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Convergence of Solutions of General Dispersive Equations Along Curve*

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Abstract In this paper, the authors give the local L^2 estimate of the maximal operator $S^*_{\phi,\gamma}$ of the operator family $\{S_{t,\phi,\gamma}\}$ defined initially by

$$S_{t,\phi,\gamma}f(x) := \mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})}f(\gamma(x,t)) = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t)\cdot\xi+\mathrm{i}t\phi(|\xi|)}\widehat{f}(\xi)\mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}),$$

which is the solution (when $n = 1$) of the following dispersive equations $(*)$ along a curve γ:

$$\begin{cases} \mathrm{i}\partial_t u + \phi(\sqrt{-\Delta})u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}(\mathbb{R}^n), \end{cases} \tag{$*$}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfies some suitable conditions and $\phi(\sqrt{-\Delta})$ is a pseudo-differential operator with symbol $\phi(|\xi|)$. As a consequence of the above result, the authors give the pointwise convergence of the solution (when $n = 1$) of the equation $(*)$ along curve γ.

Moreover, a global L^2 estimate of the maximal operator $S^*_{\phi,\gamma}$ is also given in this paper.

Keywords L^2 estimate, Global maximal operator, Dispersive equation, Curve
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1 Introduction and Main Results

Let f be a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$ and set

$$S_t f(x) = \mathrm{e}^{\mathrm{i}t\Delta}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi+\mathrm{i}t|\xi|^2}\widehat{f}(\xi)\mathrm{d}\xi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x}f(x)\mathrm{d}x$ is the Fourier transform of f. It is well known that $u(x,t) := S_t f(x)$ is the solution of the following Cauchy problem for the Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}(\mathbb{R}^n). \end{cases} \tag{1.1}$$

In 1979, Carleson [2] proposed a problem: Determining the optimal exponents s for which

$$\lim_{t\to 0} \mathrm{e}^{\mathrm{i}t\Delta}f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n \tag{1.2}$$

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holds whenever $f \in H^s(\mathbb{R}^n)$. Here $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ denotes the non-homogeneous Sobolev space, which is defined by

$$H^s(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}' : \|f\|_{H^s} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \mathrm{d}\xi \Big)^{\frac{1}{2}} < \infty \Big\}.$$

When $n = 1$, Carleson [2] proved that the convergence (1.2) holds for $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$. Dahlberg and Kenig [6] showed that Carleson's result is sharp. For $n \geq 2$, Bourgain [1] proved that (1.2) holds for $s > \frac{1}{2} - \frac{1}{4n}$ and the necessary condition is $s \geq \frac{1}{2} - \frac{1}{n}$ for $n \geq 4$. For more results on the convergence (1.2), see [13, 19, 23, 25], for example.

One may also consider the problem of nontangential convergence of $\mathrm{e}^{\mathrm{i}t\Delta} f \to f$. That is, for $\alpha > 0$ and $f \in H^s$ $(\mathbb{R}^n)$, for which s such that

$$\lim_{\substack{(y,t)\in\Gamma_\alpha(x)\\(y,t)\to(x,0)}} \mathrm{e}^{\mathrm{i}t\Delta} f(y) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1.3}$$

where $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$. If $s > \frac{n}{2}$, by Sobolev imbedding, then

$$\sup_{\substack{x\in\mathbb{R}^n\\t\in\mathbb{R}}} |\mathrm{e}^{\mathrm{i}t\Delta} f(x)| \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

Thus, by a standard argument, (1.3) holds for $s > \frac{n}{2}$. However, Sjögren and Sjölin [17] proved that (1.3) fails in general when $s \leq \frac{n}{2}$. In fact, in [17], the authors proved that there is an $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and a strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(0) = 0$, such that

$$\limsup_{\substack{(y,t)\to(x,0)\\|x-y|<\gamma(t)\\t>0}} |\mathrm{e}^{\mathrm{i}t\Delta} f(y)| = \infty \quad \text{for all } x \in \mathbb{R}^n.$$

Lee and Rogers [14], Cho, Lee and Vargas [3] considered the pointwise convergence problems along the curve $(\gamma(x,t), t)$. Suppose that $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the following conditions: There exist constants C_i $(i = 1, 2, 3)$, independent of x, y and t, t', such that

(A1) Hölder condition of order α $(0 < \alpha \leq 1)$ in t :

$$|\gamma(x,t) - \gamma(x,t')| \leq C_1 |t-t'|^\alpha;$$

(A2) Bilipschitz condition in x :

$$C_2|x-y| \leq |\gamma(x,t) - \gamma(y,t)| \leq C_3|x-y|.$$

For γ satisfying the above conditions and $f \in \mathcal{S}(\mathbb{R}^n)$, define the operator family by

$$\mathrm{e}^{\mathrm{i}t\Delta} f(\gamma(x,t)) := (2\pi)^{-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t)\cdot\xi + \mathrm{i}t|\xi|^2} \widehat{f}(\xi)\mathrm{d}\xi, \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

For $x_0, t_0 \in \mathbb{R}$, and $R, T > 0$, denote

$$B(x_0, R) := \{x \in \mathbb{R}; |x - x_0| \leq R\}, \quad I_T(t_0) := \{t \in \mathbb{R}; |t - t_0| \leq T\}.$$

Cho, Lee and Vargas obtained the following result.

Theorem A (see [3]) *Let $0 < \alpha \le 1$. Assume that γ satisfies the conditions* (A1)–(A2) *for $x, y \in B(x_0, R)$ and $t, t' \in I_T(t_0)$. If $s > \max\left\{\frac{1}{2} - \alpha, \frac{1}{4}\right\}$, then*

$$\Big\| \sup_{t \in I_T(t_0)} |\mathrm{e}^{\mathrm{i}t\Delta} f(\gamma(x,t))| \Big\|_{L^2(B(x_0,R))} \le C\|f\|_{H^s(\mathbb{R})}. \tag{1.4}$$

As a consequence of Theorem A, they obtained the pointwise convergence along curve γ.

Theorem B (see [3]) *Let $0 < \alpha \le 1$. Suppose that for every $x_0 \in \mathbb{R}$, there exists a neighborhood V of $(x_0, 0)$ such that* (A1) *holds for $(x,t),\ (x,t') \in V$ and* (A2) *holds for all $(x,t),\ (y,t) \in V$. Then for $s > \max\left\{\frac{1}{2} - \alpha, \frac{1}{4}\right\}$,*

$$\lim_{t \to 0} \mathrm{e}^{\mathrm{i}t\Delta} f(\gamma(x,t)) = f(x), \quad \text{a.e. } x \in \mathbb{R}. \tag{1.5}$$

In the present paper, under more general conditions, we will discuss the local and global L^2 maximal estimates of the operator family $\{S_{t,\phi,\gamma}\}_{t\in\mathbb{R}}$ which is defined by

$$S_{t,\phi,\gamma} f(x) := \mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(\gamma(x,t)) = (2\pi)^{-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t)\cdot\xi + \mathrm{i}t\phi(|\xi|)} \widehat{f}(\xi) \mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}), \tag{1.6}$$

where γ and $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfy some suitable conditions and $\phi(\sqrt{-\Delta})$ is a pseudo-differential operator with symbol $\phi(|\xi|)$.

Recently, for a curve γ which satisfies the conditions (A1)–(A2), in [7] we gave some weighted local and global L^q maximal estimate for the operator family (1.6) when symbol ϕ satisfies some growth conditions. In particular, taking $\phi(r) = r^2$, we showed that (1.5) holds if $f \in H^s(\mathbb{R})$ for $s \ge \frac{1}{4}$ and $\frac{1}{2} \le \alpha \le 1$ (see [7]), which improves a conclusion in Theorem B, where (1.5) holds for $s > \frac{1}{4}$ only.

We would like to point out that the operator $S_{t,\phi,\gamma}$ is associated closely with a class of general dispersive equations. In fact, when $\gamma(x,t) = x$ for any $t \in \mathbb{R}$, then $\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(x)$ is the formal solution of the following general dispersive equation defined by

$$\begin{cases} \mathrm{i}\partial_t u + \phi(\sqrt{-\Delta}) u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}(\mathbb{R}^n). \end{cases} \tag{1.7}$$

Many dispersive equations can be reduced to this type, for instance, the half-wave equation ($\phi(r) = r$), the fractional Schrödinger equation ($\phi(r) = r^a$ $(0 < a, a \ne 1)$), the Beam equation ($\phi(r) = \sqrt{1 + r^4}$), Klein-Gordon or semirelativistic equation ($\phi(r) = \sqrt{1 + r^2}$), iBq ($\phi(r) = r\sqrt{1 + r^2}$), imBq $\left(\phi(r) = \frac{r}{\sqrt{1+r^2}}\right)$ and the fourth-order Schrödinger equation ($\phi(r) = r^2 + r^4$) (see [4–5, 8–12] and references therein).

An important motivation of discussing the operator family $\{S_{t,\phi,\gamma}\}_{t\in\mathbb{R}}$ is to give the pointwise convergence along curve γ of the solution of the equation (1.7). To be precise, we will identify the exponents s for which

$$\lim_{t \to 0} \mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(\gamma(x,t)) = f(x), \quad \text{a.e. } x \in \mathbb{R} \tag{1.8}$$

holds whenever $f \in H^s(\mathbb{R})$ and ϕ, γ satisfy some suitable conditions.

It is well known that, by a fundamental idea in harmonic analysis, the problem whether the pointwise convergence (1.8) holds can be reduced to a local L^2 estimate for a local maximal operator of the family $\{S_{t,\phi,\gamma}\}$ defined by

$$S^*_{\phi,\gamma} f(x) = \sup_{t \in I_T(t_0)} |S_{t,\phi,\gamma} f(x)|. \tag{1.9}$$

Therefore, the first aim of the present paper is to give a local L^2 estimate of the maximal operator $S^*_{\phi,\gamma}$ for the γ and ϕ satisfying some growth conditions.

Theorem 1.1 *Let $0 < \alpha \le 1$. Assume that γ satisfies the conditions* (A1)–(A2) *for $x, y \in B(x_0, R)$ and $t, t' \in I_T(t_0)$, and ϕ satisfies the following conditions*:

(H1) *There exists $m \ge 2$, such that $|\phi'(r)| \sim r^{m-1}$ and $|\phi''(r)| \sim r^{m-2}$ for all $r \ge 1$.*

(H2) *There exists $m \ge 2$, such that $|\phi^{(3)}(r)| \lesssim r^{m-3}$ for all $r \ge 1$.*

If $s > \max\{\frac{1}{2} - \alpha, \frac{1}{4}\}$, then

$$\|S^*_{\phi,\gamma} f\|_{L^2(B(x_0,R))} \le C_R \|f\|_{H^s(\mathbb{R})}. \tag{1.10}$$

Remark 1.1 There are many elements ϕ satisfying the conditions (H1)–(H2), for instance, r^a $(a \ge 2)$, $(1 + r^2)^{\frac{a}{2}}$ $(a \ge 2)$, $\sqrt{1 + r^4}$ and $r^2 + r^4$, $r\sqrt{1 + r^2}$ and so on. Hence, Theorem 1.1 is an extension of Theorem A. The following result is an immediate consequence of Theorem 1.1.

Theorem 1.2 *Let $0 < \alpha \le 1$, and ϕ satisfies the conditions in Theorem* 1.1*. Suppose that for every $x_0 \in \mathbb{R}$, there exists a neighborhood V of $(x_0, 0)$ such that γ satisfies* (A1) *for (x, t), $(x, t') \in V$ and* (A2) *for all (x, t), $(y, t) \in V$. Then* (1.8) *holds for $s > \max\{\frac{1}{2} - \alpha, \frac{1}{4}\}$.*

Remark 1.2 Obviously, Theorem 1.2 is an extension of Theorem B. In fact, let $\phi(r) = r^2$. Then (1.8) is just (1.5).

Remark 1.3 Let $\gamma(x, t) = x$ for any $t \in \mathbb{R}$. Then (1.8) is just

$$\lim_{t\to 0} \mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}. \tag{1.11}$$

By the result of Theorem 1.2, when ϕ satisfying the conditions (H1)–(H2), the pointwise convergence (1.11) holds for $f \in H^s(\mathbb{R})$ with $s > \frac{1}{4}$. In this sense, Theorem 1.2 is an extension of Carleson's classical problem on the pointwise convergence (1.2) in [2]. In particular, when $\phi(r) = r^a$ $(a \ge 2)$, in one spatial dimension $(n = 1)$, the pointwise convergence of the fractional Schrödinger equation

$$\lim_{t\to 0} \mathrm{e}^{\mathrm{i}t(-\Delta)^{\frac{a}{2}}} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n \tag{1.12}$$

holds for $f \in H^s(\mathbb{R})$ with $s > \frac{1}{4}$. Recently, in the case when dimension $n = 2$ and $a > 1$, Miao, Yang and Zheng [15] showed that (1.12) holds for $f \in H^s(\mathbb{R}^2)$ with $s > \frac{3}{8}$, which improved the result that (1.12) holds for $s \ge \frac{1}{2}$ in [19].

On the other hand, we discuss the necessity of local maximal estimate (1.10) and give the following result.

Theorem 1.3 *Suppose that $0 < \alpha \le 1$, γ satisfies the conditions in Theorem* 1.1 *and ϕ satisfies the following condition*:

(H3) *There exists $m \ge 2$, such that $|\phi(r)| \lesssim r^m$ for all $r \ge 1$.*

Then (1.10) *holds only if $s \ge \max\{\frac{1}{2} - \alpha, \frac{1}{4}\}$.*

Our second aim in this paper is to strengthen the local estimate (1.10) to the global L^2 estimates for the maximal operator $S^*_{\phi,\gamma}$.

Theorem 1.4 *Let $0 < \alpha \le 1$ and γ satisfies the conditions* (A1) *and*
(A3) $|\gamma(x,t) - \gamma(y,t)| \ge C_2|x-y|$ *for* $x, y \in \mathbb{R}$ *and* $t, t' \in I_T(t_0)$.
Suppose that function ϕ satisfies the following conditions:
(K1) *There exists $m_1 > 1$ such that* $|\phi^{(\beta)}(r)| \lesssim r^{m_1-\beta}$ $(\beta = 0,1,2)$ *for all* $0 < r < 1$.
(K2) *There exists $m_2 > 1$ such that* $|\phi^{(\beta)}(r)| \lesssim r^{m_2-\beta}$ $(\beta = 0,1,3)$ *for all* $r \ge 1$.
(K3) *There exists $m_2 > 1$ such that* $|\phi''(r)| \sim r^{m_2-2}$ *for all* $r \ge 1$.
If $f \in H^s(\mathbb{R})$ with $s > \frac{m_2}{4}$ for $\frac{1}{2} < \alpha \le 1$ or $s > \min\big\{\frac{m_2}{2}, \frac{m_2}{4}\big(\frac{1}{\alpha} - 1\big)\big\}$ for $0 < \alpha \le \frac{1}{2}$, then

$$\|S^*_{\phi,\gamma}\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}. \tag{1.13}$$

Remark 1.4 Assume that ϕ and γ satisfy the conditions in Theorem 1.4. From the proof of Theorem 1.4, for $0 < \alpha \le 1$, (1.13) holds when $s > \frac{m_2}{2}$.

Remark 1.5 There are some functions ϕ which satisfies the conditions (K1)–(K3), for instance, r^a $(a > 1)$, $(1+r^2)^{\frac{a}{2}}$ $(a > 1)$, $\sqrt{1+r^4}$ and $r^2 + r^4$, $r\sqrt{1+r^2}$ and so on. In particular, taking $\phi(r) = r^a$ $(a > 1)$ and $\gamma(x,t) = x$, which satisfies the conditions (A1) with $\alpha = 1$ and (A3) for $x, y \in \mathbb{R}$ and $t, t' \in I_T(t_0) = [0,1]$, we may get that

$$\Big\| \sup_{t \in [0,1]} |\mathrm{e}^{\mathrm{i}t(-\Delta)^{\frac{a}{2}}} f(x)| \Big\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}$$

holds for $f \in H^s(\mathbb{R})$ with $s > \frac{a}{4}$ which is consistent with Sjölin's result in [20].

The proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.4 are given in Section 2, Section 3 and Section 4, respectively.

2 Proof of Theorem 1.1

2.1 Proof of Theorem 1.1 based on Lemma 2.1

In this subsection, we first complete the proof of Theorem 1.1 by using Lemma 2.1. The latter will be proved in the next subsection. Performing a change of variables, $x = x_0 + Rx'$ and $t = t_0 - T + 2Tt'$, we may assume $B(x_0, R) = [-1,1]$ and $I_T(t_0) = [0,1]$. In this case, the local maximal operator $S^*_{\phi,\gamma}$ defined in (1.9) is

$$S^*_{\phi,\gamma} f(x) = \sup_{0 \le t \le 1} |S_{t,\phi,\gamma} f(x)|, \quad x \in \mathbb{R}.$$

Thus, to get (1.10) it suffices to show that for $s > \max\big\{\frac{1}{2} - \alpha, \frac{1}{4}\big\}$,

$$\|S^*_{\phi,\gamma} f\|_{L^2([-1,1])} \le C\|f\|_{H^s(\mathbb{R})}. \tag{2.1}$$

Choose a nonnegative function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset \big\{\xi : \frac{1}{2} < |\xi| < 2\big\}$ and

$$\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1, \quad \xi \ne 0.$$

Set $\varphi_0(\xi) = 1 - \sum\limits_{k=1}^{\infty} \varphi(2^{-k}\xi)$, then $\varphi_0 \in C_0^\infty(\mathbb{R})$. Denote $\widehat{P_0 f}(\xi) = \varphi_0(\xi)\widehat{f}(\xi)$ and $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\widehat{f}(\xi)$ for $k \ge 1$. Rewrite

$$S_{t,\phi,\gamma} f(x) = S_{t,\phi,\gamma}(P_0 f)(x) + \sum_{k=1}^{\infty} S_{t,\phi,\gamma}(P_k f)(x). \tag{2.2}$$

Therefore, by (2.2), we obtain

$$S^*_{\phi,\gamma}f(x) \le S^*_{\phi,\gamma}(P_0f)(x) + \sum_{k=1}^{\infty} S^*_{\phi,\gamma}(P_kf)(x). \tag{2.3}$$

Noting that $\varphi_0 \in C_0^\infty(\mathbb{R})$, by Hölder's inequality, we have

$$|S_{t,\phi,\gamma}(P_0f)(x)| \le C\int_{|\xi|<2} |\widehat{P_0f}(\xi)|\mathrm{d}\xi \le C\|P_0f\|_{L^2(\mathbb{R})}.$$

Thus, we get $|S^*_{\phi,\gamma}(P_0f)| \le C\|P_0f\|_{L^2(\mathbb{R})}$ from which it follows

$$\|S^*_{\phi,\gamma}(P_0f)\|_{L^2([-1,1])} \le C\|P_0f\|_{L^2(\mathbb{R})}. \tag{2.4}$$

Lemma 2.1 *Assume that* $0 < \alpha \le 1$ *and* $\gamma : [0,1] \to \mathbb{R}$ *satisfies the conditions* (A1)–(A2). *Suppose that* ϕ *satisfies the conditions* (H1)–(H2). *If* $s \ge \max\left\{\frac{1}{2} - \alpha, \frac{1}{4}\right\}$, *then for all* $k \ge 1$,

$$\|S^*_{\phi,\gamma}(P_kf)\|_{L^2([-1,1])} \le C2^{ks}\|P_kf\|_{L^2(\mathbb{R})}. \tag{2.5}$$

Hence, by the estimates (2.3)–(2.4), to get (2.1) it is sufficient to prove Lemma 2.1.

2.2 Proof of Lemma 2.1

The proof of Lemma 2.1 is based on Lemma 2.2, which is an immediate consequence of Lemma 2.1 in [3, p. 979]. Recall that

$$S_{t,\phi,\gamma}f(x) = (2\pi)^{-1}\int \mathrm{e}^{\mathrm{i}\gamma(x,t)\xi + \mathrm{i}t\phi(|\xi|)}\widehat{f}(\xi)\mathrm{d}\xi,$$

where γ is a continuous function defined on $B(x_0,R) \times I_T(t_0)$.

Lemma 2.2 *Suppose that* $\lambda \ge 1$, $\sigma = \max\{\frac{1}{2} - \alpha, \frac{1}{4}\}$ *and* $q, r \ge 2$. *Let* $D = \{I\}$ *be a collection of intervals of length* λ^{1-m} *such that* $I \subset I_T(t_0)$ *and* $\sum_{I\in D}\chi_I \le 4$. *Assume that* ϕ *satisfies* (H1)–(H2), *and assume that*

$$\|S_{\phi,\gamma}f\|_{L^q_x(B(x_0,R),L^r_t(I))} \le C\lambda^\sigma\|f\|_{L^2(\mathbb{R})} \tag{2.6}$$

with C *uniform in* $I \in D$ *provided that* $\widehat{f}$ *is supported in* $A(\lambda) := \left\{\xi : \frac{\lambda}{2} \le |\xi| \le 2\lambda\right\}$. *Then, there exists* $C = C(B, \|\gamma\|_{L^\infty(B(x_0,R)\times I_T(t_0))})$ *such that*

$$\|S_{(\cdot),\phi,\gamma}f(\cdot)\|_{L^q_x(B(x_0,R),L^r_t(\bigcup_{I\in D} I))} \le C\lambda^\sigma\|f\|_{L^2(\mathbb{R})}, \tag{2.7}$$

whenever $\widehat{f}$ *is supported in* $A(\lambda)$.

We now give the proof of Lemma 2.1. For $\lambda \ge 2$, let $A(\lambda) = \left\{\xi : \frac{\lambda}{2} \le |\xi| \le 2\lambda\right\}$. Define an operator $L_{\phi,\gamma}$ by

$$L_{\phi,\gamma}g(x,t) = (2\pi)^{-1}\int \mathrm{e}^{\mathrm{i}\gamma(x,t)\xi + \mathrm{i}t\phi(|\xi|)}g(\xi)\mathrm{d}\xi \quad \text{for } \operatorname{supp}(g) \subset A(\lambda).$$

To prove Lemma 2.1, it suffices to show that

$$\|L_{\phi,\gamma}g\|_{L^2_xL^\infty_t([-1,1]\times I)} \le C\lambda^{\max\{\frac{1}{2}-\alpha,\frac{1}{4}\}}\|g\|_{L^2(\mathbb{R})} \tag{2.8}$$

holds for any $I \in D$ in Lemma 2.2. Since $\operatorname{supp} g \subset A(\lambda) = \{\xi : \frac{\lambda}{2} \le |\xi| \le 2\lambda\}$ with $\lambda \ge 2$, we may choose $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\frac{1}{\lambda}(\operatorname{supp} g) \subset \operatorname{supp} \psi \subset A(1) = \Big\{\xi : \frac{1}{2} \le |\xi| \le 2\Big\}$$

and $\psi(\xi) = 1$ if $\xi \in \frac{1}{\lambda}(\operatorname{supp} g)$. Thus, write

$$L_{\phi,\gamma} g(x,t) = (2\pi)^{-1} \int e^{i\gamma(x,t)\xi + it\phi(|\xi|)} g(\xi) \psi\Big(\frac{\xi}{\lambda}\Big) d\xi.$$

Denote by $L'_{\phi,\gamma}$ the adjoint operator of $L_{\phi,\gamma}$. Then it is easy to see that

$$L'_{\phi,\gamma} h(\xi) = (2\pi)^{-1} \psi\Big(\frac{\xi}{\lambda}\Big) \iint e^{-i\gamma(x,t)\xi - it\phi(|\xi|)} h(x,t) dx dt, \quad \lambda \ge 2$$

and

$$L_{\phi,\gamma} L'_{\phi,\gamma} F(x,t) = \iint k(x,y,t,t') F(y,t') dy dt',$$

where

$$k(x,y,t,t') = \int e^{i(\gamma(x,t)-\gamma(y,t'))\xi + i(t-t')\phi(|\xi|)} \psi^2\Big(\frac{\xi}{\lambda}\Big) d\xi.$$

By duality argument, to get (2.8), it remains to show that

$$\|L_{\phi,\gamma} L'_{\phi,\gamma} F(x,t)\|_{L_x^2 L_t^\infty([-1,1]\times I)} \le C\lambda^{2(\max\{\frac{1}{2}-\alpha, \frac{1}{4}\})} \|F\|_{L_x^2 L_t^1([-1,1]\times I)}, \quad \lambda \ge 2. \tag{2.9}$$

2.3 Proof of (2.9) based on Lemmas 2.3–2.4

Making a change of variables, we have

$$k(x,y,t,t') = \lambda \int e^{i\lambda(\gamma(x,t)-\gamma(y,t'))\xi + i(t-t')\phi(\lambda|\xi|)} \psi^2(\xi) d\xi, \tag{2.10}$$

where $\lambda \ge 2$ and $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \psi \subset A(1) = \big\{\xi : \frac{1}{2} \le |\xi| \le 2\big\}$. Now we give two lemmas, which play an important role in verifying (2.9). Their proofs will be given in the next subsections.

Lemma 2.3 *Assume that* $0 < \alpha \le 1$, $\gamma : [0,1] \to \mathbb{R}$ *satisfies the conditions* (A1)–(A2) *and* ϕ *satisfies* (H1). *Let* $\lambda \ge 2$ *and* $I \subset [0,1]$ *be an interval of sidelength* λ^{1-m}. *If* $|x-y| \ge C_8 \lambda^{-\alpha}$ *for some* $C_8 > \max\big\{\frac{4C_1}{C_2}, 1\big\}$, *then for* $t, t' \in I$,

$$|k(x,y,t,t')| \le C\lambda(1+\lambda|x-y|)^{-\frac{1}{2}}. \tag{2.11}$$

Lemma 2.4 *Assume that* $0 < \alpha \le 1$, $\gamma : [0,1] \to \mathbb{R}$ *satisfies the conditions* (A1)–(A2) *and* ϕ *satisfies* (H1). *Let* $\lambda \ge 2$ *and* $I \subset [0,1]$ *be an interval of sidelength* λ^{1-m}. *Then for* $t, t' \in I$,

$$|k(x,y,t,t')| \le C \max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, \frac{1}{|x-y|^{\frac{1}{2\alpha}}}\Big\}. \tag{2.12}$$

The following lemma is an immediate consequence of Proposition 0.5.A. in [24, p. 16].

Lemma 2.5 *Assume that $K(x,y)$ be measurable function on $\mathbb{R}^n\times\mathbb{R}^n$. Denote*

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)\mathrm{d}y.$$

Assume $1\le p\le\infty$. If there exist $A>0$ and $B>0$ such that

$$\int_{\mathbb{R}^n}|K(x,y)|\mathrm{d}x\le A,\quad a.e.\ y\in\mathbb{R}^n$$

and

$$\int_{\mathbb{R}^n}|K(x,y)|\mathrm{d}y\le B,\quad a.e.\ x\in\mathbb{R}^n,$$

then

$$\|Tf\|_{L^p(\mathbb{R}^n)}\le A^{\frac{1}{p}}B^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}^n)},\tag{2.13}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Now we verify (2.9). We divide the interval $[-1,1]$ into essentially intervals $\{J_k\}$ of sidelength with $C_8\lambda^{-\alpha}$ so that $[-1,1]=\cup J_k$. Thus, we have

$$\begin{aligned}
&\|L_{\phi,\gamma}L'_{\phi,\gamma}F(x,t)\|^2_{L^2_xL^\infty_t([-1,1]\times I)}\\
&\le\sum_k\Big\|\sum_{k'}\iint\chi_{J_k}(x)k(x,y,t,t')\chi_{J_{k'}}(y)F(y,t')\mathrm{d}y\mathrm{d}t'\Big\|^2_{L^2_xL^\infty_t(J_k\times I)}\\
&\le2\sum_k\Big\|\iint\chi_{J_k}(x)k(x,y,t,t')\chi_{\widetilde{J_k}}(y)F(y,t')\mathrm{d}y\mathrm{d}t'\Big\|^2_{L^2_xL^\infty_t(J_k\times I)}\\
&\quad+2\sum_k\Big\|\sum_{k'\not\sim k}\iint\chi_{J_k}(x)k(x,y,t,t')\chi_{J_{k'}}(y)F(y,t')\mathrm{d}y\mathrm{d}t'\Big\|^2_{L^2_xL^\infty_t(J_k\times I)}\\
&=:\mathrm{E}_1+\mathrm{E}_2,
\end{aligned}\tag{2.14}$$

where $\widetilde{J_k}$ is an interval containing J_k and the length of $\widetilde{J_k}$ is bigger than $4C_8\lambda^{-\alpha}$. Moreover, the notation $k'\not\sim k$ means that the distance $\mathrm{dist}\,(J_k,J_{k'})>4C_8\lambda^{-\alpha}$ between the intervals J_k and $J_{k'}$.

Case I $\frac12\le\alpha\le1$. In this case, (2.9) is in the following form:

$$\|L_{\phi,\gamma}L'_{\phi,\gamma}F(x,t)\|_{L^2_xL^\infty_t([-1,1]\times I)}\le C\lambda^{\frac12}\|F\|_{L^2_xL^1_t([-1,1]\times I)}.\tag{2.15}$$

We first estimate E_1. Since $k(x,y,t,t')\le C\lambda$ by (2.10) and the length of $J_k\sim\lambda^{-\alpha}$. As [3, p. 988], applying the disjoint of the supports, by Lemma 2.5 and noting that $0\le2(1-\alpha)\le1$, we have

$$\mathrm{E}_1\le C\lambda^{2(1-\alpha)}\|F\|^2_{L^2_xL^1_t([-1,1]\times I)}\le C\lambda\|F\|^2_{L^2_xL^1_t([-1,1]\times I)}.\tag{2.16}$$

As for E_2, note that $\mathrm{dist}\,(J_k,J_{k'})>4C_8\lambda^{-\alpha}$ when $k'\not\sim k$. Similar to [3, p. 988–989], by Lemma 2.3 and Lemma 2.5 and note that the fact $\|\lambda(1+\lambda|\cdot|)^{-\frac12}\|_{L^1_{[-2,2]}}\le C\lambda^{\frac12}$, we have

$$\mathrm{E}_2\le C\lambda\|F\|^2_{L^2_xL^1_t([-1,1]\times I)}.\tag{2.17}$$

Hence, by (2.14) and (2.16)–(2.17), we get (2.15).

Case II $0<\alpha<\frac{1}{2}$. In this case, we need only to show that

$$\|L_{\phi,\gamma}L'_{\phi,\gamma}F(x,t)\|_{L^2_xL^\infty_t([-1,1]\times I)}\le C\lambda^{\max\{1-2\alpha,\frac{1}{2}\}}\|F\|_{L^2_xL^1_t([-1,1]\times I)}. \tag{2.18}$$

Noting that

$$\int_{-1}^{1}\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}\mathrm{d}x\le C\lambda^{\frac{1}{2}},\quad \forall y\in[-1,1], \tag{2.19}$$

and when $0<\alpha<\frac{1}{2}$,

$$\int_{-1}^{1}\min\{|x-y|^{-\frac{1}{2\alpha}},\lambda\}\mathrm{d}x\le C\lambda^{1-2\alpha},\quad \forall y\in[-1,1]. \tag{2.20}$$

Thus, by (2.12), (2.19)–(2.20) and the fact $|k(x,y,t,t')|\le C\lambda$, we have

$$\int_{-1}^{1}\sup_{t,t'\in I}|k(x,y,t,t')|\mathrm{d}x\le C\lambda^{\max\{1-2\alpha,\frac{1}{2}\}},\quad \forall y\in[-1,1] \tag{2.21}$$

and

$$\int_{-1}^{1}\sup_{t,t'\in I}|k(x,y,t,t')|\mathrm{d}y\le C\lambda^{\max\{1-2\alpha,\frac{1}{2}\}},\quad \forall x\in[-1,1]. \tag{2.22}$$

Thus, (2.18) follows from Lemma 2.5 combining (2.21)–(2.22). Summing up all the above estimates, we complete the proof of the estimate (2.9).

Hence, to finish the proof of Theorem 1.1, it remains to prove Lemmas 2.3–2.4.

2.4 Proof of Lemma 2.3

We need the following two lemmas.

Lemma 2.6 (Van der corput' Lemma) (see [22, p. 309]) *Let $\psi\in C_0^\infty(\mathbb{R})$ and $\phi\in C^\infty(\mathbb{R})$ satisfy that $|\phi''(\xi)|>\lambda>0$ on the support of ψ. Then*

$$\Big|\int \mathrm{e}^{\mathrm{i}\phi(\xi)}\psi(\xi)\mathrm{d}\xi\Big|\le 10\lambda^{-\frac{1}{2}}\{\|\psi\|_\infty+\|\psi'\|_1\}.$$

Lemma 2.7 (see [18]) *Let I denote an open integral in $\mathbb{R}$. For $g\in C_0^\infty(I)$ and real valued function $F\in C^\infty(I)$ with $F'\ne 0$, if $k\in\mathbb{N}$, then*

$$\int_I \mathrm{e}^{\mathrm{i}F(x)}g(x)\mathrm{d}x=\int_I \mathrm{e}^{\mathrm{i}F(x)}h_k(x)\mathrm{d}x,$$

where h_k is a linear combination of functions of the form $g^{(s)}(F')^{-k-r}\prod\limits_{q=1}^{r}F^{(j_q)}$ with $0\le s\le k$, $0\le r\le k$ and $2\le j_q\le k+1$.

Proof of Lemma 2.3 By the condition (H1), there exist positive constants C_i ($i=4,5,\cdots,7$) so that for $r\ge 1$ and $m\ge 2$,

$$C_4r^{m-1}\le|\phi'(r)|\le C_5r^{m-1},\quad C_6r^{m-1}\le|\phi''(r)|\le C_7r^{m-2}. \tag{2.23}$$

Denote $g(\xi)=\psi^2(\xi)$ and $F(\xi)=\lambda(\gamma(x,t)-\gamma(y,t'))\xi+(t-t')\phi(\lambda|\xi|)$. We have

$$K(x,y,t,t')=\lambda\int \mathrm{e}^{\mathrm{i}F(\xi)}g(\xi)\mathrm{d}\xi.$$

Note that

$$F'(\xi)=\lambda(\gamma(x,t)-\gamma(y,t'))+\lambda(t-t')\phi'(\lambda|\xi|)\mathrm{sgn}(\xi)$$

and

$$F''(\xi)=\lambda^2(t-t')\phi''(\lambda|\xi|).$$

Choose a large positive constant C_9 such that $C_9>\max\left\{\frac{4C_5 2^{m-1}}{C_2},1\right\}$, and a small positive constant C_{10} such that $C_{10}<\min\left\{\frac{C_4 2^{2-m}}{4C_3+C_2},1\right\}$. We verify (2.11) by dividing three cases according to the value of $|x-y|$, respectively.

Case I $|x-y|\geq C_9\lambda^{m-1}|t-t'|$. Noting that γ satisfies the condition (A1), $t,t'\in I$, $m\geq 2$ and $C_8>\frac{4C_1}{C_2}$, $\lambda^{-\alpha}\leq\frac{1}{C_8}|x-y|$ since $|x-y|\geq C_8\lambda^{-\alpha}$, we have

$$|\gamma(x,t')-\gamma(x,t)|\leq C_1|t-t'|^{\alpha}\leq C_1\lambda^{(1-m)\alpha}\leq C_1\lambda^{-\alpha}\leq\frac{C_1}{C_8}|x-y|\leq\frac{C_2}{4}|x-y|\leq\frac{C_2}{4}|x-y|.$$

Since $|\gamma(y,t')-\gamma(x,t')|\geq C_2|x-y|$ by the condition (A2), we get

$$|\gamma(x,t)-\gamma(y,t')|\geq|\gamma(y,t')-\gamma(x,t')|-|\gamma(x,t')-\gamma(x,t)|\geq\frac{3C_2}{4}|x-y|. \tag{2.24}$$

Note that $\lambda|\xi|\geq 1$ since $\lambda\geq 2$ and $\operatorname{supp} g=\operatorname{supp}\psi\subset A(1)=\left\{\xi:\frac{1}{2}\leq|\xi|\leq 2\right\}$. Applying (2.23) with $m\geq 2$, we have

$$\begin{aligned}&|\lambda(t-t')\phi'(\lambda|\xi|)\mathrm{sgn}(\xi)|\\ &\leq C_5\lambda|t-t'|\lambda^{m-1}|\xi|^{m-1}\\ &\leq C_5 2^{m-1}\lambda^m|t-t'|\leq\frac{C_5 2^{m-1}}{C_9}\lambda|x-y|\leq\frac{C_2}{4}\lambda|x-y|.\end{aligned} \tag{2.25}$$

By (2.24)–(2.25), we get

$$|F'(\xi)|\geq\lambda|\gamma(x,t)-\gamma(y,t')|-|\lambda(t-t')\phi'(\lambda|\xi|)\mathrm{sgn}(\xi)|\geq\frac{C_2}{2}\lambda|x-y|. \tag{2.26}$$

On the other hand, using (2.23) again with $m\geq 2$, and noting that $\lambda|x-y|\geq C_9\lambda^m|t-t'|$, we have

$$|F''(\xi)|\leq C_7\lambda^2|\lambda\xi|^{m-2}|t-t'|\leq C_7 2^{m-2}\lambda^m|t-t'|=C\lambda^m|t-t'|\leq C\lambda|x-y|. \tag{2.27}$$

Applying Lemma 2.7 for $k=1$ and (2.26)–(2.27), we obtain

$$\Big|\int \mathrm{e}^{\mathrm{i}F(\xi)}g(\xi)\mathrm{d}\xi\Big|\leq\int\frac{|g'(\xi)|}{|F'(\xi)|}\mathrm{d}\xi+\int\frac{|F''(\xi)||g(\xi)|}{|F'(\xi)|^2}\mathrm{d}\xi\leq C(\lambda|x-y|)^{-1}.$$

Note that $\lambda|x-y|\geq C_8\lambda^{1-\alpha}>1$ since $|x-y|\geq C_8\lambda^{-\alpha}$, $0<\alpha\leq 1$, $\lambda\geq 2$ and $C_8>1$. Hence it follows that

$$|k(x,y,t,t')|\leq C\lambda(\lambda|x-y|)^{-1}\leq C\lambda(1+\lambda|x-y|)^{-1}\leq C\lambda(1+\lambda|x-y|)^{-\frac{1}{2}},$$

which is the desired inequality.

Case II $|x-y| \le C_{10}\lambda^{m-1}|t-t'|$. From the above estimate, we have

$$|\gamma(x,t') - \gamma(x,t)| \le \frac{C_1}{C_8}\lambda|x-y|.$$

Applying the condition (A2) and noting that $C_8 > \frac{4C_1}{C_2}$, $C_{10} < \frac{C_4 2^{2-m}}{4C_3+C_2}$ and $\lambda|x-y| \le C_{10}\lambda^m|t-t'|$, we get

$$\begin{aligned}\lambda|\gamma(x,t) - \gamma(y,t')| &\le \lambda|\gamma(y,t') - \gamma(x,t')| + \lambda|\gamma(x,t') - \gamma(x,t)| \\ &\le C_3\lambda|x-y| + \frac{C_1}{C_8}\lambda|x-y| \\ &\le \frac{4C_3+C_2}{4}\lambda|x-y| \\ &\le \frac{(4C_3+C_2)C_{10}}{4}\lambda^m|t-t'| \le C_4 2^{-m}\lambda^m|t-t'|. \end{aligned} \tag{2.28}$$

Note that $\lambda|\xi| \ge 1$ by $\lambda \ge 2$ and $\operatorname{supp} g = \operatorname{supp}\psi \subset A(1) = \left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\}$. Thus by (2.23) with $m \ge 2$, we have

$$|\lambda(t-t')\phi'(\lambda|\xi|)\operatorname{sgn}(\xi)| \ge C_4\lambda|t-t'|\lambda^{m-1}|\xi|^{m-1} \ge 2C_4 2^{-m}\lambda^m|t-t'|. \tag{2.29}$$

By (2.28)–(2.29), we get

$$|F'(\xi)| \ge |\lambda(t-t')\phi'(\lambda|\xi|)\operatorname{sgn}(\xi)| - \lambda|\gamma(x,t) - \gamma(y,t')| \ge C_4 2^{-m}\lambda^m|t-t'|. \tag{2.30}$$

On the other hand, since $\lambda|\xi| \ge 1$ and $\operatorname{supp} g = \operatorname{supp}\psi \subset A(1) = \left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\}$, by (2.23) with $m \ge 2$, we have

$$|F''(\xi)| = \lambda^2|t-t'||\phi''(\lambda|\xi|)| \le C_7\lambda^m|t-t'||\xi|^{m-2} \le C\lambda^m|t-t'|. \tag{2.31}$$

Applying Lemma 2.7 for $k=1$, by (2.30)–(2.31), we obtain

$$\Big|\int e^{iF(\xi)}g(\xi)d\xi\Big| \le \int \frac{|g'(\xi)|}{|F'(\xi)|}d\xi + \int \frac{|F''(\xi)||g(\xi)|}{|F'(\xi)|^2}d\xi \le C(\lambda^m|t-t'|)^{-1}.$$

Noting that $\lambda|x-y| \ge C_8\lambda^{1-\alpha} > 1$ and $\lambda|x-y| \le C_{10}\lambda^m|t-t'|$, we get

$$|k(x,y,t,t')| \le C\lambda(\lambda|x-y|)^{-1} \le C\lambda(1+\lambda|x-y|)^{-1} \le C\lambda(1+\lambda|x-y|)^{-\frac{1}{2}}.$$

Case III $C_{10}\lambda^{m-1}|t-t'| < |x-y| < C_9\lambda^{m-1}|t-t'|$. Noting that $\lambda|\xi| \ge 1$ and $\operatorname{supp} g = \operatorname{supp}\psi \subset A(1) = \left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\}$, applying (2.23) with $m \ge 2$, we have

$$|F''(\xi)| = \lambda^2|t-t'||\phi''(\lambda|\xi|)| \ge C\lambda^m|t-t'|\xi|^{m-2} \ge C\lambda^m|t-t'|. \tag{2.32}$$

Noting that $\|g\|_\infty \le C$ and $\|g'\|_1 \le C$, by Lemma 2.6 and (2.32) we have

$$\Big|\int e^{iF(\xi)}g(\xi)d\xi\Big| \le C(\lambda^m|t-t'|)^{-\frac{1}{2}}(\|g\|_\infty + \|g'\|_1) \le C(\lambda^m|t-t'|)^{-\frac{1}{2}}.$$

On the other hand, by $\lambda|x-y| \ge C_8\lambda^{1-\alpha} > 1$ and $C_{10}\lambda^m|t-t'| < \lambda|x-y| < C_9\lambda^m|t-t'|$, we have

$$|k(x,y,t,t')| \le C\lambda\big(\lambda^m|t-t'|\big)^{-\frac{1}{2}} \le C\lambda\big(\lambda|x-y|\big)^{-\frac{1}{2}} \le C\lambda(1+\lambda|x-y|)^{-\frac{1}{2}},$$

which is just (2.11).

Summing up all the above estimates, we show (2.11) and complete the proof of Lemma 2.3.

2.5 Proof of Lemma 2.4

Recall that

$$F'(\xi) = \lambda(\gamma(x,t) - \gamma(y,t')) + \lambda(t-t')\phi'(\lambda|\xi|)\mathrm{sgn}(\xi)$$

and

$$F''(\xi) = \lambda^2(t-t')\phi''(\lambda|\xi|).$$

Choose a large positive constant C_{11} such that $C_{11} > \frac{4C_1}{C_2}$.

Case I $|x-y| \le C_{11}|t-t'|^\alpha$. In this case, we need to prove that

$$|k(x,y,t,t')| \le C|x-y|^{-\frac{1}{2\alpha}}. \tag{2.33}$$

Since $\lambda|\xi| \ge 1$ and $\operatorname{supp} g = \operatorname{supp}\psi \subset A(1) = \{\xi : \frac{1}{2} \le |\xi| \le 2\}$, by (2.23) with $m \ge 2$, we have

$$|F''(\xi)| = \lambda^2|t-t'||\phi''(\lambda|\xi|)| \ge C_6\lambda^m|t-t'||\xi|^{m-2} \ge C\lambda^m|t-t'|. \tag{2.34}$$

Note that $\|g\|_\infty \le C$ and $\|g'\|_1 \le C$ on the support of g. By Lemma 2.6 and (2.34) we have

$$\Big|\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}F(\xi)}g(\xi)\mathrm{d}\xi\Big| \le C(\lambda^m|t-t'|)^{-\frac{1}{2}}(\|g\|_\infty + \|g'\|_1) \le C\lambda^{-\frac{m}{2}}|t-t'|^{-\frac{1}{2}}. \tag{2.35}$$

On the other hand, noting that $\lambda \ge 2$, $m \ge 2$ and $(C_{11})^{-\frac{1}{\alpha}}|x-y|^{\frac{1}{\alpha}} \le |t-t'|$, by (2.35) we get

$$|k(x,y,t,t')| \le C\lambda^{1-\frac{m}{2}}|t-t'|^{-\frac{1}{2}} \le C\lambda^{1-\frac{m}{2}}|x-y|^{-\frac{1}{2\alpha}} \le C|x-y|^{-\frac{1}{2\alpha}}.$$

Case II $|x-y| > C_{11}|t-t'|^\alpha$. In this case, we need to show that

$$|k(x,y,t,t')| \le C\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}. \tag{2.36}$$

Subcase II-a $|x-y| \le \frac{1}{\lambda}$. In this case, we have

$$|k(x,y,t,t')| \le C\lambda \le C\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}},$$

which is just (2.36).

Subcase II-b $|x-y| > \frac{1}{\lambda}$. In this case, we choose a large positive constant C_{12} such that $C_{12} > \max\{\frac{4C_5 2^{m-1}}{C_2}, 1\}$. We prove (2.36) by considering separately two cases $|x-y| \ge C_{12}\lambda^{m-1}|t-t'|$ and $|x-y| < C_{12}\lambda^{m-1}|t-t'|$.

If $|x-y| \ge C_{12}\lambda^{m-1}|t-t'|$, then by the condition (A1) and noting that $t,t' \in I$, $|t-t'|^\alpha < \frac{1}{C_{11}}|x-y|$ and $C_{11} > \frac{4C_1}{C_2}$, we get

$$|\gamma(x,t') - \gamma(x,t)| \le C_1|t-t'|^\alpha \le \frac{C_1}{C_{11}}|x-y| \le \frac{C_2}{4}|x-y|.$$

Thus, since γ satisfies (A2), we have

$$|\gamma(x,t) - \gamma(y,t')| \ge |\gamma(y,t') - \gamma(x,t')| - |\gamma(x,t') - \gamma(x,t)| \ge \frac{3C_2}{4}|x-y|. \tag{2.37}$$

Note that $\lambda|\xi| \geq 1$ and $\operatorname{supp} g = \operatorname{supp} \psi \subset A(1) = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Thus by (2.23) with $m \geq 2$, we have

$$\begin{aligned}|\lambda(t-t')\phi'(\lambda|\xi|)\operatorname{sgn}(\xi)| &\leq C_5\lambda|t-t'|\lambda^{m-1}|\xi|^{m-1} \leq C_5 2^{m-1}\lambda^m|t-t'| \\ &\leq \frac{C_5 2^{m-1}}{C_{12}}\lambda|x-y| \leq \frac{C_2}{4}\lambda|x-y|. \end{aligned} \tag{2.38}$$

By (2.37)–(2.38), for ξ on the support of g, we get

$$|F'(\xi)| \geq \lambda|\gamma(x,t)-\gamma(y,t')| - |\lambda(t-t')\phi'(\lambda|\xi|)\operatorname{sgn}(\xi)| \geq \frac{C_2}{2}\lambda|x-y|. \tag{2.39}$$

On the other hand, note that $\lambda|\xi| \geq 1$ and $\operatorname{supp} g \subset A(1) = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Thus by (2.23) with $m \geq 2$, and $|x-y| \geq C_{12}\lambda^{m-1}|t-t'|$, we have

$$|F''(\xi)| = \lambda^2|t-t'||\phi''(\lambda|\xi|)| \leq C_7\lambda^m|t-t'| \leq C\lambda|x-y|. \tag{2.40}$$

Applying Lemma 2.7 for $k=1$, by (2.39)–(2.40), we obtain

$$\Big|\int e^{iF(\xi)}g(\xi)d\xi\Big| \leq \int \frac{|g'(\xi)|}{|F'(\xi)|}d\xi + \int \frac{|F''(\xi)||g(\xi)|}{|F'(\xi)|^2}d\xi \leq C(\lambda|x-y|)^{-1},$$

which, combining with the fact $\lambda|x-y| > 1$, yields

$$|k(x,y,t,t')| \leq C\lambda(\lambda|x-y|)^{-1} \leq C\lambda(\lambda|x-y|)^{-\frac{1}{2}} = C\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}. \tag{2.41}$$

If $|x-y| \leq C_{12}\lambda^{m-1}|t-t'|$, noting that $\lambda|\xi| \geq 1$ and $\operatorname{supp} g \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. then by (2.23) with $m \geq 2$ we have

$$|F''(\xi)| = \lambda^2|t-t'||\phi''(\lambda|\xi|)| \geq C\lambda^m|t-t'||\xi|^{m-2} \geq C\lambda^m|t-t'|. \tag{2.42}$$

Since $\|g\|_\infty \leq C$ and $\|g'\|_1 \leq C$, by Lemma 2.6 and (2.42) we have

$$\Big|\int e^{iF(\xi)}g(\xi)d\xi\Big| \leq C(\lambda^m|t-t'|)^{-\frac{1}{2}}(\|g\|_\infty + \|g'\|_1) \leq C(\lambda^m|t-t'|)^{-\frac{1}{2}}.$$

Since $\lambda|x-y| \leq C_{12}\lambda^m|t-t'|$. Hence it follows

$$|k(x,y,t,t')| \leq C\lambda(\lambda^m|t-t'|)^{-\frac{1}{2}} \leq C\lambda(\lambda|x-y|)^{-\frac{1}{2}} = C\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}.$$

Summing up all the above estimates, we show (2.36). Hence, by (2.33) and (2.36), it follows (2.12), and we complete the proof of Lemma 2.4.

3 Proof of Theorem 1.3

To prove Theorem 1.3, we first prove the following proposition.

Proposition 3.1 *Let I be an interval and $\nu : I \to \mathbb{R}^n$ be a continuous function. Assume $\gamma(x,t) = x - v(t)$ and there exists a point $t_0 \in I$, $d_0 > 0$ and $\varepsilon > 0$ such that $(t_0, t_0+\varepsilon) \subset I$ and*

$$\operatorname{diam}\{v(\tau) : \tau \in [t_0, t]\} \geq d_0|t-t_0|^\alpha$$

for all $t \in (t_0, t_0+\varepsilon)$ *and* $0<\alpha\le 1$. *Assume that* ϕ *satisfies the condition* (H3). *Then* (1.10) *holds only if* $s \ge \max\{\frac12-\alpha, 0\}$.

Proof By the condition (H3), there exists a positive constant $C_0>0$ so that for $r\ge 1$ and $m\ge 2$, $|\phi(r)|\le C_0 r^m$. Fix $\lambda > \max\big\{\frac{1}{C_0}\varepsilon^{-\frac{m}{2}}, \frac{1}{C_0T}, \big(\frac{12}{\pi}\big)^m\big\}$. Choose a nonnegative function $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi\subset\big\{\xi : \frac{\pi}{12}<|\xi|<\frac{\pi}{6}\big\}$. Denote $\int\psi(\xi)\mathrm{d}\xi = l>0$ and $\widehat{f}(\xi) = \mathrm{e}^{-\mathrm{i}t_0\phi(|\xi|)}\psi(\lambda^{-\frac1m}\xi)$, and by simple calculation, we have

$$\|f\|_{H^s(\mathbb{R}^n)} \le C\lambda^{\frac{n}{2m}+\frac{s}{m}}, \tag{3.1}$$

where C is independent of λ. Making a change of variables, it is easy to see that

$$\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})}f(\gamma(x,t)) = (2\pi)^{-n}\lambda^{\frac{n}{m}}\int \mathrm{e}^{\mathrm{i}(t-t_0)\phi(|\lambda^{\frac1m}\xi|)}\mathrm{e}^{\mathrm{i}\lambda^{\frac1m}\gamma(x,t)\cdot\xi}\psi(\xi)\mathrm{d}\xi. \tag{3.2}$$

Therefore, if $|t-t_0|\le (C_0\lambda)^{-1}$, then

$$|(t-t_0)\phi(|\lambda^{\frac1m}\xi|)| \le |t-t_0|C_0\lambda|\xi|^m \le |\xi|^m \le \frac{\pi}{6}.$$

Thus,

$$\begin{aligned}(2\pi)^{-n}\lambda^{\frac{n}{m}}\Big|\int \mathrm{e}^{\mathrm{i}(t-t_0)\phi(|\lambda^{\frac1m}\xi|)}\psi(\xi)\mathrm{d}\xi\Big| &\ge (2\pi)^{-n}\lambda^{\frac{n}{m}}\Big|\int \cos((t-t_0)\phi(|\lambda^{\frac1m}\xi|))\psi(\xi)\mathrm{d}\xi\Big| \\ &\quad - (2\pi)^{-n}\lambda^{\frac{n}{m}}\Big|\int \sin((t-t_0)\phi(|\lambda^{\frac1m}\xi|))\psi(\xi)\mathrm{d}\xi\Big| \\ &\ge \Big(\frac{\sqrt3-1}{2}\Big)(2\pi)^{-n}\lambda^{\frac{n}{m}}l. \end{aligned} \tag{3.3}$$

If $|t-t_0|\le (C_0\lambda)^{-1}$ and x is contained in $\min\big\{d_0(C_0)^{-\alpha}, \frac{1}{10}\big\}\lambda^{-\frac1m}$ -neighborhood of the set $\big\{v(\tau) : \tau\in\big[t_0, t_0+(C_0\lambda)^{-\frac2m}\big]\big\}$, we have

$$|\lambda^{\frac1m}\gamma(x,t)\cdot\xi| \le \frac{1}{10}\lambda^{\frac1m}\lambda^{-\frac1m}|\xi| \le \frac{1}{10}|\xi| \le \frac{1}{10}.$$

Hence it follows

$$(2\pi)^{-n}\lambda^{\frac{n}{m}}\Big|\int \mathrm{e}^{\mathrm{i}(t-t_0)\phi(|\lambda^{\frac1m}\xi|)}(\mathrm{e}^{\mathrm{i}\lambda^{\frac1m}\gamma(x,t)\cdot\xi}-1)\psi(\xi)\mathrm{d}\xi\Big| \le \frac{1}{10}(2\pi)^{-n}\lambda^{\frac{n}{m}}l. \tag{3.4}$$

By (3.2)–(3.4), we get

$$|\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})}f(\gamma(x,t))| \ge \Big(\frac{\sqrt3-1}{2}-\frac{1}{10}\Big)(2\pi)^{-n}l\lambda^{\frac{n}{m}} =: c\lambda^{\frac{n}{m}}. \tag{3.5}$$

When $|t-t_0|\le (C_0\lambda)^{-1}$ and x is contained in $\min\big\{d_0(C_0)^{-\alpha}, \frac{1}{10}\big\}\lambda^{-\frac1m}$ -neighborhood of the set $\{v(\tau) : \tau\in[t_0, t_0+(C_0\lambda)^{-\frac2m}]\}$, we have $|\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})}f(\gamma(x,t))|\ge c\lambda^{\frac{n}{m}}$. Hence it follows

$$S^*_{\phi,\gamma}f(x) = \sup_{t\in I_T(t_0)}|S_{t,\phi,\gamma}f(x)| \ge c\lambda^{\frac{n}{m}}. \tag{3.6}$$

Since x is contained in $\min\big\{d_0(C_0)^{-\alpha}, \frac{1}{10}\big\}\lambda^{-\frac1m}$ -neighborhood of the set $\{v(\tau) : \tau\in[t_0, t_0+(C_0\lambda)^{-\frac2m}]\}$, (of length $\ge d_0(C_0\lambda)^{-\frac{2\alpha}{m}}$) which has measure $\gtrsim \lambda^{-\frac{n}{m}}$ if $\alpha\ge\frac12$ and $\gtrsim \lambda^{-\frac{n-1}{m}}\lambda^{-\frac{2\alpha}{m}}$ if $\alpha<\frac12$. Assuming the local estimate (1.10), then by (3.1) and (3.6), we obtain

$$\lambda^{\frac{n}{m}}\lambda^{-\frac{n-1}{2m}}\max\{\lambda^{-\frac{1}{2m}}, \lambda^{-\frac{\alpha}{m}}\} \le C\lambda^{\frac{n}{2m}+\frac{s}{m}}, \tag{3.7}$$

where C depends on n, m and l only, and does not depend on λ. Taking λ large enough in (3.7), $s \geq \max\left\{\frac{1}{2} - \alpha, 0\right\}$ is necessary since the local estimate (1.10) holds in this case. Hence, we complete the proof of Proposition 3.1.

Now let us turn to the proof of Theorem 1.3. We first prove that $s \geq \frac{1}{4}$ is necessary for the local estimate (1.10). Fix $\lambda > 10$. Assume ψ and l defined in the proof of Proposition 3.1 and denote $\widehat{f}(\xi) = \psi(\lambda^{-\frac{1}{2}}(\xi - \lambda e_1))$, and by simple calculation, we have

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C\lambda^{s+\frac{n}{4}}, \tag{3.8}$$

where C is independent of λ. Let $\gamma(x,t) = x - (t^\alpha, \cdots, 0)$ and performing a change of variables two times, we obtain

$$\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(\gamma(x,t)) = (2\pi)^{-n}\lambda^{\frac{n}{2}} \int \mathrm{e}^{\mathrm{i}t\phi(|\lambda^{\frac{1}{2}}\xi + \lambda e_1|)} \mathrm{e}^{\mathrm{i}\lambda^{\frac{1}{2}}(x_1 - t^\alpha, \overline{x})\cdot\xi} \psi(\xi)\mathrm{d}\xi, \tag{3.9}$$

where $x = (x_1, \overline{x}) \in \mathbb{R} \times \mathbb{R}^n$. By the condition (H3), there exists a positive constant $C_0 > 0$ so that for $r \geq 1$ and $m \geq 2$, $|\phi(r)| \leq C_0 r^m$. Noting that

$$|\lambda^{\frac{1}{2}}\xi + \lambda e_1| \geq |\lambda e_1| - |\lambda^{\frac{1}{2}}\xi| = \lambda^{\frac{1}{2}}(\lambda^{\frac{1}{2}} - |\xi|) > 1$$

and

$$|\lambda^{\frac{1}{2}}\xi + \lambda e_1| \leq |\lambda^{\frac{1}{2}}\xi| + |\lambda e_1| = \lambda^{\frac{1}{2}} + \lambda \leq 2\lambda$$

by the support of ψ and $\lambda > 10$, therefore, if $|t| \leq (2\lambda)^{-m}\frac{\pi}{6C_0}$, then

$$|t\phi(|\lambda^{\frac{1}{2}}\xi + \lambda e_1|)| \leq |t|C_0|\lambda^{\frac{1}{2}}\xi + \lambda e_1|^m \leq |t|C_0(2\lambda)^m \leq \frac{\pi}{6}.$$

Thus, we have

$$(2\pi)^{-n}\lambda^{\frac{n}{2}}\Big|\int \mathrm{e}^{\mathrm{i}t\phi(|\lambda^{\frac{1}{2}}\xi + \lambda e_1|)}\psi(\xi)\mathrm{d}\xi\Big| \geq \Big(\frac{\sqrt{3}-1}{2}\Big)(2\pi)^{-n}\lambda^{\frac{n}{2}} l. \tag{3.10}$$

Let c_1 is a small constant such that $c_1 < \frac{1}{10}$. When $|\xi| < \frac{\pi}{6}$, $|t| \leq (2\lambda)^{-m}\frac{\pi}{6C_0}$, $0 \leq x_1 \leq \frac{c_1}{100}$ and $|\overline{x}| \leq \frac{c_1\lambda^{-\frac{1}{2}}}{100}$, we have

$$|\lambda^{\frac{1}{2}}(x_1 - t^\alpha, \overline{x})\cdot\xi| \leq |\lambda^{\frac{1}{2}}(x_1 - t^\alpha, \overline{x})||\xi| \leq |\lambda^{\frac{1}{2}}(x_1 - t^\alpha, \overline{x})| \leq c_1 \leq \frac{1}{10}.$$

Hence it follows

$$\begin{aligned} &(2\pi)^{-n}\lambda^{\frac{n}{2}}\Big|\int \mathrm{e}^{\mathrm{i}t\phi(|\lambda^{\frac{1}{2}}\xi + \lambda e_1|)}(\mathrm{e}^{\mathrm{i}\lambda^{\frac{1}{2}}(x_1 - t^\alpha, \overline{x})\cdot\xi} - 1)\psi(\xi)\mathrm{d}\xi\Big| \\ &\leq (2\pi)^{-n}\lambda^{\frac{n}{2}} \int |\lambda^{\frac{1}{2}}(x_1 - t^\alpha, \overline{x})\cdot\xi|\psi(\xi)\mathrm{d}\xi \leq \frac{1}{10}(2\pi)^{-n}\lambda^{\frac{n}{2}} l. \end{aligned} \tag{3.11}$$

By (3.9)–(3.11), we get

$$|\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(\gamma(x,t))| \geq \Big(\frac{\sqrt{3}-1}{2} - \frac{1}{10}\Big)(2\pi)^{-n} l\lambda^{\frac{n}{2}} = c\lambda^{\frac{n}{2}}, \tag{3.12}$$

where $c = \big(\frac{\sqrt{3}-1}{2} - \frac{1}{10}\big)(2\pi)^{-n} l$. Hence, when $|t| \le (2\lambda)^{-m}\frac{\pi}{6C_0}$ and x is contained in domain $\big\{x : 0 \le x_1 \le \frac{c}{100}, |\overline{x}| \le \frac{c\lambda^{-\frac{1}{2}}}{100}\big\}$ (which has measure $\gtrsim \lambda^{-\frac{n-1}{2}}$), we have $|\mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(\gamma(x,t))| \ge c\lambda^{\frac{n}{2}}$ and it follows that

$$\sup_{0\le t\le 1} |\mathrm{e}^{\mathrm{i}t\Delta} f(\gamma(x,t))| \ge c\lambda^{\frac{n}{2}}. \tag{3.13}$$

Assuming the local estimate (1.10), then by (3.8) and (3.13), we obtain

$$\lambda^{\frac{n}{2}}\lambda^{-\frac{n-1}{4}} \le C\lambda^{s+\frac{n}{4}}, \tag{3.14}$$

where C depends on n, m and l only, and does not depend on λ. Taking λ large enough in (3.14), $s \ge \frac{1}{4}$ is necessary since the local estimate (1.10) holds in this case. Hence, we complete the proof of Theorem 1.3.

4 Proof of Theorem 1.4

4.1 Proof of Theorem 1.4 based on Lemmas 4.1–4.2

In this subsection, we give the proof of Theorem 1.4 applying Lemmas 4.1–4.2. The proof of the latter will be given in the following subsections.

Let $t(x) : \mathbb{R} \to I_T(t_0)$ be a measurable function. Set

$$Tf(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi} \mathrm{e}^{\mathrm{i}t(x)\phi(|\xi|)} \widehat{f}(\xi)\mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}).$$

By linearizing the maximal operator, to prove (1.13), we need to show that

$$\Big(\int_{\mathbb{R}} |Tf(x)|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \le C\|f\|_{H^s(\mathbb{R})}, \tag{4.1}$$

where $s > \frac{m_2}{4}$ if $\frac{1}{2} < \alpha \le 1$ or $s > \min\big\{\frac{m_2}{2}, \frac{m_2}{4}\big(\frac{1}{\alpha} - 1\big)\big\}$ if $0 < \alpha \le \frac{1}{2}$. Choose a nonnegative function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$ and

$$\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1, \quad \xi \ne 0.$$

Set $\varphi_0(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$ and it follows that $\varphi_0 \in C_0^\infty(\mathbb{R})$. Rewrite

$$\begin{aligned} Tf(x) &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi+\mathrm{i}t(x)\phi(|\xi|)} \varphi_0(\xi)\widehat{f}(\xi)\mathrm{d}\xi + \sum_{k=1}^{\infty}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi+\mathrm{i}t(x)\phi(|\xi|)} \varphi(2^{-k}\xi)\widehat{f}(\xi)\mathrm{d}\xi \\ &=: T_0 f(x) + \sum_{k=1}^{\infty} T_k f(x). \end{aligned} \tag{4.2}$$

By Minkowski's inequality, we get

$$\|Tf\|_{L^2(\mathbb{R})} \le \|T_0 f\|_{L^2(\mathbb{R})} + \sum_{k=1}^{\infty} \|T_k f\|_{L^2(\mathbb{R})}. \tag{4.3}$$

Define the operator R_N by

$$R_N g(x) = N^{-s}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi} \mathrm{e}^{\mathrm{i}t(x)\phi(|\xi|)} \varphi\Big(\frac{\xi}{N}\Big) g(\xi)\mathrm{d}\xi, \quad g \in \mathcal{S}(\mathbb{R}),\ N \ge 2. \tag{4.4}$$

Lemma 4.1 *Suppose that ϕ and γ satisfy the conditions in Theorem* 1.4. *Then*

$$\|T_0 f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}. \tag{4.5}$$

Lemma 4.2 *Suppose that ϕ and γ satisfy the conditions in Theorem* 1.4. *If $s > \frac{m_2}{4}$ for $\frac{1}{2} < \alpha \leq 1$ or $s > \min\left\{\frac{m_2}{2}, \frac{m_2}{4}\left(\frac{1}{\alpha} - 1\right)\right\}$ for $0 < \alpha \leq \frac{1}{2}$, then there exists $\delta > 0$ and $C > 0$, such that for all $N \geq 2$,*

$$\|R_N g\|_{L^2(\mathbb{R})} \leq CN^{-\delta}\|g\|_{L^2(\mathbb{R})}. \tag{4.6}$$

By estimate (4.3), to get estimate (4.1) it is sufficient to show Lemmas 4.1–4.2. Therefore, to finish the proof of Theorem 1.4, it remains to show Lemmas 4.1–4.2

4.2 Proof of Lemma 4.1 based on Lemma 4.3

Denote

$$L_0 g(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi} \mathrm{e}^{\mathrm{i}t(x)\phi(|\xi|)} \varphi_0(\xi) g(\xi) \mathrm{d}\xi, \quad g \in \mathcal{S}(\mathbb{R}).$$

We first assume that

$$\|L_0 g\|_{L^2(\mathbb{R})} \leq C\|g\|_{L^2(\mathbb{R})} \tag{4.7}$$

holds and complete the proof of Lemma 4.1. Noting that $T_0 f(x) = L_0(\widehat{f})(x)$, and by (4.7), we get

$$\|T_0 f\|_{L^2(\mathbb{R})} = \|L_0(\widehat{f})\|_{L^2(\mathbb{R})} \leq C\|\widehat{f}\|_{L^2(\mathbb{R})} = C\|f\|_{L^2(\mathbb{R})}. \tag{4.8}$$

Next, we verify (4.7). Taking function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho(x) = 1$ if $|x| \leq 1$, and $\rho(x) = 0$ if $|x| \geq 2$. For $M > 1$, define the operator

$$L_{0,M} g(x) = \rho\Big(\frac{x}{M}\Big) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi} \mathrm{e}^{\mathrm{i}t(x)\phi(|\xi|)} \varphi_0(\xi) g(\xi) \mathrm{d}\xi, \quad g \in \mathcal{S}(\mathbb{R}).$$

It is easy to see that the adjoint operator $L'_{0,M}$ of $L_{0,M}$ is given by

$$L'_{0,M} h(\xi) = \varphi_0(\xi) \int_{\mathbb{R}} \rho\Big(\frac{x}{M}\Big) \mathrm{e}^{-\mathrm{i}\gamma(x,t(x))\cdot\xi} \mathrm{e}^{-\mathrm{i}t(x)\phi(|\xi|)} h(x) \mathrm{d}x, \quad M > 1.$$

Thus, we have

$$\begin{aligned} \|L'_{0,M} h\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \Big(\varphi_0(\xi) \int_{\mathbb{R}} \rho\Big(\frac{x}{M}\Big) \mathrm{e}^{-\mathrm{i}\gamma(x,t(x))\cdot\xi} \mathrm{e}^{-\mathrm{i}t(x)\phi(|\xi|)} h(x) \mathrm{d}x\Big) \\ &\quad \times \overline{\Big(\varphi_0(\xi) \int_{\mathbb{R}} \rho\Big(\frac{y}{M}\Big) \mathrm{e}^{-\mathrm{i}\gamma(y,t(y))\cdot\xi} \mathrm{e}^{-\mathrm{i}t(y)\phi(|\xi|)} h(y) \mathrm{d}y\Big)} \mathrm{d}\xi \\ &=: \int_{\mathbb{R}} \int_{\mathbb{R}} K_0(x,y) h(x) h(y) \mathrm{d}x \mathrm{d}y, \end{aligned} \tag{4.9}$$

where

$$K_0(x,y) := \rho\Big(\frac{x}{M}\Big) \rho\Big(\frac{y}{M}\Big) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(\gamma(y,t(y)) - \gamma(x,t(x)))\xi + \mathrm{i}(t(y) - t(x))\phi(|\xi|)} \varphi_0^2(\xi) \mathrm{d}\xi.$$

Lemma 4.3 *Suppose that ϕ and γ satisfy the conditions in Theorem* 1.4. *Then*

$$|K_0(x,y)| \le CJ_0(x-y), \tag{4.10}$$

where

$$J_0(x) = \begin{cases} 1, & |x| \le \max\Big\{\dfrac{(2T)^\alpha}{C_4}, \dfrac{4}{(C_2)^2}, 1\Big\}, \\ \dfrac{1}{(1+|x|)^2}, & |x| > \max\Big\{\dfrac{(2T)^\alpha}{C_4}, \dfrac{4}{(C_2)^2}, 1\Big\} \end{cases} \tag{4.11}$$

and the constant $0 < C_4 < \frac{C_2}{2C_1}$.

We first assume that Lemma 4.3 holds and complete the proof of Lemma 4.1. Note that $\rho \in C_0^\infty(\mathbb{R})$ and $J_0 \in L^1(\mathbb{R})$. Thus by (4.9)–(4.10), invoking Hölder's inequality and Young's inequality, we obtain

$$\|L'_{0,M}h\|^2_{L^2(\mathbb{R})} \le C\int (J_0 * |h|)(x)|h(x)|\mathrm{d}x \le C\|J_0\|_{L^1(\mathbb{R})}\|h\|^2_{L^2(\mathbb{R})} \le C\|h\|^2_{L^2(\mathbb{R})}.$$

Thus, $\|L_{0,M}g\|_{L^2(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}$ by duality, where C is independent of M. Let $M \to \infty$, we obtain $\|L_0 g\|_{L^2(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}$. Thus, to complete the proof of Lemma 4.1, it remains to prove Lemma 4.3.

4.3 Proof of Lemma 4.3

Since $\varphi_0^2 \in C_0^\infty(\mathbb{R})$, (4.10) is obvious for the case $|x-y| \le \max\big\{\frac{(2T)^\alpha}{C_4}, \frac{4}{(C_2)^2}, 1\big\}$. So, below we only consider the case $|x-y| > \max\big\{\frac{(2T)^\alpha}{C_4}, \frac{4}{(C_2)^2}, 1\big\}$. Rewrite

$$\begin{aligned} K_0(x,y) &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}[\gamma(y,t(y))-\gamma(x,t(x))]\xi}\varphi_0^2(\xi)\mathrm{d}\xi \\ &\quad + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}[\gamma(y,t(y))-\gamma(x,t(x))]\xi}(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)} - 1)\varphi_0^2(\xi)\mathrm{d}\xi \\ &=: K_{0,1}(x,y) + K_{0,2}(x,y). \end{aligned}$$

The estimate $K_{0,1}$ is simple. In fact, noting that $K_{0,1}(x,y) = \widehat{\varphi_0^2}(\gamma(x,t(x)) - \gamma(y,t(y)))$ and $\varphi_0^2 \in C_0^\infty(\mathbb{R})$, it follows

$$|K_{0,1}(x,y)| \le C\frac{1}{(1+|\gamma(x,t(x)) - \gamma(y,t(y)|^2)^2}. \tag{4.12}$$

Notice that $C_4 < \frac{C_2}{2C_1}$ and $|t(y)-t(x)|^\alpha \le (2T)^\alpha < C_4|x-y|$ since $t(x), t(y) \in I_T(t_0)$, and $|x-y| > \frac{(2T)^\alpha}{C_4}$. Thus, by the conditions (A1) and (A3), we have

$$\begin{aligned} |\gamma(x,t(x)) - \gamma(y,t(y))| &\ge |\gamma(x,t(x)) - \gamma(y,t(x))| - |\gamma(y,t(x)) - \gamma(y,t(y))| \\ &\ge C_2|x-y| - C_1C_4|x-y| \\ &\ge \frac{C_2}{2}|x-y|. \end{aligned} \tag{4.13}$$

By (4.13) and $|x-y| \ge \frac{4}{(C_2)^2}$, we have

$$|\gamma(x,t(x)) - \gamma(y,t(y))|^2 \ge \frac{(C_2)^2|x-y|}{4}|x-y| \ge |x-y|.$$

From this and combining with (4.12), when $|x-y| > \max\left\{\frac{(2T)^\alpha}{C_4}, \frac{4}{(C_2)^2}, 1\right\}$, we get

$$|K_{0,1}(x,y)| \leq C\frac{1}{(1+|x-y|)^2}. \tag{4.14}$$

Next we prove

$$|K_{0,2}(x,y)| \leq C\frac{1}{(1+|x-y|)^2}. \tag{4.15}$$

Assume ρ defined as above, and set $\psi = 1-\rho$. For $0 < \varepsilon < \frac{1}{2}$, we set

$$K_{0,2,\varepsilon}(x,y) = \int \mathrm{e}^{\mathrm{i}[\gamma(y,t(y))-\gamma(x,t(x))]\xi}(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)\varphi_0^2(\xi)\psi\Big(\frac{\xi}{\varepsilon}\Big)\mathrm{d}\xi =: \int \mathrm{e}^{\mathrm{i}P(\xi)}Q(\xi)\mathrm{d}\xi,$$

where $P(\xi) = [\gamma(y,t(y))-\gamma(x,t(x))]\xi$ and $Q(\xi) = (\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)\varphi_0^2(\xi)\psi\big(\frac{\xi}{\varepsilon}\big)$. We fist prove

$$|K_{0,2,\varepsilon}(x,y)| \leq C\frac{1}{(1+|x-y|)^2}. \tag{4.16}$$

Integrating by part two times, we have

$$K_{0,2,\varepsilon}(x,y) = -\frac{1}{(\gamma(y,t(y))-\gamma(x,t(x)))^2}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}P(\xi)}Q''(\xi)\mathrm{d}\xi. \tag{4.17}$$

Noting that

$$Q''(\xi) = \sum_{\beta_1+\beta_2+\beta_3=2} (\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)^{(\beta_1)}\Big(\psi\Big(\frac{\xi}{\varepsilon}\Big)\Big)^{(\beta_2)}(\varphi_0^2(\xi))^{(\beta_3)}$$

and by (4.17) and (4.13), when $|x-y| > \max\left\{\frac{(2T)^\alpha}{C_4}, \frac{4}{(C_2)^2}, 1\right\}$, we have

$$\begin{aligned}
|K_{0,2,\varepsilon}(x,y)| &\leq \frac{1}{|\gamma(y,t(y))-\gamma(x,t(x))|^2}\int |\mathrm{e}^{\mathrm{i}P(\xi)}||Q''(\xi)|\mathrm{d}\xi \\
&\leq C\frac{1}{|x-y|^2}\sum_{\beta_1+\beta_2+\beta_3=2}\int |(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)^{(\beta_1)}|\Big|\Big(\psi\Big(\frac{\xi}{\varepsilon}\Big)\Big)^{(\beta_2)}\Big||(\varphi_0^2(\xi))^{(\beta_3)}|\mathrm{d}\xi \\
&\leq C\frac{1}{(1+|x-y|)^2}\sum_{\beta_1+\beta_2+\beta_3=2} I_{\beta_1,\beta_2,\beta_3},
\end{aligned} \tag{4.18}$$

where

$$I_{\beta_1,\beta_2,\beta_3} = \int |(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)^{(\beta_1)}|\Big|\Big(\psi\Big(\frac{\xi}{\varepsilon}\Big)\Big)^{(\beta_2)}\Big||(\varphi_0^2(\xi))^{(\beta_3)}|\mathrm{d}\xi.$$

Noting that the following estimate holds: For $0 < |\xi| < 1$,

$$|(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)^{(\beta_1)}| \leq C|\xi|^{m_1-\beta_1} \quad \text{for } \beta_1 = 0,1,2. \tag{4.19}$$

In fact, for $0 < |\xi| < 1$, by the condition (K1), we have

$$|\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1| \leq |t(y)-t(x)||\phi(|\xi|)| \leq C|\xi|^{m_1}, \tag{4.20}$$

$$|(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)'| = |\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}||t(y)-t(x)||\phi'(|\xi|)| \leq C|\xi|^{m_1-1} \tag{4.21}$$

and

$$\begin{aligned}|(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)''| &\le |\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}||t(y)-t(x)|^2|\phi'(|\xi|)|^2\\ &\quad+|\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}||t(y)-t(x)||\phi''(|\xi|)|\\ &\le C|\phi'(|\xi|)|^2+C|\phi''(|\xi|)|\le C|\xi|^{m_1-2}.\end{aligned}\tag{4.22}$$

Thus (4.19) holds from (4.20)–(4.22). In a way similar to the above estimate (4.19), by the conditions (K2)–(K3), for $1\le|\xi|<2$, we may get

$$|(\mathrm{e}^{\mathrm{i}(t(y)-t(x))\phi(|\xi|)}-1)^{(\beta_1)}|\le C\quad \text{for } \beta_1=0,1,2.\tag{4.23}$$

From the definition of ψ, we have

$$\Big|\Big(\psi\Big(\frac{\xi}{\varepsilon}\Big)\Big)^{(\beta_2)}\Big|\le C|\xi|^{-\beta_2}\quad \text{for } \beta_2=1,2.\tag{4.24}$$

Next, we estimate $I_{\beta_1,\beta_2,\beta_3}$ by two cases, respectively.

Case I $\beta_2=0$. By (4.19), (4.23) and noting that $\beta_1\le 2$, and $m_1>1$, we have

$$I_{\beta_1,\beta_2,\beta_3}\le C\int_{\varepsilon<|\xi|<1}|\xi|^{m_1-\beta_1}\mathrm{d}\xi+C\int_{1\le|\xi|<2}\mathrm{d}\xi\le C\int_{|\xi|<1}|\xi|^{m_1-2}\mathrm{d}\xi+C\le C.\tag{4.25}$$

Case II $\beta_2=1$ or $\beta_2=2$. Notice that $\varepsilon<|\xi|<1$ by $\varepsilon\le|\xi|\le 2\varepsilon$ and $0<\varepsilon<\frac12$. Thus, by (4.19), (4.24) and noting that $\beta_1+\beta_2\le 2$, $0<\varepsilon<\frac12$ and $m_1>1$, we have

$$I_{\beta_1,\beta_2,\beta_3}\le\int_{\varepsilon\le|\xi|\le 2\varepsilon}|\xi|^{m_1-\beta_1}|\xi|^{-\beta_2}\mathrm{d}\xi\le\varepsilon\varepsilon^{m_1-(\beta_1+\beta_2)}\le\varepsilon\varepsilon^{m_1-2}=C\varepsilon^{m_1-1}\le C.\tag{4.26}$$

Hence, (4.16) holds from (4.18) and (4.25)–(4.26). Letting $\varepsilon\to 0$ in (4.16), we get (4.15). Then by (4.14)–(4.15), when $|x-y|>\max\big\{\frac{(2T)^\alpha}{C_4},\frac{4}{(C_2)^2},1\big\}$, we get

$$|K_0(x,y)|\le C\frac{1}{(1+|x-y|)^2}.\tag{4.27}$$

Thus, we complete the proof of Lemma 4.3.

4.4 Proof of Lemma 4.2 based on Lemmas 4.4–4.5

Recall that

$$R_Ng(x)=N^{-s}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi}\mathrm{e}^{\mathrm{i}t(x)\phi(|\xi|)}\varphi\Big(\frac{\xi}{N}\Big)g(\xi)\mathrm{d}\xi,\quad g\in\mathcal{S}(\mathbb{R}),\ N\ge 2.$$

We verify (4.6) now. Taking function $\rho\in C_0^\infty(\mathbb{R})$ such that $\rho(x)=1$ if $|x|\le 1$, and $\rho(x)=0$ if $|x|\ge 2$. For $M>1$, define

$$R_{N,M}g(x)=N^{-s}\rho\Big(\frac{x}{M}\Big)\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\gamma(x,t(x))\cdot\xi}\mathrm{e}^{\mathrm{i}t(x)\phi(|\xi|)}\varphi\Big(\frac{\xi}{N}\Big)g(\xi)\mathrm{d}\xi,\quad g\in\mathcal{S}(\mathbb{R}).$$

It is easy to see that the adjoint operator $R'_{N,M}$ of $R_{N,M}$ is given by

$$R'_{N,M}h(\xi)=N^{-s}\varphi\Big(\frac{\xi}{N}\Big)\int_{\mathbb{R}}\rho\Big(\frac{x}{M}\Big)\mathrm{e}^{-\mathrm{i}\gamma(x,t(x))\cdot\xi}\mathrm{e}^{-\mathrm{i}t(x)\phi(|\xi|)}h(x)\mathrm{d}x,\quad M>1,$$

since

$$\|R'_{N,M}h\|^2_{L^2(\mathbb{R})} =: \int_{\mathbb{R}}\int_{\mathbb{R}} K_N(x,y)h(x)h(y)\mathrm{d}x\mathrm{d}y, \tag{4.28}$$

where

$$K_N(x,y) := \rho\Big(\frac{x}{M}\Big)\rho\Big(\frac{y}{M}\Big)N^{-2s}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(\gamma(y,t(y))-\gamma(x,t(x)))\xi+\mathrm{i}(t(y)-t(x))\phi(|\xi|)}\varphi^2\Big(\frac{\xi}{N}\Big)\mathrm{d}\xi.$$

Denote $\omega = t(y) - t(x)$ and let

$$I_N(x,y,\omega) = N^{-2s}\int \mathrm{e}^{\mathrm{i}(\gamma(y,t(y))-\gamma(x,t(x)))\xi+\mathrm{i}\omega\phi(|\xi|)}\varphi^2\Big(\frac{\xi}{N}\Big)\mathrm{d}\xi$$

for $x, y \in \mathbb{R}$, $-2T \le \omega \le 2T$ and $N \ge 2$. To prove Lemma 4.2, we need the following lemma.

Lemma 4.4 *Let $I_N(x,y,\omega)$ be defined as above. Suppose that ϕ and γ satisfy the conditions in Theorem* 1.4.

For $0 < \alpha \le 1$, we have the following estimate:

$$\sup_{|\omega|\le 2T} |I_N(x,y,\omega)| \le CJ_N(x-y), \tag{4.29}$$

where

$$J_N(x) = \begin{cases} N^{1-2s}, & |x| \le C_7C_8N^{m_2-1},\\ (N|x|)^{-2}N^{1-2s}, & |x| > C_7C_8N^{m_2-1}. \end{cases} \tag{4.30}$$

Moreover, for $\frac14 \le \alpha \le 1$, we have the following estimate:

$$\sup_{|\omega|\le 2T} |I_N(x,y,\omega)| \le CJ_N(x-y), \tag{4.31}$$

where

$$J_N(x) = \begin{cases} N^{1-2s}, & 0 < |x| \le \dfrac{1}{N},\\ N^{\frac{m_2}{2}(\frac{1}{\alpha}-1)}(N|x|)^{-\frac{1}{2\alpha}}N^{1-2s}, & \dfrac{1}{N} < |x| \le C_7C_8N^{m_2-1},\\ (N|x|)^{-2}N^{1-2s}, & |x| > C_7C_8N^{m_2-1}. \end{cases} \tag{4.32}$$

Here C_7 and C_8 are large constants independent of N.

Lemma 4.5 *Let J_N be defined as above. Suppose that ϕ and γ satisfy the conditions in* Theorem 1.4. *If $s > \frac{m_2}{4}$ for $\frac12 < \alpha \le 1$ or $s > \min\big\{\frac{m_2}{2}, \frac{m_2}{4}\big(\frac{1}{\alpha}-1\big)\big\}$ for $0 < \alpha \le \frac12$, then there exists $\delta > 0$ and $C > 0$, such that for all $N \ge 2$,*

$$\|J_N\|_{L^1(\mathbb{R})} \le CN^{-2\delta}. \tag{4.33}$$

We first finish the proof of Lemma 4.2 using Lemmas 4.4–4.5. The latter will be proved in the following subsections. Noting that $\rho \in C_0^\infty(\mathbb{R})$, and by (4.28)–(4.29), (4.31) and (4.33), invoking Hölder's inequality and Young's inequality, we have

$$\int |R'_{N,M}h(\xi)|^2\mathrm{d}\xi \le C\int (J_N * |h|)(x)|h(x)|\mathrm{d}x \le C\|J_N\|_{L^1(\mathbb{R})}\|h\|^2_{L^2(\mathbb{R})} \le CN^{-2\delta}\|h\|^2_{L^2(\mathbb{R})}.$$

From this we get $\|R'_{N,M}h\|_{L^2(\mathbb{R})} \le CN^{-\delta}\|h\|_{L^2(\mathbb{R})}$. Thus, $\|R_{N,M}g\|_{L^2(\mathbb{R})} \le CN^{-\delta}\|g\|_{L^2(\mathbb{R})}$ by duality, where C is independent of N and M. Letting $M \to \infty$, we obtain $\|R_Ng\|_{L^2(\mathbb{R})} \le CN^{-\delta}\|g\|_{L^2(\mathbb{R})}$. (4.6) follows. Thus, we complete the proof of Lemma 4.2 based on Lemmas 4.4–4.5.

4.5 Proof of Lemma 4.4

Now we verify the estimate (4.29) and (4.31). Recall that

$$I_N(x,y,\omega)=N^{-2s}\int \mathrm{e}^{\mathrm{i}(\gamma(y,t(y))-\gamma(x,t(x)))\xi+\mathrm{i}\omega\phi(|\xi|)}\varphi^2\Big(\frac{\xi}{N}\Big)\mathrm{d}\xi$$

for $x,y\in\mathbb{R}, -2T\le\omega\le 2T$ and $N\ge 2$. Performing a change of variables, we have

$$I_N(x,\omega)=N^{1-2s}\int \mathrm{e}^{\mathrm{i}N(\gamma(y,t(y))-\gamma(x,t(x)))\xi+\mathrm{i}\omega\phi(N|\xi|)}G(\xi)\mathrm{d}\xi,$$

where $x,y\in\mathbb{R}, -2T\le\omega\le 2T, N\ge 2$ and $G(\xi)=\varphi^2(\xi)$. It is obvious that for all $x,y\in\mathbb{R}, -2T\le\omega\le 2T$ and $N\ge 2$,

$$|I_N(x,y,\omega)|\le CN^{1-2s}. \tag{4.34}$$

Below we give more estimates of $I_N(x,y,\omega)$. By the condition (K2), there exists $m_2>1$ and $C_5>0$ such that $|\phi'(r)|\le C_5 r^{m_2-1}$ for $r\ge 1$. Denote

$$C_6=\max_{\frac12\le|\xi|\le 2}\{|\xi|^{m_2-1}\},\quad C_7=\max\{C_5C_6,1\},\quad C_8=\max\Big\{\frac{8T}{C_2},\frac{4C_1(2T)^\alpha}{C_7C_2},1\Big\}.$$

Now we give the following estimates of $I_N(x,y,\omega)$ for $x,y\in\mathbb{R}$, $-2T\le\omega\le 2T$ and $N\ge 2$:

$$|I_N(x,y,\omega)|\le\begin{cases} C(N|x-y|)^{-2}N^{1-2s}, & \omega:\Big|\dfrac{\omega}{2T}\Big|^\alpha<\dfrac{N|x-y|}{C_7C_8N^{m_2}},\\ CN^{\frac{m_2}{2}(\frac1\alpha-1)}(N|x-y|)^{-\frac{1}{2\alpha}}N^{1-2s}, & \omega:\Big|\dfrac{\omega}{2T}\Big|^\alpha\ge\dfrac{N|x-y|}{C_7C_8N^{m_2}}.\end{cases} \tag{4.35}$$

Let $F(\xi)=N(\gamma(y,t(y))-\gamma(x,t(x)))\xi+\omega\phi(N|\xi|)$. Thus, we have

$$\begin{aligned} &I_N(x,y,\omega)=N^{1-2s}\int \mathrm{e}^{\mathrm{i}F(\xi)}G(\xi)\mathrm{d}\xi,\\ &F'(\xi)=N(\gamma(y,t(y))-\gamma(x,t(x)))+N\mathrm{sgn}(\xi)\omega\phi'(N|\xi|),\\ &F''(\xi)=N^2\omega\phi''(N|\xi|),\\ &F^{(3)}(\xi)=N^3\mathrm{sgn}(\xi)\omega\phi^{(3)}(N|\xi|).\end{aligned}$$

If $\big|\frac{\omega}{2T}\big|^\alpha\le\frac{N|x-y|}{C_7C_8N^{m_2}}$, that is $|t(y)-t(x)|^\alpha\le\frac{(2T)^\alpha N|x-y|}{C_7C_8N^{m_2}}$. From γ satisfying the conditions (A1), (A3) and noting that $m_2>1,\ N\ge 2$ and $C_8\ge\frac{4C_1(2T)^\alpha}{C_7C_2}$, we have

$$\begin{aligned} N|\gamma(y,t(y))-\gamma(x,t(x))|&\ge N|\gamma(y,t(y))-\gamma(x,t(y))|-N|\gamma(x,t(y))-\gamma(x,t(x))|\\ &\ge C_2N|x-y|-\frac{C_1(2T)^\alpha N|x-y|}{C_7C_8N^{m_2-1}}\ge\frac{3C_2}{4}N|x-y|. \end{aligned}\tag{4.36}$$

Noting that $N|\xi|>1$ since $N\ge 2$ and $\frac12<|\xi|<2$, by (K2) we get

$$|N\mathrm{sgn}(\xi)\omega\phi'(N|\xi|)|\le C_5N|\omega|(N|\xi|)^{m_2-1}\le C_5C_6N^{m_2}|\omega|\le C_7N^{m_2}|\omega|. \tag{4.37}$$

Since $|\frac{\omega}{2T}|\le 1$ and $0<\alpha\le 1$, we have $|\frac{\omega}{2T}|^\alpha\le\frac{N|x-y|}{C_7C_8N^{m_2}}$. Hence it follows $|\frac{\omega}{2T}|\le\frac{N|x-y|}{C_7C_8N^{m_2}}$ (equivalently, $C_7N^{m_2}|\omega|\le\frac{2T}{C_8}N|x-y|$), and noting that $C_8>\frac{8T}{4C_2}$ we have

$$|N\mathrm{sgn}(\xi)\omega\phi'(N|\xi|)|\le\frac{2T}{C_8}N|x-y|\le\frac{C_2}{4}N|x-y|.$$

Therefore, by(4.36)–(4.37), we have

$$|F'(\xi)| \geq N|\gamma(y,t(y)) - \gamma(x,t(x))| - |N\,\mathrm{sgn}(\xi)\,\omega\phi'(N|\xi|)| \geq \frac{C_2}{2}N|x-y|. \tag{4.38}$$

Since ϕ satisfies (K2)–(K3), we have

$$|F^{(j)}(\xi)| \leq CN^{m_2}|\omega| \quad \text{for } j=2,3. \tag{4.39}$$

By the fact $\frac{N^{m_2}|\omega|}{N|x-y|} \leq \frac{2T}{C_7C_8}$ and Lemma 2.7 for $k=2$ and (4.38)–(4.39), we get

$$\begin{aligned}\Big|\int \mathrm{e}^{\mathrm{i}F(\xi)}G(\xi)\mathrm{d}\xi\Big| &\leq C\int_{\frac{1}{2}<|\xi|<2}\frac{1}{|F'(\xi)|^2}\Big(1+\frac{|F''(\xi)|}{|F'(\xi)|}++\Big(\frac{|F''(\xi)|}{|F'(\xi)|}\Big)^2+\frac{|F^{(3)}(\xi)|}{|F'(\xi)|}\Big)\mathrm{d}\xi\\ &\leq C(N|x-y|)^{-2}\sum_{r=0}^{2}\Big(\frac{N^{m_2}|\omega|}{N|x-y|}\Big)^r \leq C(N|x-y|)^{-2},\end{aligned}$$

from which it follows the first estimate in (4.35). On the other hand, when $\big|\frac{\omega}{2T}\big|^\alpha \geq \frac{N|x-y|}{C_7C_8N^{m_2}}$, noting that $N^{m_2}|\omega| \geq \frac{2T}{(C_7C_8)^{\frac{1}{\alpha}}}N^{m_2(1-\frac{1}{\alpha})}(N|x-y|)^{\frac{1}{\alpha}}$ by $|\omega| \geq \big(\frac{(2T)^\alpha N|x-y|}{C_7C_8N^{m_2}}\big)^{\frac{1}{\alpha}}$, combining with this, we get

$$|F''(\xi)| \geq CN^2|\omega|(N|\xi|)^{m_2-2} > CN^{m_2}|\omega| \geq CN^{m_2(1-\frac{1}{\alpha})}(N|x-y|)^{\frac{1}{\alpha}} > 0.$$

Noting that $\|G\|_\infty \leq C$ and $\|G'\|_1 \leq C$ on the support of φ, by Lemma 2.6, we have

$$|I_N(x,y,\omega)| \leq CN^{\frac{m_2}{2}(\frac{1}{\alpha}-1)}(N|x-y|)^{-\frac{1}{2\alpha}}N^{1-2s}.$$

This is just the second estimate in (4.35).

We now give the proof of Lemma 4.4. We divide the verification of (4.29) and (4.31) into three cases according to the value of $|x-y|$.

Case I $0<|x-y|\leq \frac{1}{N}$. Note that (4.34) holds for all x,y and $0<\alpha\leq 1$. Hence it follows (4.29) and (4.31) when $0<|x-y|\leq\frac{1}{N}$.

Case II $|x-y|>C_7C_8N^{m_2-1}$. Since $\big|\frac{\omega}{2T}\big|\leq 1$, it follows that $|x-y|>C_7C_8N^{m_2-1}\big|\frac{\omega}{2T}\big|^\alpha$ for all $0<\alpha\leq 1$. Equivalently, $\big|\frac{\omega}{2T}\big|^\alpha < \frac{N|x-y|}{C_7C_8N^{m_2}}$ for all $0<\alpha\leq 1$. Thus, the first inequality in (4.35) holds for all $0<\alpha\leq 1$. Hence we get (4.29) and (4.31).

Case III $\frac{1}{N}<|x-y|\leq C_7C_8N^{m_2-1}$, that is $1<N|x-y|\leq C_7C_8N^{m_2}$.

Subcase III-a Noting that (4.34) holds for all x,y and $0<\alpha\leq 1$, it follows (4.29).

Subcase III-b $\frac{1}{4}\leq\alpha\leq 1$. Note that $\frac{1}{\alpha}\geq 1$ and $-\frac{1}{2\alpha}\geq -2$ by $\frac{1}{4}\leq\alpha\leq 1$. Thus, by (4.35), it follows (4.31).

Summing up all the above estimates, we complete the proof of Lemma 4.4.

4.6 Proof of Lemma 4.5

We first prove that for $0<\alpha\leq 1$, when $s>\frac{m_2}{2}$, there exists $\delta>0$ and $C>0$, such that for all $N\geq 2$,

$$\|J_N\|_{L^1(\mathbb{R})} \leq CN^{-2\delta}. \tag{4.40}$$

Noting that $m_2 > 1$, $N \geq 2$ and $C_7C_8 > 1$ by $C_7 > 1$ and $C_8 > 1$, we write

$$\int |J_N(x)|\mathrm{d}x = \int_{|x|\leq C_7C_8N^{m_2-1}} |J_N(x)|\mathrm{d}x + \int_{|x|>C_7C_8N^{m_2-1}} |J_N(x)|\mathrm{d}x =: \mathrm{D}_1 + \mathrm{D}_2.$$

By (4.30), we see that

$$\mathrm{D}_1 \leq C \int_{0<|x|\leq C_7C_8N^{m_2-1}} N^{1-2s}\mathrm{d}x \leq C_7C_8N^{m_2-1}N^{1-2s} = C_7C_8N^{m_2-2s}. \tag{4.41}$$

As for D_2, by (4.30), we obtain

$$\mathrm{D}_2 \leq CN^{-1-2s} \int_{|x|>C_7C_8N^{m_2-1}} |x|^{-2}\mathrm{d}x \leq CN^{-m_2-2s}. \tag{4.42}$$

Since $m_2 > 1$, thus, by (4.41)–(4.42), when $0 < \alpha \leq 1$, we have

$$\|J_N\|_{L^1(\mathbb{R})} \leq CN^{m_2-2s} =: CN^{-2\delta},$$

where $2\delta = s - \frac{m_2}{2} > 0$ since $s > \frac{m_2}{2}$ and $m_2 > 1$. Thus, we complete the proof of estimate (4.40).

Let us now return to the proof of Lemma 4.5. Noting that $m_2 > 1$, $N \geq 2$ and $C_7C_8 > 1$ by $C_7 > 1$ and $C_8 > 1$, we write

$$\begin{aligned}\int |J_N(x)|\mathrm{d}x &= \int_{0<|x|\leq \frac{1}{N}} |J_N(x)|\mathrm{d}x \\ &\quad + \int_{\frac{1}{N}<|x|\leq C_7C_8N^{m_2-1}} |J_N(x)|\mathrm{d}x + \int_{|x|>C_7C_8N^{m_2-1}} |J_N(x)|\mathrm{d}x \\ &=: \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.\end{aligned}$$

Case I $\frac{1}{2} < \alpha \leq 1$ and $s > \frac{m_2}{4}$. By (4.32), we have

$$\mathrm{E}_1 \leq C \int_{0<|x|\leq \frac{1}{N}} N^{1-2s}\mathrm{d}x \leq CN^{-2s}. \tag{4.43}$$

As for E_2, by (4.32) we get

$$\begin{aligned}\mathrm{E}_2 &\leq C \int_{|x|\leq C_7C_8N^{m_2-1}} N^{m_2(\frac{1}{2\alpha}-\frac{1}{2})}(N|x|)^{-\frac{1}{2\alpha}}N^{1-2s}\mathrm{d}x \\ &= CN^{m_2(\frac{1}{2\alpha}-\frac{1}{2})-\frac{1}{2\alpha}+1-2s} \int_{|x|\leq C_7C_8N^{m_2-1}} |x|^{-\frac{1}{2\alpha}}\mathrm{d}x \\ &\leq CN^{m_2(\frac{1}{2\alpha}-\frac{1}{2})-\frac{1}{2\alpha}+1-2s}N^{(m_2-1)(1-\frac{1}{2\alpha})} = CN^{\frac{m_2}{2}-2s}.\end{aligned} \tag{4.44}$$

Finally, we consider E_3. By (4.32), we obtain

$$\begin{aligned}\mathrm{E}_3 &\leq C \int_{|x|>C_7C_8N^{m_2-1}} (N|x|)^{-2}N^{1-2s}\mathrm{d}x \\ &= CN^{-1-2s} \int_{|x|>C_7C_8N^{m_2-1}} |x|^{-2}\mathrm{d}x \leq CN^{-m_2-2s}.\end{aligned} \tag{4.45}$$

Since $m_2 > 1$, thus, by (4.43)–(4.45), when $\frac{1}{2} < \alpha \le 1$, we have

$$\|J_N\|_{L^1(\mathbb{R})} \le CN^{\frac{m_2}{2}-2s} =: CN^{-2\delta},$$

where $2\delta = 2s - \frac{m_2}{2} > 0$ since $s > \frac{m_2}{4}$ and $m_2 > 1$.

Case II $\frac{1}{4} \le \alpha < \frac{1}{2}$ and $s > \frac{m_2}{4}\big(\frac{1}{\alpha} - 1\big)$. As for E_2, when $\frac{1}{4} \le \alpha < \frac{1}{2}$ by (4.32), we get

$$\begin{aligned} E_2 &\le C \int_{\frac{1}{N}<|x|\le C_7C_8N^{m_2-1}} N^{m_2(\frac{1}{2\alpha}-\frac{1}{2})}(N|x|)^{-\frac{1}{2\alpha}}N^{1-2s}\mathrm{d}x \\ &= CN^{m_2(\frac{1}{2\alpha}-\frac{1}{2})-\frac{1}{2\alpha}+1-2s}\int_{\frac{1}{N}<|x|\le C_7C_8N^{m_2-1}} |x|^{-\frac{1}{2\alpha}}\mathrm{d}x \\ &\le CN^{m_2(\frac{1}{2\alpha}-\frac{1}{2})-\frac{1}{2\alpha}+1-2s}N^{\frac{1}{2\alpha}-1} = CN^{\frac{m_2}{2}(\frac{1}{\alpha}-1)-2s}. \end{aligned} \tag{4.46}$$

By (4.32), from the proof above, we know (4.43) and (4.45) also hold. Since $m_2 > 1$, by (4.43) and (4.45)–(4.46), when $\frac{1}{4} \le \alpha < \frac{1}{2}$, we have

$$\|J_N\|_{L^1(\mathbb{R})} \le CN^{\frac{m_2}{2}(\frac{1}{\alpha}-1)-2s} =: CN^{-2\delta},$$

where $2\delta = 2s - \frac{m_2}{2}\big(\frac{1}{\alpha} - 1\big) > 0$ since $s > \frac{m_2}{4}\big(\frac{1}{\alpha} - 1\big)$ and $m_2 > 1$.

Case III $\alpha = \frac{1}{2}$ and $s > \frac{m_2}{4}(\frac{1}{\alpha} - 1) = \frac{m_2}{4}$.

When γ satisfies the condition (A1) with $\alpha = \frac{1}{2}$, then γ satisfies also the condition (A1) with $\alpha = \frac{1}{2} - \theta$ for any $0 < \theta < \frac{1}{6}$. Thus, from the above proof, it is easy to check that Lemma 4.5 holds for $s > \frac{m_2}{4}\big(\frac{1}{\frac{1}{2}-\theta} - 1\big)$ with $0 < \theta < \frac{1}{6}$. Hence, when $\alpha = \frac{1}{2}$, Lemma 4.5 follows for $s > \frac{m_2}{4}$.

Summing up all the above estimates, we complete the proof of Lemma 4.5.

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