Chin. Ann. Math. Ser. B
40(3), 2019, 321–330
DOI: 10.1007/s11401-019-0135-7

Chinese Annals of Mathematics, Series B

Problems of Lifts in Symplectic Geometry

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Abstract Let (M,ω) be a symplectic manifold. In this paper, the authors consider the notions of musical (bemolle and diesis) isomorphisms $\omega^b : TM \to T^*M$ and $\omega^\sharp : T^*M \to TM$ between tangent and cotangent bundles. The authors prove that the complete lifts of symplectic vector field to tangent and cotangent bundles is ω^b-related. As consequence of analyze of connections between the complete lift ${}^c\omega_{TM}$ of symplectic 2-form ω to tangent bundle and the natural symplectic 2-form $\mathrm{d}p$ on cotangent bundle, the authors proved that $\mathrm{d}p$ is a pullback of ${}^c\omega_{TM}$ by $\omega^\sharp$. Also, the authors investigate the complete lift ${}^c\varphi_{T^*M}$ of almost complex structure φ to cotangent bundle and prove that it is a transform by $\omega^\sharp$ of complete lift ${}^c\varphi_{TM}$ to tangent bundle if the triple (M,ω,φ) is an almost holomorphic $\mathfrak{A}$-manifold. The transform of complete lifts of vector-valued 2-form is also studied.

Keywords Symplectic manifold, Tangent bundle, Cotangent bundle, Transform of tensor fields, Pullback, Pure tensor, Holomorphic manifold
2000 MR Subject Classification 53D05, 53C12, 55R10

1 Introduction

Let M be an n-dimensional C^∞-manifold and $T_P(M)$ ($T_P^*(M)$) be the tangent (cotangent) vector space at a point $P \in M$. Then the set

$$T(M) = \bigcup_{P\in M} T_P(M)\ \Big(T^*(M) = \bigcup_{P\in M} T_P^*(M)\Big)$$

is, by definition, the tangent (cotangent) bundle over the manifold M. For any point $\widetilde{P}$ of $T_P(M)$ ($T_P^*(M)$) such that $\widetilde{P} \in T_P(M)$ ($\widetilde{P} \in T_P^*(M)$), the correspondence $\widetilde{P} \to P$ determines the natural tensor bundle projection $\pi : T(M) \to M$ ($\pi : T^*(M) \to M$), that is, $\pi(\widetilde{P}) = P$. Suppose that the base space M is covered by a system of coordinate neighborhoods (U, x^i), where $x^i, i = 1, \cdots, n$ are local coordinates in the neighborhood U. The open set $\pi^{-1}(U) \subset T(M)$ ($\pi^{-1}(U) \subset T^*(M)$) is naturally diffeomorphic to the direct product $U \times \mathbb{R}^n$ in such a way that a point $\widetilde{P} \in T_P(M)$ ($\widetilde{P} \in T_P^*(M)$) is represented by an ordered pair (P,ν) ((P,p)) of the point $P \in M$ and a vector (covector) $\nu \in \mathbb{R}^n$ ($p \in \mathbb{R}^n$) whose components are given by $\nu^i(p_i)$ of $\widetilde{P}$ in $T_P(M)$ ($T_P^*(M)$) with respect to the frame (coframe) ∂_i ($\mathrm{d}x^i$). Denoting (x^i) by the coordinates of $P = \pi(\widetilde{P})$ in U and establishing the correspondence $(x^i, \nu^i) \to \widetilde{P} \in$

Manuscript received August 23, 2017. Revised May 15, 2018.
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$\pi^{-1}(U)$ $((x^i,p_i) \to \widetilde{P} \in \pi^{-1}(U))$, we can introduce a system of local coordinates $(x^i, x^{\overline{i}}) = (x^i, \nu^i)$ $((x^i, \widetilde{x}^{\overline{i}}) = (x^i, p_i))$, $\overline{i} = n+1, \cdots, 2n$ in the open set $\pi^{-1}(U) \subset T(M)$ $(\pi^{-1}(U) \subset T^*(M))$. We call $(x^i, x^{\overline{i}}) = (x^J)$, $J = 1, \cdots, 2n$ $((x^i, \widetilde{x}^{\overline{i}}) = (\widetilde{x}^J))$, $J = 1, \cdots, 2n$ the induced coordinates in $\pi^{-1}(U) \subset T(M)$ $(\pi^{-1}(U) \subset T^*(M))$.

A manifold M of dimension $n = 2m$ is symplectic if it possesses a nondegenerate 2-form ω which is closed (i.e., $\mathrm{d}\omega = 0$). For any manifold M of dimension n, the cotangent bundle $T^*_P(M)$ is a natural symplectic $2n$-manifold with symplectic 2-form $\widetilde{\omega} = -\mathrm{d}p = \mathrm{d}x^i \wedge \mathrm{d}p_i$, where $p = p_i \mathrm{d}x^i$ is the Liouville form (basic 1-form) on $T^*(M)$.

In Riemannian geometry, the musical isomorphism (or canonical isomorphism) is an isomorphism between the tangent and cotangent bundles of a Riemannian manifold given by its metric. There are similar isomorphisms on symplectic manifolds. Let now (M, ω) be a symplectic manifold. Then the musical isomorphisms $\omega^b : TM \to T^*M$ and $\omega^\sharp : T^*M \to TM$ are given by

$$\omega^b : x^I = (x^i, x^{\overline{i}}) = (x^i, \nu^i) \to \widetilde{x}^K = (x^k, \widetilde{x}^{\overline{k}}) = (x^k = \delta^k_i x^i, p_k = \omega_{ki}\nu^i)$$

and

$$\omega^\sharp : \widetilde{x}^K = (x^k, \widetilde{x}^{\overline{k}}) = (x^k, p_k) \to x^I = (x^i, x^{\overline{i}}) = (x^i = \delta^i_k x^k, \nu^i = \omega^{ik} p_k),$$

where $\omega^{ik}\omega_{kj} = \delta^i_j$, δ^i_j is the Kronecker symbol. The Jacobian matrices of ω^b and $\omega^\sharp$ are given, respectively, by

$$\begin{aligned}(\omega^b)_* = \widetilde{A} = (\widetilde{A}^K_I) &= \begin{pmatrix} \widetilde{A}^k_i & \widetilde{A}^k_{\overline{i}} \\ \widetilde{A}^{\overline{k}}_i & \widetilde{A}^{\overline{k}}_{\overline{i}} \end{pmatrix} \\ &= \left(\frac{\partial \widetilde{x}^K}{\partial x^I}\right) = \begin{pmatrix} \delta^k_i & 0 \\ \nu^s \partial_i \omega_{ks} & \omega_{ki} \end{pmatrix} \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}(\omega^\sharp)_* = A = (A^I_K) &= \begin{pmatrix} A^i_k & A^i_{\overline{k}} \\ A^{\overline{i}}_k & A^{\overline{i}}_{\overline{k}} \end{pmatrix} \\ &= \left(\frac{\partial x^I}{\partial \widetilde{x}^K}\right) = \begin{pmatrix} \delta^i_k & 0 \\ p_s \partial_k \omega^{is} & \omega^{ik} \end{pmatrix}. \end{aligned} \tag{1.2}$$

The theory of prolongations (lifts) of tensor fields from base manifold to its tangent and cotangent bundles was developed by Yano and Ishihara [18] (also, see for example [1–2, 4]). The main purpose of this paper is to study the transform (pullback and pushforward) of lifts via the musical symplectic isomorphisms. A similar problem for Riemannian manifolds was solved in [3] (also, see [13]).

2 ω^b-Related Vector Fields

Let f be any function on symplectic manifold (M, ω). If ${}^C X_T$ is the complete lift of vector field X from manifold M to its tangent bundle $T(M)$ which is defined by

$${}^C X_T \, {}^C f = {}^C (Xf), \quad {}^C f = v^s \partial_s f,$$

then has components (see [18, p.15])

$$ {}^C X_T = \begin{pmatrix} X^i \\ v^s \partial_s X^i \end{pmatrix} \tag{2.1} $$

with respect to the coordinates $(x^i, x^{\bar{i}}) = (x^i, \nu^i)$.

Using (1.1) and (2.1) we have

$$ \begin{aligned} (\omega^b)_* \, {}^C X_T = (\widetilde{A}_I^K \, {}^C X_T^I) &= \begin{pmatrix} \delta_i^k & 0 \\ \nu^s \partial_i \omega_{ks} & \omega_{ki} \end{pmatrix} \begin{pmatrix} X^i \\ v^s \partial_s X^i \end{pmatrix} \\ &= \begin{pmatrix} X^k \\ X^i v^s \partial_i \omega_{ks} + \omega_{ki} v^s \partial_s X^i \end{pmatrix} \\ &= \begin{pmatrix} X^k \\ v^s (X^i \partial_i \omega_{ks} + \omega_{is} \partial_k X^i + \omega_{ki} \partial_s X^i) - v^s \omega_{is} \partial_k X^i \end{pmatrix} \\ &= \begin{pmatrix} X^k \\ v^s L_X \omega_{ks} - p_i \partial_k X^i \end{pmatrix}, \end{aligned} \tag{2.2} $$

where L_X denotes the Lie derivations.

On the other hand, the complete lift ${}^C X_{T^*}$ of vector field X from manifold M to its cotangent bundle $T^*(M)$ is defined by

$$ {}^C X_{T^*}(\gamma Z) = \gamma(L_X Z), $$

where γZ and $\gamma(L_X Z)$ are functions in $T^*(M)$ with local expressions

$$ \gamma Z = p_i Z^i, \quad \gamma(L_X Z) = p_i [X, Z]^i, $$

and the complete lift ${}^C X_{T^*}$ has components (see [18, p.236])

$$ {}^C X_{T^*} = \begin{pmatrix} X^k \\ -p_i \partial_k X^i \end{pmatrix} $$

with respect to the coordinates $(x^i, x^{\bar{i}}) = (x^i, p_i)$.

From (2.2) we obtain

$$ (\omega^b)_* \, {}^C X_T = {}^C X_{T^*} + \begin{pmatrix} 0 \\ v^s L_X \omega_{ks} \end{pmatrix}, $$

i.e., if $L_X \omega_{ks} = 0$, then $(\omega^b)_* \, {}^C X_T = {}^C X_{T^*}$. A symplectic vector field X is a vector field on (M, ω) which preserves the symplectic form, i.e., $L_X \omega = 0$. Thus we have the following theorem.

Theorem 2.1 *Let (M, ω) be a symplectic manifold, ${}^C X_T$ and ${}^C X_{T^*}$ be complete lifts of a vector field X to tangent bundle $T(M)$ and cotangent bundle $T^*(M)$, respectively. If X is a symplectic vector field, then ${}^C X_T$ and ${}^C X_{T^*}$ is ω^b-related, i.e., $(\omega^b)_* \, {}^C X_T = {}^C X_{T^*}$.*

Since every Hamiltonian vector fields X_H ($\iota_{X_H} \omega = \mathrm{d}H$) is a symplectic vector field ($L_{X_H} \omega = d \circ \iota_{X_H} \omega + \mathrm{d}\iota_{X_H} \circ \mathrm{d}\omega = \mathrm{d}^2 H = 0$). From Theorem 2.1 we immediately have the following corollary.

Corollary 2.1 *If X_H is a Hamiltonian vector field, then ${}^C(X_H)_T$ and ${}^C(X_H)_{T^*}$ are ω^b-related.*

3 Pullback of ${}^C\omega_T$

Let (M,ω) be a symplectic manifold of dimension $n = 2m$. It is well known that in the cotangent bundle $T^*(M)$, there exists a closed 2-form

$$\widetilde{\omega} = \mathrm{d}p = \mathrm{d}p_i \wedge \mathrm{d}x^i,$$

where $p = p_i \mathrm{d}x^i$, i.e., $T^*(M)$ is a symplectic $4m$-manifold. If we write $\widetilde{\omega} = \frac{1}{2}\widetilde{\omega}_{KL}\mathrm{d}x^K \wedge \mathrm{d}x^L$, then we have

$$\widetilde{\omega} = (\widetilde{\omega}_{KL}) = \begin{pmatrix} 0 & \delta_k^l \\ -\delta_l^k & 0 \end{pmatrix}.$$

The complete lift ${}^C\omega_T$ of ω to tangent bundle $T(M)$ is a 2-form and has components of the form (see [18, p.38])

$${}^C\omega_T = \begin{pmatrix} v^s\partial_s\omega_{ij} & \omega_{ij} \\ \omega_{ij} & 0 \end{pmatrix} \tag{3.1}$$

with respect to the coordinates $(x^i, x^{\bar{i}}) = (x^i, v^i)$.

We now consider the musical isomorphism $\omega^\sharp : T^*M \to TM$. Using

$$\begin{gathered}(\mathrm{d}\omega)_{skl} = \frac{1}{3}(\partial_s\omega_{kl} + \partial_k\omega_{ls} + \partial_l\omega_{sk}) = 0, \\ \omega_{ij} = -\omega_{ji}, \quad \omega^{ij} = -\omega^{ji}, \quad \omega^{is}\omega_{sj} = \delta_j^i,\end{gathered} \tag{3.2}$$

from (1.2) and (3.1) we see that the pullback of ${}^C\omega$ by $\omega^\sharp$ is a 2-form $(\omega^\sharp)^*\ {}^C\omega$ on $T^*(M)$ and has components

$$\begin{aligned}
((\omega^\sharp)^*\ {}^C\omega_T)_{kl} &= A_k^I A_l^J ({}^C\omega_T)_{IJ} \\
&= A_k^i A_l^j ({}^C\omega_T)_{ij} + A_k^{\bar{i}} A_l^j ({}^C\omega_T)_{\bar{i}j} + A_k^i A_l^{\bar{j}} ({}^C\omega_T)_{i\bar{j}} \\
&= \delta_k^i \delta_l^j v^s \partial s\omega_{ij} + p_s(\partial_k\omega^{is})\delta_l^j\omega_{ij} + \delta_k^i p_s(\partial_l\omega^{js})\omega_{ij} \\
&= v^s\partial_s\omega_{kl} + p_s((\partial_k\omega^{is})\omega_{il} - (\partial_l\omega^{sj})\omega_{kj}) \\
&= p_t\omega^{ts}\partial_s\omega_{kl} - p_s(\omega^{si}\partial_k\omega_{il} - \omega^{sj}\partial_l\omega_{jk}) \\
&= p_t\omega^{ts}(\partial_s\omega_{kl} - \partial_k\omega_{sl} + \partial_l\omega_{sk}) \\
&= 3p_t\omega^{ts}(\mathrm{d}\omega)_{skl} = 0, \\
((\omega^\sharp)^*\ {}^C\omega_T)_{k\bar{l}} &= A_k^i A_{\bar{l}}^{\bar{j}} ({}^C\omega_T)_{i\bar{j}} = \delta_k^i \omega^{jl}\omega_{ij} = \delta_k^i \delta_i^l = \delta_k^l\ , \\
((\omega^\sharp)^*\ {}^C\omega_T)_{\bar{k}l} &= A_{\bar{k}}^{\bar{i}} A_l^j ({}^C\omega_T)_{\bar{i}j} = \omega^{ik}\delta_l^j\omega_{ij} = -\delta_j^k\delta_l^j = -\delta_l^k\ , \\
((\omega^\sharp)^*\ {}^C\omega_T)_{\bar{k}\bar{l}} &= 0
\end{aligned}$$

or

$$(\omega^\sharp)^*\ {}^C\omega_T = (((\omega^\sharp)^*\ {}^C\omega_T)_{KL}) = \begin{pmatrix} 0 & \delta_k^l \\ -\delta_l^k & 0 \end{pmatrix}.$$

From here follows that the pullback $(\omega^\sharp)^*\ {}^C\omega_T$ coincides with the symplectic form $\widetilde{\omega} = \mathrm{d}p = \mathrm{d}p_i \wedge \mathrm{d}x^i$. Thus we have the following theorem.

Theorem 3.1 *Let (M,ω) be a symplectic manifold. The natural symplectic structure* $\mathrm{d}p = \mathrm{d}p_i \wedge \mathrm{d}x^i$ *on cotangent bundle T^*M is a pullback by $\omega^\sharp$ of complete lift of ω to tangent bundle TM, i.e.,* $(\omega^\sharp)^{*}\ {}^{C}\omega_T = \mathrm{d}p$.

A diffeomorphism between any two symplectic manifolds $f : (M,\omega) \to (N,\acute{\omega})$ is called symplectomorphism if $f^*\acute{\omega} = \omega$, where f^* is the pullback of f. Since $\mathrm{d}^C\omega_T =^C (\mathrm{d}\omega)_T = 0$ (see [18, p.25]), from Theorem 3.1 we have the following corollary.

Corollary 3.1 *The musical isomorphism* $\omega^\sharp : (T^*M, \mathrm{d}p) \to (TM,^C \omega_T)$ *is a symplectomorphism.*

4 Transform of Tensor Fields of Type $(1,1)$

Let (M,ω) be a symplectic manifold with almost complex structure φ ($\varphi^2 = -I$). If the 2-form ω satisfies the purity condition $\omega(\varphi X, Y) = \omega(X, \varphi Y)$, i.e., $(\omega\circ\varphi)(X,Y) = -(\omega\circ\varphi)(Y,X)$, then the triple (M,ω,φ) is called $\mathfrak{A}$-manifold according to the terminology accepted in [6] (also, see [16, p.31]). We call

$$\Omega(X,Y) = (\omega\circ\varphi)(X,Y) = \omega(\varphi Y, X)$$

the twin 2-form associated with ω.

Let $\mathbb{C}$ be a complex algebra and $\omega^* = (\omega^*_{v_1 v_2}),\ v_1, v_2 = 1,\cdots,r$ be a complex tensor field of type $(0,2)$ on holomorphic (analytic) complex manifold $\mathfrak{X}_r(\mathbb{C})$. Then the real model of ω^* is a tensor field $\omega = (\omega_{j_1 j_2}),\ j_1, j_2 = 1,\cdots,2r$ on M such that

$$\omega(\varphi X_1, X_2) = \omega(X_1, \varphi X_2)$$

for any vector fields X_1, X_2. Such tensor fields are said to be pure with respect to . They were studied by many authors (see [5, 7–9, 12–14]).

The Φ_φ-operator applied to a pure tensor field ω is defined by (see [10, 15])

$$\begin{aligned}(\Phi_\varphi\omega)(X,Y_1,Y_2) &= (\varphi X)(\omega(Y_1,Y_2)) - X(\omega(\varphi Y_1, Y_2)) \\ &\quad + \omega((L_{Y_1}\varphi)X, Y_2) + \omega(Y_1, (L_{Y_2}\varphi)X)\end{aligned}$$

and has the local expression

$$\begin{aligned}(\Phi_\varphi\omega)_{kij} &= \varphi_k^m \partial_m \omega_{ij} - \partial_k(\omega\circ\varphi)_{ij} \\ &\quad + \omega_{mj}\partial_i\varphi_k^m + \omega_{im}\partial_j\varphi_k^m,\end{aligned} \tag{4.1}$$

where $\Phi_\varphi\omega$ is a tensor field of type $(0,3)$, L_X is the Lie derivation with respect to X and

$$(\omega\circ\varphi)_{ij} = \varphi_i^m \omega_{mj}.$$

Let M on be given the integrable almost complex structure φ. For complex tensor field, ω^* of type $(0,2)$ on $\mathfrak{X}_r(\mathbb{C})$ to be $\mathbb{C}$-holomorphic tensor field, it is necessary and sufficient that $\Phi_\varphi\omega = 0$ (see [11, p.57]). Let M now be a manifold with non-integrable almost complex structure φ. In this case, when $\Phi_\varphi\omega = 0$, ω is said to be almost holomorphic. If the symplectic 2-form ω of $\mathfrak{A}$-manifold (M,ω,J) satisfies the almost holomorphicity condition $\Phi_\varphi\omega = 0$, then it is called an almost holomorphic symplectic 2-form. We call $\mathfrak{A}$-manifold admitting such a 2-form an almost holomorphic $\mathfrak{A}$-manifold.

Let $\varphi = \varphi_j^i \partial_i \otimes \mathrm{d}x^j$ be a field of tensor field of type $(1,1)$ in $U \subset M$. The complete lift ${}^C\varphi_{TM}$ of φ to tangent bundle is completely determined by ${}^C\varphi_{TM}({}^CX) = {}^C(\varphi(X))_{TM}$. In an analogous way, the complete lift ${}^C\varphi_{T^*M}$ of φ to cotangent bundle is completely determined by

$$ {}^C\varphi_{T^*M}({}^CX) = {}^C(\varphi(X))_{T^*M} + \gamma(L_X\varphi), $$

where $\gamma(L_X\varphi)$ is a vertical vector field on T^*M with components

$$ \gamma(L_X\varphi) = \sum_{i=1}^{n} p_s(L_X\varphi)_i^s \partial_{\overline{i}}. $$

The complete lift of φ to tangent and cotangent bundles are given, respectively, by (see [18])

$$ {}^C\varphi_{TM} = (({}^C\varphi_{TM})_J^I) = \begin{pmatrix} \varphi_j^i & 0 \\ v^s\partial_s\varphi_j^i & \varphi_j^i \end{pmatrix} $$

and

$$ {}^C\varphi_{T^*M} = (({}^C\varphi_{T^*M})_J^I) = \begin{pmatrix} \varphi_j^i & 0 \\ p_s(\partial_j\varphi_i^s - \partial_i\varphi_j^s) & \varphi_i^j \end{pmatrix} $$

with respect to the induced coordinates $(x^j, x^{\overline{j}}) = (x^j, v^j)$ and $(x^j, x^{\overline{j}}) = (x^j, p_j)$.

Using (1.1)–(1.2), (3.2) and $\varphi_j^m\omega_{mk} = \varphi_k^m\omega_{jm}$, for transform of ${}^C\varphi_{TM}$ by $\omega^\sharp : T^*M \to TM$ we have

$$ (\omega^\sharp)^* \, {}^C\varphi_{TM} = ((\widetilde{\varphi}_{T^*M})_L^J) = (\widetilde{A}\,{}_I^J A_L^K ({}^C\varphi_{TM})_K^I) $$

or

$$
\begin{aligned}
(\widetilde{\varphi}_{T^*M})_l^j &= \varphi_l^j, \quad (\widetilde{\varphi}_{T^*M})_{\overline{l}}^j = 0, \\
(\widetilde{\varphi}_{T^*M})_{\overline{l}}^{\overline{j}} &= \omega_{ji}\omega^{kl}\varphi_k^i = \varphi_j^l, \\
(\widetilde{\varphi}_{T^*M})_l^{\overline{j}} &= v^s(\partial_i\omega_{js})\varphi_l^i + \omega_{ji}v^s\partial_s\varphi_l^i + \omega_{ji}p_s(\partial_l\omega^{ks})\varphi_k^i \\
&= v^s((\Phi_\varphi\omega)_{ljs} + \partial_l(\omega\circ\varphi)_{js} - \omega_{is}\partial_j\varphi_l^i) + \omega_{ji}p_s(\partial_l\omega^{ks})\varphi_k^i \\
&= v^s(\Phi_\varphi\omega)_{ljs} - p_i\partial_j\varphi_l^i + v^s\partial_l(\varphi_j^m\omega_{ms}) + \omega_{ji}p_s(\partial_l\omega^{ks})\varphi_k^i \\
&= v^s(\Phi_\varphi\omega)_{ljs} - p_i\partial_j\varphi_l^i + v^s(\partial_l\varphi_j^m)\omega_{ms} \\
&\quad + v^s\varphi_j^m(\partial_l\omega_{ms}) + \omega_{jm}p_s(\partial_l\omega^{ks})\varphi_k^m \\
&= v^s(\Phi_\varphi\omega)_{ljs} - p_i\partial_j\varphi_l^i + p_m\partial_l\varphi_j^m \\
&\quad + v^s(\partial_l\omega_{ms})\varphi_j^m + \omega_{mk}p_s(\partial_l\omega^{ks})\varphi_j^m \\
&= v^s(\Phi_\varphi\omega)_{ljs} + p_m(\partial_l\varphi_j^m - \partial_j\varphi_l^m) \\
&\quad + v^s(\partial_l\omega_{ms})\varphi_j^m - \omega^{ks}p_s(\partial_l\omega_{mk})\varphi_j^m \\
&= v^s(\Phi_\varphi\omega)_{ljs} + p_m(\partial_l\varphi_j^m - \partial_j\varphi_l^m) \\
&\quad + v^s(\partial_l\omega_{ms})\varphi_j^m - v^k(\partial_l\omega_{mk})\varphi_j^m \\
&= v^s(\Phi_\varphi\omega)_{ljs} + p_m(\partial_l\varphi_j^m - \partial_j\varphi_l^m).
\end{aligned}
$$

Thus, if $\Phi_\varphi \omega = 0$, then the transform $(\omega^\sharp)^* \ {}^C\varphi_{TM}$ of ${}^C\varphi_{TM}$ coincides with ${}^C\varphi_{T^*M}$. Thus we have the following theorem.

Theorem 4.1 *Let (M, ω, φ) be a symplectic $\mathfrak{A}$-manifold and $\omega^\sharp : T^*M \to TM$ be a musical isomorphism between cotangent and tangent bundles. If the symplectic $\mathfrak{A}$-manifold is an almost holomorphic ($\Phi_\varphi \omega = 0$), then the complete lift ${}^C\varphi_{T^*M}$ is a transform of ${}^C\varphi_{TM}$ by $\omega^\sharp$, i.e., $(\omega^\sharp)^* \ {}^C\varphi_{TM} = {}^C \varphi_{T^*M}$.*

In the case of integrability of φ, the complete lifts ${}^C\varphi_{TM}$ and ${}^C\varphi_{T^*M}$ are complex structures on tangent and cotangent bundles, respectively (see [18, p.37, p.256]), i.e., $(T^*M, {}^C \varphi_{T^*M})$ and $(TM, {}^C \varphi_{TM})$ are complex manifolds. Since $\widetilde{A}^{-1} = A$ (see (1.1)–(1.2)), the condition

$$(\omega^\sharp)^* \ {}^C\varphi_{TM} = (\widetilde{A}\ _I^J A_L^K ({}^C\varphi_{TM})_K^I) = ((\widetilde{\varphi}_{T^*M})_L^J) = {}^C \varphi_{T^*M}$$

in the Theorem 4.1 can be written in the following form

$${}^C\varphi_{TM} \circ (\omega^\sharp)_* = (\omega^\sharp)_* \circ \ {}^C\varphi_{T^*M},$$

where $(\omega^\sharp)_* = (A_J^I)$. From here it is clear that the mapping $\omega^\sharp : T^*M \to TM$ is a holomorphic. Thus we have the following corollary.

Corollary 4.1 *Let (M, ω, φ) be a holomorphic symplectic $\mathfrak{A}$-manifold. If φ is an integrable almost complex structure, then the musical isomorphism $\omega^\sharp$ (or ω^b) is a holomorphic mapping.*

On the other hand, from (4.1) we obtain

$$\begin{aligned}
(\Phi_\varphi \omega)_{kij} &= \varphi_k^m \partial_m \omega_{ij} - \partial_k (\omega \circ \varphi)_{ij} + \omega_{mj} \partial_i \varphi_k^m + \omega_{im} \partial_j \varphi_k^m \\
&= \varphi_k^m (\partial_m \omega_{ij} - \partial_i \omega_{mj} - \partial_j \omega_{im}) + (\partial_i \omega_{mj}) \varphi_k^m \\
&\quad + (\partial_j \omega_{im}) \varphi_k^m + \omega_{mj} \partial_i \varphi_k^m + \omega_{im} \partial_j \varphi_k^m - \partial_k (\varphi_i^m \omega_{mj}) \\
&= \varphi_k^m (\partial_m \omega_{ij} + \partial_i \omega_{jm} + \partial_j \omega_{mi}) \\
&\quad + \partial_i (\varphi_k^m \omega_{mj}) + \partial_j (\varphi_k^m \omega_{im}) - \partial_k (\varphi_i^m \omega_{mj}) \\
&= 3 \varphi_k^m (\mathrm{d}\omega)_{mij} + \partial_i \Omega_{kj} + \partial_j (\varphi_i^m \omega_{mk}) - \partial_k \Omega_{ij} \\
&= 3 \varphi_k^m (\mathrm{d}\omega)_{mij} + \partial_i \Omega_{kj} + \partial_j \Omega_{ik} + \partial_k \Omega_{ji} \\
&= 3 (\varphi_k^m (\mathrm{d}\omega)_{mij} + (\mathrm{d}\Omega)_{ikj}),
\end{aligned}$$

which on symplectic $\mathfrak{A}$-manifold ($\mathrm{d}\omega = 0$) has the form

$$(\Phi_\varphi \omega)(X, Y_1, Y_2) = 3 (\mathrm{d}\Omega)(Y_1, X, Y_2),$$

where $\Omega = \omega \circ \varphi$ is the twin 2-form. Thus we have the following theorem.

Theorem 4.2 *A symplectic $\mathfrak{A}$-manifold (M, ω, φ) is holomorphic if and only if the twin 2-form $\Omega = \omega \circ \varphi$ is closed.*

From Theorems 4.1–4.2 we have the following corollary.

Corollary 4.2 *If $\Omega = \omega \circ \varphi$ is a closed twin 2-form on $\mathfrak{A}$-manifold (M, ω, φ), then ${}^C\varphi_{T^*M}$ is a transform of ${}^C\varphi_{TM}$ by musical isomorphism $\omega^\sharp : T^*M \to TM$.*

5 Transform of Skew-Symmetric Tensor Field of Type $(1, 2)$

Let $S = S_{ij}^k \partial_k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j$ be a vector valued 2-form on a symplectic manifold (M, ω). The symplectic 2-form ω is called a pure with respect to if

$$\omega(S(X,Z),Y) = \omega(X, S(Y,Z)), \quad S_{il}^m \omega_{mj} = S_{jl}^m \omega_{im}. \tag{5.1}$$

Since

$$S(X,Y) = -S(Y,X),$$

from here follows

$$\omega(S(Z,X,),Y) = -\omega(S(X,Z),Y) = -\omega(X,S(Y,Z)) = \omega(X,S(Z,Y)),$$

or

$$S_{li}^m \omega_{mj} = S_{lj}^m \omega_{im}. \tag{5.2}$$

The Yano-Ako operator applied to a pure tensor field is defined by (see [10, 17])

$$\begin{aligned}
&(\Phi_S \omega)(X_1, X_2, Y_1, Y_2) \\
=&(L_{S(X_1,X_2)}\omega)(Y_1,Y_2) - (L_{X_1}(\omega \circ S))(Y_1, X_2, Y_2) \\
&- (L_{X_2}(\omega \circ S))(X_1, Y_1, Y_2)) + (\omega \circ S)([X_1, X_2], Y_1, Y_2)
\end{aligned}$$

and has the local expression

$$\begin{aligned}
(\Phi_S \omega)_{jihs} = S_{ji}^m \partial_m \omega_{hs} - (\partial_j S_{hi}^m)\omega_{ms} - S_{hi}^m \partial_j \omega_{ms} - (\partial_i S_{jh}^m)\omega_{ms} \\
- S_{jh}^m \partial_i \omega_{ms} + \omega_{ms} \partial_h S_{ji}^m + \omega_{hm} \partial_s S_{ji}^m \ .
\end{aligned}$$

The objects $(\Phi_S \omega)_{jihs}$ are components of tensor field of type $(0, 4)$ if and only if ω is pure with respect to S.

The complete lift ${}^C S_{TM}$ of S to tangent bundle is completely determined by (see [18])

$${}^C S_{TM}({}^C X, {}^C Y) = {}^C\ (S(X,Y))_{TM}$$

for any vector fields X,Y on M, and has non-zero components

$$({}^C S_{TM})_{ji}^h = ({}^C S_{TM})_{\overline{j}i}^{\overline{h}} = ({}^C S_{TM})_{j\overline{i}}^{\overline{h}} = S_{ji}^h, \quad ({}^C S_{TM})_{ji}^{\overline{h}} = v^{\overline{m}} \partial_m S_{ji}^h$$

with respect to the induced coordinates

$$(x^j, x^{\overline{j}}) = (x^j, v^j).$$

Now, we consider the transform of ${}^C S_{TM}$ by

$$\omega^\sharp : T^*M \to TM,$$

i.e.,

$$(\omega^\sharp)^*\ {}^C S_{TM} = ((\widetilde{S}_{T^*M})_{JI}^H) = (\widetilde{A}\ {}_M^H A_J^K A_I^P ({}^C S_{TM})_{KP}^M) \ .$$

Using (3.2) and (5.1)–(5.2) we have

$$
\begin{aligned}
(\widetilde{S}_{T^*M})^h_{ji} &= S^h_{ji}\ ,\\
({}^C\widetilde{S}_{T^*M})^h_{\bar{j}i} &= ({}^C\widetilde{S}_{T^*M})^h_{j\bar{i}} = ({}^C\widetilde{S}_{T^*M})^h_{\bar{j}\,\bar{i}} = ({}^C\widetilde{S}_{T^*M})^{\bar{h}}_{\bar{j}\,\bar{i}} = 0,\\
({}^C\widetilde{S}_{T^*M})^{\bar{h}}_{j\bar{i}} &= \widetilde{A}\,{}^{\bar{h}}_{\overline{m}}A^k_j A^{\bar{p}}_{\bar{i}}({}^C S_{TM})^{\overline{m}}_{k\bar{p}} = \omega_{hm}\delta^k_j\omega^{pi}S^m_{kp}\\
&= \omega_{hm}\omega^{pi}S^m_{ki} = \omega^{kj}\omega_{mk}S^m_{hi} = \delta^j_m S^m_{hi} = S^i_{jh},\\
({}^C\widetilde{S}_{T^*M})^{\bar{h}}_{\bar{j}i} &= \widetilde{A}\,{}^{\bar{h}}_{\overline{m}}A^{\bar{k}}_{\bar{j}}A^p_i({}^C S_{TM})^{\overline{m}}_{\bar{k}p} = \omega_{hm}\omega^{kj}\delta^p_i S^m_{kp}\\
&= \omega_{hm}\omega^{kj}S^m_{jp} = \omega^{pi}\omega_{mp}S^m_{jh} = \delta^i_m S^m_{jh} = S^j_{hi},\\
({}^C\widetilde{S}_{T^*M})^{\bar{h}}_{ji} &= \widetilde{A}\,{}^{\bar{h}}_{m}A^k_j A^p_i({}^C S_{TM})^m_{kp} + \widetilde{A}\,{}^{\bar{h}}_{\overline{m}}A^k_j A^p_i({}^C S_{TM})^{\overline{m}}_{kp}\\
&\quad + \widetilde{A}\,{}^{\bar{h}}_{\overline{m}}A^k_j A^{\bar{p}}_i({}^C S_{TM})^{\overline{m}}_{k\bar{p}} + \widetilde{A}\,{}^{\bar{h}}_{\overline{m}}A^{\bar{k}}_j A^p_i({}^C S_{TM})^{\overline{m}}_{\bar{k}p}\\
&= v^s(\partial_m\omega_{hs})\delta^k_j\delta^p_i S^m_{kp} + \omega_{hm}\delta^k_j\delta^p_i v^s\partial_s S^m_{kp}\\
&\quad + \omega_{hm}\delta^k_j p_s(\partial_i\omega^{ps})S^m_{kp} + \omega_{hm}p_s(\partial_j\omega^{ks})\delta^p_i S^m_{kp}\\
&= v^s(S^m_{ji}\partial_m\omega_{hs} + \omega_{hm}\partial_s S^m_{ji}) + \omega_{mt}p_s(\partial_i\omega^{ts})S^m_{jh} \qquad (5.3)\\
&\quad + \omega_{mk}p_s(\partial_j\omega^{ks})S^m_{hi}\\
&= v^s(S^m_{ji}\partial_m\omega_{hs} + \omega_{hm}\partial_s S^m_{ji}) - p_s\omega^{ts}(\partial_i\omega_{mt})S^m_{jh}\\
&\quad - \omega^{ks}p_s(\partial_j\omega_{mk})S^m_{hi}\\
&= v^s(S^m_{ji}\partial_m\omega_{hs} + \omega_{hm}\partial_s S^m_{ji})\\
&\quad - v^t(\partial_i\omega_{mt})S^m_{jh} - v^k(\partial_j\omega_{mk})S^m_{hi}\\
&= v^s(S^m_{ji}\partial_m\omega_{hs} + \omega_{hm}\partial_s S^m_{ji}\\
&\quad - (\partial_i\omega_{ms})S^m_{jh} - (\partial_j\omega_{ms})S^m_{hi})\\
&= v^s(\Phi_S\omega)_{jihs} + v^s(\partial_j S^m_{hi})\omega_{ms}\\
&\quad + v^s(\partial_i S^m_{jh})\omega_{ms} - v^s(\partial_h S^m_{ji})\omega_{ms}\\
&= v^s(\Phi_S\omega)_{jihs} - p_m(\partial_j S^m_{ih} + \partial_i S^m_{hj} + \partial_h S^m_{ji}).
\end{aligned}
$$

On the other hand, it is well known that the complete lift ${}^C S_{T^*M}$ of skew-symmetric tensor field of type $(1,2)$ to cotangent bundle has components of the form (see [18 , p.245]):

$$
\begin{aligned}
({}^C S_{T^*M})^h_{ji} &= S^h_{ji}\ , ({}^C S_{T^*M})^h_{\bar{j}i} = ({}^C S_{T^*M})^h_{j\bar{i}}\\
&= ({}^C S_{T^*M})^h_{\bar{j}\,\bar{i}} = ({}^C S_{T^*M})^{\bar{h}}_{\bar{j}\,\bar{i}} = 0,\\
({}^C S_{T^*M})^{\bar{h}}_{j\bar{i}} &= S^i_{jh}, \quad ({}^C S_{T^*M})^{\bar{h}}_{\bar{j}i} = S^j_{hi},\\
({}^C S_{T^*M})^{\bar{h}}_{ji} &= -p_m(\partial_j S^m_{ih} + \partial_i S^m_{hj} + \partial_h S^m_{ji}).
\end{aligned}
$$

From here and (5.3) follows that if $(\Phi_S\omega)_{jihs} = 0$, then $(\omega^\sharp)^*\ {}^C S_{TM} = ^C S_{T^*M}$. Thus we have the following theorem.

Theorem 5.1 *Let ω be a pure symplectic 2-form with respect to the skew-symmetric tensor S of type $(1,2)$ on a symplectic manifold (M,ω), and let ${}^C S_{TM}$ and ${}^C S_{T^*M}$ be complete lifts of S to the tangent and cotangent bundles, respectively. If the symplectic 2-form satisfies the*

Yano-Ako equation

$$S^m_{ji}\partial_m\omega_{hs} - (\partial_j S^m_{hi})\omega_{ms} - S^m_{hi}\partial_j\omega_{ms} - (\partial_i S^m_{jh})\omega_{ms}$$
$$- S^m_{jh}\partial_i\omega_{ms} + \omega_{ms}\partial_h S^m_{ji} + \omega_{hm}\partial_s S^m_{ji} = 0,$$

then the complete lift ${}^C S_{T^*M}$ *is a transform of* ${}^C S_{TM}$ *by a musical isomorphism* $\omega^\sharp : T^*M \to TM$.

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