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Blow up for Systems of Wave Equations in Exterior Domain

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Abstract In this paper the author studies the initial boundary value problem of semilinear wave systems in exterior domain in high dimensions ($n \geq 3$). Blow up result is established and what is more, the author gets the upper bound of the lifespan. For this purpose the test function method is used.

Keywords Wave equations, Exterior domain, Blow up, Lifespan
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1 Introduction and Main Results

In this paper, we consider the initial boundary value problem of semilinear wave equations in exterior domain:

$$\begin{cases} u_{tt} - \Delta u = |v|^p, & \text{in} \mathbb{R}^+ \times \Omega^{\rm c}, \\ v_{tt} - \Delta v = |u|^q, & \text{in} \mathbb{R}^+ \times \Omega^{\rm c}, \\ u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), & \text{in } \Omega^{\rm c}, \\ v(0,x) = \varepsilon v_0(x), \quad v_t(0,x) = \varepsilon v_0(x), & \text{in } \Omega^{\rm c}, \\ u(t,x) \mid_{\partial\Omega} = 0, \\ v(t,x) \mid_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $u(t,x), v(t,x)$ are unknown functions of the variable $t \in \mathbb{R}^+$ and $x \in \Omega^{\rm c}$. Ω is a smooth compact obstacle in $\mathbb{R}^n (n \geq 3)$, and $\Omega^{\rm c}$ is its complement. Without loss of generality, we assume that $0 \in \Omega \subsetneqq B_R$, where $B_R = \{x \mid |x| \leq R\}$ is a ball of radius R centered at the origin. The smallness of initial data can be measured by the constant ε satisfying $0 < \varepsilon \leq 1$ and u_0, u_1, v_0, v_1 are compactly supported nonnegative functions, i.e.,

$$\begin{cases} u_0, u_1, v_0, v_1 \geq 0, \quad \forall x \in \Omega^{\rm c}, \\ u_0, v_0 \not\equiv 0, \quad \forall x \in \Omega^{\rm c}, \\ \text{supp}\{u_0, u_1, v_0, v_1\} \in B_R. \end{cases} \tag{1.2}$$

This problem can date back to the famous Strauss' conjecture (see [19]): For the Cauchy problem of the semilinear wave equation with small initial date

$$u_{tt} - \Delta u = |u|^p, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

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there exists a critical power $p_c(n)$, which is the positive root of the following quadratic equation:

$$(n-1)p^2-(n+1)p-2=0, \quad n\geq 2,$$

which means that blow up happens when $1<p\leq p_c(n)$, global solutions exist if $p>p_c(n)$. Strauss' conjecture was first investigated by John [7] and recently ended after decades by a series of hardworking (see [4–5, 9, 14, 16–17, 20–24]).

For the initial boundary value problem of a single wave equation in exterior domain, Zhou and Han [25] gave the lifespan estimation of the solutions when $1<p<p_c(n)$ and $n\geq 3$. The case $1<p<p_c(2)$ was considered by Li and Wang [13]. Lai and Zhou [10–12] showed the blow up of $p=p_c(3),\ p=p_c(n),\ n\geq 5$ and $p=p_c(2)$, moreover, they gave the lifespan estimate from above for the first two cases. For dimension one, Lai [8] got the blow up for initial boundary value problem with a nonlinear memory. For global existence we refer the reader to [1, 6, 18].

The Cauchy problem for high dimension ($n\geq 4$) systems was considered by Georgiev, Takamura and Zhou [3].

For high dimension ($n\geq 3$) systems, there exists a curve $F(p,q)=0$ which separates the domains of existence and blow-up of the solutions in the (p,q) plane (see [15]). And we are concerned for the blow-up and the lifespan $T(\varepsilon)$ of the solutions. More precisely, the blow-up happens when

$$F(p,q)=F(p,q,n)=\max\Big\{\frac{q+2+\frac{1}{p}}{pq-1},\frac{p+2+\frac{1}{q}}{pq-1}\Big\}-\frac{n-1}{2}>0.$$

We establish our theorem as follows.

Theorem 1.1 *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ satisfy the exterior ball conditions and the exponents $p,q>1$ satisfy $F(p,q)>0$. Suppose that problem (1.1)–(1.2) has a solution $(u,u_t),(v,v_t)\in C([0,T),H^1(\Omega^c)\times L^2(\Omega^c))$ such that*

$$\operatorname{supp}(u,u_t,v,v_t)\subset\{(t,x)\mid |x|\leq t+R\}\cap(\mathbb{R}^+\times\Omega^c).$$

Then $T<\infty$, and there exists a positive constant C which is independent of ε such that

$$T(\varepsilon)\leq C\varepsilon^{-\frac{1}{F(p,q)}}. \tag{1.3}$$

Here and hereafter, C and C_i denote positive constants and may change from line to line.

2 Preliminaries

To prove our theorem, we first introduce a lemma.

Lemma 2.1 *Let $1<p\leq q$ and $X,Y\in C^2([0,\tau_0])$. Assume that τ_0 is large enough and X,Y satisfies the following ordinary differential inequalities: $\forall\tau\leq\tau_0$*

$$\begin{cases} X''(\tau)\geq C_0\tau^{-n(p-1)}|Y(\tau)|^p,\\ Y''(\tau)\geq C_0\tau^{-n(q-1)}|X(\tau)|^q,\\ X(\tau)\geq C_0\tau^{n+1-\frac{p(n-1)}{2}},\\ Y(\tau)\geq C_1\tau. \end{cases} \tag{2.1}$$

Then there exists a positive constant C_2 independent of C_1 such that

$$\tau_0 \leq C_2,$$

when $F(p,q) > 0$.

It is easy to prove this lemma by exchanging p, q and set $T = \tau,\ F(t) = Y(t),\ G(t) = X(t),\ \alpha = n(q-1), \beta = n(p-1),\ s = n + 1 - \frac{(n-1)p}{2} \geq 1$ in Lemma 2.1 of [15]. Then the blow-up condition can be rewritten as

$$p(n(q-1)-2) + n(p-1) - 2 < \Big(n + 1 - \frac{(n-1)p}{2}\Big)(pq-1),$$

which is equivalent to $F(p,q) > 0$ when $p \leq q$.

The following lemmas are from [2, 21, 25].

Lemma 2.2 *There exists a function $\phi_0(x) \in C^2(\Omega^{\mathrm{c}})$ satisfying the following boundary value problem*:

$$\begin{cases} \Delta\phi_0(x) = 0, \quad \textit{in } \Omega^{\mathrm{c}}, \\ \phi_0(x) \mid_{\partial\Omega} = 0, \\ |x| \to \infty, \quad \phi_0(x) \to 1. \end{cases} \tag{2.2}$$

Moreover, $\phi_0(x)$ satisfies that $\forall x \in \Omega^{\mathrm{c}},\ 0 < \phi_0(x) < 1$.

Lemma 2.3 *There exists a function $\phi_1(x) \in C^2(\Omega^{\mathrm{c}})$ satisfying the following boundary value problem*:

$$\begin{cases} \Delta\phi_1(x) = \phi_1(x), \quad \textit{in } \Omega^{\mathrm{c}}, \\ \phi_1(x) \mid_{\partial\Omega} = 0, \\ |x| \to \infty, \quad \phi_1(x) \to \displaystyle\int_{S^{n-1}} \mathrm{e}^{x\cdot\omega}\mathrm{d}\omega. \end{cases} \tag{2.3}$$

Moreover, there exists a positive constant C_3 such that $\phi_1(x)$ satisfies that $\forall x \in \Omega^{\mathrm{c}},\ 0 < \phi_1(x) < C_3(1+|x|)^{-\frac{n-1}{2}} \cdot \mathrm{e}^{|x|}$.

We define a test function $\psi_1(x,t)$ as

$$\psi_1(t,x) = \phi_1(x)\mathrm{e}^{-t}, \quad \forall t \geq 0,\ x \in \Omega^{\mathrm{c}}. \tag{2.4}$$

Then we have the following lemma.

Lemma 2.4 *Let $p > 1,\ p' = \frac{p}{p-1}$. Then $\forall t \geq 0$,*

$$\int_{\Omega^{\mathrm{c}}\cap\{|x|\leq t+R\}} [\psi_1(t,x)]^{p'}\mathrm{d}x \leq C_4(t+R)^{n-1-\frac{(n-1)p'}{2}}. \tag{2.5}$$

Moreover, we have the following lemmas.

Lemma 2.5 *Let $p > 1,\ p' = \frac{p}{p-1}$ and Ω satisfy the condition in Theorem 1.1. Then $\forall t \geq 0$,*

$$\int_{\Omega^{\mathrm{c}}\cap\{|x|\leq t+R\}} [\phi_0(x)]^{-\frac{1}{p-1}}[\psi_1(t,x)]^{p'}\mathrm{d}x \leq C_5(t+R)^{n-1-\frac{(n-1)p'}{2}}. \tag{2.6}$$

Lemma 2.6 *Let $p > 1$. Then $\forall t \geq 0$,*

$$\int_{\Omega^c \cap \{|x| < \leq t+R\}} \psi_1(t,x)\mathrm{d}x \leq C_6(t+R)^{\frac{n-1}{2}}. \tag{2.7}$$

.

3 Proof of Theorem 1.1

We give the proof of Theorem 1.1 in this section. We first define functions

$$\begin{cases} F_0(t) = \displaystyle\int_{\Omega^c} u(t,x)\phi_0(x)\mathrm{d}x, \\ F_1(t) = \displaystyle\int_{\Omega^c} u(t,x)\psi_1(x,t)\mathrm{d}x, \\ G_0(t) = \displaystyle\int_{\Omega^c} v(t,x)\phi_0(x)\mathrm{d}x, \\ G_1(t) = \displaystyle\int_{\Omega^c} v(t,x)\psi_1(x,t)\mathrm{d}x, \end{cases} \tag{3.1}$$

where $u(t,x), v(t,x), \phi_0(x), \psi_1(t,x)$ are defined as before. The assumptions on $u(t,x), v(t,x)$ imply that $F_0(t), F_1(t), G_0(t), G_1(t)$ are well-defined C^2 functions for all $t \geq 0$. We want to derive nonlinear ordinary differential inequalities of those new functions as the form in Lemma 2.1, then we can easily proof Theorem 1.1.

We need one more lemma as below.

Lemma 3.1 *Let* $(u_0,u_1),(v_0,v_1)$ *satisfy* (1.2). *Suppose that problem* (1.1) *has a solution* $(u,u_t),(v,v_t) \in C([0,T), H^1(\Omega^c) \times L^2(\Omega^c))$*, such that*

$$\mathrm{supp}(u,u_t,v,v_t) \subset \{(t,x) \mid |x| \leq t+R\} \cap (\mathbb{R}^+ \times \Omega^c).$$

Then we have for all $t \geq 0$,

$$F_1(t) \geq \frac{1}{2}(1-\mathrm{e}^{-2t})\varepsilon \int_{\Omega^c} [u_0(x)+u_1(x)]\phi_1(x)\mathrm{d}x + \mathrm{e}^{-2t}\varepsilon \int_{\Omega^c} u_0(x)\phi_1(x)\mathrm{d}x \geq \varepsilon C_7 > 0, \tag{3.2}$$

$$G_1(t) \geq \frac{1}{2}(1-\mathrm{e}^{-2t})\varepsilon \int_{\Omega^c} [v_0(x)+v_1(x)]\phi_1(x)\mathrm{d}x + \mathrm{e}^{-2t}\varepsilon \int_{\Omega^c} v_0(x)\phi_1(x)\mathrm{d}x \geq \varepsilon C_8 > 0. \tag{3.3}$$

Proof We multiply the first equation in (1.1) by the test function $\psi_1(t,x)$ and integrate over $[0,t] \times \Omega^c$, then use integration by parts and Lemma 2.3.

First, it is easy to verify that

$$\int_0^t \int_{\Omega^c} \psi_1(u_{tt} - \Delta u - |v|^p)\mathrm{d}x\mathrm{d}\tau = 0.$$

Noticing that $\psi_{1t} = -\psi_1$, we have

$$\begin{aligned} \int_0^t \int_{\Omega^c} \psi_1 u_{tt}\mathrm{d}x\mathrm{d}\tau &= \int_{\Omega^c} \int_0^t [(\psi_1 u_t)_t - \psi_{1t}u_t]\mathrm{d}\tau\mathrm{d}x \\ &= \int_{\Omega^c} \int_0^t \psi_1 u_t \mathrm{d}\tau\mathrm{d}x + \int_{\Omega^c} \psi_1 u_t\Big|_0^t \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega^c} \int_0^t [(\psi_1 u)_t - \psi_{1t} u] \mathrm{d}\tau \mathrm{d}x + \int_{\Omega^c} \psi_1 u_t \Big|_0^t \mathrm{d}x \\
&= \int_{\Omega^c} \int_0^t \psi_1 u \mathrm{d}\tau \mathrm{d}x + \int_{\Omega^c} \psi_1 u \Big|_0^t \mathrm{d}x + \int_{\Omega^c} \psi_1 u_t \Big|_0^t \mathrm{d}x \\
&= \int_{\Omega^c} \int_0^t \psi_1 u \mathrm{d}\tau \mathrm{d}x + \int_{\Omega^c} \psi_1 (u + u_t) \mathrm{d}x - \varepsilon \int_{\Omega^c} \phi_1 (u_0 + u_1) \mathrm{d}x.
\end{aligned}
$$

And

$$
\begin{aligned}
\int_0^t \int_{\Omega^c} \psi_1 \Delta u \mathrm{d}x \mathrm{d}\tau &= \int_0^t \int_{\Omega^c} [\nabla(\psi_1 \nabla u) - \nabla \psi_1 \cdot \nabla u] \mathrm{d}x \mathrm{d}\tau \\
&= \int_0^t \int_{\partial\Omega} \psi_1 \nabla u \cdot n \mathrm{d}S \mathrm{d}\tau - \int_0^t \int_{\Omega^c} \nabla \psi_1 \cdot \nabla u \mathrm{d}x \mathrm{d}\tau \\
&= - \int_0^t \int_{\Omega^c} \nabla \psi_1 \cdot \nabla u \mathrm{d}x \mathrm{d}\tau \\
&= - \int_0^t \int_{\Omega^c} [\nabla(\nabla \psi_1 u) - \Delta \psi_1 u] \mathrm{d}x \mathrm{d}\tau \\
&= \int_0^t \int_{\Omega^c} \Delta \psi_1 u \mathrm{d}x \mathrm{d}\tau - \int_0^t \int_{\partial\Omega} \nabla \psi_1 u \cdot n \mathrm{d}S \mathrm{d}\tau \\
&= \int_0^t \int_{\Omega^c} \psi_1 u \mathrm{d}x \mathrm{d}\tau.
\end{aligned}
$$

So we get

$$
\int_0^t \int_{\Omega^c} \psi_1 |v|^p \mathrm{d}x \mathrm{d}\tau = \int_{\Omega^c} \psi_1 (u + u_t) \mathrm{d}x - \varepsilon \int_{\Omega^c} \phi_1 (u_0 + u_1) \mathrm{d}x,
$$

where

$$
\begin{aligned}
\int_{\Omega^c} \psi_1 (u + u_t) \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^c} (\psi_1 u) \mathrm{d}x - \int_{\Omega^c} \psi_{1t} u \mathrm{d}x + \int_{\Omega^c} \psi_1 u \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^c} (\psi_1 u) \mathrm{d}x + 2 \int_{\Omega^c} \psi_1 u \mathrm{d}x \\
&= \frac{\mathrm{d}F_1(t)}{\mathrm{d}t} + 2F_1(t).
\end{aligned}
$$

Because of $\psi_1 > 0$, we have

$$
\begin{aligned}
\frac{\mathrm{d}F_1(t)}{\mathrm{d}t} + 2F_1(t) &= \int_0^t \int_{\Omega^c} \psi_1 |v|^p \mathrm{d}x \mathrm{d}\tau + \varepsilon \int_{\Omega^c} \phi_1 (u_0 + u_1) \mathrm{d}x \\
&\geq \varepsilon \int_{\Omega^c} \phi_1 (u_0 + u_1) \mathrm{d}x.
\end{aligned}
$$

Multiplying the above inequality with e^{2t}, one has

$$
\frac{\mathrm{d}(\mathrm{e}^{2t} F_1(t))}{\mathrm{d}t} \geq \mathrm{e}^{2t} \varepsilon \int_{\Omega^c} \phi_1 (u_0 + u_1) \mathrm{d}x.
$$

Then integrating over $[0, t]$, we get

$$
\mathrm{e}^{2t} F_1(t) - F_1(0) \geq \frac{1}{2} (\mathrm{e}^{2t} - 1) \varepsilon \int_{\Omega^c} \phi_1 (u_0 + u_1) \mathrm{d}x,
$$

where $F_1(0) = \int_{\Omega^c} u_0\psi_1(x,0)\mathrm{d}x = \varepsilon \int_{\Omega^c} \phi_1 u_0 \mathrm{d}x$. Hence we get (3.2), and (3.3) can be obtained in a similar way.

Next we show that $F_0(t), G_0(t)$ satisfy the ordinary differential inequalities as in (2.1). For this purpose, multiplying the first equation in (1.1) by $\phi_0(x)$ and integrating over Ω^c. We note that for a fixed t, $u(\cdot,t) \in H_0^1(D_t)$ where D_t is the support of $u(\cdot,t)$. Hence we can use integration by parts and Lemma 2.2.

First

$$\int_{\Omega^c} \phi_0[u_{tt} - \Delta u - |v|^p]\mathrm{d}x = 0,$$

where

$$\begin{aligned}
\int_{\Omega^c} \phi_0 u_{tt}\mathrm{d}x &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\Omega^c} \phi_0 u\mathrm{d}x = \frac{\mathrm{d}^2 F_0(t)}{\mathrm{d}t^2},\\
\int_{\Omega^c} \phi_0 \Delta u\mathrm{d}x &= \int_{\Omega^c} [\nabla(\phi_0\nabla u) - \nabla\phi_0\cdot\nabla u]\mathrm{d}x\\
&= \int_{\partial\Omega} \phi_0\nabla u\cdot n\mathrm{d}S - \int_{\Omega^c} \nabla\phi_0\cdot\nabla u\mathrm{d}x\\
&= -\int_{\Omega^c} \nabla\phi_0\cdot\nabla u\mathrm{d}x\\
&= -\int_{\Omega^c} [\nabla(\nabla\phi_0 u) - \Delta\phi_0 u]\mathrm{d}x\\
&= -\int_{\partial\Omega} \nabla\phi_0 u\cdot n\mathrm{d}S + \int_{\Omega^c} \Delta\phi_0 u\mathrm{d}x\\
&= 0.
\end{aligned}$$

So we get

$$\frac{\mathrm{d}^2 F_0(t)}{\mathrm{d}t^2} = \int_{\Omega^c} |v|^p\phi_0\mathrm{d}x.$$

By the Hölder inequality, we get the estimate of the right hand side

$$\begin{aligned}
\Big|\int_{\Omega^c} v\phi_0\mathrm{d}x\Big| &= \Big|\int_{\Omega^c\cap\{|x|\le t+R\}} v\phi_0^{\frac{1}{p}}\phi_0^{\frac{p-1}{p}}\mathrm{d}x\Big|\\
&\le \Big(\int_{\Omega^c\cap\{|x|\le t+R\}} |v\phi_0^{\frac{1}{p}}|^p\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_{\Omega^c\cap\{|x|\le t+R\}} |\phi_0^{\frac{p-1}{p}}|^{p'}\mathrm{d}x\Big)^{\frac{1}{p'}}\\
&= \Big(\int_{\Omega^c\cap\{|x|\le t+R\}} |v|^p\phi_0\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_{\Omega^c\cap\{|x|\le t+R\}} \phi_0\mathrm{d}x\Big)^{\frac{1}{p'}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\Big|\int_{\Omega^c} v\phi_0\mathrm{d}x\Big|^p &\le \Big(\int_{\Omega^c\cap\{|x|\le t+R\}} |v|^p\phi_0\mathrm{d}x\Big)\Big(\int_{\Omega^c\cap\{|x|\le t+R\}} \phi_0\mathrm{d}x\Big)^{p-1}\\
&\le \Big(\int_{\Omega^c} |v|^p\phi_0\mathrm{d}x\Big)\Big(\int_{\Omega^c\cap\{|x|\le t+R\}} \phi_0\mathrm{d}x\Big)^{p-1}.
\end{aligned}$$

So we have

$$\int_{\Omega^c} |v|^p\phi_0\mathrm{d}x \ge \frac{\Big|\int_{\Omega^c} v\phi_0\mathrm{d}x\Big|^p}{\Big(\int_{\Omega^c\cap\{|x|\le t+R\}} \phi_0\mathrm{d}x\Big)^{p-1}},$$

where

$$\int_{\Omega^c\cap\{|x|\le t+R\}}\phi_0\mathrm{d}x\le\int_{\{|x|\le t+R\}}1\mathrm{d}x\le\mathrm{Vol}\{B_{t+R}\}=\mathrm{Vol}(B_1^n)(t+R)^n.$$

Therefore

$$\int_{\Omega^c}|v|^p\phi_0\mathrm{d}x\ge\frac{\Big|\displaystyle\int_{\Omega^c}v\phi_0\mathrm{d}x\Big|^p}{(\mathrm{Vol}(B_1^n)(t+R)^n)^{p-1}}=\frac{k|G_0(t)|^p}{(t+R)^{n(p-1)}},$$

where $k=(\mathrm{Vol}(B_1^n))^{-(p-1)}>0$. Thus

$$\frac{\mathrm{d}^2F_0(t)}{\mathrm{d}t^2}\ge k(t+R)^{-n(p-1)}|G_0(t)|^p. \tag{3.4}$$

Similarly, we get

$$\frac{\mathrm{d}^2G_0(t)}{\mathrm{d}t^2}\ge k'(t+R)^{-n(q-1)}|F_0(t)|^q, \tag{3.5}$$

where $k'=(\mathrm{Vol}(B_1^n))^{-(q-1)}>0$. So $F_0(t),G_0(t)$ satisfy the ordinary differential inequalities as in (2.1). We use the Hölder inequality again to estimate the lower bound of $F_0(t)$ and $G_0(t)$.

$$\begin{aligned}\Big|\int_{\Omega^c}v\psi_1\mathrm{d}x\Big|&=\Big|\int_{\Omega^c\cap\{|x|\le t+R\}}v\phi_0^{\frac1p}\phi_0^{-\frac1p}\cdot\psi_1\mathrm{d}x\Big|\\&\le\Big(\int_{\Omega^c\cap\{|x|\le t+R\}}|v|^p\phi_0\mathrm{d}x\Big)^{\frac1p}\Big(\int_{\Omega^c\cap\{|x|\le t+R\}}|\phi_0^{-\frac1p}\psi_1|^{p'}\Big)^{\frac{1}{p'}}\\&\le\Big(\int_{\Omega^c}|v|^p\phi_0\mathrm{d}x\Big)^{\frac1p}\Big(\int_{\Omega^c\cap\{|x|\le t+R\}}|\phi_0^{-\frac{1}{p-1}}\psi_1|^{p'}\Big)^{\frac{1}{p'}},\end{aligned}$$

which implies

$$\Big|\int_{\Omega^c}v\psi_1\mathrm{d}x\Big|^p\le\Big(\int_{\Omega^c}|v|^p\phi_0\mathrm{d}x\Big)\cdot\Big(\int_{\Omega^c\cap\{|x|\le t+R\}}|\phi_0^{-\frac{1}{p-1}}\psi_1|^{p'}\Big)^{p-1}.$$

So we have

$$\begin{aligned}\frac{\mathrm{d}^2F_0(t)}{\mathrm{d}t^2}&=\int_{\Omega^c}|v|^p\phi_0\mathrm{d}x\\&\ge\frac{\Big|\displaystyle\int_{\Omega^c}v\psi_1\mathrm{d}x\Big|^p}{\Big(\displaystyle\int_{\Omega^c\cap\{|x|\le t+R\}}|\phi_0^{-\frac{1}{p-1}}\psi_1|^{p'}\Big)^{p-1}}\\&=\frac{|G_1(t)|^p}{\Big(\displaystyle\int_{\Omega^c\cap\{|x|\le t+R\}}|\phi_0^{-\frac{1}{p-1}}\psi_1|^{p'}\Big)^{p-1}}.\end{aligned}$$

Noticing $G_1(t)\ge\varepsilon C_8>0$ by (3.3) and Lemma 2.5, we get

$$\frac{\mathrm{d}^2F_0(t)}{\mathrm{d}t^2}\ge C_9\varepsilon^p(t+R)^{-(n-1)\left(\frac{p}{2}-1\right)}, \tag{3.6}$$

where $C_9 = C_8^p C_5^{-(p-1)} > 0$. Integrating on $[0,t]$, we have

$$\begin{aligned}\frac{\mathrm{d}F_0(t)}{\mathrm{d}t} &= \frac{\mathrm{d}F_0(t)}{\mathrm{d}t}\Big|_{t=0} + \frac{C_9\varepsilon^p}{n-\dfrac{(n-1)p}{2}}[(t+R)^{n-\frac{(n-1)p}{2}} - R^{n-\frac{(n-1)p}{2}}] \\ &\geq \frac{\mathrm{d}F_0(t)}{\mathrm{d}t}\Big|_{t=0} + \frac{C_9\varepsilon^p}{n-\dfrac{(n-1)p}{2}}(t+R)^{n-\frac{(n-1)p}{2}}.\end{aligned}$$

Integrating once more and we reach the final estimate

$$F_0(t) \geq \delta\varepsilon^p (t+R)^{n+1-\frac{(n-1)p}{2}} + \frac{\mathrm{d}F_0(t)}{\mathrm{d}t}\Big|_{t=0}\cdot t + F_0(0), \tag{3.7}$$

where $\delta = \frac{C_9}{[n-\frac{(n-1)p}{2}][n+1-\frac{(n-1)p}{2}]} > 0$. It is easy to check that

$$p > 1 \Rightarrow n+1-\frac{(n-1)p}{2} > 1. \tag{3.8}$$

Hence the following estimation is valid when t is large enough:

$$F_0(t) \geq \frac{1}{2}\delta\varepsilon^p (t+R)^{n+1-\frac{(n-1)p}{2}} \geq C_{10}\varepsilon^p t^{n+1-\frac{(n-1)p}{2}}. \tag{3.9}$$

Similarly, we have

$$G_0(t) \geq \frac{1}{2}\delta'\varepsilon^q (t+R)^{n+1-\frac{(n-1)q}{2}} \geq C_{11}\varepsilon t. \tag{3.10}$$

Therefore, we establish the following inequalities:

$$\begin{cases} F_0''(t) \geq C_{10}t^{-n(p-1)}|G_0(t)|^p, \\ G_0''(t) \geq C_{10}t^{-n(q-1)}|F_0(t)|^q, \\ F_0(t) \geq C_{10}\varepsilon^p t^{n+1-(n-1)q-2}, \\ G_0(t) \geq C_{11}\varepsilon t. \end{cases} \tag{3.11}$$

Set $X(\tau) = \varepsilon^a F_0(t),\ Y(\tau) = \varepsilon^b G_0(t),\ \tau = \varepsilon^d t$ in (3.11), where a, b, d are constants which should be determined later. (3.11) becomes

$$\begin{cases} X''(\tau) \geq C_{10}\varepsilon^{a-2d-pb+nd(p-1)}\tau^{-n(p-1)}|Y(\tau)|^p, \\ Y''(\tau) \geq C_{10}\varepsilon^{b-2d-pa+nd(q-1)}\tau^{-n(q-1)}|X(\tau)|^q, \\ X(\tau) \geq C_{10}\varepsilon^{p+a-d(n+1-\frac{(n-1)p}{2})}\tau^{n+1-(n-1)q-2}, \\ Y(\tau) \geq C_{11}\varepsilon^{b+1-d}\tau. \end{cases}$$

To use Lemma 2.1, we need

$$\begin{cases} a-2d-pb+nd(p-1) = 0, \\ b-2d-pa+nd(q-1) = 0, \\ p+a-d\Big(n+1-\dfrac{(n-1)p}{2}\Big) = 0, \end{cases}$$

and take $C_0 = C_{10},\ C_1 = C_{11}\varepsilon^{b+1-d}$. Solving this system, we get

$$\begin{cases} a = \dfrac{n(pq-1)-2(1+p)}{(pq-1)F(p,q)}, \\ b = \dfrac{n(pq-1)-2(1+q)}{(pq-1)F(p,q)}, \\ d = \dfrac{1}{F(p,q)}. \end{cases}$$

And by Lemma 2.1, we have

$$\tau_0 = \varepsilon^d T_\varepsilon \le C_2,$$

i.e.,

$$T_\varepsilon \le C_2 \varepsilon^{-\frac{1}{F(p,q)}}.$$

This completes the proof of Theorem 1.1.

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