Chin. Ann. Math. Ser. B
40(2), 2019, 199–212
DOI: 10.1007/s11401-019-0126-8

Chinese Annals of Mathematics, Series B

Strong Embeddability for Groups Acting on Metric Spaces*

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Abstract The strong embeddability is a notion of metric geometry, which is an intermediate property lying between coarse embeddability and property A. In this paper, the permanence properties of strong embeddability for groups acting on metric spaces are studied. The authors show that a finitely generated group acting on a finitely asymptotic dimension metric space by isometries whose K-stabilizers are strongly embeddable is strongly embeddable. Moreover, they prove that the fundamental group of a graph of groups with strongly embeddable vertex groups is also strongly embeddable.

Keywords Strong embeddability, Groups action, Graph of groups, Relative hyperbolic groups
2010 MR Subject Classification 20H15, 20E06, 20F65

1 Introduction

The notion of coarse embeddability was introduced by Gromov [9] in relation to the Novikov conjecture (see [8]) on the homotopy invariance of higher signatures for closed manifolds. Yu [19] subsequently proved that the coarse Baum-Connes conjecture holds for metric spaces with bounded geometry which are coarsely embeddable into a Hilbert space and in a particular case when this space is a finitely generated group with the word length metric, the Novikov conjecture holds for the group. In [19], Yu introduced a weak version of amenability for metric spaces, which he called property A, a discrete metric space with this property may be coarsely embeddable into a Hilbert space. Since the appearance of Yu's work, the permanence properties of coarse embeddability and property A have been intensively studied (see [2, 7, 13, 16–17]). In [7], Dadarlat and Guentner showed that property A is stable under group extensions, by a recent result of Arzhantseva and Tessera [1], coarse embeddability is not preserved under group extensions.

Recently, the notion of strong embeddability (see Definition 2.1 below) for metric spaces was introduced by Ji, Ogle and Ramsey [12], they showed that strong embeddability is implied by property A and implies coarse embeddability, so strong embeddability should share many permanence properties with coarse embeddability and property A. In [18], the authors showed

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*This work was supported by the National Natural Science Foundation of China (Nos. 11231002, 11771061).

that strong embeddability for metric spaces is coarsely invariant and it is closed under taking subspaces, direct products, direct limits and finite unions. Moreover, they proved a permanence property linking finite decomposition complexity and strong embeddability: A discrete metric space with bounded geometry has weak finite decomposition complexity with respect to strong embeddability if and only if itself is strongly embeddable.

The concept of stabilizer was introduced by Bell and Dranishnikov [3–5] for studying the permanence properties of the asymptotic dimension. In [16], Tu proved that a discrete group acting on a tree with finite quotient has property A if and only if the stabilizer of each vertex group has property A. By using the equivalent characterization of property A which given by Higson and Roe [10], Bell [2] extended this result to conclude that a group acting by isometries on a metric space with finite asymptotic dimension whose stabilizers have property A also has property A. As a result, Bell concluded a theorem of [16], a fundamental group of a finite graph of groups whose vertices groups have property A also has property A. By using Bell's arguments, a parallel statement concerning operator norm localization property was obtained by Chen and Wang [6] (In fact, Sako [14] proved that operator norm localization property is equivalent to property A for metric spaces with bounded geometry). On the other hand, the permanence property for coarse embeddability under relative hyperbolic groups was studied by Dadarlat and Guentner [7], they proved that a finitely generated group which is hyperbolic relative to a finite family of subgroups is coarsely embeddable if and only if each subgroup is coarsely embeddable.

Inspired by the above works, we investigate permanence properties of strong embeddability for groups acting on metric spaces. By using the technique of gluing spaces in [7], we first prove that certain infinite unions property holds for strong embeddability (see Theorem 3.1). By applying this infinite unions property, we show that a group acting on a metric space with finite asymptotic dimension by isometries whose stabilizers are strongly embeddable is also strongly embeddable (see Theorem 4.1). Moreover, we point out that our argument is also suitable for property A. Thus, our approach in this paper gives an alternative proof to Bell's. By the works of Bell [2] and Chen-Wang [6], we conclude that the fundamental group of a finite graph of groups whose vertices groups are strongly embeddable is also strongly embeddable (see Proposition 5.1). As a corollary, amalgamated free products and HNN extensions of groups from groups with strong embeddability will be strongly embeddable by the Bass-Serre Theory (see Corollary 5.1). In [12], Ji, Ogle and Ramsey proved that if a finitely generated group is hyperbolic relative to a finite family of subgroups, and if each subgroup is strongly embeddable, then the group itself is strongly embeddable. The main tool in their proof is relative property A (see [11]). In this paper, our approach to this result, which is based on the works of Osin [13] and Dadarlat-Guentner [7], hence, is quite different from [12] (see Theorem 5.1).

2 Preliminaries

Let X be a metric space, and let $|B_S(x)|$ denote the number of points of the ball $B_S(x)$.

Then X is said to have bounded geometry if for every $S > 0$ there exists a number $N > 0$ such that for every $x \in X$ we have $|B_S(x)| \leq N$. The strong embeddability is defined for the metric spaces with bounded geometry, therefore, throughout this paper, all metric spaces are assumed to be discrete with bounded geometry.

Let B be a Banach space and $B_1 = \{\xi \in B \mid \|\xi\| = 1\}$. For every $R > 0$ and $\epsilon > 0$, a map $\alpha : X \to B_1$ defined by $x \mapsto \alpha_x$ will be said to have (R, ϵ) variation if $d(x, y) \leq R$ implies $\|\alpha_x - \alpha_y\| \leq \epsilon$.

Definition 2.1 (see [12]) *Let X be a metric space. Then X is strongly embeddable if and only if for every $R, \epsilon > 0$, there exists a map $\beta : X \to \ell^2(X)_1$ satisfying*

(1) *β has (R, ϵ) variation;*

(2) $\lim\limits_{S\to\infty} \sup\limits_{x\in X} \sum\limits_{z\notin B_S(x)} |\beta_x(z)|^2 = 0.$

We give an equivalent formulation of strong embeddability which will be useful to prove the main theorem.

Lemma 2.1 *Let X be a metric space. Then X is strongly embeddable if and only if for every $R, \epsilon > 0$, there exists a Hilbert space $\mathscr{H}$ and a map $\xi : X \to \ell^2(X, \mathscr{H})_1$ satisfying*

(1) *ξ has (R, ϵ) variation;*

(2) $\lim\limits_{S\to\infty} \sup\limits_{x\in X} \sum\limits_{z\notin B_S(x)} \|\xi_x(z)\|^2 = 0.$

Proof Suppose that X is strongly embeddable. Let $\mathscr{H} = \mathbb{C}$, where $\mathbb{C}$ is the set of all complex numbers. We can conclude the result by the fact that $\ell^2(X) \cong \ell^2(X, \mathbb{C})$.

Conversely, let $R, \epsilon > 0$ be given. By assumption, there exists a Hilbert space $\mathscr{H}$ and a map ξ from X to $\ell^2(X, \mathscr{H})_1$ satisfying (1) and (2).

Define

$$\beta : X \to \ell^2(X)$$

by $\beta_x(w) = \|\xi_x(w)\|$ for any $x, w \in X$.

Notice that for each $x \in X$,

$$\|\beta_x\|^2 = \sum_{w\in X} |\beta_x(w)|^2 = \sum_{w\in X} \|\xi_x(w)\|^2 = \|\xi_x\|^2 = 1.$$

For any $x, x' \in X$, if $d(x, x') \leq R$,

$$\begin{aligned}
\|\beta_x - \beta_{x'}\|^2 &= \sum_{w\in X} |\beta_x(w) - \beta_{x'}(w)|^2 = \sum_{w\in X} |\|\xi_x(w)\| - \|\xi_{x'}(w)\||^2 \\
&\leq \sum_{w\in X} \|\xi_x(w) - \xi_{x'}(w)\|^2 \\
&= \|\xi_x - \xi_{x'}\|^2.
\end{aligned}$$

By (1), we have $\|\beta_x - \beta_{x'}\| \leq \epsilon$. Moreover, by (2) we also have

$$\lim_{S\to\infty} \sup_{x\in X} \sum_{z\notin B_S(x)} |\beta_x(z)|^2 = \lim_{S\to\infty} \sup_{x\in X} \sum_{z\notin B_S(x)} \|\xi_x(z)\|^2 = 0.$$

Definition 2.2 (see [12]) *A family of metric spaces $(X_i)_{i\in \mathrm{I}}$ is equi-strongly embeddable if for every $R, \epsilon > 0$ there exists a family of maps $\xi^i: X_i \to \ell^2(X_i)_1$ satisfying*

(1) *for any $i \in I$, ξ^i has (R, ϵ) variation;*

(2) $\lim_{S\to\infty} \sup_{i\in I} \sup_{x\in X_i} \sum_{z\notin B_S(x)} |\xi^i_x(z)|^2 = 0.$

Definition 2.3 *Let X be a metric space and let $\mathscr{U} = (U_i)_{i\in I}$ be a cover of X. A partition of unity is a family of continuous functions $\phi_i : X \to [0, 1]$ such that for every $x \in X$ there is a neighborhood V of x for which $\phi_i|_V \neq 0$ only for finitely many $i \in I$ and $\sum_{i\in I} \phi_i(x) = 1$ for every $x \in X$. A partition of unity $(\phi_i)_{i\in I}$ is said to be subordinated to $\mathscr{U}$ if each ϕ_i vanishes outside U_i and $\sum_{i\in I} \phi_i(x) = 1$ for any $x \in X$. A partition of unity $(\phi_i)_{i\in I}$ is subordinated to a cover $\mathscr{U}$ will be denoted by $(\phi_U)_{U\in\mathscr{U}}$.*

Let $\mathscr{U} = (U_i)_{i\in I}$ be a cover of a metric space X. The Lebesgue number for $\mathscr{U}$ is a number $L > 0$ with the property that any subset $B \subset X$ of diameter less than L is contained in some $U_i \in \mathscr{U}$. The multiplicity of $\mathscr{U}$ is the smallest integer n such that every point $x \in X$ is contained in at most n elements of $\mathscr{U}$. We call $\mathscr{U}$ is L-disjoint if $d(U, V) > L$ for all $U, V \in \mathscr{U}$ with $U \neq V$.

Definition 2.4 *Let X be a metric space. Then X has asymptotic dimension at most k if and only if for every $R > 0$ there exists a uniformly bounded cover $\mathscr{U}$ of X with Lebesgue number at least R and multiplicity at most $k + 1$.*

The following lemma states that given a cover of a metric space with multiplicity at most $k + 1$ and Lebesgue number L, we can construct a partition of unity with Lipschitz properties subordinated to the cover. The proof can be found in [2, 5].

Lemma 2.2 *Let $\mathscr{U}$ be a cover of a metric space X with multiplicity at most $k+1$ $(k \geq 0)$, and Lebesgue number $L > 0$. For $U \in \mathscr{U}$, define*

$$\phi_U(x) = \frac{d(x, X\backslash U)}{\sum_{V\in\mathscr{U}} d(x, X\backslash V)}.$$

Then $(\phi_U)_{U\in\mathscr{U}}$ is a partition of unity on X subordinated to the cover $\mathscr{U}$. Moreover each ϕ_U satisfies

$$|\phi_U(x) - \phi_U(y)| \leq \frac{2k + 3}{L} d(x, y), \quad \forall\ x, y \in X,$$

and the family $(\phi_U)_{U\in\mathscr{U}}$ satisfies

$$\sum_{U\in\mathscr{U}} |\phi_U(x) - \phi_U(y)| \leq \frac{(2k + 2)(2k + 3)}{L} d(x, y), \quad \forall\ x, y \in X.$$

Let Γ be a countable discrete group. We can view Γ as a metric space if we endow Γ with a length function.

Definition 2.5 *Let Γ be a countable discrete group. A length function on Γ is a non-negative real-valued function, denoted by l, satisfying that for all $x, y \in \Gamma$,*

(1) $l(xy) \leq l(x) + l(y)$,

(2) $l(x^{-1}) = l(x)$,

(3) $l(x) = 0$ *if and only if* $x = e$, *where* e *is the identity element in* Γ.

This defines a metric on Γ by $d_\Gamma(f, g) = l_\Gamma(f^{-1}g)$. If Γ is a finitely generated group and its generating set S is symmetric, i.e., $S = S^{-1}$, then the length $l_\Gamma(g)$ of an element $g \in \Gamma$ is defined to be the length of the shortest word in S representing g. In this case, d_Γ is left invariant in the sense that $d_\Gamma(hf, hg) = d_\Gamma(f, g)$ for any $h \in \Gamma$.

3 Infinite Unions

In this section, we prove the following infinite unions property for strong embeddability by using the gluing technique in [7].

Theorem 3.1 *Let* X *be a metric space. Suppose that for all* $R, \epsilon > 0$ *there is a partition of unity* $(\phi_i)_{i\in I}$ *on* X *such that*

(1) *for any* $x, y \in X$, *if* $d(x, y) \leq R$, *then* $\sum\limits_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$;

(2) $(\phi_i)_{i\in I}$ *is subordinated to an equi-strongly embeddable cover* $\mathscr{U} = (U_i)_{i\in I}$ *of* X. *Then* X *is strongly embeddable.*

To prove this, we will use the following lemma. Given $R > 0$, let $\mathscr{U} = (U_i)_{i\in I}$ be a cover of a metric space X and $\mathscr{U}_R = (U_i(R))_{i\in I}$ be R-neighbourhood of $\mathscr{U}$.

Lemma 3.1 *Let* $\mathscr{U}$ *be a cover of a metric space* X. *If* $\mathscr{U}$ *is equi-strongly embeddable, so is* $\mathscr{U}_R$.

Proof First, we construct a map from $U_i(R)$ to U_i for each $i \in I$. To do this, we define $f : U_i(R) \to U_i$ by

$$f(x) = \begin{cases} x, & x \in U_i, \\ y, & x \in U_i(R) \setminus U_i, \end{cases}$$

where y is any point in $\omega_i(x)$ and $\omega_i(x) = \{z \in U_i \mid d(x, z) \leq R\}$.

By triangle inequality, for each $i \in I$ and all $x, y \in U_i(R)$, we have

$$d(f(x), f(y)) \leq d(f(x), x) + d(x, y) + d(y, f(y)) \leq d(x, y) + 2R.$$

Let $L, \epsilon > 0$ be given. Since $\mathscr{U} = (U_i)_{i\in I}$ is equi-strongly embeddable, there exists a family of maps

$$\beta^i : U_i \to \ell^2(U_i)_1$$

such that

(1) for each $i \in I$, β^i has $(L + 2R, \epsilon)$ variation;

(2) for every $\delta > 0$, any $i \in I$ and any $x \in U_i(R)$, there exists an $S_0 > 0$ such that if $S > S_0$, $\sum\limits_{z\notin B_S(x)} |\beta^i_x(z)|^2 < \delta$.

Now, for each $i \in I$, we define an isometry

$$g : \ell^2(U_i) \to \ell^2(U_i(R))$$

by $g(\xi) = \eta$ for each $\xi \in \ell^2(U_i)$, where

$$\eta(x) = \begin{cases} \xi(x), & x \in U_i, \\ 0, & x \in U_i(R) \setminus U_i. \end{cases}$$

Finally, for each $i \in I$, we define

$$\alpha^i : U_i(R) \to \ell^2(U_i(R))$$

by taking $\alpha^i = g \circ \beta^i \circ f$ according to the following diagram:

$$U_i(R) \xrightarrow{f} U_i \xrightarrow{\beta^i} \ell^2(U_i) \xrightarrow{g} \ell^2(U_i(R)).$$

Notice that for each $x \in U_i(R)$,

$$\|\alpha^i_x\| = \|g \circ \beta^i \circ f(x)\| = \|g \circ \beta^i_{f(x)}\| = \|\beta^i_{f(x)}\| = 1.$$

For each $i \in I$ and any $x, y \in U_i(R)$, if $d(x, y) \leq L$, then $d(f(x), f(y)) \leq L + 2R$. So by (1) we have

$$\|\alpha^i_x - \alpha^i_y\| = \|g \circ \beta^i \circ f(x) - g \circ \beta^i \circ f(y)\| = \|\beta^i_{f(x)} - \beta^i_{f(y)}\| \leq \epsilon.$$

Moreover, let $S_1 = S_0 + 2R$. For every $\delta > 0$, each $i \in I$ and all $x \in U_i(R)$, by (2) we also have: If $S > S_1$,

$$\begin{aligned} \sum_{z \notin B_S(x)} |\alpha^i_x(z)|^2 &= \sum_{z \notin B_S(x)} |g \circ \beta^i \circ f(x)(z)|^2 \\ &\leq \sum_{z \notin B_{S-R}(f(x))} |\beta^i_{f(x)}(z)|^2 \\ &\leq \delta. \end{aligned}$$

Thus, a such family of maps $(\alpha^i)_{i \in I}$ is as desired. We complete the proof.

Proof of Theorem 3.1 Let $R, \epsilon > 0$ be given. By assumption, there is an equi-strongly embeddable cover $\mathscr{U} = (U_i)_{i \in I}$ of X and a partition of unity $(\phi_i)_{i \in I}$ subordinated to $\mathscr{U}$ such that for any $x, y \in X$, if $d(x, y) \leq R$, then $\sum_{i \in I} |\phi_i(x) - \phi_i(y)| \leq \frac{\epsilon^2}{4}$.

By Lemma 3.1, we know that $\mathscr{U}_R$ is equi-strongly embeddable. Thus, there exists a family of maps $\xi^i : U_i(R) \to \ell^2(U_i(R))_1$ satisfying

(1) for any $i \in I$, ξ^i has $(R, \frac{\epsilon}{2})$ variation;

(2) $\forall \delta > 0$, $\forall i \in I$, $\forall x \in U_i(R)$, $\exists S_0 > 0$, if $S > S_0$, $\sum_{z \notin B_S(x)} |\xi^i_x(z)|^2 \leq \delta$.

Next, we will construct a map θ from X to $\ell^2(X, \ell^2(I))_1$ such that θ satisfies the standards of Lemma 2.1, then we conclude the result.

For any $x \in U_i(R)$, each $i \in I$, we can extend each ξ_x^i to X by setting $\xi_x^i(z) = 0$ for $z \in X \setminus U_i(R)$. Thus, $\xi_x^i \in \ell^2(X)$ for any $x \in U_i(R)$, $i \in I$.

Firstly, define

$$\eta : X \to \bigoplus_{i\in I} \ell^2(X)$$

by taking $\eta_x = (\eta_x^i)_{i\in I},\ \eta_x^i = \phi_i(x)^{\frac{1}{2}}\xi_x^i$ for each $x \in X$.

Now, define

$$\sigma : \bigoplus_{i\in I} \ell^2(X) \to \ell^2(X, \ell^2(I))$$

by $\sigma(\mu) = g,\ g(z) = \nu(z,\cdot),\ \nu(z,i) = \mu_i(z)$, where $\mu = (\mu_i)_{i\in I} \in \bigoplus_{i\in I} \ell^2(X),\ z \in X$.

Finally, define

$$\theta : X \to \ell^2(X, \ell^2(I))$$

by $\theta = \sigma \circ \eta$.

We first check that $\|\theta_x\| = 1$. Notice that for any $x \in X$, $\|\eta_x\|^2 = \sum_{i\in I} \|\eta_x^i\|^2 = \sum_{i\in I} \phi_i(x)\|\xi_x^i\|^2 = \sum_{i\in I} \phi_i(x) = 1$. Hence, we have

$$\begin{aligned}\|\theta_x\|^2 &= \sum_{z\in X} \|\theta_x(z)\|^2_{\ell^2(I)} = \sum_{z\in X} \|\nu_x(z,\cdot)\|^2_{\ell^2(I)} = \sum_{z\in X}\sum_{i\in I} |\nu_x(z,i)|^2 \\ &= \sum_{z\in X}\sum_{i\in I} |\eta_x^i(z)|^2 = \|\eta_x\|^2 = 1.\end{aligned}$$

Next, we verify that θ satisfies the remaining two conditions of Lemma 2.1. If $x, y \in X$ with $d(x,y) \le R$, we have

$$\begin{aligned}\|\eta_x - \eta_y\|^2 &= \sum_{i\in I} \|\phi_i(x)^{\frac{1}{2}}\xi_x^i - \phi_i(y)^{\frac{1}{2}}\xi_y^i\|^2 \\ &= \sum_{i\in I}\sum_{z\in X} |\phi_i(x)^{\frac{1}{2}}\xi_x^i(z) - \phi_i(y)^{\frac{1}{2}}\xi_y^i(z)|^2 \\ &\le 2\sum_{i\in I}\sum_{z\in X} |\phi_i(x)^{\frac{1}{2}}(\xi_x^i(z) - \xi_y^i(z))|^2 + 2\sum_{i\in I}\sum_{z\in X} |(\phi_i(x)^{\frac{1}{2}} - \phi_i(y)^{\frac{1}{2}})\xi_y^i(z)|^2 \\ &= 2\sum_{i\in I} \phi_i(x)\|\xi_x^i - \xi_y^i\|^2 + 2\sum_{i\in I} |\phi_i(x)^{\frac{1}{2}} - \phi_i(y)^{\frac{1}{2}}|^2.\end{aligned}$$

If $x \in U_i$, then $y \in U_i(R)$, so by (1), $\|\xi_x^i - \xi_y^i\| \le \frac{\epsilon}{2}$ for each $i \in I$. Thus

$$2\sum_{i\in I} \phi_i(x)\|\xi_x^i - \xi_y^i\|^2 \le \frac{\epsilon^2}{2}.$$

By the inequality $|x^{\frac{1}{2}} - y^{\frac{1}{2}}|^2 \le |x - y|$, we have

$$2\sum_{i\in I} |\phi_i(x)^{\frac{1}{2}} - \phi_i(y)^{\frac{1}{2}}|^2 \le 2\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \le \frac{\epsilon^2}{2}.$$

Therefore $\|\eta_x - \eta_y\| \le \epsilon$. Since

$$\begin{aligned}\|\theta_x - \theta_y\|^2 &= \sum_{z\in X} \|\theta_x(z) - \theta_y(z)\|^2_{\ell^2(I)} = \sum_{z\in X} \|\nu_x(z,\cdot) - \nu_y(z,\cdot)\|^2_{\ell^2(I)} \\ &= \sum_{z\in X}\sum_{i\in I} |\nu_x(z,i) - \nu_y(z,i)|^2 = \sum_{z\in X}\sum_{i\in I} |\eta^i_x(z) - \eta^i_y(z)|^2 \\ &= \|\eta_x - \eta_y\|^2,\end{aligned}$$

we have $\|\theta_x - \theta_y\| \le \epsilon$.

Moreover, for above $\delta > 0$ and any $x \in X$, if $S > S_0$, then by (2) we have

$$\begin{aligned}\sum_{z\notin B_S(x)} \|\theta_x(z)\|^2_{\ell^2(I)} &= \sum_{z\notin B_S(x)}\sum_{i\in I} \|\nu_x(z,\cdot)\|^2_{\ell^2(I)} = \sum_{z\notin B_S(x)}\sum_{i\in I} |\nu_x(z,i)|^2 \\ &= \sum_{z\notin B_S(x)}\sum_{i\in I} |\eta^i_x(z)|^2 = \sum_{z\notin B_S(x)}\sum_{i\in I} \phi_i(x)|\xi^i_x(z)|^2 \\ &= \sum_{i\in I} \phi_i(x) \sum_{z\notin B_S(x)} |\xi^i_x(z)|^2 \le \delta.\end{aligned}$$

Thus, the proof is completed.

The following result is immediate from Theorem 3.1, and relative versions for property A, asymptotic dimension and coarse embeddability were proved in [2–3, 7], respectively.

Corollary 3.1 *Let $X = \bigcup_{i\in I} X_i$ be a metric space with $\{X_i\}_{i\in I}$ being equi-strongly embeddable. If for every $L > 0$ there exists a strongly embeddable subspace Y of X such that $\{X_i \setminus Y\}_{i\in I}$ is L-disjoint. Then X is strongly embeddable.*

Proof Let R, ϵ be given and let $L > 0$ such that $\frac{20R}{L} \le \epsilon$. By assumption,

$$\mathscr{U} = \bigcup_{i\in I} (X_i \setminus Y) \cup Y$$

is a cover of X. Let $\mathscr{V} = (V_j)_j$ be the L-neighborhood of $\mathscr{U}$.

By Lemma 3.1, $\mathscr{V}$ is an equi-strongly embeddable cover of X, with multiplicity at most 2 and Lebesgue number at least L. Moreover, Lemma 2.2 tells us that there exists a partition of unity $(\phi_{V_j})_{V_j\in\mathscr{V}}$ on X subordinate to $\mathscr{V}$ with the following property:

$$\sum_{V_j\in\mathscr{V}} |\phi_{V_j}(x) - \phi_{V_j}(y)| \le \frac{(2+2)(2+3)}{L} d(x,y) = \frac{20d(x,y)}{L}, \quad \forall\, x, y \in X.$$

For all $x, y \in X$ with $d(x,y) \le R$, we have $\sum_{V_j\in\mathscr{V}} |\phi_{V_j}(x) - \phi_{V_j}(y)| \le \frac{20R}{L} \le \epsilon$. Thus, we complete the proof by Theorem 3.1.

4 Groups Acting on Metric Spaces

In this section, we will prove the following theorem.

Theorem 4.1 *Let Γ be a finitely generated group acting on metric space X by isometries. If X has asymptotic dimension less than k and there exists $x_0 \in X$ such that $W_K(x_0)$ is strongly embeddable for every $K > 0$. Then Γ is strongly embeddable.*

We first recall the concept of the K-stabilizers. Let Γ be a group acting on a metric space X. For every $K > 0$, the K-stabilizer $W_K(x_0)$ of $x_0 \in X$ is defined to be the set of all $g \in \Gamma$ such that $gx_0 \in B_K(x_0)$.

Proof of Theorem 4.1 We define a map $\pi : \Gamma \to X$ by $\pi(\gamma) = \gamma x_0$ for all $\gamma \in \Gamma$. Let S be the finite symmetric generating set in the definition of the word metric for Γ and let

$$\lambda = \max_{s \in S} d(sx_0, x_0).$$

It is easy to see that π is λ-Lipschitz.

Let $L > 0$ be given and take a uniformly bounded cover $\mathscr{U} = (U_i)_{i \in I}$ of X with Lebesgue number L and multiplicity $k + 1$ such that the L-neighbourhood $\mathscr{V} = (V_i)_{i \in I}$ of $\mathscr{U}$ is also a cover of X with multiplicity $k + 1$.

Now, take $K > 0$ and $x_i \in X$ such that $V_i \subset B_K(x_i)$ for each $i \in I$, then take $\gamma_i \in \Gamma$ such that $x_i = \gamma_i x_0$.

By the definition of K-stabilizer, it is clear that $W_K(x_0) = \pi^{-1}(B_K(x_0))$. Moreover, by the isometric action of Γ, we have $\gamma\pi^{-1}(B_K(x_0)) = \pi^{-1}(B_K(\gamma x_0))$ for all $\gamma \in \Gamma$. Since $\pi^{-1}(B_K(\gamma x_0))$ is isometric to $\pi^{-1}(B_K(x_0))$ and $\pi^{-1}(B_K(x_0)) = W_K(x_0)$ is strongly embeddable, we know that $\pi^{-1}(B_K(\gamma_i x_0))$ is strongly embeddable.

Since $\pi^{-1}(U_i) \subset \pi^{-1}(V_i) \subset \pi^{-1}(B_K(x_i))$, by the closure of taking subgroups, we conclude that $\pi^{-1}(U_i)$ is strongly embeddable for each $i \in I$. Thus, we obtain an equi-strongly embeddable cover $\mathscr{D} = (D_i)_{i \in I}$ of Γ, where $D_i = \pi^{-1}(U_i)$.

Next, we will construct a partition of unity $(\varphi_{D_i})_{i \in I}$ on Γ satisfying the conditions of Theorem 3.1, then we will conclude the result.

Let $R, \epsilon > 0$ be given and let L be sufficiently large such that

$$\frac{\lambda(2k+2)(2k+3)R}{L} \leq \frac{\epsilon}{2}. \tag{4.1}$$

By the equi-strong embeddability of $\{\pi^{-1}(V_i)\}_{i \in I}$, there exists a family of maps

$$\xi^i : \pi^{-1}(V_i) \to \ell^2(\pi^{-1}(V_i))_1$$

such that each ξ^i has $(R, \frac{\epsilon}{4})$ variation.

Since $\mathscr{U}$ is a uniformly bounded cover of X with Lebesgue number L and multiplicity $k+1$, Lemma 2.2 tells us that there exists a partition of unity $(\phi_{U_i})_{U_i \in \mathscr{U}}$ subordinated to it with the property: For all $x, y \in X$,

$$\sum_{U_i \in \mathscr{U}} |\phi_{U_i}(x) - \phi_{U_i}(y)| \leq \frac{(2k+2)(2k+3)}{L} d(x, y). \tag{4.2}$$

Finally, for any $\gamma \in \Gamma$, we define

$$\varphi_{D_i}(\gamma) = \sum_{v \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(\gamma)) |\xi^i_\gamma(v)|^2.$$

Notice first that for each $\gamma \in \Gamma$,

$$\sum_{i \in I} \varphi_{D_i}(\gamma) = \sum_{i \in I} \sum_{v \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(\gamma)) |\xi^i_\gamma(v)|^2 = \sum_{i \in I} \phi_{U_i}(\pi(\gamma)) \|\xi^i_\gamma\|^2 = 1.$$

Since each ϕ_{U_i} vanishes outside U_i, each φ_{D_i} vanishes outside D_i. This implies that $\{\varphi_{D_i}\}_{i \in I}$ is a partition of unity on Γ and subordinated to $\mathscr{D}$.

Moreover, for all $\gamma, \gamma' \in \Gamma$ with $d(\gamma, \gamma') \leq R$, if $\gamma \in \pi^{-1}(U_i)$, then $\gamma' \in \pi^{-1}(V_i)$. So we have

$$\begin{aligned}
\sum_{i \in I} |\varphi_{D_i}(\gamma) - \varphi_{D_i}(\gamma')| &= \sum_{i \in I} \Big| \sum_{v \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(\gamma)) |\xi^i_\gamma(v)|^2 - \sum_{v \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(\gamma')) |\xi^i_{\gamma'}(v)|^2 \Big| \\
&= \sum_{i \in I} \Big| \sum_{v \in \pi^{-1}(V_i)} (\phi_{U_i}(\pi(\gamma)) |\xi^i_\gamma(v)|^2 - \phi_{U_i}(\pi(\gamma')) |\xi^i_{\gamma'}(v)|^2) \Big| \\
&\leq \sum_{i \in I} \sum_{v \in \pi^{-1}(V_i)} |\phi_{U_i}(\pi(\gamma)) |\xi^i_\gamma(v)|^2 - \phi_{U_i}(\pi(\gamma')) |\xi^i_{\gamma'}(v)|^2| \\
&\leq \sum_{i \in I} \phi_{U_i}(\pi(\gamma)) \sum_{v \in \pi^{-1}(V_i)} |(\xi^i_\gamma(v))^2 - (\xi^i_{\gamma'}(v))^2| \\
&\quad + \sum_{i \in I} |\phi_{U_i}(\pi(\gamma)) - \phi_{U_i}(\pi(\gamma'))|.
\end{aligned}$$

For the second term, by (4.1)–(4.2) we have

$$\begin{aligned}
\sum_{i \in I} |\phi_{U_i}(\pi(\gamma)) - \phi_{U_i}(\pi(\gamma'))| &\leq \frac{(2k+2)(2k+3)}{L} d(\pi(\gamma), \pi(\gamma')) \\
&\leq \frac{\lambda(2k+2)(2k+3)}{L} d(\gamma, \gamma') \\
&\leq \frac{\epsilon}{2}.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
\sum_{v \in \pi^{-1}(V_i)} |(\xi^i_\gamma(v))^2 - (\xi^i_{\gamma'}(v))^2| &\leq \sum_{v \in \pi^{-1}(V_i)} |\xi^i_\gamma(v) + \xi^i_{\gamma'}(v)| |\xi^i_\gamma(v) - \xi^i_{\gamma'}(v)| \\
&\leq \Big(\sum_{v \in \pi^{-1}(V_i)} |\xi^i_\gamma(v) + \xi^i_{\gamma'}(v)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{v \in \pi^{-1}(V_i)} |\xi^i_\gamma(v) - \xi^i_{\gamma'}(v)|^2 \Big)^{\frac{1}{2}} \\
&= \|\xi^i_\gamma + \xi^i_{\gamma'}\| \|\xi^i_\gamma - \xi^i_{\gamma'}\| \\
&\leq 2 \|\xi^i_\gamma - \xi^i_{\gamma'}\|.
\end{aligned}$$

This term is bounded by $\frac{\epsilon}{2}$ since each ξ^i has $(R, \frac{\epsilon}{4})$ variation. Therefore $\sum_{i \in I} |\varphi_{D_i}(\gamma) - \varphi_{D_i}(\gamma')| \leq \epsilon$. Thus, we complete the proof by Theorem 3.1.

Remark 4.1 Bell [2] proved that a group acting by isometries on a metric space with finite asymptotic dimension whose R-stabilizers have property A also has property A. We point out that our argument of Theorem 4.1 is also suitable for Bell's result. In fact, the partition of unity constructed in this result coincides with Theorem 4.1. Thus, our approach in this paper has given an alternative proof to Bell's.

In [7], the infinite unions property for coarse embeddability and exactness (it is equivalent to property A for the metric spaces with bounded geometry) were shown by Dadarlat-Guentner.

Proposition 4.1 (see [7]) *Let X be a metric space. Suppose that for all $R, \epsilon > 0$ there is a partition of unity $(\phi_i)_{i\in I}$ on X such that*

(1) *for any $x, y \in X$, if $d(x, y) \leq R$, then $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$;*

(2) *$(\phi_i)_{i\in I}$ is subordinated to an equi-coarsely embeddable (resp. equi-exact) cover.*

Then X is coarsely embeddable (resp. exact).

The following result is natural.

Proposition 4.2 *Let Γ be a finitely generated group which acts on the metric space X by isometries. If X has asymptotic dimension less than k and there exists $x_0 \in X$ such that $W_K(x_0)$ is coarsely embeddable for every $K > 0$. Then Γ is coarsely embeddable.*

Proof The proof is analogous to Theorem 4.1 with the partition of unity constructed by

$$\varphi_{D_i}(\gamma) = \phi_{U_i}(\pi(\gamma))\|\alpha^i(\gamma)\|^2,$$

where $\{\alpha^i\}_{i\in I}$ is similarly defined by the coarse embeddability of each $\pi^{-1}(V_i)$ in the proof of Theorem 4.1.

5 Graph of Groups and Relative Hyperbolic Groups

In this section, we will focus on particular examples: Graph of groups and relative hyperbolic groups.

In [2], Bell proved that a fundamental group of a finite graph of groups whose vertex groups have property A, also has property A, and a same statement concerning operator norm localization property was obtained by Chen-Wang [6]. In their both proofs, the relative versions of Theorem 4.1 for property A and operator norm localization property play an important role. Hence, we have the following parallel result for strong embeddability in the same way.

Proposition 5.1 *Let (G, Y) be a finite graph of groups with finitely generated vertex groups $\{G_P\}_P$ which is equi-strongly embeddable. Then for any vertex P_0 the fundamental group $\pi_1(G, Y, P_0)$ is strongly embeddable.*

Proof For the proof we refer to [2, Lemma 3] and [6, Theorem 4.6].

Corollary 5.1 *Any amalgamated free products and HNN extensions of groups from groups with strong embeddability will be strongly embeddable.*

Proof They are both the special graph of groups according to Bass-Serre Theory (see [15]). In fact, if Y is the graph with two vertices P, Q and one edge y, then

$$\pi_1(G, Y, P) = \pi_1(G, Y, Q) = G_P *_{G_y} G_Q$$

is the free product of G_P and G_Q amalgamated over G_y.

If Y is the graph with one vertex P and one edge y, then $\pi_1(G, Y, P)$ is the HNN extension of G_p over the subgroup $\phi_{\bar{y}}(G_y)$ by means of $\phi_y \phi_{\bar{y}}^{-1}$.

So we conclude the result by Proposition 5.1.

Let Γ be a finitely generated group which is hyperbolic relative to a finite family of subgroups $\{H_1, \cdots, H_n\}$. We will prove that Γ is strongly embeddable if and only if each subgroup H_k is strongly embeddable.

If A is a symmetric set of finite generators of Γ, we denote by d_A the corresponding left invariant metric on Γ. If B is another such set with $A \subset B$, then the identity map $p : (\Gamma, d_A) \to (\Gamma, d_B)$ is equivariant and $d_B(p(\gamma), p(\gamma')) \le d_A(\gamma, \gamma')$.

Let S be a finite symmetric set generating Γ. Denote

$$H = \bigcup_k (H_k - e).$$

Let d_S and $d_{S \cup H}$ be the left invariant metrics on Γ induced by S and $S \cup H$, respectively. For $n \ge 1$, denote

$$B(n) = \{\gamma \in \Gamma \mid d_{S \cup H}(\gamma, e) \le n\}.$$

In this section, we always view $B(n)$ as a subspace of Γ equipped with the metric d_S. The following useful recursive decomposition of $B(n)$ is contained in the proof of theorem in [13]:

$$B(1) = S \cup \Big(\bigcup_k H_k\Big), \tag{5.1}$$

$$B(n) = \Big(\bigcup_k B(n-1)H_k\Big) \cup \Big(\bigcup_{x \in S} B(n-1)x\Big), \tag{5.2}$$

$$B(n-1)H_k = \bigsqcup_{g \in R(n-1)} gH_k, \tag{5.3}$$

where (5.3) represents a partition of $B(n-1)H_k$ into disjoint cosets according to a fixed set $R(n-1)$ of coset representative, $R(n-1) \subset B(n-1)$.

Proposition 5.2 (see [7, 13]) *For every $L > 0$ there exists $C(L) > 0$ such that if*

$$Y = \{\gamma \in \Gamma \mid d_S(\gamma, B(n-1)) \le C(L)\},$$

then for each k,

$$B(n-1)H_k \subset Y \cup \Big(\bigcup_{g \in R(n-1)} gH_k \setminus Y\Big) \tag{5.4}$$

and the subspaces $gH_k \setminus Y$, $g \in R(n-1)$, are L-disjoint with the metric d_S.

Proposition 5.3 *If each H_k is strongly embeddable, so is $B(n)$.*

Proof The proof is by induction. For the basis, observe that $B(1)$ is strongly embeddable by (5.1) and the finite union theorem of strong embeddability. For the induction step, assume that $B(n-1)$ is strongly embeddable. Using again the finite union theorem and (5.2) we are reduced to verifying that each $B(n-1)H_k$ is strongly embeddable. This follows from the Corollary 3.1 and Proposition 5.2.

The following proposition is contained in [7], proved by Osin in [13].

Proposition 5.4 (see [7, 13]) *The metric space $(\Gamma, d_{S\cup H})$ has finite asymptotic dimension.*

Now we can prove the following theorem which was also obtained by Ji, Ogle and Ramsey [12, Corollary 4.8].

Theorem 5.1 *Let Γ be a finitely generated group which is hyperbolic relative to a finite family $\{H_1, \cdots, H_n\}$ of subgroups. Then Γ is strongly embeddable if and only if each subgroup H_k is strongly embeddable.*

Proof If Γ is strongly embeddable, then so are its subgroups H_k by the closure of taking subspaces. For the inverse, assume that each subgroup H_k is strongly embeddable. Let $X = (\Gamma, d_S)$ and $Y = (\Gamma, d_{S\cup H})$. We choose $x_0 = e$ to be the unit in Γ and define

$$\pi : X \to Y, \quad \gamma \mapsto \gamma x_0.$$

Then $W_n(x_0) = B(n) = \pi^{-1}(B_Y(e, n))$ is strongly embeddable by Proposition 5.3. Note that X acts isometrically on Y. The conclusion follows from Theorem 4.1.

Acknowledgement The authors are indebted to referees for their useful comments.

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