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An Extension of the Win Theorem: Counting the Number of Maximum Independent Sets*

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Abstract Win proved a well-known result that the graph G of connectivity $\kappa(G)$ with $\alpha(G) \leq \kappa(G) + k - 1$ $(k \geq 2)$ has a spanning k-ended tree, i.e., a spanning tree with at most k leaves. In this paper, the authors extended the Win theorem in case when $\kappa(G) = 1$ to the following: Let G be a simple connected graph of order large enough such that $\alpha(G) \leq k + 1$ $(k \geq 3)$ and such that the number of maximum independent sets of cardinality $k + 1$ is at most $n - 2k - 2$. Then G has a spanning k-ended tree.

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1 Introduction

A graph is traceable if it contains a Hamilton path, and hamiltonian if it contains a Hamilton cycle. In 1972, Chvátal and Erdős gave the following well-known sufficient condition for a graph to be traceable. Given a graph G, let $\kappa(G)$ and $\alpha(G)$ denote the connectivity and the independence number of G, respectively.

Theorem 1.1 (see [6]) *If G is a graph on at least 3 vertices such that $\alpha(G) \leq \kappa(G) + 1$, then G is traceable.*

Theorem 1.1 has been extended in many different directions (see [1–2, 7, 9–11]). For the recent results, see [4, 8, 12].

A Hamiltonian path is a spanning tree having exactly two leaves. From this point of view, some sufficient conditions for a graph to be traceable are modified to those for a spanning tree having at most k leaves. A tree having at most k leaves is called a k-ended tree, and we now turn our attention to spanning k-ended trees. It is clear that if $s \leq t$ then a spanning s-ended tree is also a spanning t-ended tree. Theorem 1.1 says that every graph G satisfying $\alpha(G) \leq \kappa(G) + 1$ is traceable. Las Vergnas conjectured the following theorem, which is a generalization of Theorem 1.1. This conjecture was proved by Win, who introduced a new proof technique called a k-ended system.

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Theorem 1.2 (see [14]) *Let $k \geq 2$ be an integer and G be a connected simple graph. If $\alpha(G) \leq \kappa(G) + k - 1$, then G has a spanning k-ended tree.*

Let $m(H)$ denote the number of maximum independent sets of H for a subgraph $H \subseteq G$. In [5], Chen et al. proved that it does not change the traceability of those graphs G with a slight larger independence number (i.e., $\alpha(G) \leq \kappa(G)+2$) when we bound $m(G)$. The complete graph with s vertices is denoted by K_s and its complement is denoted by $\overline{K_s}$, i.e., sK_1. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the join of G and H, denote by $G \vee H$.

Theorem 1.3 (see [5]) *Let G be a connected graph of order $n \geq 2\kappa^2(G)$ such that $\alpha(G) \leq \kappa(G) + 2$, $\kappa(G) = \kappa \geq 1$ and $m(G) \leq n - 2\kappa(G) - 1$. Then either G is traceable or a subgraph of $K_\kappa \vee ((\kappa K_1) \cup K_{n-2\kappa})$.*

In this paper, we extend Theorem 1.2 in the case when $\kappa(G) = 1$ by the following direction.

Theorem 1.4 *Let $k \geq 3$ and G be a connect graph of order $n \geq 2k+2$ such that $\alpha(G) \leq 1+k$ and $m(G) \leq n - 2k - 2$. Then G has a spanning k-ended tree.*

2 Notation and Terminology

For graph-theoretic notation not explained in this paper, we refer the reader to [13]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $N(v)$ the neighborhood of vertex v in G. For a nonempty subset X of $V(G)$, we write $N(X) = \bigcup_{x \in X} N(x)$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. Let H_1 and H_2 be two vertex disjoint subgraphs of G, $x, y \in V(G)$, and P is a path of G. A path xPy in G with end vertices x and y is called a path from H_1 to H_2 if $V(xPy) \cap V(H_1) = \{x\}$ and $V(xPy) \cap V(H_2) = \{y\}$. A path from $\{x\}$ to a vertex set U is also called an (x, U)-path. A subgraph F of G is called an (x, U)-fan of width k if F is a union of (x, U)-paths $P_1, P_2, \cdots, P_k$, where $V(P_i) \cap V(P_j) = \{x\}$ for $i \neq j$. Let G_0 be a subgraph of G. For the convenience, if $G[\{x\} \cup V(G_0)]$ has a spanning path one of whose end vertices is x, then we denote by xG_0 (G_0x, respectively) the spanning path of $G[\{x\} \cup V(G_0)]$, which starts at x (terminates at x, respectively). If $G[V(G_0) \cup \{x, y\}]$ has a spanning path, then we denote by xG_0y the spanning path of $G[V(G_0) \cup \{x, y\}]$, which starts at x and terminates at y. Let G_1 and G_2 be two subgraphs of G. If $G[V(G_1) \cup \{x\} \cup V(G_2)]$ has a spanning path, then we denote by G_1xG_2 the spanning path of $G[V(G_1) \cup \{x\} \cup V(G_2)]$.

We now introduce the concept of a k-ended system in order to prove Theorem 1.4. We denote the set of all simple graph by $\mathcal{G}$. Define a function $f : \mathcal{G} \to \{0, 1, 2\}$ as follows: For each element X of $\mathcal{G}$,

$$f(X) = \begin{cases} 2, & \text{if } X \text{ is a path of order at least three,} \\ 1, & \text{if } X \text{ is } K_1,\ K_2 \text{ or a cycle,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $\mathcal{S}$ be a set of vertex-disjoint subgraphs X of G with $f(X) \geq 1$, and

$$\mathcal{S}_P = \{X \in \mathcal{S} : f(X) = 2\}, \quad \mathcal{S}_C = \{X \in \mathcal{S} : f(X) = 1\}.$$

Then

$$\mathcal{S} = \mathcal{S}_P \cup \mathcal{S}_C, \quad V(\mathcal{S}) = \bigcup_{X \in \mathcal{S}} V(X).$$

Moreover, $V(\mathcal{S}_P)$ and $V(\mathcal{S}_C)$ can be defined analogously, and $|\mathcal{S}|, |\mathcal{S}_P|, |\mathcal{S}_C|$ denote the number of elements in $\mathcal{S}, \mathcal{S}_P$ and $\mathcal{S}_C$, respectively. For each path $P \in \mathcal{S}_P$, let $v_L(P)$ and $v_R(P)$ denote the two end-vertices of P. For each element $C \in \mathcal{S}_C$, let v_C be an arbitrarily chosen vertex of C. Once we choose a vertex v_C, each C corresponds the unique vertex v_C. Then define

$$\begin{aligned} &\mathrm{End}(\mathcal{S}_P) = \bigcup_{P \in \mathcal{S}_P} \{v_L(P), v_R(P)\}, \quad \mathrm{End}(\mathcal{S}_C) = \bigcup_{C \in \mathcal{S}_C} \{v_C\}, \\ &\mathrm{End}(\mathcal{S}) = \mathrm{End}(\mathcal{S}_P) \cup \mathrm{End}(\mathcal{S}_C). \end{aligned}$$

Then $\mathcal{S}$ is called a k-ended system if $\sum_{X \in \mathcal{S}} f(X) = 2|\mathcal{S}_P| + |\mathcal{S}_C| \le k$. If $V(\mathcal{S}) = V(G)$, then $\mathcal{S}$ is called a spanning k-ended system of G.

For any $P \in \mathcal{S}_P$, we orient P from $v_L(P)$ to $v_R(P)$, say $\overrightarrow{P}$ for the oriented path (from $v_R(P)$ to $v_L(P)$, say $\overleftarrow{P}$, respectively). With a given orientation $\overrightarrow{P}$ and for every vertex x of P, we will denote the first, second and ith predecessor (successor, respectively) of x as x^-, x^{--} and x^{i-} (x^+, x^{++}, and x^{i+}, respectively). If $x = v_R(P)$ ($x = v_L(P)$, respectively), we have only predecessor of $v_R(P)$ (successor of $v_L(P)$, respectively). Given x and y on P, a section $P(x, y)$ is a path $x^+x^{2+}x^{3+} \cdots x^{s+}(= y^-)$ of consecutive vertices of P, and a section $P[x, y]$ is a path $xx^+x^{2+} \cdots x^{s+}(= y)$ of consecutive vertices of P. The section $P[x, y]$ is trivial if $x = y$.

For each element $C \in \mathcal{S}_C$ and $|V(C)| \ge 3$, with a given orientation $\overrightarrow{C}$ and for every vertex v of C, let v^+ and v^- denote the successor and the predecessor of v, respectively.

The following lemma shows why a k-ended system is important for spanning k-ended trees.

Lemma 2.1 (see [14]) *Let $k \ge 3$ be an integer, and G be a connected simple graph. If G has a spanning k-ended system, then G has a spanning k-ended tree.*

We call a k-ended system $\mathcal{S}$ of G a maximal k-ended system if there exists no k-ended system $\mathcal{S}'$ in G such that $V(\mathcal{S}) \subset V(\mathcal{S}')$. The following lemma expresses some nice properties of k-ended systems. Note that we say that two distinct elements of $\mathcal{S}$ are connected by a path in $G - V(\mathcal{S})$ if there exists a path in G whose end-vertices are in $\mathcal{S}$ and whose inner vertices are all contained in $V(G) \setminus V(\mathcal{S})$, where a path may has no inner vertex.

Lemma 2.2 (see [3]) *Let $k \ge 3$ be an integer, and G be a connected simple graph. Suppose that G has no spanning k-ended system, and let $\mathcal{S}$ be a maximal k-ended system of G such that $|\mathcal{S}_P|$ is maximum subject to the maximality of $V(\mathcal{S})$. Then the following statements hold.*

(i) *No two elements of $\mathcal{S}_C$ are connected by a path whose inner vertices are in $V(G) \setminus V(\mathcal{S})$.*

(ii) *No element of $\mathcal{S}_C$ is connected to an end-vertex of an element of $\mathcal{S}_P$ by a path whose inner vertices are in $V(G) \setminus V(\mathcal{S})$.*

(iii) *No end-vertex of an element of $\mathcal{S}_P$ is connected to an end-vertex of another element of $\mathcal{S}_P$ by a path whose inner vertices are in $V(G) \setminus V(\mathcal{S})$.*

(iv) *No vertex u in $V(G)\backslash V(\mathcal{S})$ is connected to two distinct elements of $\mathcal{S}_C$ by two internally disjoint paths Q_1 and Q_2 in $G-V(\mathcal{S})$ such that $V(Q_1)\cap V(Q_2)=\{u\}$.*

3 Preparatory Works for Proving Theorem 1.4

In the following section, for the convenience, we assume the following: Let $k\geq 3$ be an integer, and G be a simple connected graph such that $\alpha(G)=1+k$. Suppose that G has no spanning k-ended system, and let $\mathcal{S}$ be a maximal k-ended system of G such that

(I) $|V(\mathcal{S})|$ is maximized.

(II) $|\mathcal{S}_P|$ is maximized subject to (I).

(III) $|V(\mathcal{S}_P)|$ is maximized subject to (I) and (II).

Then $\mathcal{S}$ is a set of subgraphs of G satisfying the hypothesis of Lemma 2.2. Let $H=G-V(\mathcal{S})$. Then $|V(H)|\geq 1$.

For the convenience, we assume that $x\in V(P)$. For any $P\in\mathcal{S}_P$, let $\overrightarrow{\mathcal{Q}}(x,P)=v_R(P)$ if $x=v_R(P)$, and $\overrightarrow{\mathcal{Q}}(x,P)=xv_R(P)$ if $x^+=v_R(P)$. In the case when $x\neq v_R(P)$ and $x^+\neq v_R(P)$, we also let $\overrightarrow{\mathcal{Q}}(x,P)=x\overrightarrow{P}v_R(P)x$ if $xv_R(P)\in E(G)$. Otherwise, we do not define $\overrightarrow{\mathcal{Q}}(x,P)$. Let $\overleftarrow{\mathcal{Q}}(x,P)=v_L(P)$ if $x=v_L(P)$, and $\overleftarrow{\mathcal{Q}}(x,P)=xv_L(P)$ if $x^-=v_L(P)$. In the case when $x\neq v_L(P)$ and $x^-\neq v_L(P)$, we also let $\overleftarrow{\mathcal{Q}}(x,P)=x\overleftarrow{P}v_L(P)x$ if $xv_L(P)\in E(G)$. Otherwise, we do not define $\overleftarrow{\mathcal{Q}}(x,P)$. Then, $f(\overrightarrow{\mathcal{Q}}(x,P))=1$ ($f(\overleftarrow{\mathcal{Q}}(x,P))=1$, respectively). Let G_0 be a subgraph of G, $C(G_0)$ is called a spanning subgraph of G_0 such that $f(C(G_0))=1$.

Lemma 3.1 *G has no k'-ended system $\mathcal{T}$ such that $V(\mathcal{T})\supseteq V(\mathcal{S})$ and $k'<k$.*

Lemma 3.2 *Suppose that there exists a path L from $v\in V(H)$ to $\mathcal{S}$ such that $V(L)\cap V(\mathcal{S})=\{x\}$ and $x\in V(P)$ for some $P\in\mathcal{S}_P$. Then $N(x^+)\cap(\mathrm{End}(\mathcal{S}_P)\setminus\{v_R(P)\})=\emptyset$.*

Proof By contradiction, suppose that $N(x^+)\cap(\mathrm{End}(\mathcal{S}_P)\setminus\{v_R(P)\})\neq\emptyset$, say $y\in N(x^+)\cap(\mathrm{End}(\mathcal{S}_P)\setminus\{v_R(P)\})$. It is easy to check that $x\notin\{v_L(P),v_R(P)\}$. Let $N(x)\cap V(L)=\{v'\}$. Then, we distinguish the following two cases to obtain a contradiction:

(1) Suppose that $y\in\mathrm{End}(\mathcal{S}_P)\setminus\{v_L(P),v_R(P)\}$. Without loss of generality, assume that $y=v_L(P')$ for $P'\in\mathcal{S}_P\setminus\{P\}$. Then $v_R(P)\overleftarrow{P}x^+v_L(P')\overrightarrow{P'}v_R(P')$ and $v_L(P)\overrightarrow{P}xv'$ in G cover $V(P')\cup V(P)\cup\{v'\}$, contradicting (I).

(2) Suppose that $y=v_L(P)$. Then $v'x\overleftarrow{P}v_L(P)x^+\overrightarrow{P}v_R(P)$ in G covers $V(P)\cup\{v'\}$, contradicting (I).

This contradiction proves Lemma 3.2.

Lemma 3.3 *For any $v\in V(H)$, G has no $(v,V(\mathcal{S}))$-fan of width 2.*

Proof By contradiction, suppose that there exists a $(v,V(\mathcal{S}))$-fan $\{L_1,L_2\}$ of width 2 for some $v\in V(H)$. Let $V(L_i)\cap V(\mathcal{S})=\{u_i\}$ for $i\in\{1,2\}$. Denote $U=\{u_1,u_2\}$. If $U\cap V(\mathcal{S}_C)\neq\emptyset$, then there exists at least one vertex $y\in U\cap V(\mathcal{S}_C)$. Without loss of generality, we assume that $y=u_1$. By Lemma 2.2 (iv), $|\{C: C\in\mathcal{S}_C$ and $U\cap V(C)\neq\emptyset\}|=1$, say $C_1\in\mathcal{S}_C$ and $U\cap V(C_1)\neq\emptyset$. For each isolated vertex $\{x\}$(say) of $\mathcal{S}_C$, $N(x)\cap(V(G)\setminus V(\mathcal{S}))=\emptyset$. Thus no isolated vertex of $\mathcal{S}_C$ is adjacent to $V(G)\setminus V(\mathcal{S})$. Then $|V(C_1)|\geq 2$. If $|V(C_1)|=2$, then we assume that C_1 contains a vertex of U, say v_{C_1}. Let $V(C_1) = \{v_{C_1}, v'_{C_1}\}$, where $v_{C_1} \in \mathrm{End}(\mathcal{S}_C)$. Obviously v'_{C_1} is not contained in U. Denote

$$u_1^* = \begin{cases} u_1^+, & \text{if either } U \cap V(\mathcal{S}_C) \neq \emptyset \text{ and } |V(C_1)| > 2, \text{ or } U \cap V(\mathcal{S}_C) = \emptyset, \\ v'_{C_1}, & \text{if } U \cap V(\mathcal{S}_C) \neq \emptyset \text{ and } |V(C_1)| = 2 \end{cases}$$

and $U^+ = \{u_1^*, u_2^+\}$. Let $Y = \mathrm{End}(\mathcal{S}) \cup U^+ \cup \{v\}$. By Lemma 2.2, $\mathrm{End}(\mathcal{S})$ is an independent set of G. It is easy to check that $u_i \notin \mathrm{End}(\mathcal{S}_P)$ for $i \in \{1, 2\}$.

We distinguish the following three cases to prove that Y includes an independent set of G with size at least $k+2$.

Case 1 $|U \cap V(\mathcal{S}_P)| = |U \cap V(\mathcal{S}_C)| = 1$.

Suppose that $u_1 \in V(C_1)$ and $u_2 \in V(P)$ for some $P \in \mathcal{S}_P$. By Lemma 2.2(iv), $\{u_1^*\} \cup (\mathrm{End}(\mathcal{S}) \setminus \{v_{C_1}\})$ is an independent set of G. Let $Q_1 = u_1^* \overleftarrow{C_1} u_1 L_1 v L_2 u_2$, $Q_2 = C u_2^+ \overrightarrow{P} v_R(P)$ and $Q_3 = \overrightarrow{Q}(u_2^+, P)$. Then, we distinguish the following four cases to prove that $Y \setminus \{v_{C_1}\}$ is an independent set of G with size $k+2$:

(1) Suppose that $u_2^+ = v_R(P)$. Then Q_1 and $v_R(P)$ cover $V(C_1) \cup V(P) \cup \{v\}$, contradicting (I).

(2) Suppose that $N(u_2^+) \cap (\mathrm{End}(\mathcal{S}_C) \setminus \{v_{C_1}\}) \neq \emptyset$, say $v_C \in N(u_2^+) \cap (\mathrm{End}(\mathcal{S}_C) \setminus \{v_{C_1}\})$. Then Q_1 and Q_2 cover $V(C_1) \cup V(C) \cup V(P) \cup \{v\}$, contradicting (I).

(3) Suppose that $N(u_2^+) \cap \mathrm{End}(\mathcal{S}_P) \neq \emptyset$. By Lemma 3.2, $N(u_2^+) \cap \mathrm{End}(\mathcal{S}_P) = \{v_R(P)\}$. Then Q_1 and Q_3 cover $V(C_1) \cup V(P) \cup \{v\}$, contradicting (I).

(4) Suppose that $u_1^* u_2^+ \in E(G)$. Then $v_R(P) \overleftarrow{P} u_2^+ Q_1$ covers $V(C_1) \cup V(P) \cup \{v\}$, contradicting Lemma 3.1.

These contradictions show that $Y \setminus \{v_{C_1}\}$ is an independent set of G with size $k+2$.

Case 2 $U \subseteq V(\mathcal{S}_C)$.

If $u_1^+ \in V(C_1)$ and $u_2^+ \in V(C_1)$ are adjacent in G, then $v L_1 u_1 \overleftarrow{C_1} u_2^+ u_1^+ \overrightarrow{C_1} u_2 L_2 v$ covers $V(C_1) \cup \{v\}$, contradicting (I). This contradiction shows that U^+ is an independent set of G. Combining this with Lemma 2.2(iv), $Y \setminus \{v_{C_1}\}$ is an independent set of G with size $k+2$.

Case 3 $U \cap V(\mathcal{S}_C) = \emptyset$, i.e., $U \subseteq V(\mathcal{S}_P)$.

Suppose first that $\mathrm{End}(\mathcal{S}) \cap U^+ = \emptyset$. Then $|Y| = k+3$. By the assumption of this case, $\mathrm{End}(\mathcal{S}) \cup \{v\}$ is an independent set of G. Let $Q_4 = v_L(P) \overrightarrow{P} u_1 L_1 v L_2 u_2$ and $Q_5 = Q_4 u_2^- \overleftarrow{P'} v_L(P')$. We shall show the following two claims.

Claim 1 $G[Y]$ is triangle-free.

Proof We shall prove that U^+ is an independent set of G.

If $u_1^+ \in V(P)$ and $u_2^+ \in V(P)$, where $P \in \mathcal{S}_P$, are adjacent in G, without loss of generality, assume $P[v_L(P), u_1] \subset P[v_L(P), u_2]$, then $Q_4 u_2^- \overleftarrow{P} u_1^+ u_2^+ \overrightarrow{P} v_R(P)$ covers $V(P) \cup \{v\}$, contradicting (I). If $u_1^+ \in V(P)$ and $u_2^+ \in V(P')$, where $P, P' \in \mathcal{S}_P$ and $P \neq P'$, are adjacent in G, then Q_5, $v_R(P) \overleftarrow{P} u_1^+ u_2^+ \overrightarrow{P'} v_R(P')$ cover $V(P) \cup V(P') \cup \{v\}$, contradicting (I). It is shown that U^+ is an independent set of G. Then $(U^+, \mathrm{End}(\mathcal{S}) \cup \{v\})$ is a bipartition of the $G[Y]$. Therefore, $G[Y]$ is triangle-free.

Claim 2 Y does not contain four distinct vertices y_1, y_2, y_3, y_4 such that $\{y_1y_2, y_3y_4\} \subseteq E(G)$.

Proof By contradiction, suppose that Y contains four distinct vertices y_1, y_2, y_3, y_4 such that $\{y_1y_2, y_3y_4\} \subseteq E(G)$. Without loss of generality, assume $y_2 = u_1^+ \in V(P)$, $y_4 = u_2^+ \in V(P')$, and if they are in the same path P (say), then $P[v_L(P), u_1] \subset P[v_L(P), u_2]$(say). Then, we distinguish the following three cases to obtain a contradiction.

(1) Suppose that $y_1 = v_C \in \mathrm{End}(\mathcal{S}_C)$, $y_3 = v_{C'} \in \mathrm{End}(\mathcal{S}_C)$ and $v_C \neq v_{C'}$. If $P \neq P'$, then Q_5, $Cu_1^+\overrightarrow{P}v_R(P)$ and $C'u_2^+\overrightarrow{P'}v_R(P')$ cover $V(P) \cup V(P') \cup V(C) \cup V(C') \cup \{v\}$, contradicting (I). If $P = P'$, then $Q_4u_2^-\overleftarrow{P}u_1^+C$, $C'u_2^+\overrightarrow{P}v_R(P)$ cover $V(P) \cup V(C) \cup V(C') \cup \{v\}$, contradicting (I).

(2) Suppose that $y_1 = v_C \in \mathrm{End}(\mathcal{S}_C)$ and $y_3 = v_R(P')$. If $P \neq P'$, then Q_5, $Cu_1^+\overrightarrow{P}v_R(P)$ and $\overrightarrow{Q}(u_2^+, P')$ cover $V(P) \cup V(P') \cup V(C) \cup \{v\}$, contradicting (I). If $P = P'$, then $Q_4u_2^-\overleftarrow{P}u_1^+C$, Q_3 cover $V(P) \cup V(C) \cup \{v\}$, contradicting (I).

(3) Suppose that $y_1 = v_R(P)$ and $y_3 = v_R(P')$, where $P \neq P'$. Then Q_5, $\overrightarrow{Q}(u_1^+, P)$ and $\overrightarrow{Q}(u_2^+, P')$ cover $V(P) \cup V(P') \cup \{v\}$, contradicting (I).

This contradiction proves Claim 2.

By Claims 1–2, Y includes an independent set of G with size at least $k + 2$.

Next, suppose that $\mathrm{End}(\mathcal{S}) \cap U^+ \neq \emptyset$. Then, there exists a path $P \in \mathcal{S}_P$ such that $v_R(P) = u_i^+$ for $i \in \{1, 2\}$. Without loss of generality, say $v_R(P) = u_1^+$. If there exists another path $Q \in \mathcal{S}_P \setminus \{P\}$ such that $v_R(Q) = u_2^+$, then Q_5, $v_R(P)$, $v_R(Q)$ cover $V(P) \cup V(Q) \cup \{v\}$, contradicting (I). Thus $U^+ \cap \mathrm{End}(\mathcal{S}) = \{v_R(P)\}$ for some $P \in \mathcal{S}_P$, and $|Y| = k + 2$. we will show that Y is an independent set of G with size $k + 2$.

Assume that $v_C \in \mathrm{End}(\mathcal{S}_C)$ and $u_2^+ \in V(P')$ are adjacent in G. If $P \neq P'$, then Q_5, $Cu_2^+\overrightarrow{P'}v_R(P')$, $v_R(P)$ cover $V(P) \cup V(P') \cup V(C) \cup \{v\}$, contradicting (I). If $P = P'$, then $v_L(P)\overrightarrow{P}u_2L_2vL_1u_1\overleftarrow{P}u_2^+C$, $v_R(P)$ cover $V(P) \cup V(C) \cup \{v\}$, contradicting (I).

Assume that $v_R(P')$ and $u_2^+ \in V(P')$ are adjacent in G, where $P \neq P'$. Then Q_5, $\overrightarrow{Q}(u_2^+, P')$, $v_R(P)$ cover $V(P) \cup V(P') \cup \{v\}$, contradicting (I).

These contradictions shows that Y is an independent set of G with size $k + 2$.

In all cases, Y includes an independent set of G with size at least $k + 2$, contradicting $\alpha(G) = k + 1$. This contradiction shows that Lemma 3.3 holds.

Let w be a vertex in $V(G) \setminus V(\mathcal{S})$. Since G is connected, by Lemma 3.3, there exists exactly one $(w, V(\mathcal{S}))$-path L such that $V(L) \cap V(\mathcal{S}) = \{\mu_w\}$. Then w is connected to μ_w by the path L.

Lemma 3.4 *$N(v) \cap \mathrm{End}(\mathcal{S}_P) = \emptyset$ for any $v \in V(H)$.*

C_{μ_w} always means the vertex $\mu_w \in V(C_{\mu_w})$ where $C_{\mu_w} \in \mathcal{S}_C$. Then $|V(C_{\mu_w})| \geq 2$, since no isolated vertex of $\mathcal{S}_C$ is adjacent to $V(G) \setminus V(\mathcal{S})$. If $|V(C_{\mu_w})| = 2$, say $V(C_{\mu_w}) = \{v_{C_{\mu_w}}, v'_{C_{\mu_w}}\}$, without loss of generality, then we assume that $\mu_w = v_{C_{\mu_w}}$. It means that $v_{C_{\mu_w}} \in \mathrm{End}(\mathcal{S}_C)$.

Denote

$$\mu_w^* = \begin{cases} \mu_w^+, & \text{if } |V(C_{\mu_w})| \geq 3 \text{ and } \mu_w \in V(C_{\mu_w}), \\ v'_{C_{\mu_w}}, & \text{if } |V(C_{\mu_w})| = 2 \text{ and } V(C_{\mu_w}) = \{v_{C_{\mu_w}}, v'_{C_{\mu_w}}\}. \end{cases}$$

P_{μ_w} always means that the vertex $\mu_w \in V(P_{\mu_w})$, where $P_{\mu_w} \in \mathcal{S}_P$. Let

$$X = \begin{cases} (\mathrm{End}(\mathcal{S}) \cup \{\mu_w^*, w\}) \setminus \{v_{C_{\mu_w}}\}, & \text{if } \ \mu_w \in V(\mathcal{S}_C), \\ \mathrm{End}(\mathcal{S}) \cup \{w\}, & \text{if } \ \mu_w \in V(\mathcal{S}_P). \end{cases} \tag{3.1}$$

Lemma 3.5 *X is an independent set of G with size $k+1$.*

Proof By Lemma 2.2, $\mathrm{End}(\mathcal{S})$ is an independent set of G. Since $\mu_w \in V(\mathcal{S}_P)$ or $V(\mathcal{S}_C)$, by Lemmas 2.2 and 3.4, X is an independent set of G with size $k+1$.

By Lemma 3.5, we have

$$N(v) \cap X \neq \emptyset \quad \text{for any } v \in V(G) \setminus X. \tag{3.2}$$

Otherwise, there exists a vertex $v_0 \in V(G) \setminus X$ such that $X \cup \{v_0\}$ is an independent set of cardinality $k+2$, contradicting $\alpha(G) = k+1$.

Lemma 3.6 *Let $S \subset V(G)$ such that $S \cap X$ has exactly one vertex, say z, i.e, $S \cap X = \{z\}$. If $N(x) \cap X = \{z\}$ for any $x \in S \setminus \{z\}$, then $G[S]$ is a clique.*

Proof By contradiction, suppose that $x_1x_2 \notin E(G)$ for some pair of vertices x_1, $x_2 \in S$ with $x_1 \neq x_2$, then $(X \setminus \{z)\}) \cup \{x_1, x_2\}$ is an independent set of G with size $k+2$, contradicting $\alpha(G) = 1 + k$. Hence, $G[S]$ is a clique.

Denote

$$\mathcal{S}' = \mathcal{S} \setminus \{C_{\mu_w}, P_{\mu_w}\}, \quad \mathcal{S}'_C = \mathcal{S}' \cap \mathcal{S}_C, \quad \mathcal{S}'_P = \mathcal{S}' \cap \mathcal{S}_P.$$

Lemma 3.7 *$G[V(C)]$ is a clique for each $C \in \mathcal{S}'_C$.*

Proof Since $G[V(C)]$ is connected, it suffices to consider the case when $|V(C)| \geq 3$. Note that $V(C) \cap X = \{v_C\}$. By Lemma 2.2(i)–(ii), $N(x) \cap (X \setminus \{v_C\}) = \emptyset$ for each vertex $x \in V(C) \setminus \{v_C\}$. Note that $N(x) \cap X \neq \emptyset$. Therefore $N(x) \cap X = \{v_C\}$. Let $S = V(C)$. Then, by Lemma 3.6, $G[V(C)]$ is a clique.

Lemma 3.8 *For any $P \in \mathcal{S}_P$, there is no pair of adjacent vertices x, y in P such that $N(x) \cap V(\mathcal{S}'_C) \neq \emptyset$ and $N(y) \cap V(\mathcal{S}'_C) \neq \emptyset$.*

Proof By contradiction, suppose that there exists a pair of vertices x_0, y_0 in $P \in \mathcal{S}_P$ such that $x_0y_0 \in E(P)$, $N(x_0) \cap V(\mathcal{S}'_C) \neq \emptyset$ and $N(y_0) \cap V(\mathcal{S}'_C) \neq \emptyset$. By Lemma 2.2(i)–(ii), $x_0 \notin \{v_L(P), v_R(P)\}$ and $y_0 \notin \{v_L(P), v_R(P)\}$. Without loss of generality, assume that $y_0 = x_0^+$. Suppose that $N(x_0) \cap V(\mathcal{S}'_C) = \{x'\}$ and $N(y_0) \cap V(\mathcal{S}'_C) = \{y'\}$. We distinguish the following two cases to obtain a contradiction.

(1) Suppose that $\{x', y'\} \subseteq V(C)$ with $C \in \mathcal{S}'_C$. If either $|V(C)| = 1$ or $x' \neq y'$, then by Lemma 3.7, $v_L(P)\overrightarrow{P}x_0Cx_0^+\overrightarrow{P}v_R(P)$ in G covers $V(P) \cup V(C)$, contradicting Lemma 3.1.

If $x' = y'$ and $|V(C)| > 1$, then $v_L(P)\overrightarrow{P}x_0x'x_0^+\overrightarrow{P}v_R(P)$ and $C(G[V(C) \setminus \{x'\}])$ in G cover $V(P) \cup V(C)$, satisfying (I),(II) but not (III), a contradiction.

(2) Suppose that $x' \in V(C)$ and $y' \in V(C')$ such that $\{C, C'\} \subseteq \mathcal{S}'_C$ and $V(C) \cap V(C') = \emptyset$. Then, $Cx_0\overleftarrow{P}v_L(P)$ and $C'y_0\overrightarrow{P}v_R(P)$ in G cover $V(P) \cup V(C) \cup V(C')$, satisfying (I) but not (II), a contradiction.

This contradiction proves Lemma 3.8.

Denote

$$T_{P,1} := \{x \in V(P) : \ P \in \mathcal{S}_P, \ xv_L(P) \in E(G), \ x^+v_L(P) \notin E(G), \ x^+ \neq \mu_w\},$$
$$T_{P,2} := \{x \in V(P) : \ P \in \mathcal{S}_P, \ N(x) \cap V(\mathcal{S}'_C) \neq \emptyset, \ x^+ \neq \mu_w\}.$$

Lemma 3.9 *Let $P \in \mathcal{S}_P$. Then the following three statements hold.*

(1) *If $x \in T_{P,1} \cup T_{P,2} \cup \{\mu_w\}$, then either $N(x^+) \cap X \subseteq (\mathrm{End}(\mathcal{S}'_C) \cup \{v_R(P)\})$ or $x^+ = v_R(P)$.*

(2) *If $x \in T_{P,2}$, then either $N(x^+) \cap X = \{v_R(P)\}$ or $x^+ = v_R(P)$.*

(3) *If $x = \mu_w$, then either $N(x^-) \cap X \subseteq (\mathrm{End}(\mathcal{S}'_C) \cup \{v_L(P)\})$ or $x^- = v_L(P)$.*

Proof First, we will prove Lemma 3.9(1). We denote set $\mathrm{End}(\mathcal{S}'_C) \cup \{v_R(P)\}$ by B_1. If $x^+ \neq v_R(P)$, then we will show that $N(x^+) \cap X \subseteq B_1$. Since $B_1 \subseteq X$, we denote $X - B_1$ by B_2. By the assumption of this case, $w \notin N(x^+) \cap X$.

Suppose that $x = \mu_w$. Then by Lemma 3.2, $N(x^+) \cap (\mathrm{End}(\mathcal{S}_P) \setminus \{v_R(P)\}) = \emptyset$. Combining this with (3.1), $N(x^+) \cap B_2 = \emptyset$. Hence, $N(x^+) \cap X \subseteq B_1$. By symmetry, Lemma 3.9(3) holds.

Suppose that $x \in T_{P,1} \cup T_{P,2}$. Then, suppose that there exists a vertex $x' \in N(x^+) \cap B_2$. We distinguish the following three cases to obtain a contradiction by the definition of X.

(1) Suppose that $x' \in \mathrm{End}(\mathcal{S}_P) \setminus \{v_L(P), v_R(P)\}$. Without loss of generality, assume that $x' = v_L(P')$ for $P' \in \mathcal{S}_P \setminus \{P\}$. If $x \in T_{P,1}$, then $v_R(P)\overleftarrow{P}x^+v_L(P')\overrightarrow{P'}v_R(P')$ and $\overleftarrow{Q}(x, P)$ in G cover $V(P') \cup V(P)$, contradicting Lemma 3.1. If $x \in T_{P,2}$, say $v_C \in N(x) \cap V(\mathcal{S}'_C)$, then $v_R(P)\overleftarrow{P}x^+v_L(P')\overrightarrow{P'}v_R(P')$ and $Cx\overleftarrow{P}v_L(P)$ in G cover $V(P') \cup V(P) \cup V(C)$, contradicting Lemma 3.1.

(2) Suppose that $x' = v_L(P)$. If $x \in T_{P,2}$, say $v_C \in N(x) \cap V(\mathcal{S}'_C)$, then $Cx\overleftarrow{P}v_L(P)x^+\overrightarrow{P}$ $v_R(P)$ in G covers $V(P) \cup V(C)$, contradicting Lemma 3.1.

(3) Suppose that $x' = \mu_w^*$. Then, by Lemma 3.5 and (3.1), $\mu_w \in V(\mathcal{S}_C)$. If $x \in T_{P,1}$, then $wL\mu_wC_{\mu_w}x^+\overrightarrow{P}v_R(P)$ and $\overleftarrow{Q}(x, P)$ in G cover $V(P) \cup V(C_{\mu_w}) \cup \{w\}$, contradicting (I). If $x \in T_{P,2}$, say $v_C \in N(x) \cap V(\mathcal{S}'_C)$, then $wL\mu_wC_{\mu_w}x^+\overrightarrow{P}v_R(P)$ and $Cx\overleftarrow{P}v_L(P)$ in G cover $V(P) \cup V(C) \cup V(C_{\mu_w}) \cup \{w\}$, contradicting (I).

This contradiction proves that $N(x^+) \cap B_2 = \emptyset$. Hence, $N(x^+) \cap X \subseteq B_1$. Lemma 3.9(1) is proved.

If $x \in T_{P,2}$ then by Lemma 3.9(1), $N(x^+) \cap X \subseteq (\mathrm{End}(\mathcal{S}'_C) \cup \{v_R(P)\})$ or $x^+ = v_R(P)$. By Lemma 3.8, $N(x^+) \cap V(\mathcal{S}'_C) = \emptyset$. Note that (3.2). Therefore Lemma 3.9(2) holds.

Lemma 3.10 *Let $P \in \mathcal{S}_P$ and $y \in V(P)$ with $yv_R(P) \in E(G)$, $|V(P[y, v_R(P)])| \geq 3$ and $\mu_w \notin V(P[y, v_R(P)])$. Then it holds that $N(y^+) \cap X = \{v_R(P)\}$.*

Proof By contradiction, suppose that there exists at least one vertex $x \in N(y^+) \cap X$ such that $x \neq v_R(P)$. By the definition of X, we distinguish the following four cases to obtain a contradiction.

(1) Suppose that $x \in \mathrm{End}(\mathcal{S}_C)$, say $x = v_C$. Then $Cy^+\overrightarrow{P}v_R(P)y\overleftarrow{P}v_L(P)$ in G covers $V(P) \cup V(C)$, contradicting Lemma 3.1.

(2) Suppose that $x = v_L(P) \in \mathrm{End}(\mathcal{S}_P)$. Then $v_R(P)\overleftarrow{P}y^+v_L(P)\overrightarrow{P}yv_R(P) \in \mathcal{S}_C$ in G covers $V(P)$, contradicting Lemma 3.1.

(3) Suppose that $x \in \mathrm{End}(\mathcal{S}_P)\setminus\{v_L(P), v_R(P)\}$. Without loss of generality, assume that $x = v_L(P')$ for $P' \in \mathcal{S}_P\setminus\{P\}$. Then $v_R(P')\overleftarrow{P'}v_L(P')y^+\overrightarrow{P}v_R(P)y\overleftarrow{P}v_L(P)$ in G covers $V(P)\cup V(P')$, contradicting Lemma 3.1.

(4) Suppose that $x = \mu_w^*$. Then by Lemma 3.5 and (3.1), $\mu_w \in V(\mathcal{S}_C)$. Then $C_{\mu_w}y^+\overrightarrow{P}$ $v_R(P)y\overleftarrow{P}v_L(P)$ in G covers $V(P) \cup V(C_{\mu_w})$, contradicting Lemma 3.1.

This contradiction shows that $N(y^+) \cap X \subseteq \{v_R(P)\}$. By (3.2), $N(y^+) \cap X = \{v_R(P)\}$.

Lemma 3.11 *The following two statements hold.*

(1) *Let $P \in \mathcal{S}_P$ and $y \in V(P)$ with $yv_R(P) \in E(G)$ and $\mu_w \notin V(P[y, v_R(P)])$. Then it holds that $G[V(P[y^+, v_R(P)])]$ is a clique. Furthermore, if $N(y) \cap X = \{v_R(P)\}$, then $G[V(P[y, v_R(P)])]$ is a clique.*

(2) *Let $P \in \mathcal{S}_P$ and $x \in V(P)$ with $xv_L(P) \in E(G)$ and $\mu_w \notin V(P[v_L(P), x])$. Then it holds that $G[V(P[v_L(P), x^-])]$ is a clique. Furthermore, if $N(x) \cap X = \{v_L(P)\}$, then $G[V(P[v_L(P), x])]$ is a clique.*

Proof By symmetry, we may only prove that (1) is true. Since $G[V(P[y^+, v_R(P)])]$ is connected, it suffices to consider the case when $|V(P[y^+, v_R(P)])| \geq 3$. Note that $V(P[y^+, v_R(P)]) \cap X = \{v_R(P)\}$. Let $S = V(P[y^+, v_R(P)])$. Then by Lemma 3.6, it suffices to prove the following statement

$$N(y') \cap X = \{v_R(P)\} \quad \text{for each vertex } y' \in V(P[y^+, v_R(P)]). \tag{3.3}$$

We repeatedly apply Lemma 3.10 to obtain (3.3).

Furthermore, we will prove that if $N(y) \cap X = \{v_R(P)\}$, then $G[V(P[y, v_R(P)])]$ is a clique. Since $G[V(P[y, v_R(P)])]$ is connected, it suffices to consider the case when $|V(P[y, v_R(P)])| \geq 3$. Note that $N(y) \cap X = \{v_R(P)\}$. Combining this with (3.3), we have $N(x) \cap X = \{v_R(P)\}$ for each vertex $x \in V(P[y, v_R(P)))$. Let $S = V(P[y, v_R(P)])$. Then by Lemma 3.6, $G[V(P[y, v_R(P)])]$ is a clique.

Lemma 3.12 *Let $P \in \mathcal{S}_P$ and $x \in V(P)\setminus\{v_L(P), v_R(P)\}$ with $N(x) \cap \mathrm{End}(\mathcal{S}'_C) \neq \emptyset$. Then the following two statements hold.*

(1) *If $\mu_w \notin V(P[x^+, v_R(P)])$, then $G[V(P[x^+, v_R(P)])]$ is a clique.*

(2) *If $\mu_w \notin V(P[v_L(P), x^-])$, then $G[V(P[v_L(P), x^-])]$ is a clique.*

Proof By symmetry, we may only prove that $G[V(P[x^+, v_R(P)])]$ is a clique. Since $G[V(P[x^+, v_R(P)])]$ is connected, it suffices to consider the case when $|V(P[x^+, v_R(P)])| \geq 3$. By Lemma 3.9(2), $N(x^+) \cap X = \{v_R(P)\}$. By Lemma 3.11(1), $G[V(P[x^+, v_R(P)])]$ is a clique.

Lemma 3.13 *For any pair of paths $P, P' \in \mathcal{S}_P$, suppose that there exist two vertices $x \in V(P) \setminus \{v_L(P),\ v_R(P)\}$ and $y \in V(P') \setminus \{v_L(P'), v_R(P')\}$ such that $N(x) \cap \mathrm{End}(\mathcal{S}'_C) \neq \emptyset$, $N(y) \cap \mathrm{End}(\mathcal{S}'_C) \neq \emptyset$ and $\mu_w \notin (V(P) \setminus \{x\}) \cup (V(P') \setminus \{y\})$. Then any pair of vertices in $V(P) \setminus \{x\}$ and $V(P') \setminus \{y\}$ respectively are not adjacent.*

Proof By symmetry, we only prove that any pair of vertices in $V(P[x^+, v_R(P)])$ and $V(P'[y^+, v_R(P')])$ are not adjacent.

By contradiction, suppose that there exists a pair of vertices $x_0 \in V(P[x^+, v_R(P)])$, $y_0 \in V(P'[y^+, v_R(P')])$ such that $x_0y_0 \in E(G)$. By Lemma 3.12(1), both $G[V(P[x^+, v_R(P)])]$ and $G[V(P'[y^+, v_R(P')])]$ are cliques. Let $Q_6 = G[V(P'[y^+, v_R(P')])]x_0G[V(P[x^+, v_R(P)] \setminus \{x_0\})]$ and $Q_7 = v_L(P)\overrightarrow{P}xv_Cy\overleftarrow{P'}v_L(P')$. To obtain our contradiction, we consider the following two cases:

(1) Suppose that $N(x) \cap \mathrm{End}(\mathcal{S}'_C) = N(y) \cap \mathrm{End}(\mathcal{S}'_C) = \{v_C\}$. If $|V(C)| = 1$, then by Lemma 3.12(1), Q_6 and Q_7 in G cover $V(P) \cup V(P') \cup V(C)$, contradicting Lemma 3.1. If $|V(C)| > 1$, then by Lemmas 3.7 and 3.12(1), Q_6, Q_7 and $C(G[V(C) \setminus \{v_C\}])$ in G cover $V(P) \cup V(P') \cup V(C)$, satisfying (I),(II) but not (III), a contradiction.

(2) Suppose that there exist two distinct vertices $v_C \in V(C)$ and $v_{C'} \in V(C')$ such that $v_C \in N(x) \cap \mathrm{End}(\mathcal{S}_C)$ and $v_{C'} \in N(y) \cap \mathrm{End}(\mathcal{S}_C)$. By Lemma 3.12(1), Q_6, $Cx\overleftarrow{P}v_L(P)$ and $C'y\overleftarrow{P'}v_L(P')$ in G cover $V(P) \cup V(P') \cup V(C) \cup V(C')$, satisfying (I) but not (II), a contradiction.

This contradiction shows that any pair of vertices in $V(P[x^+, v_R(P)])$ and $V(P'[y^+, v_R(P')])$ are not adjacent.

Lemma 3.14 *Let $P \in \mathcal{S}_P$ and $x \in V(P) \setminus \{v_L(P), v_R(P)\}$. Then the following two statements hold.*

(1) *Suppose that $\mu_w \notin V(P[v_L(P), x])$ and $N(x) \cap (\mathrm{End}(\mathcal{S}'_C) \cup \{v_L(P)\}) \neq \emptyset$. Then $N(V(C)) \cap V(P[v_L(P), x)) = \emptyset$ for any $C \in \mathcal{S}_C$.*

(2) *Suppose that $\mu_w \notin V(P[x, v_R(P)])$ and $N(x) \cap (\mathrm{End}(\mathcal{S}'_C) \cup \{v_R(P)\}) \neq \emptyset$. Then $N(V(C)) \cap V(P(x, v_R(P)]) = \emptyset$ for any $C \in \mathcal{S}_C$.*

Proof By symmetry, we may only prove that (1) holds. By contradiction, suppose that there exists some element $C_0 \in \mathcal{S}_C$ such that $N(C_0) \cap V(P[v_L(P), x)) \neq \emptyset$, say $z \in N(V(C_0)) \cap V(P[v_L(P), x))$. By Lemma 2.2(ii), $z \notin \mathrm{End}(\mathcal{S}_P)$. If $xv_L(P) \in E(G)$ or $N(x) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$, then by Lemmas 3.11(2) and 3.12(2), $G[V(P[v_L(P), x^-])]$ is a clique. Suppose that $z = x^-$. By Lemma 3.8, $N(x) \cap V(\mathcal{S}'_C) = \emptyset$. Then, $xv_L(P) \in E(G)$. Hence, if $z \in V(P(v_L(P), x^-])$, then by Lemmas 3.11(2) and 3.12(2), $C_0z\overleftarrow{P}v_L(P)z^+\overrightarrow{P}v_R(P)$ in G covers $V(P) \cup V(C_0)$, contradicting Lemma 3.1. This contradiction proves Lemma 3.14(1).

For any path $P \in S_P$, we know $v_L(P)v_R(P) \notin E(G)$ by Lemma 3.1. We may take the vertex x_P of $V(P)$ such that $V(P[v_L(P), x_P^-]) \subseteq N(v_L(P))$ and $x_P \notin N(v_L(P))$.

Lemma 3.15 *For any $P \in \mathcal{S}'_P$, the following two statements hold.*

(1) *Suppose that $N(V(\mathcal{S}'_C)) \cap V(P) = \emptyset$. Then $f(P[v_L(P), x_P^-]v_L(P)) = 1$ and $f(P[x_P, v_R(P)]x_P) = 1$.*

(2) *Suppose that* $N(V(\mathcal{S}'_C)) \cap V(P) \neq \emptyset$. *Then* $N(V(\mathcal{S}'_C)) \cap V(P) = \{x\} \subseteq \{x^-_P, x_P\}$, $f(P[v_L(P),\, x^-_P]v_L(P)) = 1$ *and* $f(P[x^+, v_R(P)]x^+) = 1$.

(3) $f(P[x^+_P, v_R(P)]x^+_P) = 1$.

Proof Since $P \in \mathcal{S}'_P$, $\mu_w \notin V(P)$. By Lemma 3.14(1), $N(V(C)) \cap V(P[v_L(P), x^-_P)) = \emptyset$ for any $C \in \mathcal{S}'_C$. We distinguish the following two cases to prove Lemma 3.15(1)–(2).

(1) Suppose that $N(V(\mathcal{S}'_C)) \cap V(P) = \emptyset$. If $x_P = v_R(P)$, then $f(P[v_L(P), x^-_P]v_L(P)) = 1$ and $f(P[x_P, v_R(P)]x_P) = 1$. If $x_P \neq v_R(P)$, then by Lemma 3.9(1), $N(x_P) \cap X \subseteq (\mathrm{End}(\mathcal{S}'_C) \cup \{v_R(P)\})$. By (3.2), $N(x_P) \cap X = \{v_R(P)\}$. So $f(P[v_L(P), x^-_P]v_L(P)) = 1$ and $f(P[x_P, v_R(P)]x_P) = 1$. Lemma 3.15(1) holds.

(2) Suppose that $N(V(\mathcal{S}'_C)) \cap V(P) \neq \emptyset$. If $x_P = v_R(P)$, then by Lemmas 2.2(ii) and 3.14, $N(V(\mathcal{S}'_C)) \cap V(P) = \{x^-_P\}$, $f(P[v_L(P), x^-_P]v_L(P)) = 1$ and $f(P[x_P, v_R(P)]x_P) = 1$. If $x_P \neq v_R(P)$, then we distinguish the following two cases to prove Lemma 3.15(2).

• Suppose that $N(x_P) \cap V(\mathcal{S}'_C) \neq \emptyset$. By Lemma 3.9(2), $f(P[x^+_P, v_R(P)]x_P) = 1$. By Lemmas 3.8 and 3.14, $N(\mathcal{S}'_C) \cap V(P) = \{x_P\}$.

• Suppose that $N(x_P) \cap V(\mathcal{S}'_C) = \emptyset$. Then by Lemma 3.9(1) and (3.2), $f(P[x_P, v_R(P)\,]x_P) = 1$. By the assumption of this case and Lemma 3.14, $N(V(\mathcal{S}'_C)) \cap V(P) = \{x^-_P\}$.

Lemma 3.15(2) holds.

Next, we will prove Lemma 3.15(3). By Lemma 3.15(1)–(2), $f(P[x_P, v_R(P)]x_P) = 1$ or $f(P[x^+_P,\, v_R(P)]x^+_P) = 1$. If $f(P[x^+_P,\, v_R(P)]x^+_P) = 1$, then we are done. Otherwise, $f(P[x^+_P, v_R(P)] \neq 1$. Then we assume that $f(P[x_P, v_R(P)]x_P) = 1$. Since $G[V(P[x_P, v_R(P)])]$ is connected, it suffices to consider the case when $|V(P[x_P, v_R(P)])| \geq 2$. By Lemma 3.11(1), $G[V(P[x_P, v_R(P)])]$ is a clique. Then $f(P[x^+_P, v_R(P)]x^+_P) = 1$. Lemma 3.15(3) holds.

For any $P \in \mathcal{S}'_P$, by Lemma 3.15, $f(P[v_L(P), x^-_P]v_L(P)) = 1$ and $f(P[x^+_P, v_R(P)]\, x^+_P) = 1$. $\mathcal{S}'_P$ be partitioned into classes $\mathcal{S}'_P = \mathcal{S}'_{P_1} \cup \mathcal{S}'_{P_2}$ as follows:

(1) $\mathcal{S}'_{P_1} = \{P : P \in \mathcal{S}'_P, N(V(\mathcal{S}'_C)) \cap V(P) = \{x_P\}\}$;

(2) $\mathcal{S}'_{P_2} = \{P : P \in \mathcal{S}'_P, N(V(\mathcal{S}'_C)) \cap V(P) \neq \{x_P\}\}$.

For any path $P \in \mathcal{S}'_{P_2}$, if $N(V(\mathcal{S}'_C)) \cap V(P) = \emptyset$, then by Lemma 3.15(1), $f(P[x_P, v_R(P)]x_P) = 1$; if $N(V(\mathcal{S}'_C)) \cap V(P) \neq \emptyset$, then by Lemma 3.15(2), $N(V(\mathcal{S}'_C)) \cap V(P) = \{x\} \subseteq \{x^-_P, x_P\}$. Note that $N(V(\mathcal{S}'_C)) \cap V(P) \neq \{x_P\}$. Then $N(V(\mathcal{S}'_C)) \cap V(P) = \{x^-_P\}$. By Lemma 3.9(2), $f(P[x_P, v_R(P)]x_P) = 1$. Hence, for any $P \in \mathcal{S}'_{P_2}$, $f(P[x_P, v_R(P)]x_P) = 1$ and $N(V(\mathcal{S}'_C)) \cap V(P) \subseteq \{x^-_P\}$ (i.e., $N(V(\mathcal{S}'_C)) \cap V(P) = \emptyset$ or $N(V(\mathcal{S}'_C)) \cap V(P) = \{x^-_P\}$).

For any $P \in \mathcal{S}_P$, $P^{\{v_e,x\}}$ is a subpath of P between x and $v_e \in \{v_L(P), v_R(P)\}$. Denote

$$x^\star = \begin{cases} x^+, & \text{if } \ v_e = v_L(P) \text{ for } x \in V(P), \\ x^-, & \text{if } \ v_e = v_R(P) \text{ for } x \in V(P). \end{cases}$$

Lemma 3.16 *For any pair of* $\{P, P'\} \subseteq \mathcal{S}_P$, *suppose there exists a pair of subpaths* $P^{\{v_e,x\}} \subseteq P$ *and* $P'^{\{v'_e,y\}} \subseteq P'$. *If* $xv_e \in E(G)$, $yv'_e \in E(G)$ *and* $\mu_w \notin (V(P^{\{v_e,x\}}) \cup V(P'^{\{v'_e,y\}}))$. *Then any pair of vertices in* $V(P^{\{v_e,x\}}) \cup \{x^\star\}$ *and* $V(P'^{\{v'_e,y\}}) \setminus \{y\}$ *respectively are not adjacent.*

Proof By symmetry, we only prove that any pair of vertices in $V(P[v_L(P), x^+])$ and $V(P'(y, v_R(P')])$ are not adjacent.

By contradiction, suppose that there exists a pair of vertices $x_0 \in V(P[v_L(P), x^+])$ and $y_0 \in V(P'(y, v_R(P')])$ with $x_0y_0 \in E(G)$. By Lemma 3.11(1)–(2), we may obtain that $G[V(P[v_L(P), x^-])]$ and $G[V(P'[y^+, v_R(P')])]$ are cliques. To obtain our contradiction, we distinguish the following two cases:

(1) Suppose that $x_0 = v_L(P)$. Then by Lemma 3.11(1), $v_L(P')\overrightarrow{P'}y^-G[V(P'[y, v_R(P')])]x_0 \overrightarrow{P}v_R(P)$ in G covers $V(P) \cup V(P')$, contradicting Lemma 3.1.

(2) Suppose that $x_0 \in V(P(v_L(P), x^+])$. Then by Lemma 3.11(1), we may obtain that $\overleftarrow{Q}(x_0^-, P)$ and $v_L(P')\overrightarrow{P'}y^-G[V(P'[y, v_R(P')])]x_0 \overrightarrow{P}v_R(P)$ in G cover $V(P) \cup V(P')$, contradicting Lemma 3.1.

This contradiction proves Lemma 3.16.

4 Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4.

Let $k \geq 3$ and G be a graph of order $n > 2k+2$ such that $\alpha(G) \leq k+1$ and $m(G) \leq n-2k-2$. We assume on the contrary that G has no spanning k-ended tree. The assumption that G has no spanning k-ended tree and Theorem 1.2 imply the following two equations

$$\kappa(G) = 1 \tag{4.1}$$

and

$$\alpha(G) = k + 1. \tag{4.2}$$

Then G have no spanning k-ended system by Lemma 2.1. Choose a maximal k-ended system $\mathcal{S}$ of G satisfying (I)–(III). Let $H = G - V(\mathcal{S})$. Then $|V(H)| \geq 1$.

Fact 1 $m(G) \geq n - 2k - 1$.

Proof Denote $G_0 = G[V(\mathcal{S}')]$, $|V(G_0)| = n_0$ and $|\mathrm{End}(\mathcal{S}'_P) \cup \mathrm{End}(\mathcal{S}'_C)| = k_0$. Let $X_0 = X \cap V(G_0) = \mathrm{End}(\mathcal{S}'_P) \cup \mathrm{End}(\mathcal{S}'_C)$,

$$\widehat{x}_P = \begin{cases} x_P^+, & \text{if } \ P \in \mathcal{S}'_{P_1}, \\ x_P, & \text{if } \ P \in \mathcal{S}'_{P_2}. \end{cases}$$

Claim 1 Let $P \in \mathcal{S}'_P$ and $y \in N_{G_0}(v_R(P))$. Then the following two statements hold:

(1) Suppose that $y = \widehat{x}_P$. Then $N_{G_0}(y') \subseteq V(P[y^-, v_R(P)])$ for any $y' \in V(P(y, v_R(P)])$, $N_{G_0}(x') \subseteq V(P[v_L(P), x_P])$ for any $x' \in V(P[v_L(P), x_P^-))$.

(2) Suppose that $y = \widehat{x}_P^-$. If $P \in \mathcal{S}'_{P_1}$, then $N_{G_0}(y') \subseteq V(P[y^-, v_R(P)])$ for any $y' \in V(P(y, v_R(P)])$, $N_{G_0}(x') \subseteq V(P[v_L(P), x_P])$ for any $x' \in V(P[v_L(P), x_P^-))$. If $P \in \mathcal{S}'_{P_2}$, then $N_{G_0}(y') \subseteq V(P[y, v_R(P)])$ for any $y' \in V(P(y, v_R(P)])$, $N_{G_0}(x') \subseteq V(P[v_L(P), x_P^-])$ for any $x' \in V(P[v_L(P), x_P^-))$.

Proof By symmetry, we only consider y''s neighbourhood. Suppose that $y \in \{\widehat{x}_P, \widehat{x}_P^-\}$. Taking any $z \in N_{G_0}(y')$, we know $z \notin V(\mathcal{S}_C')$ by Lemma 3.15(1)–(2). By Lemmas 3.15(1)–(2) and 3.16, $z \notin V(P')$ for any $P' \in \mathcal{S}_P' \setminus \{P\}$. Hence, $z \in V(P)$. We claim that $z \notin V(P[v_L(P), x_P^-))$. Suppose otherwise that $z \in V(P[v_L(P), x_P^-))$. Then by Lemma 3.11(1)–(2), $y'G[V(P[v_L(P), x_P^-])]x_P\overrightarrow{P}y'^-v_R(P)\overleftarrow{P}y'$ in G covers $V(P)$, contradicting Lemma 3.1. This contradiction shows that our claim hold. By our claim, if $y \in \{\widehat{x}_P, \widehat{x}_P^-\}$, then

$$z \in V(P[x_P^-, v_R(P)]). \tag{4.3}$$

By (4.3), Claim 1(2) holds. Suppose that $y = \widehat{x}_P$. If $P \in \mathcal{S}_{P_1}'$ and $z = x_P^-$. Then there exists at least one vertex $v_C \in N(x) \cap V(\mathcal{S}_C')$, by Lemma 3.11(2), $Cx_P\overrightarrow{P}y'^-v_R(P)\overleftarrow{P}y'z\overleftarrow{P}v_L(P)$ in G covers $V(P) \cup V(C)$, contradicting Lemma 3.1. This contradiction shows that $z \in V(P[y^-, v_R(P)])$. Claim 1(1) is proved.

Denote $\mathcal{S}_{P_{21}}' = \{P : P \in \mathcal{S}_{P_2}' \text{ and } x_P \neq v_R(P)\}$, $\mathcal{S}_{P_{22}}' = \{P : P \in \mathcal{S}_{P_2}' \text{ and } x_P = v_R(P)\}$. Then $\mathcal{S}_{P_2}'$ can be partitioned into two subsets $\mathcal{S}_{P_{21}}'$ and $\mathcal{S}_{P_{22}}'$. Define $A_P = \{x_P^- \mid P \in \mathcal{S}_{P_1}' \cup \mathcal{S}_{P_{21}}'\} \cup \{x_P \mid P \in \mathcal{S}_{P_1}' \cup \mathcal{S}_{P_{21}}'\} \cup \{x_P^- \mid P \in \mathcal{S}_{P_{22}}'\}$, $A_1 = \bigcup\limits_{P \in \mathcal{S}_P'} A_P$ and

$$A_2 = \begin{cases} \{\mu_w\}, & \text{if either } \mu_w \in V(\mathcal{S}_C) \text{ or } \mu_w \in V(\mathcal{S}_P), v_L(P_{\mu_w}) = \mu_w^-, v_R(P_{\mu_w}) = \mu_w^+, \\ \{\mu_w^-, \mu_w, \mu_w^+\}, & \text{if } \mu_w \in V(\mathcal{S}_P),\ v_L(P_{\mu_w}) \neq \mu_w^- \text{ and } v_R(P_{\mu_w}) \neq \mu_w^+, \\ \{\mu_w, \mu_w^+\}, & \text{if } \mu_w \in V(\mathcal{S}_P),\ v_L(P_{\mu_w}) = \mu_w^- \text{ and } v_R(P_{\mu_w}) \neq \mu_w^+, \\ \{\mu_w^-, \mu_w\}, & \text{if } \mu_w \in V(\mathcal{S}_P),\ v_L(P_{\mu_w}) \neq \mu_w^- \text{ and } v_R(P_{\mu_w}) = \mu_w^+. \end{cases}$$

Let $A = A_1 \cup A_2$. Then $X \cap A = \emptyset$ and $X_0 \cap A_1 = \emptyset$.

Claim 2 $\omega(G_0 - A_1) = |X_0|$ and each component of $G_0 - A_1$ is a clique.

Proof Denote $\mathcal{Z} = \mathcal{S}_C' \cup \{P[v_L(P), x_P^-) \mid P \in \mathcal{S}_P'\} \cup \{v_R(P) \mid \widehat{x}_P = v_R(P), P \in \mathcal{S}_P'\} \cup \{P(x_P, v_R(P)] \mid \widehat{x}_P \neq v_R(P), P \in \mathcal{S}_{P_2}'\} \cup \{P[x_P^+, v_R(P)] \mid \widehat{x}_P \neq v_R(P), P \in \mathcal{S}_{P_1}'\}$. Let $Z \in \mathcal{Z}$. Then $G_0 - A_1 = \bigcup\limits_{Z \in \mathcal{Z}} Z$. We will prove that $N_{G_0}(V(Z)) \cap V(G_0 \setminus Z) \subseteq A_1$ by considering the following two cases:

(1) Suppose that $Z \in (\mathcal{Z} \setminus \{P[x_P^+, v_R(P)] \mid \widehat{x}_P \neq v_R(P), P \in \mathcal{S}_{P_1}'\})$. By Lemma 3.15(1)–(2) and Claim 1(1)–(2), $N_{G_0}(V(Z)) \cap V(G_0 \setminus Z) \subseteq A_1$.

(2) Suppose that $Z = \{P[x_P^+, v_R(P)] \mid \widehat{x}_P \neq v_R(P), P \in \mathcal{S}_{P_1}'\}$. By Claim 1(1), $N(V(Z \setminus \{x_P^+\})) \cap V(G_0 \setminus Z) \subseteq A_1$. Hence, we consider the vertex x_P^+'s neighbourhood. By the definition of $\mathcal{S}_{P_1}'$, $N(x_P) \cap V(\mathcal{S}_C') \neq \emptyset$, say $v_C \in (N(x_P) \cap V(\mathcal{S}_C'))$. Taking any $z \in N_{G_0 - Z}(x_P^+)$, we know that $z \notin V(\mathcal{S}_C')$ by Lemma 3.8. For any $P' \in \mathcal{S}_P' \setminus \{P\}$, if $P' \in \mathcal{S}_{P_1}'$, then by Lemma 3.13, $z \notin V(P') \setminus A_1$; if $P' \in \mathcal{S}_{P_2}'$, then by Lemma 3.16, $z \notin V(P') \setminus A_1$. We claim that $z \notin V(P[v_L(P), x_P^-))$. Suppose otherwise that $z \in V(P[v_L(P), x_P^-))$. Then, by Lemma 3.12(2), $Cx_P\overleftarrow{P}z^+v_L(P)\overrightarrow{P}zG[V(P[x_P^+, v_R(P)])]$ in G covers $V(P) \cup V(C)$, contradicting Lemma 3.1. Hence, $z \in A_1$. Then $N_{G_0}(V(Z)) \cap V(G_0 \setminus Z) \subseteq A_1$.

Hence, Z is a component of $G_0 - A_1$. By Lemmas 3.7, 3.11(1)–(2) and 3.12(1)–(2), $G[V(Z)]$ is a clique. By the definition of G_0, it is easy to check that $|X_0 \cap V(Z)| = 1$, then $\omega(G_0 - A_1) = |X_0|$.

Suppose that $|\mathcal{S}'_C| = x \geq 0$, $|\mathcal{S}'_{P_1}| = y \geq 0$, $|\mathcal{S}'_{P_{21}}| = z \geq 0$, $|\mathcal{S}'_{P_{22}}| = t \geq 0$. Then we obtain that

$$x + 2(y + z + t) = k_0. \tag{4.4}$$

$$\begin{aligned} m(G_0) \geq & \prod_{P\in\mathcal{S}'_{P_1}} (|V(P[v_L(P), x_P^{2-}])| \cdot |V(P[x_P^+, v_R(P)])|) \cdot \prod_{P\in\mathcal{S}'_{P_{21}}} (|V(P[v_L(P), x_P^{2-}])| \cdot \\ & |V(P[x_P^+, v_R(P)])|) \cdot \prod_{P\in\mathcal{S}'_{P_{22}}} |V(P[v_L(P), x_P^{2-}])| \cdot \prod_{C\in\mathcal{S}'_C} |V(C)| \\ \geq & \underbrace{1\cdot 1\cdot\cdots\cdot 1\cdot 1}_{k_0-1}\cdot[n_0 - (4y + 4z + 3t + x - 1)] \\ \geq & n_0 - (4y + 4z + 3t + x) + 1. \end{aligned}$$

Combining this with (4.4), we have

$$m(G_0) \geq n_0 - (y + z) - \frac{3}{2}k_0 + \frac{1}{2}x + 1 \geq n_0 - 2k_0 + 1. \tag{4.5}$$

Denote

$$v_C^* = \begin{cases} v_C^+, & \text{if } |V(C)| \geq 3 \text{ and } v_C \in V(C), \\ v'_C, & \text{if } |V(C)| = 2 \text{ and } V(C) = \{v_C, v'_C\}. \end{cases}$$

By Lemma 2.2, we distinguish the following two cases to prove Fact 1.

Case 1 $\mu_w \in V(\mathcal{S}_C)$, i.e., $\mu_w \in V(C_{\mu_w})$.

By the choice of $\mathcal{S}$, $|\ \mathcal{S}_P\ | \geq 1$. By (3.1) and Lemma 3.5, $X = \{\mu_w^*, w\} \cup (\mathrm{End}(\mathcal{S}) \setminus \{\mu_w\})$ is an independent set of G with size $k + 1$.

Claim 3 $G[V(C_{\mu_w}) \setminus \{\mu_w\}]$ is a clique.

Proof Since $G[V(C_{\mu_w}) \setminus \{\mu_w\}]$ is connected, it suffices to consider the case when $|V(C_{\mu_w}) \setminus \{\mu_w\}| \geq 3$. For each vertex $v \in V(C_{\mu_w}) \setminus \{\mu_w, \mu_w^+\}$, $v \notin \mathrm{End}(\mathcal{S}) \setminus \{\mu_w\}$ by Lemma 2.2(i)–(ii). Hence, by the definition of X and (3.2), $N(v) \cap X = \{\mu_w^+\}$ for each vertex $v \in V(C_{\mu_w}) \setminus \{\mu_w, \mu_w^+\}$. Note that $(V(C_{\mu_w}) \setminus \{\mu_w\}) \cap X = \{\mu_w^+\}$. Let $S = V(C_{\mu_w}) \setminus \{\mu_w\}$. By Lemma 3.6, $G[V(C_{\mu_w}) \setminus \{\mu_w\}]$ is a clique.

Claim 4 If $N(V(P)) \cap V(C_{\mu_w}) \neq \emptyset$ for any $P \in \mathcal{S}_P$, then $N(V(P)) \cap V(C_{\mu_w}) = \{\mu_w\}$.

Proof By contradiction, suppose that there exists some $P \in \mathcal{S}_P$ such that $N(V(P)) \cap V(C_{\mu_w}) \neq \emptyset$ and $N(V(P)) \cap V(C_{\mu_w}) \neq \{\mu_w\}$, say $v' \in N(V(P)) \cap V(C_{\mu_w})$ satisfying $v' \neq \mu_w$. Then there exists a vertex $z \in V(P)$ such that $v'z \in E(G)$. By Lemma 2.2(ii), $z \notin \mathrm{End}(\mathcal{S}_P)$. Suppose that $z \in V(P) \setminus \{v_L(P), v_R(P)\}$. Then, by Lemmas 3.11(1)–(2) and 3.15(1)–(2), there exists $Q' \in \{P[v_L(P), z^-]v_L(P), P[z^+, v_R(P)]z^+\}$ such that $f(Q') = 1$. By Claim 3, $G[E(P \setminus Q')]v'C_{\mu_w}\mu_w Lw$ and Q' in G cover $V(P) \cup V(C_{\mu_w}) \cup \{w\}$, contradicting (I). This contradiction shows that Claim 4 holds.

Claim 5 $G[V(H)]$ is a clique.

Proof First, we will show that $G[V(H)]$ is connected.

By contradiction, suppose that H has at least two components. Choose any vertex v such that w and v belong to different components of H. We claim that $N(v)\cap V(C_{\mu_w}) \neq \emptyset$. Suppose otherwise that $N(v) \cap V(C_{\mu_w}) = \emptyset$. Then $v\mu_w^* \notin E(G)$. By Lemma 3.4, $N(v) \cap \mathrm{End}(\mathcal{S}_P) = \emptyset$. Suppose that $N(v)\cap\mathrm{End}(\mathcal{S}'_C) \neq \emptyset$, say $vv_C \in E(G)$ ($v_C \neq v_{C_{\mu_w}}$). Then $|V(C)| \geq 2$. Otherwise, $v_Cv \in \mathcal{S}_C$, contradicting (I). By Lemma 3.3, $wv_C^* \notin E(G)$. Combining this with Lemma 2.2(i)–(ii), $\{v, v_C^*\} \cup (X \setminus \{v_C\})$ is an independent set of G with cardinality $k+2$, contradicting (4.2). Hence, $N(v)\cap\mathrm{End}(\mathcal{S}'_C) = \emptyset$. Then $N(v)\cap X = \emptyset$, contradicting (3.2). Hence, $N(v)\cap V(C_{\mu_w}) \neq \emptyset$.

By our claim and Lemma 3.3, $N(V(P)) \cap V(H) = \emptyset$ for any $P \in \mathcal{S}_P$. Since G is connected, there exists at least one path $P \in \mathcal{S}_P$ such that $N(V(P)) \cap V(C_{\mu_w}) \neq \emptyset$. By the arbitrariness of vertex w and Claim 4, $N(v)\cap V(C_{\mu_w}) = \{\mu_w\}$. By Lemma 3.5, $\{v, w, \mu_w^*\}\cup(\mathrm{End}(\mathcal{S}) \setminus \{\mu_w\})$ is an independent set of G with cardinality $k+2$, contradicting (4.2). This contradiction proves that $G[V(H)]$ is connected.

Next, we will show that $G[V(H)]$ is a clique. Since $G[V(H)]$ is connected, it suffices to consider the case when $|V(H)| \geq 3$. Since $G[V(H)]$ is connected, by Lemma 3.3, $N(v) \cap (V(\mathcal{S}) \setminus \{\mu_w\}) = \emptyset$ for each vertex $v \in V(H) \setminus \{w\}$. Note that (3.2), therefore, $N(v) \cap X = \{w\}$ for every vertex $v \in V(H) \setminus \{w\}$. Note that $V(H) \cap X = \{w\}$. Let $S = V(H)$, then by Lemma 3.6, $G[V(H)]$ is a clique.

Since G is connected, by Lemma 3.3 and Claims 4–5, we have the following claim.

Claim 6 $N(V(H)) \cap V(\mathcal{S}) = \{\mu_w\}$ and $N(V(\mathcal{S}')) \cap V(C_{\mu_w}) = \{\mu_w\}$.

By Claims 2–6, $\omega(G - A) = k + 1$ and each component of $G - A$ is a clique.
In this case, $n_0 = n - |V(C_{\mu_w})| - |V(H)|$ and $k_0 = k - 1$. Then,

$$\begin{aligned}
m(G) &\geq |V(C_{\mu_w}) \setminus \{\mu_w\}| \cdot |V(H)| \cdot m(G_0)\\
&\geq |V(C_{\mu_w}) \setminus \{\mu_w\}| \cdot |V(H)| \cdot [(n - |V(C_{\mu_w})| - |V(H)|) - 2(k - 1) + 1]\\
&\geq 1 \cdot 1 \cdot [(n - |V(C_{\mu_w})| - |V(H)|) - 2(k - 1) + 1 + (|V(C_{\mu_w})| - 2) + |V(H)| - 1]\\
&= n - 2k,
\end{aligned}$$

which proves Fact 1 in this case.

Case 2 $\mu_w \in V(\mathcal{S}_P)$, i.e., $\mu_w \in V(P_{\mu_w})$.
By (3.1) and Lemma 3.5, $X = \mathrm{End}(\mathcal{S}) \cup \{w\}$ is an independent set of G with size $k + 1$.

Claim 7 $G[V(H)]$ is a clique.

Proof It suffices to consider the case when $|V(H)| \geq 2$. By (3.2), $N(v) \cap X \neq \emptyset$ for each vertex $v \in V(H) \setminus \{w\}$. Suppose that there exists a vertex $x \in N(v) \cap X$ such that $x \neq w$. We claim that $x \notin \mathrm{End}(\mathcal{S}_C)$. Otherwise suppose that $x \in \mathrm{End}(\mathcal{S}_C)$, say $x = v_C$. Then $|V(C)| \geq 2$. Otherwise, $xv \in \mathcal{S}_C$, contradicting (I). By Lemma 3.5, $\{v, v_C^*\} \cup (\mathrm{End}(\mathcal{S}) \setminus \{v_C\})$ is an independent set of G with size $k + 1$. By Lemma 3.3, $N(w) \cap (V(\mathcal{S}) \setminus \{\mu_w\}) = \emptyset$. Hence,

$\{w, v, v_C^*\} \cup (\mathrm{End}(\mathcal{S}) \setminus \{v_C\})$ would have an independent set of cardinality $k+2$, contradicting (4.2). This contradiction shows that our claim holds.

By Lemma 3.4 and our claim, $N(v) \cap X \subseteq \{w\}$. Note that (3.2), therefore, $N(v) \cap X = \{w\}$ for each vertex $v \in V(H) \setminus \{w\}$. Let $S = V(H)$. Then by Lemma 3.6, $G[V(H)]$ is a clique.

Claim 8 The following two statements hold:

(1) If $\mu_w^+ \neq v_R(P_{\mu_w})$, then $G[V(P_{\mu_w}[\mu_w^{2+}, v_R(P_{\mu_w})])]$ is a clique.

(2) If $\mu_w^- \neq v_L(P_{\mu_w})$, then $G[V(P_{\mu_w}[v_L(P_{\mu_w}), \mu_w^{2-}])]$ is a clique.

Proof By symmetry, we may only prove that $G[V(P_{\mu_w}[\mu_w^{2+}, v_R(P_{\mu_w})])]$ is a clique. Since $G[V(P_{\mu_w}[\mu_w^{2+}, v_R(P_{\mu_w})])]$ is connected, it suffices to consider the case when $|V(P_{\mu_w}[\mu_w^{2+}, v_R(P_{\mu_w})])| \geq 3$. By Lemma 3.9(1) and (3.2), $N(\mu_w^+) \cap (\mathrm{End}(\mathcal{S}'_C) \cup \{v_R(P_{\mu_w})\}) \neq \emptyset$. Then by Lemmas 3.11(1) and 3.12(1), $G[V(P_{\mu_w}[\mu_w^{2+}, v_R(P_{\mu_w})])]$ is a clique.

Since G is connected, by Lemma 3.3 and Claim 7, we have the following claim.

Claim 9 $N(V(H)) \cap V(\mathcal{S}) = \{\mu_w\}$.

By Claim 9, μ_w is the unique vertex μ (say) for any $w \in V(H)$. Then denote

$$\mathcal{B}_1 = V(P_\mu[v_L(P_\mu), \mu]) \setminus A_2,$$
$$\mathcal{B}_2 = V(P_\mu[\mu, v_R(P_\mu)]) \setminus A_2.$$

Claim 10 $xy \notin E(G)$ for any pair of vertices $x \in \mathcal{B}_1$ and $y \in \mathcal{B}_2$.

Proof By contradiction, suppose that there exists a pair of vertices $x_0 \in \mathcal{B}_1$ and $y_0 \in \mathcal{B}_2$ such that $x_0y_0 \in E(G)$. To obtain our contradiction, we distinguish the following three cases.

(1) Suppose that $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$, say $v_C \in (N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C))$, and $N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$, say $v_{C'} \in (N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C))$. By Claim 8, $G[V(P_\mu[v_L(P_\mu), \mu^{2-}])]$ and $G[V(P_\mu[\mu^{2+}, v_R(P_\mu)])]$ are cliques. Let $Q^1 = G[V(P_\mu[v_L(P_\mu), \mu^{2-}])]y_0\overleftarrow{P_\mu}\mu^+C\mu^-$ and $Q^2 = G[V(P_\mu[v_L(P_\mu), \mu^{2-}])]y_0G[V(P_\mu[\mu^{2+}, v_R(P_\mu)]) \setminus \{y_0\}]$. To obtain our contradiction, we distinguish the following two cases:

• Suppose that $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) = N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) = \{v_C\}$. To obtain our contradiction, we distinguish the following two cases:

– Either $|V(C)| = 1$ or $|V(C)| > 1$ and $\mu^+v_C^* \in E(G)$ is true. If $y_0 = v_R(P_\mu)$, then by Claim 8, $Q^1\mu$ in G covers $V(P_\mu) \cup V(C)$, contradicting Lemma 3.1. If $y_0 \neq v_R(P_\mu)$, then by Claim 8, $Q^1\mu Lw$ and $\overrightarrow{\mathcal{Q}}(y_0^+, P_\mu)$ in G cover $V(P_\mu) \cup V(C) \cup \{w\}$, contradicting (I).

– Suppose that $|V(C)| > 1$ and $\mu^+v_C^* \notin E(G)$. Note that $N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) = \{v_C\}$. If $\mu^+v_R(P_\mu) \notin E(G)$, then by Lemma 3.9(1) and (3.2), $N(\mu^+) \cap X = \{v_C\}$. By Lemma 2.2(ii), $N(v_C^*) \cap X = \{v_C\}$. The set $(X \setminus \{v_C\}) \cup \{\mu^+, v_C^*\}$ would be an independent set of cardinality $k+2$, contradicting (4.2). But if $\mu^+v_R(P_\mu) \in E(G)$, then $Q^2\mu^+v_C\mu^-\mu Lw$ and $C(G[V(C) \setminus \{v_C\}])$ in G cover $V(P_\mu) \cup V(C) \cup \{w\}$, contradicting (I).

• Suppose that there exist two distinct vertices $v_C \in V(C)$ and $v_{C'} \in V(C')$ such that $v_C \in N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C)$ and $v_{C'} \in N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C)$. Then, Q^2 and $C\mu^-\mu\mu^+C'$ in G cover $V(P_\mu) \cup V(C) \cup V(C')$, satisfying (I) but not (II), a contradiction.

(2) Suppose that either $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$ and $N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) = \emptyset$, or $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) = \emptyset$ and $N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$. By symmetry, assume that $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$ and $N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) = \emptyset$. By Lemma 3.9(1) and (3.2), $f(\overrightarrow{\mathcal{Q}}(\mu^+, P_\mu)) = 1$. By Lemma 3.11(1), then $G[V(P_\mu[v_L(P_\mu), \mu^{2-}])]y_0 G[V(P_\mu[\mu^+, v_R(P_\mu)]) \setminus \{y_0\}]\mu\mu^- C$ in G covers $V(P_\mu) \cup V(C)$, contradicting Lemma 3.1.

(3) Suppose that $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) = \emptyset$ and $N(\mu^+) \cap \mathrm{End}(\mathcal{S}_C) = \emptyset$. By Lemma 3.9(1)–(3) and (3.2), $f(\overleftarrow{\mathcal{Q}}(\mu^-, P_\mu)) = 1$ and $f(\overrightarrow{\mathcal{Q}}(\mu^+, P_\mu)) = 1$. By Lemma 3.11(1)–(2), then $\mu G[V(P_\mu [v_L(P_\mu)\mu^-])]y_0 G[V(P_\mu[\mu^+, v_R(P_\mu)]) \setminus \{y_0\}]\mu$ in G covers $V(P_\mu)$, contradicting Lemma 3.1.

This contradiction shows that Claim 10 holds.

Claim 11 $N(V(P) \setminus A_1) \cap \mathcal{B}_i = \emptyset$ for any $P \in \mathcal{S}'_P$ and for any $i \in \{1, 2\}$.

Proof By the symmetry, we may only prove that $N(V(P) \setminus A_1) \cap \mathcal{B}_1 = \emptyset$ for any $P \in \mathcal{S}'_P$. We distinguish the following two cases to prove Claim 11.

(1) Suppose that $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) = \emptyset$. By Lemma 3.9(3) and (3.2), $f(\overleftarrow{\mathcal{Q}}(\mu^-, P_\mu)) = 1$. By Lemmas 3.15(1)–(2) and 3.16, $N(V(P) \setminus A_1) \cap \mathcal{B}_1 = \emptyset$.

(2) Suppose that $N(\mu^-) \cap \mathrm{End}(\mathcal{S}_C) \neq \emptyset$. By Claim 8, $G[V(P_\mu[v_L(P_\mu), \mu^{2-}])]$ is a clique. If $P \in \mathcal{S}'_{P_1}$, then by Lemma 3.13, $N(V(P) \setminus A_1) \cap \mathcal{B}_1 = \emptyset$. If $P \in \mathcal{S}'_{P_2}$, then by Lemma 3.16, $N(V(P) \setminus A_1) \cap \mathcal{B}_1 = \emptyset$.

Hence, Claim 11 is proved.

By Lemma 3.14, $N(V(C)) \cap (V(P_\mu) \setminus A_2) = \emptyset$ for any $C \in \mathcal{S}_C$. Combining this with Claims 2 and 7–11, $\omega(G - A) = k + 1$ and each component of $G - A$ is a clique.

In this case, $n_0 = n - |V(P_\mu)| - |V(H)|$ and $k_0 + 2 = k$. We distinguish the following three cases to prove Fact 1.

(1) Suppose that $v_L(P_\mu) \neq \mu^-$ and $v_R(P_\mu) \neq \mu^+$. By Claims 7–11, we have the following:

$$\begin{aligned} m(G) &\geq |V(H)| \cdot |V(P_\mu[v_L(P_\mu), \mu^{2-}])| \cdot |V(P_\mu[\mu^{2+}, v_R(P_\mu)])| \cdot m(G_0) \\ &\geq 1 \cdot 1 \cdot 1 \cdot [(n - |V(P_\mu)| - |V(H)| - 2(k-2) + 1 + (|V(P_\mu)| - 5) + (|V(H)| - 1)] \\ &= n - 2k - 1. \end{aligned}$$

(2) Suppose that either $v_L(P_\mu) = \mu^-$ and $v_R(P_\mu) \neq \mu^+$, or $v_L(P_\mu) \neq \mu^-$ and $v_R(P_\mu) = \mu^+$. By symmetry, we assume that $\mu^- = v_L(P_\mu)$ and $v_R(P_\mu) \neq \mu^+$. By Claims 7–11, we have the following:

$$\begin{aligned} m(G) &\geq |V(H)| \cdot |V(P_\mu[\mu^{2+}, v_R(P_\mu)])| \cdot m(G_0) \\ &\geq 1 \cdot 1 \cdot [(n - |V(P_\mu)| - |V(H)|) - 2(k-2) + 1 + (|V(P_\mu)| - 4) + (|V(H)| - 1)] \\ &= n - 2k. \end{aligned}$$

(3) Suppose that $v_L(P_\mu) = \mu^-$ and $v_R(P_\mu) = \mu^+$. By Claims 7–11, we have the following:

$$\begin{aligned} m(G) &\geq |V(H)| \cdot m(G_0) \\ &\geq 1 \cdot [(n - |V(P_\mu)| - |V(H)|) - 2(k - 2 - x) + 1 + (|V(P_\mu)| - 3) + (|V(H)| - 1)] \end{aligned}$$

$$= n - 2k + 1.$$

Therefore, $m(G) \geq n - 2k + 1 > n - 2k > n - 2k - 1$. This completes the proof of Fact 1 and also the proof of Theorem 1.4.

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