delta\le 0$ and $(E_{10},E_{20},E_{30})\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, the Cauchy problem* (1.1)–(1.4) *admits a unique solution*

$$(E_1,E_2,E_3)\in X^1_{4,\mathrm{loc}}([0,T))\times X^1_{4,\mathrm{loc}}([0,T))\times X^1_{4,\mathrm{loc}}([0,T))$$

for some $T=T(E_{10},E_{20},E_{30})$, and for any $0\le T_1<T_2<T$, the mapping

$$
\begin{aligned}
&(E_{10},E_{20},E_{30})(\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3))\\
&\mapsto(E_1,E_2,E_3)(t)(\in X^1_{4,\mathrm{loc}}([0,T))\times X^1_{4,\mathrm{loc}}([0,T))\times X^1_{4,\mathrm{loc}}([0,T)))
\end{aligned}
$$

is continuous. In addition, there holds either $T=+\infty$ or $T<+\infty$ and

$$\lim_{t\to T}(\|E_1\|_{H^1(\mathbb{R}^3)}+\|E_2\|_{H^1(\mathbb{R}^3)}+\|E_3\|_{H^1(\mathbb{R}^3)})=+\infty.$$

Here, for any interval $I\subset\mathbb{R}$, $0\le\frac{2}{q}=3\big(\frac12-\frac1\theta\big)<1$, $s\in\mathbb{R}$,

$$
\begin{aligned}
&X^s_\theta(I)=(C\cap L^\infty)(I;H^s)\cap L^q(I;H^s_\theta),\\
&X^s_{\theta,\mathrm{loc}}(I)=\{u;\ u\in X^s_\theta(J),\ \forall J\subset\subset I\},\\
&H^s_\theta=J_s(L^\theta),\quad J_s=(I-\Delta)^{-\frac{s}{2}}.
\end{aligned}
$$

Motivated by the works in [3, 6, 11, 14, 19, 22], applying Proposition 2.1 we have the following result.

Corollary 2.1 *The Cauchy problem* (1.1)–(1.4), *for* $\eta > 0$, $\delta \le 0$ *and*

$$(E_{10}, E_{20}, E_{30}) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3),$$

admits a unique solution

$$(E_1, E_2, E_3) \in C([0,T); H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3))$$

for some $T \in (0, +\infty]$ *with* $T = +\infty$ *or* $T < +\infty$ *and*

$$\lim_{t\to T} (\|E_1\|_{H^1(\mathbb{R}^3)} + \|E_2\|_{H^1(\mathbb{R}^3)} + \|E_3\|_{H^1(\mathbb{R}^3)}) = +\infty.$$

2.2 Conservation laws of the mass and of the energy

According to the structure of the system (1.1)–(1.3), we conclude the conservation laws of the total mass and of the total energy.

Lemma 2.1 *Let* (E_1, E_2, E_3) *be a smooth solution to the Cauchy problem* (1.1)–(1.4). *Then the total mass and the total energy are conserved*:

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x \\
&= \int_{\mathbb{R}^3} (|E_{10}|^2 + |E_{20}|^2 + |E_{30}|^2)\mathrm{d}x, \qquad (2.4)\\
&\mathcal{H}(E_1, E_2, E_3) \\
&= \int_{\mathbb{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3} (|E_1|^4 + |E_2|^4 + |E_3|^4)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} (|E_1|^2|E_2|^2 + |E_1|^2|E_3|^2 + |E_2|^2|E_3|^2)\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3} \frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta} |\mathcal{F}(E_1\overline{E}_2 - \overline{E}_1 E_2)|^2 \mathrm{d}\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3} \frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta} |\mathcal{F}(\overline{E}_1 E_3 - E_1\overline{E}_3)|^2 \mathrm{d}\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3} \frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta} |\mathcal{F}(E_2\overline{E}_3 - \overline{E}_2 E_3)|^2 \mathrm{d}\xi \\
&\quad + \mathrm{Re}\int_{\mathbb{R}^3} \frac{\eta\xi_1\xi_2}{|\xi|^2-\delta} \mathcal{F}(E_2\overline{E}_3 - \overline{E}_2 E_3)\overline{\mathcal{F}(\overline{E}_1 E_3 - E_1\overline{E}_3)} \mathrm{d}\xi \\
&\quad + \mathrm{Re}\int_{\mathbb{R}^3} \frac{\eta\xi_1\xi_3}{|\xi|^2-\delta} \mathcal{F}(E_1\overline{E}_2 - \overline{E}_1 E_2)\overline{\mathcal{F}(E_2\overline{E}_3 - \overline{E}_2 E_3)} \mathrm{d}\xi \\
&\quad + \mathrm{Re}\int_{\mathbb{R}^3} \frac{\eta\xi_2\xi_3}{|\xi|^2-\delta} \mathcal{F}(\overline{E}_1 E_3 - E_1\overline{E}_3)\overline{\mathcal{F}(E_1\overline{E}_2 - \overline{E}_1 E_2)} \mathrm{d}\xi \\
&= \mathcal{H}(E_{10}, E_{20}, E_{30}). \qquad (2.5)
\end{aligned}$$

Proof Multiplying (1.1) by $\overline{E}_1$, (1.2) by $\overline{E}_2$ and (1.3) by $\overline{E}_3$, taking the imaginary part and then integrating with respect to $x \in \mathbb{R}^3$, we get

$$\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^3}(\mathrm{i}\partial_t E_1\overline{E}_1+\mathrm{i}\partial_t E_2\overline{E}_2+\mathrm{i}\partial_t E_3\overline{E}_3)\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}(\Delta E_1\overline{E}_1+\Delta E_2\overline{E}_2+\Delta E_3\overline{E}_3)\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\overline{E}_1+E_2\overline{E}_2+E_3\overline{E}_3)\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_1A_1(E_1,E_2,E_3)\mathrm{d}x+\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_1A_2(E_1,E_2,E_3)\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_2B_1(E_1,E_2,E_3)\mathrm{d}x+\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_2B_2(E_1,E_2,E_3)\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_3C_1(E_1,E_2,E_3)\mathrm{d}x+\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_3C_2(E_1,E_2,E_3)\mathrm{d}x\\
=\ &0,
\end{aligned}\tag{2.6}$$

where $A_1(E_1,E_2,E_3)$, $A_2(E_1,E_2,E_3)$, $B_1(E_1,E_2,E_3)$, $B_2(E_1,E_2,E_3)$, $C_1(E_1,E_2,E_3)$ and $C_2(E_1, E_2,E_3)$ are defined by (1.5)–(1.10) (see Section 1).

Direct calculation and rearrangement for these terms in (2.6) yield

$$\mathrm{Im}\int_{\mathbb{R}^3}\mathrm{i}\partial_t E_j\overline{E}_j\mathrm{d}x=\mathrm{Re}\int_{\mathbb{R}^3}\partial_t E_j\overline{E}_j\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|E_j|^2\mathrm{d}x,\tag{2.7}$$

$$\begin{aligned}
&-\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}E_1\overline{E}_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
=\ &\mathrm{Im}\int_{\mathbb{R}^3}\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\Big]\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}E_1\overline{E}_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
=\ &\mathrm{Im}\Big\{2\mathrm{Re}\int_{\mathbb{R}^3}\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\Big]\Big\}\\
=\ &0.
\end{aligned}\tag{2.8}$$

We can make the similar estimates for the other terms in (2.6) to (2.8). Hence the mass identity (2.4) follows. We are now in the position to verify the energy identity (2.5).

Multiplying (1.1) by $2\partial_t\overline{E}_1$, (1.2) by $2\partial_t\overline{E}_2$ and (1.3) by $2\partial_t\overline{E}_3$, taking the real part and then integrating the resulting equations with respect to $x \in \mathbb{R}^3$, we obtain

$$\begin{aligned}
&2\mathrm{Re}\int_{\mathbb{R}^3}(\mathrm{i}\partial_t E_1\partial_t\overline{E}_1+\mathrm{i}\partial_t E_2\partial_t\overline{E}_2+\mathrm{i}\partial_t E_3\partial_t\overline{E}_3)\mathrm{d}x\\
=\ &2\mathrm{Re}\int_{\mathbb{R}^3}[(-\Delta E_1)\partial_t\overline{E}_1+(-\Delta E_2)\partial_t\overline{E}_2+(-\Delta E_3)\partial_t\overline{E}_3]\mathrm{d}x\\
&-2\mathrm{Re}\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\partial_t\overline{E}_1+E_2\partial_t\overline{E}_2+E_3\partial_t\overline{E}_3)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-2\mathrm{Re}\int_{\mathbb{R}^3}\partial_t\overline{E}_1A_1(E_1,E_2,E_3)\mathrm{d}x-2\mathrm{Re}\int_{R^3}\partial_t\overline{E}_1A_2(E_1,E_2,E_3)\mathrm{d}x\\
&-2\mathrm{Re}\int_{R^3}\partial_t\overline{E}_2B_1(E_1,E_2,E_3)\mathrm{d}x-2\mathrm{Re}\int_{\mathbb{R}^3}\partial_t\overline{E_2}B_2(E_1,E_2,E_3)\mathrm{d}x\\
&-2\mathrm{Re}\int_{R^3}\partial_t\overline{E}_3C_1(E_1,E_2,E_3)\mathrm{d}x-2\mathrm{Re}\int_{R^3}\partial_t\overline{E}_3C_2(E_1,E_2,E_3)\mathrm{d}x. \qquad (2.9)
\end{aligned}
$$

We now calculate (2.9) term by term.

$$
2\mathrm{Re}\int_{\mathbb{R}^3}(\mathrm{i}\partial_tE_1\partial_t\overline{E}_1+\mathrm{i}\partial_tE_2\partial_t\overline{E}_2+\mathrm{i}\partial_tE_3\partial_t\overline{E}_3)\mathrm{d}x=0, \qquad (2.10)
$$

$$
\begin{aligned}
&2\mathrm{Re}\int_{\mathbb{R}^3}[(-\triangle E_1)\partial_t\overline{E}_1+(-\triangle E_2)\partial_t\overline{E}_2+(-\triangle E_3)\partial_t\overline{E}_3]\mathrm{d}x\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)\mathrm{d}x, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
&2\mathrm{Re}\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\partial_t\overline{E}_1+E_2\partial_t\overline{E}_2+E_3\partial_t\overline{E}_3)\mathrm{d}x\\
=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)\mathrm{d}x\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)\mathrm{d}x, \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
&2\mathrm{Re}\int_{\mathbb{R}^3}\partial_t\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
&+2\mathrm{Re}\int_{\mathbb{R}^3}\partial_t\overline{E}_2E_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big]\mathrm{d}x\\
&-2\mathrm{Re}\int_{\mathbb{R}^3}\partial_t\overline{E}_2E_1\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
&-2\mathrm{Re}\int_{\mathbb{R}^3}\partial_t\overline{E}_3E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big]\mathrm{d}x\\
=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi. \qquad (2.13)
\end{aligned}
$$

Making the similar estimates to (2.13) for the other terms in (2.9) and employing (2.10)–(2.13), we conclude the energy identity (2.5).

This finishes the proof of Lemma 2.1.

2.3 Variational structures

Let $(u_1,u_2,u_3)\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ be a pair of complex-valued functions. We define two functionals $S(u_1,u_2,u_3)$ and $R(u_1,u_2,u_3)$ as follows:

$$
\begin{aligned}
&S(u_1,u_2,u_3)\\
=&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&+\frac{\omega}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)\mathrm{d}x\\
&-\frac{1}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&-\frac{1}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{1}{4}\int_{\mathbb{R}^3}F_1(u_1,u_2,u_3)\mathrm{d}\xi+\frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3}F_2(u_1,u_2,u_3)\mathrm{d}\xi, \qquad (2.14)\\
&R(u_1,u_2,u_3)\\
=&\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x\\
&-\frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&-\frac{3}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{3}{4}\int_{\mathbb{R}^3}F_1(u_1,u_2,u_3)\mathrm{d}\xi+\frac{1}{2}\delta\int_{\mathbb{R}^3}\frac{F_1(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi\\
&+\frac{3}{2}\mathrm{Re}\int_{\mathbb{R}^3}F_2(u_1,u_2,u_3)\mathrm{d}\xi-\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi, \qquad (2.15)
\end{aligned}$$

where

$$\begin{aligned}
&F_1(u_1,u_2,u_3)\\
=&\frac{\eta}{|\xi|^2-\delta}[(\xi_1^2+\xi_2^2)|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\\
&+(\xi_1^2+\xi_3^2)|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\\
&+(\xi_2^2+\xi_3^2)|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2], \qquad (2.16)\\
&F_2(u_1,u_2,u_3)\\
=&\frac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\\
&+\xi_1\xi_3\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\\
&+\xi_2\xi_3\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}]. \qquad (2.17)
\end{aligned}$$

A natural attempt to find the nontrivial solutions to the system (1.13)–(1.15) is to solve the following constrained minimization problem:

$$d:=\inf_{(u_1,u_2,u_3)\in M}S(u_1,u_2,u_3), \qquad (2.18)$$

where the set M is prescribed by

$$M=\{(u_1,u_2,u_3)\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\setminus\{(0,0,0)\},\ R(u_1,u_2,u_3)=0\}. \qquad (2.19)$$

From $(u_1,u_2,u_3)\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, $\eta>0$, $\delta\le 0$, the Sobolev's embedding theorem and some properties of Fourier transform, it follows that functionals $S(u_1,u_2,u_3)$ and $R(u_1,u_2,u_3)$ are both well defined.

For the constrained variational problem (2.18), we claim the following result.

Proposition 2.2 *For $\eta > 0$ and $\delta \le 0$, there holds $d > 0$.*

Proof (2.14)–(2.15) and (2.19) yield for $(u_1, u_2, u_3) \in M$,

$$\begin{aligned} & S(u_1,u_2,u_3) \\ & = \frac{1}{6}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x \\ & \quad + \frac{\omega}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)\mathrm{d}x \\ & \quad - \frac{\delta}{6}\int_{\mathbb{R}^3}\frac{F_1(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi + \frac{\delta}{3}\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi, \end{aligned} \tag{2.20}$$

where $F_1(u_1, u_2, u_3)$ and $F_2(u_1, u_2, u_3)$ are given by (2.16) and (2.17), respectively. Noting that (2.16)–(2.17), $\eta > 0$, $\delta \le 0$ and the inequality $\mathrm{Re}\, ab \le \frac{1}{2}(a^2 + b^2)$, then making some suitable rearrangement, we obtain for $(u_1, u_2, u_3) \in M$,

$$-\frac{\delta}{6}\int_{\mathbb{R}^3}\frac{F_1(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi + \frac{\delta}{3}\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi \ge 0. \tag{2.21}$$

Indeed, recall that $F_1(u_1, u_2, u_3)$ and $F_2(u_1, u_2, u_3)$ in (2.16)–(2.17). Since

$$\begin{aligned} & \mathrm{Re}\,\xi_1\xi_2\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)} \\ & \le \frac{1}{2}\xi_1^2|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2+\frac{1}{2}\xi_2^2|\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)|^2, \\ & \mathrm{Re}\,\xi_1\xi_3\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)} \\ & \le \frac{1}{2}\xi_1^2|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2+\frac{1}{2}\xi_3^2|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2, \\ & \mathrm{Re}\,\xi_2\xi_3\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)} \\ & \le \frac{1}{2}\xi_2^2|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2+\frac{1}{2}\xi_3^2|\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)|^2, \end{aligned}$$

through regrouping and applying some properties of Fourier transform, we conclude that for $\eta > 0$ and $\delta \le 0$,

$$\begin{aligned} & \frac{\delta}{3}\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi \\ & \ge \frac{\delta}{6}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\delta}\cdot\frac{\eta}{|\xi|^2-\delta}[(\xi_1^2+\xi_2^2)|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2 \\ & \quad (\xi_1^2+\xi_3^2)|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2(\xi_2^2+\xi_3^2)|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2]\mathrm{d}\xi \\ & = \frac{\delta}{6}\int_{\mathbb{R}^3}\frac{F_1(u_1,u_2,u_3)}{|\xi|^2-\delta}\mathrm{d}\xi. \end{aligned}$$

Hence (2.21) is valid. (2.20) and (2.21) then yield

$$\begin{aligned} S(u_1,u_2,u_3) \ge & \frac{1}{6}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x \\ & + \frac{\omega}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)\mathrm{d}x. \end{aligned} \tag{2.22}$$

On the other hand, for $j, k = 1, 2, 3$, one has

$$\frac{3}{4}\frac{\eta(\xi_j^2+\xi_k^2)}{|\xi|^2-\delta} - \frac{\delta}{2}\frac{\eta(\xi_j^2+\xi_k^2)}{(|\xi|^2-\delta)^2} \le \frac{5\eta(\xi_j^2+\xi_k^2)}{4(|\xi|^2-\delta)}, \quad 2\xi_j\xi_k \le \xi_j^2+\xi_k^2. \tag{2.23}$$

(2.23) together with $R(u_1, u_2, u_3) = 0$ and Gagliardo-Nirenberg inequality

$$\|u_i\|^4_{L^4(\mathbb{R}^3)} \leq c\|\nabla u_i\|^3_{L^2(\mathbb{R}^3)}\|u_i\|_{L^2(\mathbb{R}^3)}, \quad u_i \in H^1(\mathbb{R}^3),\ i = 1, 2, 3$$

yields

$$\begin{aligned}
&\int_{\mathbb{R}^3}(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)\mathrm{d}x \\
=& \frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^4 + |u_2|^4 + |u_3|^4)\mathrm{d}x \\
&+ \frac{3}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2 + |u_1|^2|u_3|^2 + |u_2|^2|u_3|^2)\mathrm{d}x \\
&+ \frac{3}{4}\int_{\mathbb{R}^3} F_1(u_1, u_2, u_3)\mathrm{d}\xi - \frac{1}{2}\delta\int_{\mathbb{R}^3}\frac{F_1(u_1, u_2, u_3)}{|\xi|^2 - \delta}\mathrm{d}\xi \\
&- \frac{3}{2}\mathrm{Re}\int_{\mathbb{R}^3} F_2(u_1, u_2, u_3)\mathrm{d}\xi + \delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(u_1, u_2, u_3)}{|\xi|^2 - \delta}\mathrm{d}\xi \\
\leq& C\int_{\mathbb{R}^3}(|u_1|^4 + |u_2|^4 + |u_3|^4)\mathrm{d}x \\
\leq& C\Big\{\Big(\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\Big)^{\frac{3}{2}}\Big(\int_{\mathbb{R}^3}|u_1|^2\mathrm{d}x\Big)^{\frac{1}{2}} \\
&+ \Big(\int_{\mathbb{R}^3}|\nabla u_2|^2\mathrm{d}x\Big)^{\frac{3}{2}}\Big(\int_{\mathbb{R}^3}|u_2|^2\mathrm{d}x\Big)^{\frac{1}{2}} \\
&+ \Big(\int_{\mathbb{R}^3}|\nabla u_2|^2\mathrm{d}x\Big)^{\frac{3}{2}}\Big(\int_{\mathbb{R}^3}|u_2|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big\} \\
\leq& C\Big(\int_{\mathbb{R}^3}(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)\mathrm{d}x\Big)^{\frac{3}{2}} \\
&\cdot\Big(\int_{\mathbb{R}^3}(|u_1|^2 + |u_2|^2 + |u_3|^2)\mathrm{d}x\Big)^{\frac{1}{2}}.
\end{aligned} \tag{2.24}$$

This implies

$$\Big(\int_{\mathbb{R}^3}(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}(|u_1|^2 + |u_2|^2 + |u_3|^2)\mathrm{d}x\Big)^{\frac{1}{2}} \geq C. \tag{2.25}$$

Combining Young's inequality with (2.10) and (2.23)–(2.25) claims

$$S(u_1, u_2, u_3) \geq C > 0. \tag{2.26}$$

This finishes the proof.

We further claim two identities.

Let

$$\begin{aligned}
\Sigma := \{&(E_1, E_2, E_3) : (|x|E_1, |x|E_2, |x|E_3) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\} \\
&\cap H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3).
\end{aligned}$$

Then we have the following Lemma.

Lemma 2.2 *Let* $(E_{10}, E_{20}, E_{30}) \in \Sigma$ *and* $(E_1, E_2, E_3) \in C([0, T); \Sigma)$ *be a solution to the Cauchy problem* (1.1)–(1.4) *on* $[0, T)$. *Putting*

$$J(t) =: \int_{\mathbb{R}^3}|x|^2(|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x, \tag{2.27}$$

one gets the following two identities:

$$S(E_1, E_2, E_3) = S(E_{10}, E_{20}, E_{30}), \tag{2.28}$$

$$\frac{\mathrm{d}^2 J(t)}{\mathrm{d}t^2} = 8R(E_1, E_2, E_3). \tag{2.29}$$

For convenience, we will prove this lemma in Section 4.

3 Sharp Threshold of Global Existence in $\mathbb{R}^3$

The goal of this section is to establish a sharp threshold for global existence to the Cauchy problem (1.1)–(1.4). Motivated by the works [1–2], our arguments are based on the local well-posedness theory established in Section 2.1. The main result reads as follows.

Theorem 3.1 *For $\eta > 0$ and $\delta \le 0$, suppose that $(E_{10}(x), E_{20}(x), E_{30}(x)) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and*

$$S(E_{10}, E_{20}, E_{30}) < d, \tag{3.1}$$

where d is defined by (2.16)*. Then*

(I) *Let $(|x|E_{10}, |x|E_{20}, |x|E_{30}) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfy*

$$R(E_{10}, E_{20}, E_{20}) < 0. \tag{3.2}$$

Then the solution $(E_1(t,x), E_2(t,x), E_3(t,x))$ to the Cauchy problem (1.1)–(1.4) *blows up in finite time.*

(II) *Let*

$$R(E_{10}, E_{20}, E_{30}) > 0. \tag{3.3}$$

Then the solution $(E_1(t,x), E_2(t,x), E_3(t,x))$ to the Cauchy problem (1.1)–(1.4) *exists globally on $t \in [0, +\infty)$. In addition, $(E_1(t,x), E_2(t,x), E_3(t,x))$ satisfies that for any $t \in [0, +\infty)$,*

$$\begin{aligned} &\omega(\|E_1(t)\|^2_{L^2(\mathbb{R}^3)} + \|E_2(t)\|^2_{L^2(\mathbb{R}^3)} + \|E_3(t)\|^2_{L^2(\mathbb{R}^3)}) \\ &+ \frac{1}{3}(\|\nabla E_1(t)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla E_2(t)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla E_3(t)\|^2_{L^2(\mathbb{R}^3)}) \\ &\le 2d. \end{aligned} \tag{3.4}$$

We begin with a lemma in [23] to prove Theorem 3.1.

Lemma 3.1 (see [23]) *For a scalar-valued function f, let $|x|f$ and ∇f belong to $L^2(\mathbb{R}^3)$. Then $f \in L^2(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} |f|^2 \mathrm{d}x \le \frac{2}{3}\Big(\int_{\mathbb{R}^3} |\nabla f|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^3} |x|^2 |f|^2 \mathrm{d}x\Big)^{\frac{1}{2}}.$$

The following two propositions play an important role in the proof of Theorem 3.1.

Proposition 3.1 *For $\eta > 0$ and $\delta \le 0$, let*

$$K_1 = \{(u_1, u_2, u_3) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3),\ R(u_1, u_2, u_3) > 0,\ S(u_1, u_2, u_3) < d\},$$
$$K_2 = \{(u_1, u_2, u_3) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3),\ R(u_1, u_2, u_3) < 0,\ S(u_1, u_2, u_3) < d\}.$$

Then K_1 and K_2 are invariant under the flow generated by the Cauchy problem (1.1)–(1.4).

Proof Let $(E_{10}, E_{20}, E_{30}) \in K_1$ and $(E_1(t), E_2(t), E_3(t))$ be the solution to the Cauchy problem (1.1)–(1.4) with initial data $(E_{10}(x), E_{20}(x), E_{30}(x))$. With (2.4)–(2.5) we conclude for $t \in [0, T)$,

$$S(E_1(t), E_2(t), E_3(t)) = S(E_{10}, E_{20}, E_{30}). \tag{3.5}$$

This together with $S(E_{10}, E_{20}, E_{30}) < d$ yields

$$S(E_1(t), E_2(t), E_3(t)) < d. \tag{3.6}$$

We are now in a position to show for $t \in [0, T)$,

$$R(E_1(t), E_2(t), E_3(t)) > 0. \tag{3.7}$$

This will be shown by contradiction. If (3.7) is not true, by continuity, $\eta > 0$, $\delta \le 0$ and $R(E_{10}, E_{20}, E_{30}) > 0$, there exists a $t^* \in (0, T)$ such that

$$R(E_1(t^*), E_2(t^*), E_3(t^*)) = 0, \quad (E_1(t^*), E_2(t^*), E_3(t^*)) \ne (0, 0, 0). \tag{3.8}$$

That is $(E_1(t^*), E_2(t^*), E_3(t^*)) \in M$. On the other hand, (3.6) yields

$$S(E_1(t^*), E_2(t^*), E_3(t^*)) < d.$$

This contradicts (2.16) and Proposition 2.2. Therefore, K_1 is invariant under the flow generated by the Cauchy problem (1.1)–(1.4).

Similarly, we can prove that K_2 is also invariant under the flow generated by the Cauchy problem (1.1)–(1.4).

We further state the following estimate.

Proposition 3.2 *For $\eta > 0$, $\delta \le 0$ and $\lambda > 0$, let $u_{i\lambda}(x) = \lambda^{\frac{3}{2}} u_i(\lambda x)$ $(i = 1, 2, 3)$, and let $(u_1, u_2, u_3) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \setminus \{(0, 0, 0)\}$ with $(u_1, u_2, u_3) \in K_2$ (K_2 is defined by Proposition* 3.1). *Then there exists $0 < \mu < 1$ such that $R(u_{1\mu}, u_{2\mu}, u_{3\mu}) = 0$ and*

$$S(u_1, u_2, u_3) - S(u_{1\mu}, u_{2\mu}, u_{3\mu}) \ge \frac{1}{2} R(u_1, u_2, u_3). \tag{3.9}$$

Here, $S(u_1, u_2, u_3)$ and $R(u_1, u_2, u_3)$ are defined by (2.14)–(2.15), *respectively.*

Proof By (2.14)–(2.15), it is easy to write down the expressions of $S(u_{1\lambda}, u_{2\lambda}, u_{3\lambda})$ and $R(u_{1\lambda}, u_{2\lambda}, u_{3\lambda})$,

$$\begin{aligned} &S(u_{1\lambda}, u_{2\lambda}, u_{3\lambda}) \\ = &\frac{1}{2} \lambda^2 \int_{\mathbb{R}^3} (|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{\omega}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)\mathrm{d}x\\
&-\frac{1}{4}\lambda^3\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&-\frac{1}{2}\lambda^3\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{1}{4}\lambda^3\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_1^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi\\
&-\frac{1}{4}\lambda^3\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_3^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi\\
&-\frac{1}{4}\lambda^3\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_1^2+\xi_3^2)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi\\
&+\frac{1}{2}\lambda^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_1\xi_2}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi\\
&+\frac{1}{2}\lambda^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_2\xi_3}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi\\
&+\frac{1}{2}\lambda^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_1\xi_3}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi, \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
&R(u_{1\lambda},u_{2\lambda},u_{3\lambda})\\
=\,&\lambda^2\Big\{\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x\\
&-\frac{3}{4}\lambda\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&-\frac{3}{2}\lambda\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&+\frac{3}{2}\lambda\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_1\xi_2}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi\\
&+\frac{3}{2}\lambda\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_2\xi_3}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi\\
&+\frac{3}{2}\lambda\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_1\xi_3}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi\\
&-\lambda\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_1\xi_2}{(\lambda^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi\\
&-\lambda\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_2\xi_3^2}{(\lambda^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi\\
&-\lambda\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\lambda^2\xi_1\xi_3}{(\lambda^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi\\
&-\frac{3}{4}\lambda\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_1^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi\\
&-\frac{3}{4}\lambda\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_3^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi\\
&-\frac{3}{4}\lambda\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_1^2+\xi_3^2)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\delta}{2}\lambda\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_1^2+\xi_2^2)}{(\lambda^2|\xi|^2-\delta)^2}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi\\
&+\frac{\delta}{2}\lambda\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_3^2+\xi_2^2)}{(\lambda^2|\xi|^2-\delta)^2}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi\\
&+\frac{\delta}{2}\lambda\int_{\mathbb{R}^3}\frac{\eta\lambda^2(\xi_1^2+\xi_3^2)}{(\lambda^2|\xi|^2-\delta)^2}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi\Big\}\\
=\;&\lambda^2Q_\lambda(u_1,u_2,u_3). \qquad (3.11)
\end{aligned}
$$

From $(u_1,u_2,u_3)\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\setminus\{(0,0,0)\}$ and $(u_1,u_2,u_3)\in K_2$, it follows that

$$Q_\lambda(u_1,u_2,u_3)>0 \quad \text{as } \lambda\to 0 \quad \text{and} \quad Q_\lambda(u_1,u_2,u_3)<0 \quad \text{as } \lambda\to 1.$$

By continuity, there exists $0<\mu<1$ such that $Q_\mu(u_1,u_2,u_3)=0$, which together with (3.11) implies for $0<\mu<1$,

$$R(u_{1\mu},u_{2\mu},u_{3\mu})=0. \qquad (3.12)$$

On the other hand, combining (2.14) with (3.10)–(3.12) yields

$$
\begin{aligned}
&S(u_1,u_2,u_3)-S(u_{1\mu},u_{2\mu},u_{3\mu})\\
=\;&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x\\
&-\frac{1}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&-\frac{1}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{1}{4}\int_{\mathbb{R}^3}F_1(u_1,u_2,u_3)\mathrm{d}\xi+\frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3}F_2(u_1,u_2,u_3)\mathrm{d}\xi\\
&-\frac{\mu^2}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x\\
&+\frac{\mu^3}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&+\frac{u^3}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{1}{2}\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_2}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi\\
&-\frac{1}{2}\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_2\xi_3}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi\\
&-\frac{1}{2}\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_3}{\mu^2|\xi|^2-\delta}|\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi\\
&+\frac{1}{4}\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_2^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi\\
&+\frac{1}{4}\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_3^2+\xi_2^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi\\
&+\frac{1}{4}\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_3^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}F_1(u_1,u_2,u_3)\mathrm{d}\xi + \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3}F_2(u_1,u_2,u_3)\mathrm{d}\xi \\
&\quad + \Big(\frac{3}{4}-\frac{1}{2}\Big)\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_2}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi \\
&\quad + \Big(\frac{3}{4}-\frac{1}{2}\Big)\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_2\xi_3}{u^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi \\
&\quad + \Big(\frac{3}{4}-\frac{1}{2}\Big)\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_3}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi \\
&\quad - \frac{\mu^3\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_3}{(\mu^2|\xi|^2-\delta)^2}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi \\
&\quad - \frac{\mu^3\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_2}{(\mu^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi \\
&\quad - \frac{\mu^3\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_2\xi_3}{(\mu^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi \\
&\quad - \Big(\frac{3}{8}-\frac{1}{4}\Big)\mu^3\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x \\
&\quad - \Big(\frac{3}{4}-\frac{1}{2}\Big)\mu^3\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x \\
&\quad - \Big(\frac{3}{8}-\frac{1}{4}\Big)\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_2^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi \\
&\quad - \Big(\frac{3}{8}-\frac{1}{4}\Big)\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_2^2+\xi_3^2)}{|\mu^2\xi|^2-\delta}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi \\
&\quad - \Big(\frac{3}{8}-\frac{1}{4}\Big)\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_3^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi \\
&\quad + \frac{\mu^3\delta}{4}\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_2^2)}{(\mu^2|\xi|^2-\delta)^2}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi \\
&\quad + \frac{\mu^3\delta}{4}\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_2^2+\xi_3^2)}{(\mu^2|\xi|^2-\delta)^2}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi \\
&\quad + \frac{\mu^3\delta}{4}\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_3^2)}{(\mu^2|\xi|^2-\delta)^2}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x \\
&\quad - \frac{3}{8}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x \\
&\quad - \frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}F_1(u_1,u_2,u_3)\mathrm{d}\xi + \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3}F_2(u_1,u_2,u_3)\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}\mu^3\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x\\
&-\frac{1}{8}\mu^3\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{1}{8}\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_2^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi\\
&-\frac{1}{8}\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_3^2)}{|\mu^2\xi|^2-\delta}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi\\
&-\frac{1}{8}\mu^3\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_2^2+\xi_3^2)}{\mu^2|\xi|^2-\delta}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi\\
&+\frac{\mu^3\delta}{4}\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_2^2)}{(\mu^2|\xi|^2-\delta)^2}|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2\mathrm{d}\xi\\
&+\frac{\mu^3\delta}{4}\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_2^2+\xi_3^2)}{(\mu^2|\xi|^2-\delta)^2}|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2\mathrm{d}\xi\\
&+\frac{\mu^3\delta}{4}\int_{\mathbb{R}^3}\frac{\eta\mu^2(\xi_1^2+\xi_3^2)}{(\mu^2|\xi|^2-\delta)^2}|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2\mathrm{d}\xi\\
&+\frac{1}{4}\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_2}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi\\
&+\frac{1}{4}\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_2\xi_3}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_3-u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi\\
&+\frac{1}{4}\mu^3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_3}{\mu^2|\xi|^2-\delta}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi\\
&-\frac{\mu^3\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_3}{(\mu^2|\xi|^2-\delta)^2}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}\mathrm{d}\xi\\
&-\frac{\mu^3\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_1\xi_2}{(\mu^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_2u_3-u_2\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}\mathrm{d}\xi\\
&-\frac{\mu^3\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\mu^2\xi_2\xi_3}{(\mu^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u}_1u_2-u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)}\mathrm{d}\xi. \qquad (3.13)
\end{aligned}
$$

For $j,k=1,2,3$, let

$$
G_1(\mu)=\frac{\eta\mu^5(\xi_j^2+\xi_k^2)}{\mu^2|\xi|^2-\delta},\quad G_2(\mu)=\frac{\eta\mu^5(\xi_j^2+\xi_k^2)}{(\mu^2|\xi|^2-\delta)^2}.
$$

Then

$$
G_1'(\mu)=\frac{\eta\mu^4(\xi_j^2+\xi_k^2)(3\mu^2|\xi|^2-5\delta)}{(\mu^2|\xi|^2-\delta)^2},
$$

which together with $\delta\le 0$ and $\eta>0$ implies $G_1'(\mu)\ge 0$ for $\mu>0$. That is, $G_1(\mu)$ is an increasing function of μ. Similarly, we can check that $G_2(\mu)$ is also an increasing function with respect to μ. Hence, Young's inequality and (3.13) tell us for $\mu\in(0,1)$,

$$
\begin{aligned}
&S(u_1,u_2,u_3)-S(u_{1u},u_{2u},u_{3u})\\
\ge{}&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)\mathrm{d}x\\
&-\frac{3}{8}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)\mathrm{d}x\\
&-\frac{3}{8}\int_{\mathbb{R}^3}F_1(u_1,u_2,u_3)\mathrm{d}\xi+\frac{3}{4}\mathrm{Re}\int_{\mathbb{R}^3}F_2(u_1,u_2,u_3)\mathrm{d}\xi\\
&+\frac{\delta}{4}\int_{\mathbb{R}^3}\frac{F_1(u_1,u_2,u_3)}{(|\xi|^2-\delta)}\mathrm{d}\xi-\frac{\delta}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(u_1,u_2,u_3)}{(|\xi|^2-\delta)}\mathrm{d}\xi\\
=&\frac{1}{2}R(u_1,u_2,u_3).
\end{aligned}
$$

This completes the proof of Proposition 3.2.

We are now in a position to give the proof of Theorem 3.1.

Proof of Theorem 3.1 We first show (I) by contradiction. Suppose that the maximal existence time T of the solution $(E_1(t),E_2(t),E_3(t))$ to the Cauchy problem (1.1)–(1.4) is infinity. Then from Proposition 3.1 and (3.1)–(3.2), it follows that

$$R(E_1(t),E_2(t),E_3(t))<0,\quad S(E_1(t),E_2(t),E_3(t))<d. \tag{3.14}$$

On the other hand, for $i=1,2,3$, $|x|E_{i0}\in L^2(\mathbb{R}^3)$ yields $|x|E_i\in L^2(\mathbb{R}^3)$ (see [6]), then Lemma 2.2 yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{R}^3}|x^2|(|E_1|^2+|E_2|^2+|E_3|^2)\mathrm{d}x=8R(E_1,E_2,E_3). \tag{3.15}$$

Let $\mu>0$ be such that

$$R(E_{1\mu},E_{2\mu},E_{3\mu})=0, \tag{3.16}$$

where $(E_{1\mu},E_{2\mu},E_{3\mu})$ is defined by Proposition 3.2. $R(E_1,E_2,E_3)<0$ implies $0<\mu<1$ from Proposition 3.2. In addition, by (2.15)–(2.16), (3.5), (3.16) and Proposition 2.2, we have

$$S(E_1,E_2,E_3)=S(E_{10},E_{20},E_{30}),\quad S(E_{1\mu},E_{2\mu},E_{3\mu})\geq d, \tag{3.17}$$

which together with Proposition 3.2 and (3.14) leads to

$$R(E_1,E_2,E_3)\leq 2[S(E_{10},E_{20},E_{30})-d]<0. \tag{3.18}$$

Let

$$\beta=2[S(E_{10},E_{20},E_{30})-d]<0 \tag{3.19}$$

and

$$J(t)=\int_{\mathbb{R}^3}|x|^2(|E_1|^2+|E_2|^2+|E_3|^2)\mathrm{d}x. \tag{3.20}$$

Then we observe by (3.15) and (3.18) that

$$\frac{\mathrm{d}^2J(t)}{\mathrm{d}t^2}=8R(E_1,E_2,E_3)\leq 8\beta<0. \tag{3.21}$$

Integrating (3.21) twice with respect to t, we obtain

$$J(t)\leq 8\beta t^2+J'(0)t+J(0). \tag{3.22}$$

Hence for (E_1, E_2, E_3) remaining in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, there must exist a $T^* < +\infty$ such that

$$\lim_{t\to T^*} \int_{\mathbb{R}^3} |x|^2(|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x = 0. \tag{3.23}$$

On the other hand, (2.4) and Lemma 3.1 yield

$$\begin{aligned}&\int_{\mathbb{R}^3} (|E_{10}|^2 + |E_{20}|^2 + |E_{30}|^2)\mathrm{d}x\\ &= \int_{\mathbb{R}^3} (|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x\\ &\le \frac{4}{3}\Big(\int_{\mathbb{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)\mathrm{d}x\Big)^{\frac{1}{2}}\\ &\quad\cdot\Big(\int_{\mathbb{R}^3} |x^2|(|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x\Big)^{\frac{1}{2}},\end{aligned} \tag{3.24}$$

which together with (3.23) concludes

$$\lim_{t\to T^*} \int_{\mathbb{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)\mathrm{d}x = +\infty.$$

This completes the proof of Theorem 3.1(I).

We are now in a position to give the proof of Theorem 3.1(II). Since $(E_{10}, E_{20}, E_{30}) \in K_1$ from (3.1) and (3.3), by Proposition 3.1, we have

$$(E_1(t), E_2(t), E_3(t)) \in K_1 \quad \text{for all } t \in [0, T), \tag{3.25}$$

where $(E_1(t), E_2(t), E_3(t))$ is the solution to the Cauchy problem (1.1)–(1.4). In other words,

$$R(E_1(t), E_2(t), E_3(t)) > 0, \quad S(E_1(t), E_2(t), E_3(t)) < d. \tag{3.26}$$

Then (2.4)–(2.5) conclude

$$\begin{aligned}&\omega \int_{\mathbb{R}^3} (|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x\\ &+ \frac{1}{3}\int_{\mathbb{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)\mathrm{d}x\\ &- \frac{\delta}{3}\int_{\mathbb{R}^3} \frac{\eta(\xi_1^2 + \xi_2^2)}{(|\xi|^2 - \delta)^2}|\mathcal{F}(u_1\overline{u}_2 - \overline{u}_1u_2)|^2\mathrm{d}\xi\\ &- \frac{\delta}{3}\int_{\mathbb{R}^3} \frac{\eta(\xi_2^2 + \xi_3^2)}{(|\xi|^2 - \delta)^2}|\mathcal{F}(u_2\overline{u}_3 - \overline{u}_2u_3)|^2\mathrm{d}\xi\\ &- \frac{\delta}{3}\int_{\mathbb{R}^3} \frac{\eta(\xi_1^2 + \xi_3^2)}{(|\xi|^2 - \delta)^2}|\mathcal{F}(u_1\overline{u}_3 - \overline{u}_1u_3)|^2\mathrm{d}\xi\\ &+ \frac{2\delta}{3}\mathrm{Re}\int_{\mathbb{R}^3} \frac{\eta\xi_1\xi_2}{(|\xi|^2 - \delta)^2}\mathcal{F}(\overline{u}_2u_3 - \overline{u}_3u_2)\overline{\mathcal{F}(u_1\overline{u}_3 - \overline{u}_1u_3)}\mathrm{d}\xi\\ &+ \frac{2\delta}{3}\mathrm{Re}\int_{\mathbb{R}^3} \frac{\eta\xi_2\xi_3}{(|\xi|^2 - \delta)^2}\mathcal{F}(\overline{u}_1u_3 - u_1\overline{u}_3)\overline{\mathcal{F}(u_1\overline{u}_2 - \overline{u}_1u_2)}\mathrm{d}\xi\\ &+ \frac{2\delta}{3}\mathrm{Re}\int_{\mathbb{R}^3} \frac{\eta\xi_1\xi_3}{(|\xi|^2 - \delta)^2}\mathcal{F}(\overline{u}_1u_2 - u_1\overline{u}_2)\overline{\mathcal{F}(u_2\overline{u}_3 - \overline{u}_2u_3)}\mathrm{d}\xi\end{aligned}$$

$$< 2d. \tag{3.27}$$

As $\eta > 0$ and $\delta \leqslant 0$, by Young's inequality we have

$$\begin{aligned}&\omega \int_{\mathbb{R}^3} (|E_1|^2 + |E_2|^2 + |E_3|^2)\mathrm{d}x \\ &+ \frac{1}{3}\int_{\mathbb{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)\mathrm{d}x \\ &< 2d.\end{aligned} \tag{3.28}$$

This leads to the boundedness of $(E_1(t), E_2(t), E_3(t))$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ for any $t \in [0, T)$. In particular, there must be $T = +\infty$ by the continuity argument. Therefore, the solution $(E_1(t), E_2(t), E_3(t))$ to the Cauchy problem (1.1)–(1.4) exists globally on $t \in [0, +\infty)$, and the estimate (3.4) follows immediately from (3.28).

This finishes the proof of Theorem 3.1(II).

4 Appendix

In this section, we give the detailed proof of Lemma 2.2.

Proof of Lemma 2.2 According to (2.4)–(2.5), noting the expression of functional S (see (2.14)) and (2.16)–(2.17), we can establish easily the identity (2.28).

We are now in the position to check (2.29).

Since $(E_{10}, E_{20}, E_{30}) \in \Sigma$, one has $(E_1, E_2, E_3) \in \Sigma$ by Ginibre and Velo [6], where $(E_1, E_2, E_3) \in C([0, T); \Sigma)$ is a solution to the Cauchy problem (1.1)–(1.4). In view of (1.1)–(1.3) and (2.27), one has

$$\begin{aligned}\frac{\mathrm{d}J(t)}{\mathrm{d}t} &= \int_{\mathbb{R}^3} |x|^2[(\partial_t E_1\overline{E}_1 + E_1\partial_t\overline{E}_1) + (\partial_t E_2\overline{E}_2 + E_2\partial_t\overline{E}_2) + (\partial_t E_3\overline{E}_3 + E_3\partial_t\overline{E}_3)]\mathrm{d}x \\ &= 2\mathrm{Im}\int_{\mathbb{R}^3} |x|^2(\mathrm{i}\partial_t E_1\overline{E}_1 + \mathrm{i}\partial_t E_2\overline{E}_2 + \mathrm{i}\partial_t E_3\overline{E}_3)\mathrm{d}x \\ &= 2\mathrm{Im}\int_{\mathbb{R}^3} |x|^2(-\Delta E_1\overline{E}_1 - \Delta E_2\overline{E}_2 - \Delta E_3\overline{E}_3)\mathrm{d}x \\ &\quad - 2\mathrm{Im}\int_{\mathbb{R}^3} |x|^2(|E_1|^2 + |E_2|^2 + |E_3|^2)(E_1\overline{E}_1 + E_2\overline{E}_2 + E_3\overline{E}_3)\mathrm{d}x \\ &\quad - 2\mathrm{Im}\int_{\mathbb{R}^3} |x|^2(\overline{E}_1A_1 + \overline{E}_1A_2 + \overline{E}_2B_1 + \overline{E}_2B_2 + \overline{E}_3C_1 + \overline{E}_3C_2)\mathrm{d}x,\end{aligned} \tag{A-1}$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ have the same forms in (1.5)–(1.10). We now need to calculate these terms in (A-1) carefully.

$$\begin{aligned}&\mathrm{Im}\int_{\mathbb{R}^3} |x|^2(-\Delta E_1\overline{E}_1 - \Delta E_2\overline{E}_2 - \Delta E_3\overline{E}_3)\mathrm{d}x \\ &= 2\mathrm{Im}\int_{\mathbb{R}^3} (x\nabla E_1\overline{E}_1 + x\nabla E_2\overline{E}_2 + x\nabla E_3\overline{E}_3)\mathrm{d}x,\end{aligned} \tag{A-2}$$

$$\mathrm{Im}\int_{\mathbb{R}^3} |x|^2(|E_1|^2 + ||E_2|^2 + |E_3|^2)(E_1\overline{E}_1 + E_2\overline{E}_2 + E_3\overline{E}_3)\mathrm{d}x = 0, \tag{A-3}$$

$$\mathrm{Im}\int_{\mathbb{R}^3} |x|^2\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2 - \delta}\mathcal{F}(E_2\overline{E}_3 - \overline{E}_2E_3)\Big]\mathrm{d}x$$

$$
\begin{aligned}
&+\mathrm{Im}\int_{\mathbb{R}^3}|x|^2\overline{E}_2E_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big]\mathrm{d}x\\
&-\mathrm{Im}\int_{\mathbb{R}^3}|x|^2E_1\overline{E}_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
&-\mathrm{Im}\int_{\mathbb{R}^3}|x|^2E_2\overline{E}_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big]\mathrm{d}x\\
=&\ \mathrm{Im}\int_{\mathbb{R}^3}|x|^2\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}|x|^2E_1\overline{E}_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\Big]\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}|x|^2\overline{E}_2E_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big]\mathrm{d}x\\
&+\mathrm{Im}\int_{\mathbb{R}^3}|x|^2E_2\overline{E}_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\Big]\mathrm{d}x\\
=&\ 2\mathrm{Im}\Big\{\mathrm{Re}\int_{\mathbb{R}^3}|x|^2\overline{E}_1E_2\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big]\mathrm{d}x\Big\}\\
&+2\mathrm{Im}\Big\{\mathrm{Re}\int_{\mathbb{R}^3}|x|^2\overline{E}_2E_3\mathcal{F}^{-1}\Big[\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big]\mathrm{d}x\Big\}\\
=&\ 0.
\end{aligned} \tag{A-4}
$$

Making the same estimate as (A-4) to the rest terms in

$$
-\mathrm{Im}\int_{\mathbb{R}^3}|x|^2(\overline{E}_1A_1+\overline{E}_1A_2+\overline{E}_2B_1+\overline{E}_2B_2+\overline{E}_3C_1+\overline{E}_3C_2)\mathrm{d}x,
$$

we can obtain

$$
-\mathrm{Im}\int_{\mathbb{R}^3}|x|^2(\overline{E}_1A_1+\overline{E}_1A_2+\overline{E}_2B_1+\overline{E}_2B_2+\overline{E}_3C_1+\overline{E}_3C_2)\mathrm{d}x=0. \tag{A-5}
$$

Combining (A-1)–(A-5) together leads to

$$
\frac{\mathrm{d}J(t)}{\mathrm{d}t}=4\mathrm{Im}\int_{\mathbb{R}^3}[(x\nabla E_1)\overline{E}_1+(x\nabla E_2)\overline{E}_2+(x\nabla E_3)\overline{E}_3]\mathrm{d}x. \tag{A-6}
$$

Differentiating (A-6) with respect to t, after a careful computation and proper groupings, we proceed as follows:

$$
\begin{aligned}
\frac{\mathrm{d}^2J(t)}{\mathrm{d}t^2}=&\ 4\mathrm{Im}\int_{\mathbb{R}^3}x[(\nabla E_1)\partial_t\overline{E}_1+\nabla(\partial_tE_1)\overline{E}_1]\mathrm{d}x\\
&+4\mathrm{Im}\int_{\mathbb{R}^3}x[(\nabla E_2)\partial_t\overline{E}_2+\nabla(\partial_tE_2)\overline{E}_2]\mathrm{d}x\\
&+4\mathrm{Im}\int_{\mathbb{R}^3}x[(\nabla E_3)\partial_t\overline{E}_3+\nabla(\partial_tE_3)\overline{E}_3]\mathrm{d}x\\
=&-8\mathrm{Im}\int_{\mathbb{R}^3}(x\partial_tE_1\nabla\overline{E}_1+x\partial_tE_2\nabla\overline{E}_2+x\partial_tE_3\nabla\overline{E}_3)\mathrm{d}x\\
&-12\mathrm{Im}\int_{\mathbb{R}^3}(\overline{E}_1\partial_tE_1+\overline{E}_2\partial_tE_2+\overline{E}_3\partial_tE_3)\mathrm{d}x\\
=&\ 8\mathrm{Re}\int_{\mathbb{R}^3}(x\mathrm{i}\partial_tE_1\nabla\overline{E}_1+x\mathrm{i}\partial_tE_2\nabla\overline{E}_2+x\mathrm{i}\partial_tE_3\nabla\overline{E}_3)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ 12\mathrm{Re}\int_{\mathbb{R}^3}(\overline{E}_1\mathrm{i}\partial_t E_1+\overline{E}_2\mathrm{i}\partial_t E_2+\overline{E}_3\mathrm{i}\partial_t E_3)\mathrm{d}x\\
=\,& 8\mathrm{Re}\int_{\mathbb{R}^3}x(-\Delta E_1\nabla\overline{E}_1-\Delta E_2\nabla\overline{E}_2-\Delta E_3\nabla\overline{E}_3)\mathrm{d}x\\
&- 8\mathrm{Re}\int_{\mathbb{R}^3}x(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\nabla\overline{E}_1+E_2\nabla\overline{E}_2+E_3\nabla\overline{E}_3)\mathrm{d}x\\
&+ 8\mathrm{Re}\int_{\mathbb{R}^3}[x\nabla\overline{E}_1(-A_1-A_2)+x\nabla\overline{E}_2(-B_1-B_2)+x\nabla\overline{E}_3(-C_1-C_2)]\mathrm{d}x\\
&+ 12\mathrm{Re}\int_{\mathbb{R}^3}(-\Delta E_1\overline{E}_1-\Delta E_2\overline{E}_2-\Delta E_3\overline{E}_3)\mathrm{d}x\\
&- 12\mathrm{Re}\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\overline{E}_1+E_2\overline{E}_2+E_3\overline{E}_3)\mathrm{d}x\\
&+ 12\mathrm{Re}\int_{\mathbb{R}^3}[\overline{E}_1(-A_1-A_2)+\overline{E}_2(-B_1-B_2)+\overline{E}_3(-C_1-C_2)]\mathrm{d}x. \qquad (A\text{-}7)
\end{aligned}
$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ have the same forms in (1.5)–(1.10). Using integration by parts, the Parseval identity and the properties of Fourier transforms, we attain the following estimates:

$$
\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^3}x(-\Delta E_1\nabla\overline{E}_1-\Delta E_2\nabla\overline{E}_2-\Delta E_3\nabla\overline{E}_3)\mathrm{d}x\\
=\,&-\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)\mathrm{d}x, \qquad (A\text{-}8)\\
&-\mathrm{Re}\int_{\mathbb{R}^3}x(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\nabla\overline{E}_1+E_2\nabla\overline{E}_2+E_3\nabla\overline{E}_3)\mathrm{d}x\\
=\,&\frac{3}{4}\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)\mathrm{d}x\\
&+\frac{3}{2}\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)\mathrm{d}x, \qquad (A\text{-}9)\\
&\mathrm{Re}\int_{\mathbb{R}^3}(-\Delta E_1\nabla\overline{E}_1-\Delta E_2\nabla\overline{E}_2-\Delta E_3\nabla\overline{E}_3)\mathrm{d}x\\
=\,&\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)\mathrm{d}x, \qquad (A\text{-}10)\\
&-\mathrm{Re}\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\overline{E}_1+E_2\overline{E}_2+E_3\overline{E}_3)\mathrm{d}x\\
=\,&-\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)\mathrm{d}x\\
&-2\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)\mathrm{d}x, \qquad (A\text{-}11)\\
&8\mathrm{Re}\int_{\mathbb{R}^3}[x\nabla\overline{E}_1(-A_1-A_2)+x\nabla\overline{E}_2(-B_1-B_2)+x\nabla\overline{E}_3(-C_1-C_2)]\mathrm{d}x\\
&+12\mathrm{Re}\int_{\mathbb{R}^3}[\overline{E}_1(-A_1-A_2)+\overline{E}_2(-B_1-B_2)+\overline{E}_3(-C_1-C_2)]\mathrm{d}x\\
=\,&-6\int_{\mathbb{R}^3}F_1(E_1,E_2,E_3)\mathrm{d}x+12\mathrm{Re}\int_{\mathbb{R}^3}F_2(E_1,E_2,E_3)\mathrm{d}x\\
&+4\delta\int_{\mathbb{R}^3}\frac{F_1(E_1,E_2,E_3)}{|\xi|^2-\delta}\mathrm{d}x-8\delta\int_{\mathbb{R}^3}\frac{F_2(E_1,E_2,E_3)}{|\xi|^2-\delta}\mathrm{d}x, \qquad (A\text{-}12)
\end{aligned}
$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ have the forms in (1.5)–(1.10) and F_1, F_2 are defined by (2.16) and

(2.17).

Before we show (A-12) one by one, we mention several key identities.

Lemma 4.1 *For $i, j, k, l, s, p = 1, 2, 3$, the following identities hold*:

$$\begin{aligned}
&\int_{\mathbb{R}^3} x\nabla\overline{E}_i E_j \mathcal{F}^{-1}\Big[\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\mathcal{F}(E_s\overline{E}_p)\Big]\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_i\overline{E}_j)}\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\mathcal{F}(E_s\overline{E}_p)\mathrm{d}\xi,
\end{aligned}\tag{A-13}$$

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(E_i\overline{E}_j-\overline{E}_iE_j)\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\overline{\mathcal{F}(E_s\overline{E}_i-\overline{E}_sE_i)}\mathrm{d}\xi\\
&=\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(\overline{E}_sE_i)}\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_iE_j-E_i\overline{E}_j)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_iE_j)\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_sE_i-E_s\overline{E}_i)\mathrm{d}\xi.\\
&\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_sE_j)\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\xi\cdot\partial_\xi\overline{\mathcal{F}(\overline{E}_iE_j-E_i\overline{E}_j)}\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_iE_j)\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\xi\cdot\partial_\xi\overline{\mathcal{F}(\overline{E}_sE_i-E_s\overline{E}_i)}\mathrm{d}\xi\\
&=\mathrm{Re}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_sE_j)\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_iE_j-E_i\overline{E}_j)}\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_iE_j)\frac{\eta\xi_k\xi_l}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_sE_i-E_s\overline{E}_i)}\mathrm{d}\xi.
\end{aligned}\tag{A-14}$$

We now calculate (A-12) term by term. From (1.5)–(1.10) it follows that

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^3}[\overline{E}_1(-A_1-A_2)+\overline{E}_2(-B_1-B_2)+\overline{E}_3(-C_1-C_2)]\mathrm{d}x\\
=&\,\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2)}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&-\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2)}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&-\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2)}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_3)}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_3)}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_3)}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3)}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3)}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3)}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2)}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2)}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2)}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_3)}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_3)}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_3)}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3)}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3)}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3)}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi. \qquad (A\text{-}15)
\end{aligned}
$$

Direct calculation and rearrangement for these terms in (A-15) yield

$$
(A\text{-}15)=-\int_{\mathbb{R}^3}F_1(E_1,E_2,E_3)\mathrm{d}\xi+2\mathrm{Re}\int_{\mathbb{R}^3}F_2(E_1,E_2,E_3)\mathrm{d}\xi, \qquad (A\text{-}16)
$$

where $F_1(E_1,E_2,E_3)$ and $F_2(E_1,E_2,E_3)$ are defined by (2.16) and (2.17).

We continue to calculate the other terms in (A-12):

$$
\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^3}[x\nabla\overline{E}_1(-A_1-A_2)+x\nabla\overline{E}_2(-B_1-B_2)+x\nabla\overline{E}_3(-C_1-C_2)]\mathrm{d}x\\
&=\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_1E_2\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big\}\mathrm{d}x\\
&-\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_1E_2\mathcal{F}^{-1}\Big\{\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big\}\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_1E_2\mathcal{F}^{-1}\Big\{\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\Big\}\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_1E_3\mathcal{F}^{-1}\Big\{\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\Big\}\mathrm{d}x\\
&-\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_1E_3\mathcal{F}^{-1}\Big\{\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\Big\}\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_1E_3\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\Big\}\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_2E_3\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\Big\}\mathrm{d}x\\
&-\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_2E_3\mathcal{F}^{-1}\Big\{\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big\}\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_2E_3\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big\}\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}^3}x\nabla\overline{E}_2E_1\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\Big\}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_2E_1\mathcal{F}^{-1}\Big\{\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\Big\}\mathrm{d}x\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_2E_1\mathcal{F}^{-1}\Big\{\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\Big\}\mathrm{d}x\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_3E_1\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\Big\}\mathrm{d}x\\
&- \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_3E_1\mathcal{F}^{-1}\Big\{\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\Big\}\mathrm{d}x\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_3E_1\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\Big\}\mathrm{d}x\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_3E_2\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\Big\}\mathrm{d}x\\
&- \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_3E_2\mathcal{F}^{-1}\Big\{\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\Big\}\mathrm{d}x\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} x\nabla\overline{E}_3E_2\mathcal{F}^{-1}\Big\{\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\Big\}\mathrm{d}x\\
=& \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_1\overline{E}_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&- \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_1\overline{E}_2)}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_1\overline{E}_2)}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_1\overline{E}_3)}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi\\
&- \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_1\overline{E}_3)}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_1\overline{E}_3)}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_2\overline{E}_3)}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\mathrm{d}\xi\\
&- \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_2\overline{E}_3)}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_2\overline{E}_3)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_2\overline{E}_1)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&- \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_2\overline{E}_1)}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_2\overline{E}_1)}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\mathrm{d}\xi\\
&+ \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_3\overline{E}_1)}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&- \mathrm{Re}\int_{\mathbb{R}^3} \overline{\mathcal{F}(x\nabla E_3\overline{E}_1)}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_3-E_1\overline{E}_3)\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_3\overline{E}_1)}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_3\overline{E}_2)}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_3-\overline{E}_1E_3)\mathrm{d}\xi\\
&-\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_3\overline{E}_2)}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&+\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_3\overline{E}_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi. \qquad (A\text{-}17)
\end{aligned}
$$

We first consider these terms including $\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}$. Using integration by parts, the Parseval identity and some properties of Fourier transforms, we derive that

$$
\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_1\overline{E}_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_2\overline{E}_3)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_2\overline{E}_1)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(x\nabla E_3\overline{E}_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi\\
&=\mathrm{Re}\int_{\mathbb{R}^3}-\mathrm{i}\partial_\xi\overline{\mathcal{F}(\nabla E_1\overline{E}_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}-\mathrm{i}\partial_\xi\overline{\mathcal{F}(\nabla E_2\overline{E}_3)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}-\mathrm{i}\partial_\xi\overline{\mathcal{F}(\nabla E_2\overline{E}_1)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}-\mathrm{i}\partial_\xi\overline{\mathcal{F}(\nabla E_3\overline{E}_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)\mathrm{d}\xi\\
&=\mathrm{Re}\int_{\mathbb{R}^3}\mathrm{i}\partial_\xi\mathcal{F}(\nabla\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\mathrm{i}\partial_\xi\mathcal{F}(\nabla\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\mathrm{i}\partial_\xi\mathcal{F}(\nabla\overline{E}_2E_1)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad+\mathrm{Re}\int_{\mathbb{R}^3}\mathrm{i}\partial_\xi\mathcal{F}(\nabla\overline{E}_3E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&=\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla E_1\overline{E}_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla E_2\overline{E}_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla\overline{E}_2E_1)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla E_2\overline{E}_1)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla\overline{E}_3E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla E_3\overline{E}_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi. \qquad (A\text{-}18)
\end{aligned}
$$

By rearranging these terms in (A-18), we get

$$
\begin{aligned}
(A\text{-}18)&=\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}(\partial_\xi\mathcal{F}(\nabla\overline{E}_1E_2)+\partial_\xi\mathcal{F}(\nabla E_2\overline{E}_1))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}(\partial_\xi\mathcal{F}(\nabla E_1\overline{E}_2)+\partial_\xi\mathcal{F}(E_1\nabla\overline{E}_2))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}(\partial_\xi\mathcal{F}(\nabla\overline{E}_2E_3)+\partial_\xi\mathcal{F}(\overline{E}_2\nabla E_3))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}(\partial_\xi\mathcal{F}(\nabla E_2\overline{E}_3)+\partial_\xi\mathcal{F}(E_2\nabla\overline{E}_3))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&=\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla(\overline{E}_1E_2))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla(E_1\overline{E}_2))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla(\overline{E}_2E_3))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad+\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\partial_\xi\mathcal{F}(\nabla(E_2\overline{E}_3))\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&=-\frac{1}{2}\int_{\mathbb{R}^3}\partial_\xi\cdot[\xi\mathcal{F}(\overline{E}_1E_2)]\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\partial_\xi\cdot[\xi\mathcal{F}(E_1\overline{E}_2)]\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\partial_\xi\cdot[\xi\mathcal{F}(\overline{E}_2E_3)]\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\partial_\xi\cdot[\xi\mathcal{F}(E_2\overline{E}_3)]\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&=-\frac{3}{2}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad-\frac{3}{2}\int_{\mathbb{R}^3}\mathcal{F}(E_1\overline{E}_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(E_1\overline{E}_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)}\mathrm{d}\xi\\
&\quad-\frac{3}{2}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad-\frac{3}{2}\int_{\mathbb{R}^3}\mathcal{F}(E_2\overline{E}_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\int_{\mathbb{R}^3} \xi\cdot\partial_\xi \mathcal{F}(E_2\overline{E}_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&= -3\mathrm{Re}\int_{\mathbb{R}^3}\overline{\mathcal{F}(\overline{E}_1E_2)}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)\mathrm{d}\xi\\
&\quad - 3\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad - \mathrm{Re}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad - \mathrm{Re}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi. \qquad (A\text{-}19)
\end{aligned}
$$

Let

$$
\begin{aligned}
G &= -\mathrm{Re}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad - \mathrm{Re}\int_{\mathbb{R}^3}\xi\cdot\partial_\xi\mathcal{F}(\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi. \qquad (A\text{-}20)
\end{aligned}
$$

Then

$$
\begin{aligned}
G &= \mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_1E_2)\partial_\xi\cdot\Big(\xi\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\Big)\mathrm{d}\xi\\
&\quad + \mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_2E_3)\partial_\xi\cdot\Big(\xi\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\Big)\mathrm{d}\xi\\
&= 3\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_1E_2)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad + 3\mathrm{Re}\int_{\mathbb{R}^3}\mathcal{F}(\overline{E}_2E_3)\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad - 2\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2)\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad - 2\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3)\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi\\
&\quad + \mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_1E_2)\xi\cdot\partial_\xi\overline{\mathcal{F}(\overline{E}_2E_3-E_2\overline{E}_3)}\mathrm{d}\xi\\
&\quad + \mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E}_2E_3)\xi\cdot\partial_\xi\overline{\mathcal{F}(\overline{E}_1E_2-E_1\overline{E}_2)}\mathrm{d}\xi. \qquad (A\text{-}21)
\end{aligned}
$$

In view of (A-13)–(A-14), we conclude

$$
\begin{aligned}
2G &= 3\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&\quad - 2\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi,
\end{aligned}
$$

which then gives

$$
\begin{aligned}
G &= \frac{3}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi\\
&\quad - \delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi. \qquad (A\text{-}22)
\end{aligned}
$$

Note that (A-19), we thus obtain

$$(A\text{-}18) = -\frac{3}{2}\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi - \delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(E_2\overline{E}_3-\overline{E}_2E_3)\overline{\mathcal{F}(E_1\overline{E}_2-\overline{E}_1E_2)}\mathrm{d}\xi. \quad (A\text{-}23)$$

Making the similar calculation for the other terms in (A-17) to that in the proof of (A-18), we get

$$(A\text{-}17) = \frac{3}{4}\int_{\mathbb{R}^3}F_1(E_1,E_2,E_3)\mathrm{d}\xi - \frac{3}{2}\mathrm{Re}\int_{\mathbb{R}^3}F_2(E_1,E_2,E_3)\mathrm{d}\xi + \frac{\delta}{2}\int_{\mathbb{R}^3}\frac{F_1(E_1,E_2,E_3)}{|\xi|^2-\delta}\mathrm{d}\xi - \delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(E_1,E_2,E_3)}{|\xi|^2-\delta}\mathrm{d}\xi. \quad (A\text{-}24)$$

From (A-7)–(A-12), (A-16) and (A-24), it follows that

$$\begin{aligned}\frac{\mathrm{d}^2J(t)}{\mathrm{d}t^2} &= 8\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)\mathrm{d}x - 6\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)\mathrm{d}x\\ &\quad - 12\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)\mathrm{d}x\\ &\quad - 6\int_{\mathbb{R}^3}F_1(E_1,E_2,E_3)\mathrm{d}\xi + 12\mathrm{Re}\int_{\mathbb{R}^3}F_2(E_1,E_2,E_3)\mathrm{d}\xi\\ &\quad + 4\delta\int_{\mathbb{R}^3}\frac{F_1(E_1,E_2,E_3)}{|\xi|^2-\delta}\mathrm{d}\xi - 8\delta\mathrm{Re}\int_{\mathbb{R}^3}\frac{F_2(E_1,E_2,E_3)}{|\xi|^2-\delta}\mathrm{d}\xi\\ &= 8R(E_1,E_2,E_3).\end{aligned}$$

This completes the proof of Lemma 2.2.

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References

[1] Berestycki, H., Gallouët, T. and Kavian, O., Équàtions de champs scalaires euclidiens nonlinéaires daus de plan, *C. R. Acad. Sci. Paris., Serie I*, **297**, 1983, 307–310.

[2] Berestycki, H. and Cazenave, T., Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéairees, *C. R. Acad. Sci. Paris*, **293**, 1981, 489–492.

[3] Cazenave, T., An Introduction to Nonlinear Schrodinger Equations (3rd Ed.), Textos de Metodos Matematicos, Universidade Federal do Rio de Janeiro, 1996.

[4] Dendy, R. O., Plasma Dynamics, Oxford University Press, 1990.

[5] Ginibre, J. and Ozawa, T., Long range scattering for nonlinear Schrödinger and Hartree equations in space dimensions $n \geq 2$, *Comm. Math. Phys.*, **151**, 1993, 619–645.

[6] Ginibre, J. and Velo, G., On a class of nonlinear Schrödinger equations I, The Cauchy problem, general case, *J. Funct. Anal.*, **32**, 1979, 1–32.

[7] Ginibre, J. and Velo, G., The global Cauchy problem for the nonlinear Schrödinger equation revisited, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* , **2**, 1985, 309–327.

[8] Glassey, R. T., On the blowing up of solution to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.*, **18**(9), 1977, 1794–1797.

[9] Hayashi, N., Nakamitsu, K. and Tsutsumi, M., On solutions of the initial value problem for the nonlinear Schrödinger equations, *J. Funct. Anal.*, **71**, 1987, 218–245.

[10] Hayashi, N. and Tsutsumi, Y., Scattering theory for Hartree type equations, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **46**, 1987, 187–213.

[11] Kato, T., On nonlinear Schrödinger equations, *Ann. Inst. Henri Poincaré,Physique Théorique*, **49**, 1987, 113–129.

[12] Kenig, C., Ponce, G. and Vega, L., Small solution to nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **10**, 1993, 255–288.

[13] Kono, M., Skoric, M. M. and Ter Haar, D., Spontaneous excitation of magnetic fields and collapse dynamics in a Langmuir plasma, *J. Plasma Phys.*, **26**, 1981, 123–146.

[14] Laurey, C., The Cauchy problem for a generalized Zakharov system, *Diffe. Integral Equ.*, **8**(1), 1995, 105–130.

[15] Liu, Y., Existence and blow-up of solutions of a nonlinear Pochhammer-Chree equation, *Indiana Univ. Math. J.*, 1996, **45**, 797–816.

[16] Miao, C. X., Harmonic Analysis and Applications to Partial Differential Equations, 2nd edition, Monographs on Modern Pure Mathematics, No. 89, Science Press, Beijing, 2004.

[17] Miao, C. X., The Modern Method of Nonlinear Wave Equations, Lectures in Contemporary Mathematics, No. 2., Science Press, Beijing, 2005.

[18] Miao, C. X. and Zhang, B., Harmonic Analysis Method of Partial Differential Equations, 2 edition, Monographs on Modern Pure Mathematics, No. 117, Science Press, Beijing, 2008.

[19] Nirenberg, L., On elliptic partial differential equations, *Ann. della Scuola Norm. Sup. Pisa*, **13**, 1959, 115–162.

[20] Ozawa, T. and Tsutsumi, Y., Blow-up of H^1 solution for the nonlinear Schrödinger equation, *J. Diff. Eq.*, **92**, 1991, 317–330.

[21] Ozawa, T. and Tsutsumi, Y., Blow-up of H^1 solutions for the one-dimension nonlinear Schrödinger equation with critical power nonlinearity, *Proc. A. M. S.*, **111**, 1991, 486–496.

[22] Strauss, W. A., Nonlinear Wave Equations, C. B. M. S., No. 73, Amer. Math. Soc., Providence, RI, 1989.

[23] Weinstein, M. I., Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.*, **87**, 1983, 567–576.

[24] Zakharov, V. E., The collapse of Langmuir waves, *Soviet Phys., JETP*, **35**, 1972, 908–914.

[25] Zakharov, V. E., Musher, S. L. and Rubenchik, A. M., Hamiltonian approach to the description of nonlinear plasma phenomena, *Physics Reports*, **129**(5), 1985, 285–366.

[26] Zhang, J., Sharp conditions of global existence for nonlinear Schrödinger and Klein-Gordon equations, *Nonlinear Analysis, TMA*, **48**, 2002, 191–207.

[27] Zhang, J., Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potentia, *Commun. in PDE*, **30**, 2005, 1429–1443.