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Carleson Measures and Toeplitz Operators on Doubling Fock Spaces*

Xiaofen LV[1]

Abstract Given ϕ a subharmonic function on the complex plane $\mathbb{C}$, with $\Delta\phi \mathrm{d}A$ being a doubling measure, the author studies Fock Carleson measures and some characterizations on μ such that the induced positive Toeplitz operator T_μ is bounded or compact between the doubling Fock space F^p_ϕ and F^∞_ϕ with $0 < p \leq \infty$, where μ is a positive Borel measure on $\mathbb{C}$.

Keywords Doubling Fock spaces, Carleson measure, Toeplitz operators
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1 Introduction

Suppose ν is a positive Borel measure on $\mathbb{C}$, denoted by $\nu \geq 0$. We call ν is doubling, if there exists some constant $C > 0$ such that

$$\nu(D(z, 2r)) \leq C\nu(D(z, r))$$

for $z \in \mathbb{C}$ and $r > 0$, where $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$. Throughout the paper, we assume that ϕ is a subharmonic, real-valued function on $\mathbb{C}$, and ϕ is not identically zero with $\nu = \Delta\phi \mathrm{d}A$ doubling, where $\mathrm{d}A$ is the Lebesgue area measure on $\mathbb{C}$. Denote by $\rho(\cdot)$ the positive radius such that $\nu(D(z, \rho(z))) = 1$ for $z \in \mathbb{C}$. See [15] for details.

Suppose $H(\mathbb{C})$ is the collection of all holomorphic functions on $\mathbb{C}$. For $0 < p < \infty$, the space L^p_ϕ is the family of all Lebesgue measurable functions f on $\mathbb{C}$ such that

$$\|f\|_{p,\phi} = \Big(\int_{\mathbb{C}} |f(z)|^p \mathrm{e}^{-p\phi(z)} \mathrm{d}A(z)\Big)^{\frac{1}{p}} < \infty.$$

The doubling Fock space F^p_ϕ is defined to be

$$F^p_\phi = L^p_\phi \cap H(\mathbb{C})$$

if $0 < p < \infty$ and

$$F^\infty_\phi = \Big\{f \in H(\mathbb{C}) : \|f\|_{\infty,\phi} = \sup_{z\in\mathbb{C}} |f(z)|\mathrm{e}^{-\phi(z)} < \infty\Big\}.$$

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[1]Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China.
E-mail: lvxf@zjhu.edu.cn
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It is well known that, F_ϕ^p is a Banach space under $\|\cdot\|_{p,\phi}$ for $p\geq 1$, and F_ϕ^p is an Fréchet space under $d(f,g)=\|f-g\|_{p,\phi}^p$ for $0<p<1$. The doubling Fock space has been studied in [5, 7, 11, 15–16, 19]. It is always called the generalized Fock space. When $\phi(z)=\frac{\alpha}{2}|z|^2$ with $\alpha>0$, F_ϕ^p is the classical Fock space F_α^p, which has been studied by many authors, see [6, 9, 13, 18, 24] for example. And for another special case that $\phi(z)=\frac{1}{2}|z|^2-\frac{m}{2}\ln(A+|z|^2)$ with suitable $A>0$, where m is a positive integer, F_ϕ^p is the Fock-Sobolev space $F^{p,m}$ studied in [2–4, 17, 23]. For $\phi(z)=|z|^{2m}$, F_ϕ^2 is the Fock space in [20] and [21]. If $n=1$, the Laplacian of the weight function φ in [10] and [22] satisfies the doubling measure hypothesis.

As far as we know, these doubling Fock spaces were first introduced by Christ [5]. In 2003, Marco, Massaneda and Ortega-Cerdà [15] studied the interpolating and sampling sequences for the doubling Fock spaces. After that, Marzo and Ortega-Cerdà [16] gave quite sharp pointwise estimates of the Bergman kernel associated to these spaces. Let $K(\cdot,\cdot)$ be the reproducing kernel for F_ϕ^2. The orthogonal projection P from L_ϕ^2 to F_ϕ^2 can be represented as

$$Pf(z)=\int_{\mathbb{C}}K(z,w)f(w)\mathrm{e}^{-2\phi(w)}\mathrm{d}A(w),\quad z\in\mathbb{C}.$$

Given $\mu\geq 0$, Toeplitz operator T_μ on F_ϕ^p is defined to be

$$T_\mu f(z)=\int_{\mathbb{C}}K(z,w)f(w)\mathrm{e}^{-2\phi(w)}\mathrm{d}\mu(w),\quad z\in\mathbb{C},$$

if it can be well (densely) defined.

The behaviors of positive Toeplitz operators on Fock spaces have been studied by many authors. In 2010, Isralowitz and Zhu [13] discussed the characterizations on $\mu\geq 0$ such that T_μ is bounded, compact and in Schatten classes on the classical Fock space F_α^2. Wang, Cao and Xia [23] studied the same problems on the Fock-Sobolev space $F^{2,m}$. In [9], Hu and Lv characterized the boundedness and compacteness of T_μ from one Fock space F_α^p to another F_α^q for $1<p,q<\infty$. Mengestie [18] extended them between F_α^p and F_α^∞ with $1<p<\infty$. With some weight φ satisfying $M_1dd^c|z|^2\leq dd^c\varphi\leq M_2dd^c|z|^2$ for fixed constants $M_1,M_2>0$, Schuster and Varolin [22] obtained the necessary and sufficient conditions such that T_μ is bounded or compact on F_φ^p for $1<p<\infty$. Given $0<p,q<\infty$, the corresponding problems were solved from F_φ^p to F_φ^q in [10], and between F_φ^p and F_φ^∞ for $0<p\leq\infty$ in [14]. With $1\leq p<\infty$, Oliver and Pascuas [19] studied the characterizations on μ for which T_μ is bounded or compact on the doubling Fock space F_ϕ^p.

Carleson measures have been extensively applied to various problems in Hardy (and Bergman) space theory. On the classical Fock space, Carleson measures were first introduced in [13]. The reference [4] is the first one where the so-called Carleson measures for Fock-Sobolev spaces were studied. See also [9–10, 17, 22].

In this paper, with $0<p\leq\infty$, we are going to obtain some characterizations on those $\mu\geq 0$ such that Toeplitz operators T_μ is bounded or compact from F_ϕ^p to F_ϕ^∞ and from F_ϕ^∞ to F_ϕ^p, respectively. We also introduce Fock-Carleson measures for the doubling Fock space. Our results extend those in [4, 9–10, 13–14, 17–19, 22]. In Section 2, we will introduce Fock Carleson measures with some characterizations in terms of averaging functions and Berezin transforms. In [10], on those Fock spaces induced by φ with $M_1dd^c|z|^2\leq dd^c\varphi\leq M_2dd^c|z|^2$, we proved that the (p,q)-Fock Carleson measure does not depend on the precise value $\frac{p}{q}$ when

$p \leq q$. This phenomenon seems quite inconsistent with the well-known results in Bergman (and Hardy) space theory. Our analysis in Section 2 shows that this independence of $\frac{p}{q}$ can only occur for weights ϕ for which $\rho(\cdot)$ is bounded above and below with positive constants. In Section 3, given $0 < p \leq \infty$, we are going to discuss the boundedness and compactness of T_μ from F^p_ϕ to F^∞_ϕ and from F^∞_ϕ to F^p_ϕ, respectively. It is worth to mention that, expect for [10, 14], all research mentioned above is about p-th Fock spaces with $p = 2$ or $1 < p \leq \infty$. However, since F^p_ϕ is not a Banach space for $0 < p < 1$, the Banach space technique in [9, 13, 18–19, 22] is invalid in this case. Also, the proof in [10, 14] depends strongly on two points:

$$F^p_\varphi \subset F^q_\varphi, \quad \forall 0 < p \leq q \quad \text{and} \quad Pf = f, \quad \forall f \in F^p_\varphi,\ p > 0. \tag{1.1}$$

However, these two points are not available for doubling Fock spaces. For example, for $\phi(z) = |z|^4$, Constantin and Peláez [8] concluded

$$F^p_\phi \backslash F^q_\phi \neq \emptyset, \quad F^q_\phi \backslash F^p_\phi \neq \emptyset,$$

when $p \neq q$.

We always use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. Two quantities A and B are called equivalent if there exists some C such that $C^{-1}A \leq B \leq CA$, written as "$A \simeq B$".

2 Carleson Measures

In this section, we are going to introduce the Fock Carleson measure, which will be used in the following sections. First, we list some notations and preliminary results. These results can be found in [12] and [15].

Recall that, ϕ is a subharmonic, real-valued function on $\mathbb{C}$, which satisfies $\mathrm{d}\nu = \Delta\phi \mathrm{d}A$ a doubling measure, and $\rho(\cdot)$ is the positive radius such that $\nu(D(z, \rho(z))) = 1$ for $z \in \mathbb{C}$. Given $r > 0$, write $D^r(z) = D(z, r\rho(z))$. There exists some constant $C > 0$ such that for $z \in \mathbb{C}$ and $w \in D^r(z)$,

$$\frac{1}{C}\rho(z) \leq \rho(w) \leq C\rho(z). \tag{2.1}$$

Moreover, for fixed $r > 0$ we have $m_1, m_2 > 0$ such that

$$D^r(z) \subseteq D^{m_1 r}(w), D^r(w) \subseteq D^{m_2 r}(z) \quad \text{whenever } w \in D^r(z), \tag{2.2}$$

which follows from the triangle inequality. Given $r > 0$, we say a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{C}$ is an r-lattice if $\{D^r(a_k)\}_k$ covers $\mathbb{C}$ and the disks of $\{D^{\frac{r}{5}}(a_k)\}_k$ are pairwise disjoint, see [15] for details. For $m > 0$, there exists some positive integer N such that

$$1 \leq \sum_{k=1}^\infty \chi_{D^{mr}(a_k)}(z) \leq N, \quad z \in \mathbb{C}. \tag{2.3}$$

For our later use, we need the concepts of averaging function and Berezin transform. Given $\mu \geq 0$, the average of μ is defined as

$$\widehat{\mu}_r(z) = \mu(D^r(z))/A(D^r(z)), \quad z \in \mathbb{C}.$$

For $0<p<\infty$ and $\delta>0$, there is some $C>0$ such that for $f\in H(\mathbb{C})$ and $z\in\mathbb{C}$, we obtain

$$|f(z)|\mathrm{e}^{-\phi(z)}\leq \frac{C}{A(D^\delta(z))^{\frac{1}{p}}}\Big(\int_{D^\delta(z)}|f|^p\mathrm{e}^{-p\phi}\mathrm{d}A\Big)^{\frac{1}{p}}. \tag{2.4}$$

Thus, in a way similar to Lemma 2.2 in [9], we get

$$\int_{\mathbb{C}}|f(z)\mathrm{e}^{-\phi(z)}|^p\mathrm{d}\mu(z)\leq C\int_{\mathbb{C}}|f(z)\mathrm{e}^{-\phi(z)}|^p\widehat{\mu}_r(z)\mathrm{d}A(z). \tag{2.5}$$

For $t>0$, we set the t-Berezin transform of μ to be

$$\widetilde{\mu}_t(z)=\int_{\mathbb{C}}|k_{t,z}(w)|^t\mathrm{e}^{-t\phi(w)}\mathrm{d}\mu(w),\quad z\in\mathbb{C},$$

where $k_{t,z}(w)=K(w,z)/\|K(\cdot,z)\|_{t,\phi}$ is the normalized Bergman kernel for F^t_ϕ. When $\phi(z)=\frac{1}{2}|z|^2$, the t-Berezin transform is closely connected with the heat flow as mentioned in [1].

We also need some other spaces. Let $0<p<\infty$. The space $L^p_\phi(\mathrm{d}\mu)$ is the family of all μ-measurable functions f on $\mathbb{C}$ such that

$$\|f\|_{p,\phi,\mathrm{d}\mu}=\Big(\int_{\mathbb{C}}|f(z)|^p\mathrm{e}^{-p\phi(z)}\mathrm{d}\mu(z)\Big)^{\frac{1}{p}}<\infty.$$

The space L^p is defined as

$$L^p=\Big\{f \text{ is Lebesgue measurable on } \mathbb{C}:\|f\|_{L^p}=\Big(\int_{\mathbb{C}}\|f\|^p\mathrm{d}A\Big)^{\frac{1}{p}}<\infty\Big\},$$

and l^p consists of all sequence $\{b_k\}_{k=1}^\infty\subset\mathbb{C}^n$ with

$$\|\{b_k\}_k\|_{l^p}=\Big(\sum_{k=1}^\infty|b_k|^p\Big)^{\frac{1}{p}}<\infty.$$

To prove the main results, we need some lemmas. Lemma 2.1 lists some well-known results about the Bergman kernel for F^p_ϕ. Most of them can be seen in [12, 16, 19]. We only need to show the statements of (3) and (4) for $p=\infty$. Notice that $1/p=0$ if $p=\infty$.

Lemma 2.1 *The Bergman kernel $K(\cdot,\cdot)$ satisfies*:

(1) *There exist positive constants C and ϵ such that*

$$|K(w,z)|\leq C\frac{\mathrm{e}^{\phi(w)+\phi(z)}}{\rho(w)\rho(z)}\mathrm{e}^{-\left(\frac{|z-w|}{\rho(z)}\right)^\epsilon} \tag{2.6}$$

for $w,z\in\mathbb{C}$.

(2) *There exists some $r_0>0$ such that*

$$|K(w,z)|\simeq\frac{\mathrm{e}^{\phi(w)+\phi(z)}}{\rho(z)^2}, \tag{2.7}$$

whenever $z\in\mathbb{C}$ and $w\in D^{r_0}(z)$.

(3) *For $0<p\leq\infty$, we have*

$$\|K(\cdot,z)\|_{p,\phi}\simeq\mathrm{e}^{\phi(z)}\rho(z)^{\frac{2}{p}-2},\quad z\in\mathbb{C}.$$

(4) *For $0<p\leq\infty$, $k_{p,z}\to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z\to\infty$.*

Proof Given $z \in \mathbb{C}$, by [19], there holds

$$\|K(\cdot, z)\|_{1,\phi} \simeq \mathrm{e}^{\phi(z)}.$$

This, together with (2.4), we have

$$\begin{aligned} |K(w,z)|\mathrm{e}^{-\phi(w)} &= \mathrm{e}^{\phi(z)-\phi(w)}|K(z,w)|\mathrm{e}^{-\phi(z)} \\ &\le \frac{C\mathrm{e}^{\phi(z)-\phi(w)}}{A(D^\delta(z))}\int_{D^\delta(z)}|K(\cdot,w)|\mathrm{e}^{-\phi}\mathrm{d}A \\ &\le \frac{C\mathrm{e}^{\phi(z)-\phi(w)}}{A(D^\delta(z))}\int_{\mathbb{C}}|K(\cdot,w)|\mathrm{e}^{-\phi}\mathrm{d}A \\ &\simeq \mathrm{e}^{\phi(z)}\rho(z)^{-2} \end{aligned}$$

for $z \in \mathbb{C}$. Hence, by (2.6) and (5) in [15, p. 869], if $|z|$ is large enough, we have some $\beta \in (0,1)$ such that

$$\begin{aligned} |k_{\infty,z}(w)| = |K(w,z)|\mathrm{e}^{-\phi(z)}\rho(z)^2 &= |K(z,w)|\mathrm{e}^{-\phi(z)}\rho(z)^2 \\ &\le C\rho(z)\mathrm{e}^{-\phi(w)}\rho(w)^{-1}\mathrm{e}^{-\left(\frac{|z-w|}{\rho(w)}\right)^\epsilon} \\ &\le C|z|^\beta\mathrm{e}^{\phi(w)}\rho(w)^{-1}\mathrm{e}^{-\left(\frac{|z|-|w|}{\rho(w)}\right)^\epsilon} \end{aligned}$$

for $w \in \mathbb{C}$. This tells us the statement (4) is true. The proof is completed.

Next, we are going to introduce (vanishing) (p,q)-Fock Carleson measures. When $n = 1$, all the spaces studied in [4, 9–10, 13, 17, 22] are special cases of ours here. When $p = q > 1$, this is just the Fock Carleson measure discussed in [19].

Definition 2.1 *Let $0 < p, q < \infty$ and let $\mu \ge 0$. We call μ a (p,q)-Fock Carleson measure if the embedding operator $i : F^p_\phi \to L^q_\phi(\mathrm{d}\mu)$ is bounded, i.e., there exists some constant C such that for $f \in F^p_\phi$,*

$$\Big(\int_{\mathbb{C}}|f(z)|^q\mathrm{e}^{-q\phi(z)}\mathrm{d}\mu(z)\Big)^{\frac{1}{q}} \le C\|f\|_{p,\phi}.$$

And also, we call μ a vanishing (p,q)-Fock Carleson measure if

$$\lim_{j\to\infty}\int_{\mathbb{C}}|f_j(z)|^q\mathrm{e}^{-q\phi(z)}\mathrm{d}\mu(z) = 0,$$

whenever $\{f_j\}_{j=1}^\infty$ is a bounded sequence in F^p_ϕ that converges to 0 uniformly on any compact subset of $\mathbb{C}$ as $j \to \infty$.

The following three theorems characterize (vanishing) (p,q)-Fock Carleson measures for all possible $0 < p, q < \infty$. The proof is similar to that of Theorems 3.1–3.3 in [9], we omit them here.

Theorem 2.1 *Let $0 < p \le q < \infty$, and let $\mu \ge 0$. Then the following statements are equivalent*:

(1) *μ is a (p,q)-Fock Carleson measure.*

(2) *$\widetilde{\mu}_t\rho^{2\left(1-\frac{q}{p}\right)}$ is bounded on $\mathbb{C}$ for some (or any) $t > 0$.*

(3) $\widehat{\mu}_\delta\rho^{2\left(1-\frac{q}{p}\right)}$ *is bounded on* $\mathbb{C}$ *for some (or any)* $\delta>0$.

(4) *The sequence* $\{\widehat{\mu}_r(a_k)\rho(a_k)^{2\left(1-\frac{q}{p}\right)}\}_k$ *is bounded for some (or any)* r*-lattice* $\{a_k\}_k$. *Furthermore,*

$$\|i\|^q_{F^p_\phi\to L^q(\phi,\mu)}\simeq\sup_{z\in\mathbb{C}}\widetilde{\mu}_t(z)\rho(z)^{2\left(1-\frac{q}{p}\right)}\simeq\sup_{z\in\mathbb{C}}\widehat{\mu}_\delta(z)\rho(z)^{2\left(1-\frac{q}{p}\right)}\simeq\sup_k\widehat{\mu}_r(a_k)\rho(a_k)^{2\left(1-\frac{q}{p}\right)}.$$

Theorem 2.2 *Let* $0<p\le q<\infty$, *and let* $\mu\ge0$. *Then the following statements are equivalent*:

(1) μ *is a vanishing* (p,q)*-Fock Carleson measure.*

(2) $\widetilde{\mu}_t(z)\rho(z)^{2\left(1-\frac{q}{p}\right)}\to0$ *as* $z\to\infty$ *for some (or any)* $t>0$.

(3) $\widehat{\mu}_\delta(z)\rho(z)^{2\left(1-\frac{q}{p}\right)}\to0$ *as* $z\to\infty$ *for some (or any)* $\delta>0$.

(4) $\widehat{\mu}_r(a_k)\rho(a_k)^{2\left(1-\frac{q}{p}\right)}\to0$ *as* $k\to\infty$ *for some (or any)* r*-lattice* $\{a_k\}_k$.

Theorem 2.3 *Let* $0<q<p<\infty$, *and let* $\mu\ge0$. *Then the following statements are equivalent*:

(1) μ *is a* (p,q)*-Fock Carleson measure.*

(2) μ *is a vanishing* (p,q)*-Fock Carleson measure.*

(3) $\widetilde{\mu}_t\in L^{\frac{p}{p-q}}$ *for some (or any)* $t>0$.

(4) $\widehat{\mu}_\delta\in L^{\frac{p}{p-q}}$ *for some (or any)* $\delta>0$.

(5) $\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{\frac{2(p-q)}{p}}\right\}_{k=1}^\infty\in l^{\frac{p}{p-q}}$ *for some (or any)* r*-lattice* $\{a_k\}_{k=1}^\infty$. *Furthermore,*

$$\|i\|^q_{F^p_\phi\to L^q(\phi,\mu)}\simeq\|\widetilde{\mu}_t\|_{L^{\frac{p}{p-q}}}\simeq\|\widehat{\mu}_\delta\|_{L^{\frac{p}{p-q}}}\simeq\left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{\frac{2(p-q)}{p}}\right\}_k\right\|_{l^{\frac{p}{p-q}}}.$$

Remark 2.1 In the setting of classical Fock spaces, μ is a (p,q)-Fock Carleson measure for some $p\le q$ if and only if μ is a (vanishing) (p,q)-Fock Carleson measure for all possible $p\le q$ (see [9]). This is still available for F^p_φ with $M_1dd^c|z|^2\le dd^c\varphi\le M_2dd^c|z|^2$, which can be seen in [10]. Now, Theorems 2.1–2.2 tell us that this phenomenon can only occur when $\rho(\cdot)\simeq1$. Theorems 2.1–2.3 extend the results in [4, 9–10, 13, 17, 19, 22]. From these theorems above, μ is a (p,q)-Fock Carleson measure if and only if it is a (tp,tq)-Fock Carleson measure, $t>0$. So (p,q)-Fock Carleson measure can be simply called $\frac{p}{q}$-Fock Carleson measure, and written as $\|\mu\|_{\frac{p}{q}}=\|i\|_{F^{\frac{p}{q}}_\phi\to L^1(\phi,\mu)}$ for simplicity.

3 Toeplitz Operators

In this section, for $0<p\le\infty$, we are going to characterize those $\mu\ge0$ for which Toeplitz operators T_μ are bounded and compact from F^p_ϕ to F^∞_ϕ or from F^∞_ϕ to F^p_ϕ, respectively. To study the compactness, we need Lemma 3.1. Part of this lemma can be seen in [12, Lemma 3.1].

Lemma 3.1 *Let* μ *be a* t*-Fock Carleson measure for some* $t>0$. *Toeplitz operator* T_μ *is well-defined on* F^p_ϕ *for all* $0<p\le\infty$. *Moreover,* T_{μ_R} *is compact from* F^p_ϕ *to* F^q_ϕ *for* $0<p,q\le\infty$, *where* $R>0$ *and* $\mu_R(E)=\int_{E\cap\{z:|z|\le R\}}\mathrm{d}\mu$ *for measurable set* $E\subseteq\mathbb{C}$.

Proof In a way similar to the proof of [12, Lemma 3.1], we can conclude that T_μ is well-defined on F^p_ϕ for $0<p\le\infty$, and T_{μ_R} is compact from F^p_ϕ to F^q_ϕ for $0<q<\infty$. To prove the compactness of T_{μ_R} for $q=\infty$, we suppose that $\{f_j\}_{j=1}^{\infty}\subseteq F^p_\phi$ is a bounded sequence, and f_j uniformly converges to 0 on compact subsets of $\mathbb{C}$ as $j\to\infty$. By Montel's theorem, we only need to show

$$\lim_{j\to\infty}\|T_{\mu_R}f_j\|_{\infty,\phi}=0. \tag{3.1}$$

In a way similar to the proof of (2.5), we obtain

$$\begin{aligned}&|T_{\mu_R}f_j(z)|\,\mathrm{e}^{-\phi(z)}\\ &\le \mathrm{e}^{-\phi(z)}\int_{|w|\le R}|f_j(w)||K(z,w)|\mathrm{e}^{-2\phi(w)}\mathrm{d}\mu(w)\\ &\le C\mathrm{e}^{-\phi(z)}\int_{|w|\le C_1(1+r)R}|f_j(w)||K(z,w)|\mathrm{e}^{-2\phi(w)}\widehat{\mu}_r(w)\mathrm{d}A(w).\end{aligned}$$

Since μ is a t-Fock Carleson measure, Theorems 2.1–2.3 tell us that there exists some $t_1\in\mathbb{R}$ such that

$$\sup_{z\in\mathbb{C}}\widehat{\mu}_r(z)\rho(z)^{t_1}<\infty.$$

Thus

$$\begin{aligned}&|T_{\mu_R}f_j(z)|\mathrm{e}^{-\phi(z)}\\ &\le C\Big(\sup_{z\in\mathbb{C}}\widehat{\mu}_r(z)\rho(z)^{t_1}\Big)\mathrm{e}^{-\phi(z)}\int_{|w|\le C_1(1+r)R}|f_j(w)||K(z,w)|\mathrm{e}^{-2\phi(w)}\rho(w)^{-t_1}\mathrm{d}A(w)\\ &\le C\mathrm{e}^{-\phi(z)}\int_{|w|\le C_1(1+r)R}|f_j(w)||K(z,w)|\mathrm{e}^{-2\phi(w)}\rho(w)^{-t_1}\mathrm{d}A(w).\end{aligned}$$

Notice that, for $p,s>0$ and real number k, there is $C>0$ such that

$$\int_{\mathbb{C}}\rho(w)^k\mathrm{e}^{-p\left(\frac{|z-w|}{\rho(z)}\right)^s}\mathrm{d}A(w)\le C\rho(z)^{k+2},\quad z\in\mathbb{C} \tag{3.2}$$

(see [12, Lemma 2.1]). This, together with (3.2) shows

$$\begin{aligned}&\|T_{\mu_R}f_j\|_{\infty,\varphi}\\ &\le C\sup_{z\in\mathbb{C}}\rho(z)^{-1}\int_{|w|\le C_1(1+r)R}|f_j(w)|\mathrm{e}^{-\phi(w)}\rho(w)^{-1-t_1}\mathrm{e}^{-\left(\frac{|z-w|}{\rho(z)}\right)^\epsilon}\mathrm{d}A(w)\\ &\le C\sup_{|w|\le C_1(1+r)R}\mathrm{e}^{-\phi(w)}\rho(w)^{-t_1}|f_j(w)|\sup_{z\in\mathbb{C}}\rho(z)^{-1}\int_{\mathbb{C}}\rho(w)^{-1}\mathrm{e}^{-\left(\frac{|z-w|}{\rho(z)}\right)^\epsilon}\mathrm{d}A(w)\\ &\le C\sup_{|w|\le C_1(1+r)R}|f_j(w)|\to 0\end{aligned}$$

as $j\to\infty$. Hence, (3.1) is true. The proof is ended.

In this position, we will characterize the boundedness and compactness of positive Toeplitz operators from F^p_ϕ to F^∞_ϕ or from F^∞_ϕ to F^p_ϕ with $0<p\le\infty$. Now, we state the main results as follows. These three theorems extend the main results in [9–10, 13–14, 18–19, 22]. However, the approach (Banach space technique) in [9, 13, 18–19, 22] is invalid in this case of $0<p\le 1$. Also, part of the proof here is different from that in [10, 14], because those two points are not available in the present case, see (1.1) for details.

Theorem 3.1 *Let $0 < p \le \infty$, and let $\mu \ge 0$. Then*

(1) *$T_\mu : F^p_\phi \to F^\infty_\phi$ is bounded if and only if μ is a $\frac{p}{p+1}$-Fock Carleson measure. Furthermore,*

$$\|T_\mu\|_{F^p_\phi \to F^\infty_\phi} \simeq \|\mu\|_{\frac{p}{p+1}}. \tag{3.3}$$

(2) *$T_\mu : F^p_\phi \to F^\infty_\phi$ is compact if and only if μ is a vanishing $\frac{p}{p+1}$-Fock Carleson measure.*

Proof (1) First, we assume that $T_\mu : F^p_\phi \to F^\infty_\phi$ is bounded. For $z \in \mathbb{C}$, Lemma 2.1 yields

$$\begin{aligned}\widetilde{\mu}_2(z)\rho(z)^{-\frac{2}{p}} &\le C\,|T_\mu k_{p,z}(z)|\,\mathrm{e}^{-\phi(z)}\\ &\le C\|T_\mu k_{p,z}\|_{\infty,\varphi}\\ &\le C\|T_\mu\|_{F^p_\phi \to F^\infty_\phi}\|k_{p,z}\|_{p,\phi}\\ &\le C\|T_\mu\|_{F^p_\phi \to F^\infty_\phi}.\end{aligned} \tag{3.4}$$

This, together with Theorem 2.1, shows that μ is a $\frac{p}{p+1}$-Fock Carleson measure, and

$$\|\mu\|_{\frac{p}{p+1}} \simeq \sup_{z\in\mathbb{C}} \widetilde{\mu}_2(z)\rho(z)^{-\frac{2}{p}} \le C\|T_\mu\|_{F^p(\varphi)\to F^\infty_\phi}. \tag{3.5}$$

On the other hand, suppose that μ is a $\frac{p}{p+1}$-Fock Carleson measure. Then $\widehat{\mu}_\delta\rho^{-\frac{2}{p}}$ is bounded on $\mathbb{C}$ for $\delta > 0$, which follows from Theorem 2.1. Given $f \in F^p_\phi$, by (2.4), we get

$$|f(z)|\mathrm{e}^{-\phi(z)} \le C\rho(z)^{-\frac{2}{p}}\|f\|_{p,\phi}, \quad z \in \mathbb{C}. \tag{3.6}$$

Given $z \in \mathbb{C}$, since $K(\cdot,z)f(\cdot) \in H(\mathbb{C})$, applying (2.5) to the weight 2ϕ, we have

$$|T_\mu f(z)| \le C\int_{\mathbb{C}} |K(w,z)f(w)|\mathrm{e}^{-2\phi(w)}\widehat{\mu}_\delta(w)\mathrm{d}A(w). \tag{3.7}$$

(3.6), Lemma 2.1 and Theorem 2.1 imply

$$\begin{aligned}&|T_\mu f(z)|\mathrm{e}^{-\phi(z)}\\ &\le C\|f\|_{p,\phi}\int_{\mathbb{C}} \widehat{\mu}_\delta(w)\rho(w)^{-\frac{2}{p}}|K(z,w)|\mathrm{e}^{-\phi(w)}\mathrm{d}A(w)\\ &\le C\sup_{w\in\mathbb{C}} \widehat{\mu}_\delta(w)\rho(w)^{-\frac{2}{p}}\|f\|_{p,\phi}\mathrm{e}^{-\phi(z)}\int_{\mathbb{C}} |K(z,w)|\mathrm{e}^{-\phi(w)}\mathrm{d}A(w)\\ &\le C\|\mu\|_{\frac{p}{p+1}}\|f\|_{p,\phi}.\end{aligned}$$

Therefore, T_μ is bounded from F^p_ϕ to F^∞_ϕ, and

$$\|T_\mu\|_{F^p(\varphi)\to F^\infty_\phi} \le \|\mu\|_{\frac{p}{p+1}}.$$

This, combined with (3.5), gives (3.3).

(2) Suppose that μ is a vanishing $\frac{p}{p+1}$-Fock Carleson measure. By Theorem 2.2, we know

$$\widehat{\mu}_\delta(z)\rho(z)^{-\frac{2}{p}} \to 0 \quad \text{as } z \to \infty.$$

Setting μ_R as in Lemma 3.1, T_{μ_R} is compact from F^p_ϕ to F^∞_ϕ. Moreover, $\mu - \mu_R \ge 0$, $T_{\mu-\mu_R}$ is bounded from $F^p(\phi)$ to F^∞_ϕ, and for $r > 0$,

$$\lim_{R\to\infty}\sup_{z\in\mathbb{C}} (\widehat{\mu - \mu_R})_r(z)\rho(z)^{-\frac{2}{p}} = 0.$$

Thus, (3.3) and Theorem 2.1 tell us

$$\begin{aligned}\|T_\mu - T_{\mu_R}\|_{F^p_\phi \to F^\infty(\phi)} &= \|T_{\mu-\mu_R}\|_{F^p_\phi \to F^\infty(\phi)} \\ &\simeq \|\mu - \mu_R\|_{\frac{p}{p+1}} \\ &\simeq \sup_{z\in\mathbb{C}} (\widehat{\mu - \mu_R})_r(z)\rho(z)^{-\frac{2}{p}} \\ &\to 0\end{aligned}$$

as $R \to \infty$. So we can conclude that $T_\mu : F^p_\phi \to F^\infty_\phi$ is also compact. Conversely, we assume that $T_\mu : F^p_\phi \to F^\infty_\phi$ is compact. Then $\widehat{\mu}_\delta \rho^{-\frac{2}{p}}$ is bounded for $\delta > 0$. Notice that $\{k_{p,z} : z \in \mathbb{C}\}$ is bounded in F^p_ϕ. So $\{T_\mu k_{p,z} : z \in \mathbb{C}\}$ is relatively compact in F^∞_ϕ. For any sequence $\{z_k\}_{k=1}^\infty \subset \mathbb{C}$ with $\lim\limits_{j\to\infty} z_j = \infty$, there exists a subsequence of $\{T_\mu k_{p,z_j}\}_{j=1}^\infty$ converging to some h in F^∞_ϕ. Without loss of generality, we may assume

$$\lim_{j\to\infty} \|T_\mu k_{p,z_j} - h\|_{\infty,\phi} = 0. \tag{3.8}$$

We only need to show $h \equiv 0$. For any $w \in \mathbb{C}$, (3.7) implies

$$\begin{aligned}|T_\mu k_{p,z_j}(w)| &\le \int_{\mathbb{C}} |k_{p,z_j}(u)K(w,u)\mathrm{e}^{-2\phi(u)}|\widehat{\mu}_\delta(u)\mathrm{d}A(u) \\ &\le \sup_{u\in\mathbb{C}} \widehat{\mu}_\delta(u)\rho(u)^{-\frac{2}{p}} \int_{\mathbb{C}} \rho(u)^{\frac{2}{p}} |k_{p,z_j}(u)K(w,u)\mathrm{e}^{-2\phi(u)}|\mathrm{d}A(u).\end{aligned}$$

By Lemma 2.1 and (3.2), we have

$$\int_{\mathbb{C}} |K(w,u)|^2\rho(u)^2\mathrm{e}^{-2\phi(u)}\mathrm{d}A(u) \le C\mathrm{e}^{2\phi(w)}\rho(w)^{-2} \int_{\mathbb{C}} \mathrm{e}^{-2\left(\frac{|w-u|}{\rho(w)}\right)^\epsilon} \mathrm{d}A(u) \le C\mathrm{e}^{2\phi(w)}.$$

So, for any $\varepsilon > 0$, there is some $R > 0$ such that

$$\int_{|u|>R} |K(w,u)|^2\rho(u)^2\mathrm{e}^{-2\phi(u)}\mathrm{d}A(u) < \varepsilon^2.$$

Since $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$, we get

$$\begin{aligned}|T_\mu k_{p,z_j}(w)| &\le C\Big(\int_{|u|\le R} + \int_{|u|>R}\Big)\rho(u)^{\frac{2}{p}} |k_{p,z_j}(u)K(w,u)\mathrm{e}^{-2\phi(u)}|\mathrm{d}A(u) \\ &< \varepsilon + \int_{|u|>R} \rho(u)^{\frac{2}{p}} |k_{p,z_j}(u)K(w,u)\mathrm{e}^{-2\phi(u)}|\mathrm{d}A(u),\end{aligned}$$

while j is large enough. By Hölder's inequality, Lemma 2.1 and (3.2), we obtain

$$\begin{aligned}&\int_{|u|>R} \rho(u)^{\frac{2}{p}} |k_{p,z_j}(u)K(w,u)\mathrm{e}^{-2\phi(u)}|\mathrm{d}A(u) \\ &\le \Big(\int_{|u|>R} |K(w,u)|^2\rho(u)^2\mathrm{e}^{-2\phi(u)}\mathrm{d}A(u)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{C}} |k_{p,z_j}(u)|^2\rho(u)^{\frac{2(2-p)}{p}}\mathrm{e}^{-2\phi(u)}\mathrm{d}A(u)\Big)^{\frac{1}{2}} \\ &\le C\varepsilon\rho(z_j)^{1-\frac{2}{p}} \Big(\int_{\mathbb{C}} \rho(u)^{\frac{4}{p}-4}\mathrm{e}^{-2\left(\frac{|z_j-u|}{\rho(z_j)}\right)^\epsilon} \mathrm{d}A(u)\Big)^{\frac{1}{2}} \\ &\le C\varepsilon,\end{aligned}$$

where C is independent of j and ε. Therefore

$$\lim_{j\to\infty} T_\mu k_{p,z_j}(w) = 0.$$

On the other hand, (3.8) implies

$$\lim_{j\to\infty} T_\mu k_{p,z_j}(w) = h(w)$$

for $w \in \mathbb{C}$. Hence, $h \equiv 0$, which means

$$\lim_{j\to\infty} \|T_\mu k_{p,z_j}\|_{\infty,\phi} = 0.$$

This, combined with (3.4), yields that as $j \to \infty$,

$$\widetilde{\mu}_2(z_j)\rho(z_j)^{-\frac{2}{p}} \le C|T_\mu k_{p,z_j}(z_j)|\mathrm{e}^{-\phi(z_j)} \le \|T_\mu k_{p,z_j}\|_{\infty,\phi} \to 0.$$

Thus,

$$\lim_{z\to\infty} \widetilde{\mu}_2(z)\rho(z)^{-\frac{2}{p}} = 0.$$

We conclude that μ is a vanishing $\frac{p}{p+1}$-Fock Carleson measure, which follows from Theorem 2.2. The proof is complete.

Theorem 3.2 *Let $0 < p < \infty$, and let $\mu \ge 0$. Then the following statements are equivalent:*
(1) $T_\mu : F^\infty_\phi \to F^p_\phi$ is bounded.
(2) $T_\mu : F^\infty_\phi \to F^p_\phi$ is compact.
(3) $\widetilde{\mu}_t \in L^p$ for some (or any) $t > 0$.
(4) $\widehat{\mu}_\delta \in L^p$ for some (or any) $\delta > 0$.
(5) $\big\{\widehat{\mu}_r(a_k)\rho(a_k)^{\frac{2}{p}}\big\}_{k=1}^\infty \in l^p$ for some (or any) r-lattice $\{a_k\}_{k=1}^\infty$.
Furthermore,

$$\|T_\mu\|_{F^\infty_\phi\to F^p_\phi} \simeq \|\widetilde{\mu}_t\|_{L^p} \simeq \|\widehat{\mu}_\delta\|_{L^p} \simeq \Big\|\big\{\widehat{\mu}_r(a_k)\rho(a_k)^{\frac{2}{p}}\big\}_k\Big\|_{l^p}. \tag{3.9}$$

Proof By [12, Lemma 2.4], we get the equivalence of (3), (4) and (5). To prove (1)⇒(5), we suppose that T_μ is bounded from F^∞_ϕ to F^p_ϕ. Given any bounded sequence $\{\lambda_k\}_k$ and r_0-lattice $\{a_k\}_k$, where r_0 as in Lemma 2.1, set

$$f(z) = \sum_{k=1}^\infty \lambda_k k_{2,a_k}(z)\rho(a_k), \quad z \in \mathbb{C}.$$

In a way similar to [10, Lemma 2.4], we have $f \in F^\infty_\phi$ and $\|f\|_{\infty,\phi} \le C\sup_k |\lambda_k|$. Since T_μ is bounded from F^∞_ϕ to F^p_ϕ, we have $T_\mu f \in F^p_\phi$. Khinchine's inequality and Fubini's theorem show

$$\begin{aligned}
&\int_{\mathbb{C}} \Big(\sum_{k=1}^\infty |\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^2\Big)^{\frac{p}{2}} \mathrm{e}^{-p\phi(z)}\mathrm{d}A(z)\\
&\le C\int_{\mathbb{C}}\int_0^1 \Big|\sum_{k=1}^\infty \psi_k(t)\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)\Big|^p \mathrm{d}t\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z)\\
&= C\int_0^1 \Big\|T_\mu\Big(\sum_{k=1}^\infty \psi_k(t)\lambda_k\rho(a_k)k_{2,a_k}\Big)\Big\|_{p,\phi}^p \mathrm{d}t,
\end{aligned}$$

where ψ_k is the k-th Rademacher function on $[0,1]$. Since $T_\mu : F^\infty_\phi \to F^p_\phi$ is bounded, the inequality above is no more than

$$C\int_0^1 \|T_\mu\|^p_{F^\infty_\phi\to F^p_\phi}\Big\|\sum_{k=1}^\infty \psi(t)\lambda_k\rho(a_k)k_{2,a_k}\Big\|^p_{\infty,\phi}\mathrm{d}t \le C\|T_\mu\|^p_{F^\infty_\phi\to F^p_\phi}\sup_k|\lambda_k|^p.$$

Since the balls $\{D^{r_0}(a_k)\}_k$ cover $\mathbb{C}$, (2.3) gives

$$\begin{aligned}&\int_{\mathbb{C}}\Big(\sum_{k=1}^\infty|\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^2\Big)^{\frac{p}{2}}\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z)\\ &\ge C\sum_{k=1}^\infty\int_{D^{r_0}(a_k)}\Big(\sum_{k=1}^\infty|\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^2\Big)^{\frac{p}{2}}\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z)\\ &\ge C\sum_{k=1}^\infty\int_{D^{r_0}(a_k)}|\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^p\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z).\end{aligned}$$

By (2.4), we know

$$\begin{aligned}&\rho(a_k)^{p+2}|T_\mu(k_{2,a_k})(a_k)|^p\mathrm{e}^{-p\phi(a_k)}\\ &\le C\int_{D^{r_0}(a_k)}|\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^p\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z).\end{aligned}$$

These yield

$$\begin{aligned}&\int_{\mathbb{C}}\Big(\sum_{k=1}^\infty|\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^2\Big)^{\frac{p}{2}}\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z)\\ &\ge C\sum_{k=1}^\infty|\lambda_k|^p\rho(a_k)^{p+2}|T_\mu(k_{2,a_k})(a_k)|^p\mathrm{e}^{-p\phi(a_k)}.\end{aligned}$$

Notice that Lemma 2.1 shows

$$\widehat{\mu}_{r_0}(a_k)^p \le C\rho(a_k)^p|T_\mu(k_{2,a_k})(a_k)|^p\mathrm{e}^{-p\phi(a_k)}.$$

Setting $\beta_k = |\lambda_k|^p$, we know $\{\beta_k\}_k \in l^\infty$. Hence

$$\begin{aligned}\sum_{k=1}^\infty\beta_k\widehat{\mu}_{r_0}(a_k)^p\rho(a_k)^2 &\le C\int_{\mathbb{C}}\Big(\sum_{k=1}^\infty|\lambda_k\rho(a_k)T_\mu(k_{2,a_k})(z)|^2\Big)^{\frac{p}{2}}\mathrm{e}^{-p\phi(z)}\mathrm{d}A(z)\\ &\le C\|T_\mu\|^p_{F^\infty_\phi\to F^p_\phi}\sup_k|\beta_k|.\end{aligned}$$

Therefore

$$\{\widehat{\mu}_{r_0}(a_k)^p\rho(a_k)^2\}_{k=1}^\infty \in l^1$$

and

$$\|\{\widehat{\mu}_{r_0}(a_k)\rho(a_k)^{\frac{2}{p}}\}_k\|_{l^p} \le C\|T_\mu\|_{F^\infty_\phi\to F^p_\phi}. \tag{3.10}$$

To prove (3)⇒(2), we need to give

$$\|T_\mu\|_{F^\infty_\phi\to F^p_\phi} \le C\|\widehat{\mu}_\delta\|_{L^p} \tag{3.11}$$

for some $\delta > 0$. We deal with the case $1 < p < \infty$ first. Fixed $\delta > 0$, (3.7), Lemma 2.1 and Hölder's inequality show

$$\begin{aligned}|T_\mu f(z)|^p \mathrm{e}^{-p\phi(z)} &\le C\|f\|_{\infty,\phi}^p \Big(\int_{\mathbb{C}} \widehat{\mu}_\delta(w)|K(w,z)|\mathrm{e}^{-\phi(w)}\mathrm{e}^{-\phi(z)}\mathrm{d}A(w)\Big)^p \\ &\le C\|f\|_{\infty,\phi}^p \int_{\mathbb{C}} \widehat{\mu}_\delta(w)^p |K(w,z)\mathrm{e}^{-\phi(w)}\mathrm{e}^{-\phi(z)}|\mathrm{d}A(w) \\ &\quad \times \Big(\int_{\mathbb{C}} |K(w,z)\mathrm{e}^{-\phi(w)}\mathrm{e}^{-\phi(z)}|\mathrm{d}A(w)\Big)^{p-1} \\ &\le C\|f\|_{\infty,\phi}^p \int_{\mathbb{C}} \widehat{\mu}_\delta(w)^p |K(w,z)\mathrm{e}^{-\phi(w)}\mathrm{e}^{-\phi(z)}|\mathrm{d}A(w)\end{aligned}$$

for $f \in F_\phi^\infty$. By Fubini's theorem and Lemma 2.1, we get

$$\begin{aligned}\|T_\mu f\|_{p,\phi}^p &\le C\|f\|_{\infty,\phi}^p \int_{\mathbb{C}} \widehat{\mu}_\delta(w)^p \rho(w)^{-1}\mathrm{d}A(w) \int_{\mathbb{C}} \rho(z)^{-1}\mathrm{e}^{-\left(\frac{|z-w|}{\rho(w)}\right)^\epsilon}\mathrm{d}A(z) \\ &\le C\|f\|_{\infty,\phi}^p \int_{\mathbb{C}} \widehat{\mu}_\delta(w)^p \mathrm{d}A(w).\end{aligned}$$

We now deal with the case $p \le 1$. For some r-lattice $\{a_k\}_k$ and $f \in F_\phi^\infty$, (2.1) and (2.4) show

$$\begin{aligned}|T_\mu f(z)|^p &\le \|f\|_{\infty,\phi}^p \Big(\sum_{k=1}^\infty \int_{D^r(a_k)} |K(w,z)|\mathrm{e}^{-\phi(w)}\mathrm{d}\mu(w)\Big)^p \\ &\le \|f\|_{\infty,\phi}^p \sum_{k=1}^\infty \Big(\int_{D^r(a_k)} |K(w,z)|\mathrm{e}^{-\phi(w)}\mathrm{d}\mu(w)\Big)^p \\ &\le \|f\|_{\infty,\phi}^p \sum_{k=1}^\infty \widehat{\mu}_r(a_k)^p \rho(a_k)^{2p}\Big(\sup_{w\in D^r(a_k)} |K(w,z)|\mathrm{e}^{-\phi(w)}\Big)^p \\ &\le C\|f\|_{\infty,\phi}^p \sum_{k=1}^\infty \widehat{\mu}_r(a_k)^p \rho(a_k)^{2p-2}\int_{D^{mr}(a_k)} |K(w,z)|^p\mathrm{e}^{-p\phi(w)}\mathrm{d}A(w).\end{aligned}$$

By the triangle inequality, we have $m_1 > 0$ such that $D^r(a_k) \subseteq D^{m_1 r}(w)$ if $w \in D^{mr}(a_k)$. Hence, (2.1) and (2.3) yield

$$\begin{aligned}&|T_\mu f(z)|^p \mathrm{e}^{-p\phi(z)} \\ &\le C\|f\|_{\infty,\phi}^p \mathrm{e}^{-p\phi(z)} \sum_{k=1}^\infty \int_{D^{mr}(a_k)} \widehat{\mu}_{m_1 r}(w)^p \rho(w)^{2p-2}|K(w,z)|^p \mathrm{e}^{-p\phi(w)}\mathrm{d}A(w) \\ &\le CN\|f\|_{\infty,\phi}^p \mathrm{e}^{-p\phi(z)} \int_{\mathbb{C}} \widehat{\mu}_{m_1 r}(w)^p \rho(w)^{2p-2}|K(w,z)|^p \mathrm{e}^{-p\phi(w)}\mathrm{d}A(w).\end{aligned}$$

Fubini's theorem and (3.2) give

$$\begin{aligned}\|T_\mu f\|_{p,\phi}^p &\le C\|f\|_{\infty,\phi}^p \int_{\mathbb{C}} \widehat{\mu}_{m_1 r}(w)^p \mathrm{e}^{-p\phi(w)}\rho(w)^{2p-2}\int_{\mathbb{C}} \mathrm{e}^{-p\phi(z)}|K(w,z)|^p\mathrm{d}A(z)\mathrm{d}A(w) \\ &\le C\|f\|_{\infty,\phi}^p \int_{\mathbb{C}} \widehat{\mu}_{m_1 r}(w)^p \rho(w)^{p-2}\mathrm{d}A(w)\int_{\mathbb{C}} \rho(z)^{-p}\mathrm{e}^{-p\left(\frac{|z-w|}{\rho(w)}\right)^\epsilon}\mathrm{d}A(z) \\ &\le C\|f\|_{\infty,\phi}^p \|\widehat{\mu}_{m_1 r}\|_{L^p}^p.\end{aligned}$$

Therefore, (3.11) comes true. Taking μ_R as in Lemma 3.1, then

$$\mu - \mu_R \geq 0$$

and

$$\begin{aligned}\|T_\mu - T_{\mu_R}\|_{F^\infty_\phi \to F^p_\phi} &= \|T_{\mu-\mu_R}\|_{F^\infty(\phi)\to F^p_\phi} \\ &\leq \|(\widehat{\mu - \mu_R})_\delta\|_{L^p} \\ &\to 0,\end{aligned}$$

if $R \to \infty$, since $\widehat{\mu}_\delta \in L^p$. Lemma 3.1 gives that T_{μ_R} is compact from F^∞_ϕ to $F^p(\phi)$. So $T_\mu : F^\infty_\phi \to F^p_\phi$ is also compact.

The estimate (3.9) follows from (3.10)–(3.11). The proof is completed.

By Theorems 2.3 and 3.2, we obtain the following corollary.

Corollary 3.1 *Let* $1 < p < \infty$, *and let* $\mu \geq 0$. *Then the following statements are equivalent*:
(1) $T_\mu : F^p_\phi \to F^\infty_\phi$ *is bounded.*
(2) $T_\mu : F^p_\phi \to F^\infty_\phi$ *is compact.*
(3) μ *is a* $\frac{p}{p+1}$*-Fock Carleson measure.*
(4) μ *is a vanishing* $\frac{p}{p+1}$*-Fock Carleson measure. Furthermore*

$$\|T_\mu\|_{F^p_\phi \to F^\infty_\phi} \simeq \|\mu\|_{\frac{p}{p+1}}.$$

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