Chin. Ann. Math. Ser. B
40(3), 2019, 429–438
DOI: 10.1007/s11401-019-0142-8

Chinese Annals of Mathematics, Series B

On Generalized Algebraic Cone Metric Spaces and Fixed Point Theorems*

Qing MENG[1]

Abstract In this paper, the author first introduces the concept of generalized algebraic cone metric spaces and some elementary results concerning generalized algebraic cone metric spaces. Next, by using these results, some new fixed point theorems on generalized (complete) algebraic cone metric spaces are proved and an example is given. As a consequence, the main results generalize the corresponding results in complete algebraic cone metric spaces and generalized complete metric spaces.

Keywords Generalized algebraic cone metric space, Canonical decomposition, Fixed point
2000 MR Subject Classification 47H10, 54H25

1 Introduction

Cone metric spaces were introduced by Huang and Zhang [1], as a generalization of metric spaces. The distance $d(x,y)$ of two elements x and y in a cone metric space X is defined to be a vector in an ordered Banach space E. In [1], Huang and Zhang proved that there exists a unique fixed point for contractive mappings in complete cone metric spaces. Later on, by omitting the assumption of normality in the results of [1], Rezapour and Hamlbarani [2] obtained some fixed point theorems, as the generalizations of the relevant results in [1]. Since then, many interesting results about cone metric spaces and fixed point theorems were studied by a number of mathematicians (see [3–7]).

In 2013, Liu and Xu [8] introduced the concept of cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. In this way, they proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constant L by means of spectral radius. In 2014, Xu and Radenović [9] deleted the superfluous assumption of normality in [8] and also obtained the existence and uniqueness of the fixed point for the generalized Lipschitz mappings in the setting of cone metric spaces with Banach algebras. Later on, Tootkaboni and Salec [10] introduced the concept of algebraic cone metric spaces and replaced the condition "$\rho(L)<1$" by "$\sum\limits_{n=1}^{+\infty}\|L^n\|<+\infty$" in the fixed point theorems.

Manuscript received February 24, 2016. Revised June 27, 2017.
[1]School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China.
E-mail: mengqing80@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 11871303, 11371222, 11271224), the China Postdoctoral Science Foundation (No. 2018M642633) and A Project of Shandong Province Higher Educational Science and Technology Program (No. J18KA238).

Generalized metric spaces are the generalization of metric spaces. Some fixed point theorems of contractive mappings on generalized complete metric spaces have been proved. For further information, see [11–12].

In this paper, we first introduce the concept of generalized algebraic cone metric spaces and some elementary results concerning generalized algebraic cone metric spaces. Next, by using these results, we prove some new fixed point theorems on generalized (complete) algebraic cone metric spaces and give an example. As a consequence, our main results generalize the following two fixed point theorems.

Theorem 1.1 (see [10]) *Let (X, d) be a complete algebraic cone metric space with respect to the algebraic cone P. Suppose that the mapping $T : X \to X$ satisfies the generalized Lipschitz condition*

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$, where $L \in P$ such that $\sum\limits_{n=0}^{\infty} \|L^n\|$ converges. Then T has a unique fixed point in X. For any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.2 (see [11]) *Let (X, d) be a generalized complete metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

2 Generalized Algebraic Cone Metric Space

Throughout this paper, let A always be a real Banach algebra with a unit e. The following theorem is well known.

Theorem 2.1 *Let A be a Banach algebra with a unit e and $L \in A$. If $\sum\limits_{n=0}^{\infty} \|L^n\| < +\infty$, then $e - L$ is invertible and $(e - L)^{-1} = \sum\limits_{n=0}^{\infty} L^n$.*

Remark 2.1 In Theorem 2.1, if the condition "$\sum\limits_{n=0}^{\infty} \|L^n\| < +\infty$" is replaced by $\|L\| < 1$, then the conclusion remains true.

Remark 2.2 If $\sum\limits_{n=0}^{\infty} \|L^n\| < +\infty$, then $\|L^n\| \to 0$.

Now we recall some definitions and results about cones.

Definition 2.1 *Let E be a real Banach space. A subset P of E is called a cone if*

(1) *P is non-empty closed and $P \neq \{0\}$;*
(2) *$\alpha P + \beta P \subseteq P$ for all non-negative numbers α, β;*

(3) $P \cap -P = \{0\}$.

For a given cone P, one can define a partial ordering (denoted by $\leq$) with respect to P by $u \leq v$ if and only if $v - u \in P$. The notation $u < v$ indicates that $u \leq v$ and $u \neq v$, while $u \ll v$ stands for $v - u \in \text{int}P$, where $\text{int}P$ denotes the interior of P. From now on, it is assumed that $\text{int}P \neq \emptyset$. Note that $\text{int}P + \text{int}P \subseteq \text{int}P$ and $\lambda\text{int}P \subseteq \text{int}P$ for all $\lambda > 0$.

Lemma 2.1 *Let E be a real Banach space with a cone P. If $u \leq v$ and $v \ll w$, then $u \ll w$.*

Lemma 2.2 *Let E be a real Banach space with a cone P. If $0 \leq u \ll c$ for all $0 \ll c$, then $u = 0$.*

Lemma 2.3 *Let E be a real Banach space with a cone P and $\{a_n\} \subseteq E$. If $\|a_n\| \to 0$, then for each $0 \ll c$, there exists $N \in \mathbb{N}$ such that $a_n \ll c$ for all $n \geq N$.*

Next, we introduce the definition of generalized algebraic cone metric spaces.

Definition 2.2 *A subset P of A is called a algebraic cone if*
(1) *P is a cone and $e \in P$;*
(2) *$P^2 = PP \subseteq P$.*

From now on, let P always be an algebraic cone of A. We introduce a new element "∞", which satisfies
(1) $\infty + \infty = \infty$;
(2) $a + \infty = \infty + a = \infty$ for all $a \in A$;
(3) $a \cdot \infty = \infty \cdot a = \infty$ for all $a \in P \setminus \{0\}$;
(4) $0 \cdot \infty = \infty \cdot 0 = 0$;
(5) $\infty \cdot \infty = \infty$.
And we introduce a new partial ordering (also denoted by $\leq$) such that
(1) $a < \infty$ for all $a \in A$;
(2) $\infty \leq \infty$;
(3) if $a, b \in A$, $a \leq b$ if and only if $b - a \in P$.

Definition 2.3 *Let X be a non-empty set. Suppose that the mapping $d : X \times X \to P \cup \{\infty\}$ satisfies*
(1) *$d(x, y) = 0$ if and only if $x = y$;*
(2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(3) *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*
Then d is called a generalized algebraic cone metric on X, and (X, d) is called a generalized algebraic cone metric space.

If every two points in (X, d) have a distance in P, then d is called an algebraic cone metric on X, and (X, d) is called an algebraic cone metric space.

Definition 2.4 *Let (X, d) be a generalized algebraic cone metric space and $\{x_n\}$ be a sequence in X. Then*

(1) *$\{x_n\}$ converges to x if for every $c \in A$ with $0 \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. It is denoted by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$;*

(2) $\{x_n\}$ *is a Cauchy sequence if for every* $c \in A$ *with* $0 \ll c$*, there is a natural number* N *such that* $d(x_n, x_m) \ll c$ *for all* $n, m \geq N$;

(3) (X, d) *is a generalized complete algebraic cone metric space if every Cauchy sequence is convergent.*

Lemma 2.4 *Let* (X, d) *be a generalized algebraic cone metric space. If* $\{x_n\}$ *is a convergent sequence in* X*, then the limit of* $\{x_n\}$ *is unique.*

Proof If $x_n \to x$ and $x_n \to y$, then for every $c \in A$ with $0 \ll c$, there is a natural number N such that $d(x_n, x) \ll \frac{c}{2}$ and $d(x_n, y) \ll \frac{c}{2}$ for all $n \geq N$. Thus $d(x, y) \leq d(x_n, x) + d(x_n, y) \ll c$. By Lemma 2.2, we get $d(x, y) = 0$.

3 Decomposition

Let $\{(X_\alpha, d_\alpha) \mid \alpha \in \Lambda\}$ be a family of disjoint algebraic cone metric spaces. Then there is a natural way of obtaining a generalized algebraic cone metric space (X, d) from $\{(X_\alpha, d_\alpha) \mid \alpha \in \Lambda\}$ in the following manner. Let X be the union of $\{X_\alpha \mid \alpha \in \Lambda\}$. For any $x, y \in X$, define

$$d(x, y) = \begin{cases} d_\alpha(x, y), & \text{if } x, y \in X_\alpha \text{ for some } \alpha \in \Lambda, \\ \infty, & \text{if } x \in X_\alpha,\ y \in X_\beta \text{ for some } \alpha, \beta \in \Lambda \text{ with } \alpha \neq \beta. \end{cases}$$

Clearly, (X, d) is a generalized algebraic cone metric space. Moreover, if each (X_α, d_α) is also complete, then (X, d) is a generalized complete algebraic cone metric space. The main purpose of this section is to show that the above procedure is the only way to obtain generalized (complete) algebraic cone metric spaces.

Let (X, d) be a generalized algebraic cone metric space. Define a relation $\sim$ on X as follows: $x \sim y$ if and only if $d(x, y) < \infty$. Then $\sim$ is obviously an equivalence relation on X and X is decomposed uniquely into disjoint equivalence classes X_α, $\alpha \in \Lambda$. We call this decomposition of X the canonical decomposition.

Theorem 3.1 *Let* (X, d) *be a generalized algebraic cone metric space. Suppose that*

$$X = \cup\{X_\alpha \mid \alpha \in \Lambda\}$$

is the canonical decomposition and

$$d_\alpha = d|_{X_\alpha \times X_\alpha}$$

for each $\alpha \in \Lambda$*. Then*

(1) (X_α, d_α) *is an algebraic cone metric space for each* $\alpha \in \Lambda$,

(2) *for any* $\alpha, \beta \in \Lambda$ *with* $\alpha \neq \beta$, $d(x, y) = \infty$ *for all* $x \in X_\alpha$ *and* $y \in X_\beta$,

(3) (X, d) *be a generalized complete algebraic cone metric space if and only if* (X_α, d_α) *is a complete algebraic cone metric space for each* $\alpha \in \Lambda$.

Proof (1) and (2) are clear.

(3) Suppose that (X, d) is a generalized complete algebraic cone metric space and $\{x_n\}$ is a d_α-Cauchy sequence in X_α. Then $\{x_n\}$ is a d-Cauchy sequence in X. Thus there exists $x \in X$ such that $\{x_n\}$ is d-convergent to x. Since for sufficiently large $n \in \mathbb{N}$, $d(x_n, x) < \infty$ and the limits of sequences are unique, $x \in X_\alpha$ and $\{x_n\}$ is d_α-convergent to x.

Conversely, suppose that (X_α, d_α) is a complete algebraic cone metric space for each $\alpha \in \Lambda$ and $\{x_n\}$ is a d-Cauchy sequence in X. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \infty$ for all $n, m \geq N$. Hence there exists $\alpha \in \Lambda$ such that $x_n \in X_\alpha$ for all $n \geq N$ and $\{x_n\}$ is a d_α-Cauchy sequence in X_α. Therefore, there exists $x \in X_\alpha$ such that $\{x_n\}$ is d_α-convergent to x. Obviously, $\{x_n\}$ is d-convergent to x.

4 Fixed Point Theorems

In this section we prove some new fixed point theorems on generalized (complete) algebraic cone metric spaces.

Theorem 4.1 *Let (X, d) be a generalized algebraic cone metric space and $X = \cup\{X_\alpha \mid \alpha \in \Lambda\}$ be the canonical decomposition. Suppose that $T : X \to X$ is a mapping such that*

$$d(T(x), T(y)) < \infty,$$

whenever $x, y \in X$ and $d(x, y) < \infty$. Then T has a fixed point if and only if

$$T_\alpha = T|_{X_\alpha} : X_\alpha \to X_\alpha$$

has a fixed point for some $\alpha \in \Lambda$.

Proof Let x_0 be a fixed point of T. Then there exists $\alpha \in \Lambda$ such that $x_0 \in X_\alpha$. For each $x \in X_\alpha$, we have

$$d(Tx, x_0) = d(Tx, Tx_0) < \infty.$$

This implies $Tx \in X_\alpha$ for all $x \in X_\alpha$. Therefore, x_0 is a fixed point of T_α. The converse is clear.

Furthermore, we can obtain a local version of Theorem 4.1 as follows.

Theorem 4.2 *Let (X, d) be a generalized algebraic cone metric space and $X = \cup\{X_\alpha \mid \alpha \in \Lambda\}$ be the canonical decomposition. Suppose that $T : X \to X$ is a mapping. If there exists $a \in P$ such that*

$$d(Tx, Ty) \leq a,$$

whenever $x, y \in X$ and $d(x, y) \leq a$, then T has a fixed point if and only if for some subset $Y \subseteq X$ such that $d(x, y) \leq 2a$ for all $x, y \in Y$, the restriction

$$T|_Y : Y \to Y$$

has a fixed point.

Proof If x_0 is a fixed point of T, let $Y = \{x \in X \mid d(x, x_0) \leq a\}$. Then

$$d(x, y) \leq d(x, x_0) + d(x_0, y) \leq 2a$$

for all $x, y \in Y$. For each $x \in Y$, we have

$$d(Tx, x_0) = d(Tx, Tx_0) \leq a.$$

This implies $Tx \in Y$ for all $x \in Y$. Therefore, x_0 is a fixed point of $T|_Y$. The converse is clear.

Theorem 4.3 *Let (X,d) be a generalized algebraic cone metric space and $X=\cup\{X_\alpha \mid \alpha\in\Lambda\}$ be the canonical decomposition. Suppose that the mapping $T:X\to X$ satisfies the generalized Lipschitz condition*

$$d(Tx,Ty)\le Ld(x,y),$$

whenever $x,y\in X$ and $d(x,y)<\infty$, where $L\in P$ such that $\sum\limits_{n=0}^{\infty}\|L^n\|$ converges. If there exists $x_0\in X$ such that $d(x_0,Tx_0)<\infty$, then for some $\alpha\in\Lambda$, the restriction

$$T_\alpha=T|_{X_\alpha}:X_\alpha\to X_\alpha$$

satisfies the generalized Lipschitz condition.

Proof Since $d(x_0,Tx_0)<\infty$, both x_0 and Tx_0 belong to the same X_α for some $\alpha\in\Lambda$. For each $x\in X_\alpha$,

$$d(Tx,Tx_0)\le Ld(x,x_0)<\infty.$$

Thus $TX_\alpha\subseteq X_\alpha$ and T_α satisfies the generalized Lipschitz condition.

Theorem 4.4 *Let (X,d) be a generalized complete algebraic cone metric space. Suppose that the mapping $T:X\to X$ satisfies the generalized Lipschitz condition*

$$d(Tx,Ty)\le Ld(x,y),$$

whenever $x,y\in X$ and $d(x,y)<\infty$, where $L\in P$ such that $\sum\limits_{n=0}^{\infty}\|L^n\|$ converges. Then for each given element $x\in X$, either

$$d(T^nx,T^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) *$d(T^nx,T^{n+1}x)<\infty$ for all $n\ge n_0$;*

(2) *the sequence $\{T^nx\}$ converges to a fixed point y^* of T;*

(3) *y^* is the unique fixed point of T in the set $Y=\{y\in X: d(T^{n_0}x,y)<\infty\}$;*

(4) *$d(y,y^*)\le(e-L)^{-1}d(y,Ty)$ for all $y\in Y$.*

Proof Consider the sequence $\{d(T^nx,T^{n+1}x)\}$, $n=0,1,2\cdots$. There are two mutually exclusive possibilities: Either

(a) for all nonnegative integers n,

$$d(T^nx,T^{n+1}x)=\infty,$$

or

(b) there exists a nonnegative integer n_0 such that

$$d(T^{n_0}x,T^{n_0+1}x)<\infty.$$

If (a) does not hold, let $x_0=T^{n_0}x$ and $X=\cup\{X_\alpha\mid\alpha\in\Lambda\}$ be the canonical decomposition. It follows from Theorem 4.3 that

$$T_\alpha=T|_{X_\alpha}:X_\alpha\to X_\alpha$$

satisfies the generalized Lipschitz condition, where X_α is the complete algebraic cone metric space containing x_0. By Theorem 1.1, the sequence $\{T^n x\}$ d_α-converges to a fixed point y^* of T_α. Then the sequence $\{T^n x\}$ converges to a fixed point y^* of T. Now if z is another fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$, then

$$d(y^*, z) = d(Ty^*, Tz) \leq Ld(y^*, z).$$

That is

$$(e - L)d(y^*, z) \leq 0.$$

Multiplying both sides above by $(e - L)^{-1} = \sum_{n=0}^{\infty} L^n$, we get $d(y^*, z) \leq 0$. Thus $d(y^*, z) = 0$, which implies that $y^* = z$. For all $y \in Y$,

$$d(y, y^*) \leq d(y, Ty) + d(Ty, Ty^*) \leq d(y, Ty) + Ld(y, y^*).$$

Then

$$(e - L)d(y, y^*) \leq d(y, Ty)$$

for all $y \in Y$. Multiplying both sides above by $(e - L)^{-1} = \sum_{n=0}^{\infty} L^n$, we get

$$d(y, y^*) \leq (e - L)^{-1} d(y, Ty)$$

for all $y \in Y$.

If the cone P satisfies that $a^{-1} \in P$ for every invertible element $a \in P$, we have the next theorem.

Theorem 4.5 *Let (X, d) be a generalized complete algebraic cone metric space. Suppose that the mapping $T : X \to X$ satisfies the condition*

$$d(Tx, Ty) \leq Ld(x, y),$$

whenever $x, y \in X$ and $d(x, y) \leq C$, where $C, L \in P$ such that $C \in \mathrm{int}P$ and $\sum_{n=0}^{\infty} \|L^n\|$ converges. Then for each given element $x \in X$, either

$$d(T^n x, T^{n+1}x) \nleq C$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) *$d(T^n x, T^{n+1}x) \leq C$ for all $n \geq n_0$;*

(2) *the sequence $\{T^n x\}$ converges to a fixed point y^* of T;*

(3) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(y^*, y) \leq C\}$;*

(4) *$d(y, y^*) \leq (e - L)^{-1} d(y, Ty)$ for all $y \in Y$.*

Proof Since $\sum_{n=0}^{\infty} \|L^n\|$ converges, $e - L$ is invertible and $(e - L)^{-1} = \sum_{n=0}^{\infty} L^n \geq 0$. Thus $e - L \geq 0$. This implies $a - La = (e - L)a \geq 0$, i.e., $La \leq a$ for all $a \in P$. Consider the sequence $\{d(T^n x, T^{n+1}x)\}$, $n = 0, 1, 2 \cdots$. If there exists a nonnegative integer n_0 such that

$$d(T^{n_0}x, T^{n_0+1}x) \leq C,$$

then we have

$$d(T^{n_0+1}x, T^{n_0+2}x) = d(TT^{n_0}x, TT^{n_0+1}x) \leq Ld(T^{n_0}x, T^{n_0+1}x) \leq LC \leq C.$$

By mathematical induction, we get

$$d(T^{n_0+i}x, T^{n_0+i+1}x) \leq L^i d(T^{n_0}x, T^{n_0+1}x) \leq C$$

for all $i \in \mathbb{N}$. Thus

$$\begin{aligned} d(T^n x, T^{n+l}x) &\leq \sum_{i=1}^{l} d(T^{n+i-1}x, T^{n+i}x) \\ &\leq \sum_{i=1}^{l} L^{n+i-1-n_0} d(T^{n_0}x, T^{n_0+1}x) \\ &\leq L^{n-n_0}\Big(\sum_{i=0}^{\infty} L^i\Big) d(T^{n_0}x, T^{n_0+1}x) \\ &= L^{n-n_0}(e-L)^{-1} d(T^{n_0}x, T^{n_0+1}x) \end{aligned}$$

for any $l \in \mathbb{N}$. By Lemma 2.3 and the fact that $\|L^{n-n_0}(e-L)^{-1}d(T^{n_0}x, T^{n_0+1}x)\| \to 0$, we have that for any $0 \ll c$, there exists $N \in \mathbb{N}$ such that

$$d(T^n x, T^{n+l}x) \leq L^{n-n_0}(e-L)^{-1} d(T^{n_0}x, T^{n_0+1}x) \ll c$$

for all $n \geq N$. Hence $\{T^n x\}$ is a Cauchy sequence. By the completeness of X, there exists $y^* \in X$ such that $T^n x \to y^*$. Furthermore, we have

$$\begin{aligned} d(Ty^*, y^*) &\leq d(Ty^*, T^n x) + d(T^n x, y^*) \\ &\leq Ld(y^*, T^{n-1}x) + d(T^n x, y^*) \\ &\leq d(y^*, T^{n-1}x) + d(T^n x, y^*) \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. Therefore, $d(Ty^*, y^*) \ll c$ for any $0 \ll c$. This implies that $Ty^* = y^*$. The rest proof is similar to that of Theorem 4.4, we omit it.

Remark 4.1 (a) Theorem 1.1 is a special case of Theorem 4.4. This can be seen as follows: If X is a complete algebraic cone metric space, then $d(x, y) < \infty$ for all $x, y \in X$, and

$$Y = \{y \in X : d(T^{n_0}x, y) < \infty\} = X.$$

It follows from Theorem 4.4 that T has a unique fixed point in $Y = X$.

(b) Since generalized complete metric spaces are generalized complete algebraic cone metric spaces with Banach algebra $\mathbb{R}$, Theorem 1.2 is a special case of Theorem 4.4.

Example 4.1 Let $A = \mathbb{R}^2$. For each$(x_1, x_2) \in A$,

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1).$$

Then A is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0\}$. Let $X = (\mathbb{R} \cup \{\omega\})^2$ and the metric d be defined by

$$d(x,y) = \begin{cases} (|x_1 - y_1|, |x_2 - y_2|), & \text{when } x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2, \\ (0, |x_2 - y_2|), & \text{when } x = (\omega, x_2),\ y = (\omega, y_2),\ x_2,\ y_2 \in \mathbb{R}, \\ (|x_1 - y_1|, 0), & \text{when } x = (x_1, \omega),\ y = (y_1, \omega),\ x_1,\ y_1 \in \mathbb{R}, \\ (0, 0), & \text{when } x = (\omega, \omega),\ y = (\omega, \omega), \\ \infty, & \text{other situations.} \end{cases}$$

Then (X, d) is a generalized complete algebraic cone metric space. Now we define the mapping $T : X \to X$ by

$$Tx = \begin{cases} (\log(2 + |x_1|), \arctan(3 + |x_2|) + \theta x_1), & \text{when } x = (x_1, x_2) \in \mathbb{R}^2, \\ \Big(\log(2 + |x_1|), \dfrac{\pi}{2} + \theta x_1\Big), & \text{when } x = (x_1, \omega),\ x_1 \in \mathbb{R}, \\ (\omega, \omega), & \text{when } x = (\omega, x_2),\ x_2 \in \mathbb{R}, \\ (\omega, \omega), & \text{when } x = (\omega, \omega), \end{cases}$$

where θ is a positive real number. Through calculation, we have

$$d(Tx, Ty) \leq \Big(\frac{1}{2}, \theta\Big) d(x, y)$$

for all $x, y \in X$, and

$$\Big\|\Big(\frac{1}{2}, \theta\Big)^n\Big\|^{\frac{1}{n}} = \Big\|\Big(\Big(\frac{1}{2}\Big)^n, \theta n \Big(\frac{1}{2}\Big)^{n-1}\Big)\Big\|^{\frac{1}{n}} \to \frac{1}{2} < 1.$$

It follows from Theorem 4.4 that T has a unique fixed point in $\mathbb{R}^2$. But T has another fixed point (ω, ω).

Remark 4.2 In Example 4.1, we see that

$$\Big(\frac{1}{2}, \theta\Big) \not\leq (1, 0) = e$$

and

$$\Big\|\Big(\frac{1}{2}, \theta\Big)\Big\| = \frac{1 + 2\theta}{2} > 1$$

for $\theta > 1$.

Acknowledgement The author appreciates the valuable comments made by the anonymous referee.

References

[1] Huang, L. G. and Zhang, X., Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332**, 2007, 1468–1476.

[2] Rezapour, S. and Hamlbarani, R., Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.*, **345**, 2008, 719–724.

[3] Janković, S., Kadelburg, Z. and Radenović, S., On the cone metric space: A survey, *Nonlinear Anal.*, **74**, 2011, 2591–2601.

[4] Abdeljawad, T., Turkoglu, D. and Abuloha, M., Some theorems and examples of cone metric spaces, *J. Comput. Anal. Appl.*, **12**(4), 2010, 739–753.

[5] Karapinar, E., Fixed point theorems in cone Banach spaces, *Fixed Point Theory Appl.*, **2009**, Article ID: 609281.

[6] Turkoglu, D. and Abuloha, M., Cone metric spaces and fixed point theorems in diametrically contractive mappings, *Acta Math. Sin.*, **26**, 2010, 489–496.

[7] Chaker, W., Ghribi, A., Jeribi, A. and Krichen, B., Fixed point theorems for (p, q)-quasi-contraction mappings in cone metric spaces, *Chin. Ann. Math. Ser. B*, **37**(2), 2016, 211–220.

[8] Liu, H. and Xu, S. Y., Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, *Fixed Point Theory Appl.*, 2013, DOI: 10.1186/1687-1812-2013-320.

[9] Xu, S. Y. and Radenović, S., Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, *Fixed Point Theory Appl.*, 2014, DOI: 10.1186/1687-1812-2014-102.

[10] Tootkaboni, M. A., Salec, A. B., Algebraic cone metric spaces and fixed point theorems of contractive mappings, *Fixed Point Theory Appl.*, 2014, DOI: 10.1186/1687-1812-2014-160.

[11] Diaz, J. B. and Margolis, B., A fixed point theorem of the alternative, for contractions on a generalized complete metic space, *Bull. Amer. Math. Soc.*, **74**, 1968, 305–309.

[12] Jung, C. F. K., On generalized complete metric spaces, *Bull. Amer. Math. Soc.*, **75**, 1969, 113–116.