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The Rigidity of Hypersurfaces in Euclidean Space*

Chunhe LI[1] Yanyan XU[1]

Abstract In the present paper, the rigidity of hypersurfaces in Euclidean space is revisited. The Darboux equation is highlighted and two new proofs of the rigidity are given via energy method and maximal principle, respectively.

Keywords Global rigidity, Infinitesimal rigidity, Energy method, Maximal principle
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1 Introduction

The isometric embedding problem is one of the fundamental problems in differential geometry. Since Riemannian manifold was formulated by Riemann in 1868, naturally there arose the question of whether an abstract Riemannian manifold is simply a submanifold of some Euclidean space with its induced metric. In other words, it is the question of reality of Riemannian manifold (see more details in an expository note (cf. [9])).

Mathematically, the isometric embedding problem is to solve the following system. For any given Riemannian manifold $(\mathcal{M}, g)$, there is a surface $\vec{r}: \mathcal{M} \mapsto \mathbb{R}^{n+1}$ such that

$$\mathrm{d}\vec{r} \cdot \mathrm{d}\vec{r} = g, \tag{1.1}$$

where $\cdot$ denotes the Euclidean inner product. In the present paper we assume that $\vec{r}$ is a hypersurface, i.e., $\mathcal{M}$ is a manifold of n dimension.

As is known the uniqueness of solution in PDEs is related to the existence, hence it is another important topic. The counterpart of uniqueness in isometric embedding is global rigidity. The rigidity is to characterize isometric deformation of surfaces which is closely related to the global isometric embedding of surfaces.

Definition 1.1 *An immersed surface $\vec{r}: \mathcal{M} \to \mathbb{R}^3$ is rigid if every immersion $\widetilde{r}: \mathcal{M} \to \mathbb{R}^3$, with the same induced metric, is congruent to $\vec{r}$, that is, differs from $\vec{r}$ by an isometry of $\mathbb{R}^3$.*

If $\vec{r}, \widetilde{r}$ differ from by an isometry of $\mathbb{R}^3$, they are isometric naturally. Global rigidity says that there is no other $\widetilde{r}$ which is isometric to $\vec{r}$ except such trivial $\widetilde{r}$ congruent to $\vec{r}$, hence global rigidity can be viewed as the uniqueness of the solution to isometric embedding problem.

The linearized version of global rigidity is infinitesimal rigidity. We say that $\vec{r}_t$ yields a first order isometric deformation of $\vec{r} = \vec{r}_0$ if the induced metric $g_t = \mathrm{d}\vec{r}_t \cdot \mathrm{d}\vec{r}_t$ has a critical point at

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[1]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China. E-mail: chli@fudan.edu.cn 18483680775@163.com
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$t = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\vec{r}_t \cdot \mathrm{d}\vec{r}_t) = 0 \quad \text{at } t = 0.$$

Set $\vec{\tau} = \frac{\mathrm{d}r_t}{\mathrm{d}t}$ at $t = 0$. Then the infinitesimal problem becomes

$$\mathrm{d}\vec{r} \cdot \mathrm{d}\vec{\tau} = 0. \tag{1.2}$$

As is known, the isometry group of $\mathbb{R}^{n+1}$ is orthogonal group $O(n+1)$ and translation (cf. [11]), namely affine group. Hence the $\vec{\tau} = A\vec{r} + \vec{b}$ generated by its Lie algebra is always the solution to homogeneous linearized equation, where $A \in o(n+1)$ is a skew matrix and $\vec{b}$ is a constant vector. Such $\vec{\tau}$ is called a trivial solution to (1.2). For $n = 2$, it is equivalent to $\vec{\tau} = \vec{a} \times \vec{r} + \vec{b}$ for any constant $\vec{a}$ and $\vec{b}$.

Definition 1.2 *The surface is infinitesimally rigid if* (1.2) *has only trivial solutions.*

In the present paper we will revisit several kinds of rigid surfaces and give new proof which is based on the equivalence of isometric embedding equation (1.1), Gauss-Codazzi equations and Darboux equation.

For the case of $n = 2$, Cohn-Voseen [3] and Blaschke [2] proved the following theorems.

Theorem 1.1 *Let $\mathcal{M}$ be a smooth closed surface with nonnegative curvature and let the vanishing set of the curvature have no interior points. Then $\mathcal{M}$ is globally rigid.*

Theorem 1.2 *Let $\mathcal{M}$ be a smooth closed surface with nonnegative curvature and let the vanishing set of the curvature have no interior points. Then $\mathcal{M}$ is infinitesimally rigid.*

Another rigid surface is Alexandrov's annuli (cf. [1]).

Definition 1.3 *The* 2*-dimensional multiply-connected Riemannian manifold* $(\mathcal{M}, g)$ *satisfies Alexandrov's assumption*:

$$\begin{aligned} &K > 0 \quad \textit{in } \mathcal{M}, \\ &\int_{\mathcal{M}} K \mathrm{d}g = 4\pi, \\ &K = 0, \quad \nabla K \neq 0 \qquad \textit{on } \partial\mathcal{M}. \end{aligned} \tag{1.3}$$

If $\vec{r}$ is the isometric embedding of $(\mathcal{M}, g)$ in $\mathbb{R}^3$, we call $\vec{r}$ Alexandrov's annuli.

The following rigidity theorems are due to Alexandrov [1] and Yau [15], respectively.

Theorem 1.3 *Alexandrov's annuli $\vec{r}$ is globally rigid.*

Theorem 1.4 *Alexandrov's annuli $\vec{r}$ is infinitesimally rigid.*

Ivan Izmestiev [10] proved the infinitesimal rigidity of convex surface in $\mathbb{R}^3$ via the second derivative of the Hilbert-Einstein functional. In [14], Lin and Wang proved the infinitesimal rigidity of convex surface in $\mathbb{H}^3$. Li and Wang [13] reproved Lin-Wang's theorem by Beltrami map. Li, Miao and Wang [12] reproved Lin-Wang's theorem by integral method. For the case of $n \geq 3$, Dajczer-Rodriguez [4] proved the following theorem.

Theorem 1.5 *If the rank of the matrix* (h_{ij}) *is greater than* 2, *where* $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *is the second fundamental form, then the hypersurface is globally and infinitesimally rigid.*

Remark 1.1 Compared with the case of $n = 2$, Dajczer-Rodriguez's theorem is local without any topological restriction on $\mathcal{M}$.

In [7], Guan and Shen proved a rigidity theorem for hypersurfaces in higher dimensional space forms. In [13], Li and Wang showed that if a spherically symmetric $(n+1)$-manifold with metric

$$g = \frac{1}{f^2(r)}\mathrm{d}r^2 + r^2\mathrm{d}S^n, \tag{1.4}$$

the sphere of symmetry $r = c$ is not globally rigid and infinitesimally rigid unless g is a space form.

2 Set up and Formulation

Before discussing the rigidity of Alexandrov's annuli, we need some geometric preliminaries.

We use the geodesic coordinates $(s,t) = (x^1, x^2)$ based on $\partial\mathcal{M}$,

$$\begin{aligned} &g = \mathrm{d}t^2 + B^2\mathrm{d}s^2, \\ &B(s,0) = 1, \quad B_t(s,0) = k_g, \end{aligned}$$

where $B(s,t)$ is a sufficiently smooth function and $B(s,t)$ is periodic in s, and k_g is geodesic curvature.

Under the geodesic coordinates, Alexandrov proved the following lemma (cf. [1] or [9]).

Lemma 2.1 *For Alexandrov's annuli, the coefficients of the second fundamental form of* $\vec{r}$, L, M *and* N *satisfy*: *At* $t = 0$,

$$\begin{aligned} &L = M = 0, \\ &\partial_t L = \sqrt{K_t B_t}, \quad N = \sqrt{\frac{K_t}{B_t}}. \end{aligned} \tag{2.1}$$

Since on $\partial\mathcal{M}$, $\mathrm{d}\vec{n} = 0$ and $k_n = 0$ where $\vec{n}$ and k_n are normal vector and normal curvature, respectively, we have the following lemma.

Lemma 2.2 *The components of boundary* $\vec{r}(\partial\mathcal{M})$ *are some planar curves* $\sigma_k, 1 \le k \le m$, *which are determined completely by their metric, and lie on the plane* π_k *tangential to* $\vec{r}$ *along* σ_k.

At the same time, Dong [5] proved the following lemma.

Lemma 2.3 *If there exists sufficiently smooth isometric embedding*

$$\vec{r} : \mathcal{M} \to R^3, \quad g = \mathrm{d}\vec{r}^2,$$

then we have

$$K_t k_g > 0 \quad \textit{on } \partial\mathcal{M}, \tag{2.2}$$

$$\oint_{\sigma_k} k_g \mathrm{d}s = 2\pi, \tag{2.3}$$

$$\oint_{\sigma_k} \exp\Big(\sqrt{-1}\int_0^s k_g \mathrm{d}\theta\Big)\mathrm{d}s = 0. \tag{2.4}$$

In what follows we will formulate the rigidity.

Let

$$\begin{aligned} \rho &= \frac{1}{2}\vec{r}\cdot\vec{r}, \quad \widetilde{\rho} = \frac{1}{2}\widetilde{r}\cdot\widetilde{r}, \\ \mu &= \vec{r}\cdot\vec{n}, \quad \widetilde{\mu} = \widetilde{r}\cdot\widetilde{n}. \end{aligned} \tag{2.5}$$

We have

$$\vec{r} = g^{ij}\rho_i\vec{r}_j + \mu\vec{n},$$
$$\mu^2 = 2\rho - |\nabla\rho|^2,$$

and

$$h_{ij}\mu = \rho_{i,j} - g_{ij}, \quad \widetilde{h}_{ij}\widetilde{\mu} = \widetilde{\rho}_{i,j} - g_{ij}, \tag{2.6}$$

$$\det(h_{ij}) = \det(\widetilde{h}_{ij}) = K|g|, \tag{2.7}$$

where $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $\widetilde{h} = \widetilde{h}_{ij}\mathrm{d}x^i\mathrm{d}x^j$ are the second fundamental forms, respectively, K is the Gaussian curvature.

Let $W_{ij} = \widetilde{h}_{ij} - h_{ij}$ and $\Phi = \widetilde{\rho} - \rho$. By (2.6)–(2.7) we have

$$(\widetilde{h}_{ij} - W_{ij})\mu = \widetilde{\rho}_{i,j} - \Phi_{i,j} - g_{ij} = \widetilde{h}_{ij}\widetilde{\mu} - \Phi_{i,j}, \tag{2.8}$$

$$(h_{ij} + W_{ij})\widetilde{\mu} = \rho_{i,j} + \Phi_{i,j} - g_{ij} = h_{ij}\mu + \Phi_{i,j}, \tag{2.9}$$

$$\det(\widetilde{h}_{ij} - W_{ij}) = \det(h_{ij} + W_{ij}). \tag{2.10}$$

Taking the difference of (2.8)–(2.9) and the two sides of (2.10) yields

$$W_{ij}(\mu + \widetilde{\mu}) = 2\Phi_{i,j} + (h_{ij} + \widetilde{h}_{ij})(\mu - \widetilde{\mu}), \tag{2.11}$$

$$(h_{11} + \widetilde{h}_{11})w_{22} + (h_{22} + \widetilde{h}_{22})w_{11} - 2(h_{12} + \widetilde{h}_{12})w_{12} = 0. \tag{2.12}$$

Let $\overline{h} = h + \widetilde{h}, \overline{h}_{ij} = h_{ij} + \widetilde{h}_{ij}$, then

$$W_{ij} = \frac{2\Phi_{i,j} + \overline{h}_{ij}(\mu - \widetilde{\mu})}{\mu + \widetilde{\mu}}. \tag{2.13}$$

Gauss-Codazzi equations say

$$\overline{h}^{ij}W_{ij} = 0, \tag{2.14}$$

$$W_{ij,k} = W_{ik,j}, \tag{2.15}$$

where $(\overline{h}^{ij}) = (\overline{h}_{ij})^{-1}$.

There exists an orthogonal mapping which sends the frame $\{r_1, r_2, n\}$ to $\{\widetilde{r}_1, \widetilde{r}_2, \widetilde{n}\}$. Let the associated matrix be A, if h and $\widetilde{h}$ coincide which means A is constant, i.e., $W = W_{ij}\mathrm{d}x^i\mathrm{d}x^j = 0$, $\vec{r}$ and $\widetilde{r}$ differ from an isometry and so it's globally rigid.

For the solution $\vec{\tau}$ to (1.2), let

$$u_i = \vec{n} \cdot \vec{\tau}_i \tag{2.16}$$

and

$$w = \frac{1}{2\sqrt{|g|}}(\vec{r}_2 \cdot \vec{\tau}_1 - \vec{r}_1 \cdot \vec{\tau}_2). \tag{2.17}$$

Note that $u_i\mathrm{d}x^i = \vec{n} \cdot \mathrm{d}\vec{\tau}$ is a globally well-defined 1-form, and w is a well-defined function. Then we have

$$\vec{\tau}_1 = w\sqrt{|g|}g^{2i}\vec{r}_i + u_1\vec{n}, \tag{2.18}$$

$$\vec{\tau}_2 = -w\sqrt{|g|}g^{1i}\vec{r}_i + u_2\vec{n}. \tag{2.19}$$

Then for

$$\vec{Y} = \frac{u_2\vec{r}_1 - u_1\vec{r}_2}{\sqrt{|g|}} + w\vec{n}, \quad \mathrm{d}\vec{\tau} = \vec{Y} \times \mathrm{d}\vec{r}, \tag{2.20}$$

we call $\vec{Y}$ the rotation vector. Differentiating the above equation, we have

$$\mathrm{d}^2\vec{\tau} = \mathrm{d}\vec{Y} \times \mathrm{d}\vec{r} = 0,$$

which implies that $\mathrm{d}\vec{Y}$ is parallel to the tangent plane. Let $\vec{Y}_k = g^{ij}w_{ki}n \times \vec{r}_j,\ k = 1, 2$, where $w_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is a symmetric tensor. $\mathrm{d}^2\vec{Y} = 0$ means

$$h^{ij}w_{ij} = 0, \tag{2.21}$$

$$w_{ij,k} = w_{ik,j}, \tag{2.22}$$

where $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the second fundamental form and $(h^{ij}) = (h_{ij})^{-1}$.

Remark 2.1 We note that $\vec{r}$ is infinitesimally rigid if and only if (2.21)–(2.22) have only trivial solution $w_{ij} = 0$ provided that $\mathcal{M}$ is simply connected. In fact $w_{ij} = 0$ implies that $\vec{Y}$ is a constant.

Let

$$\vec{b} = \vec{\tau} - \vec{Y} \times \vec{r}, \quad \varphi = \vec{b} \cdot \vec{r} = \vec{r} \cdot \vec{\tau}. \tag{2.23}$$

We have

$$\mathrm{d}\vec{b} = -\mathrm{d}\vec{Y} \times \vec{r}, \tag{2.24}$$

$$\vec{b} = g^{ij}\varphi_i\vec{r}_j + \frac{\varphi - g^{ij}\varphi_i\rho_j}{\mu}\vec{n}. \tag{2.25}$$

Combining (2.24)–(2.25), we have

$$\begin{aligned} w_{ij} &= \frac{\varphi_{i,j}}{\mu} + \frac{h_{ij}2(\varphi - \nabla\varphi \cdot \nabla\rho)}{\mu^2} \\ &= \frac{\varphi_{i,j}}{\mu} + \frac{h_{ij}\nu}{\mu^2}. \end{aligned} \tag{2.26}$$

If the support function $\mu \neq 0$, $w_{ij} = 0$ if and only if $\vec{b}$ is constant since $\vec{r}_1 \times \vec{r}, \vec{r}_2 \times \vec{r}$ are linearly independent, i.e., $(\vec{r}_1 \times \vec{r}) \times (\vec{r}_2 \times \vec{r}) = \sqrt{|g|}\vec{r} \cdot \vec{n} = \sqrt{|g|}\mu$. For convex surface, by a translation we can assume the support function $\mu > 0$. Throughout the paper $\mu > 0$ if not specified.

3 The Rigidity of Surfaces in $\mathbb{R}^3$

In this section we will reprove Theorem 1.1, Theorem 1.3 and Theorem 1.2, Theorem 1.4. The main ideas are from an unpublished note (cf. [12]).

To prove Theorem 1.1 and Theorem 1.3, we introduce the following inner product: For any two $(0, 2)$-symmetric tensors $\alpha = \alpha_{ik}\mathrm{d}x^i \otimes \mathrm{d}x^k, \beta = \beta_{jl}\mathrm{d}x^j \otimes \mathrm{d}x^l$,

$$(\alpha, \beta) = \int_{\mathcal{M}} \frac{\det(\overline{h})}{\det(g)} \overline{h}^{ij}\overline{h}^{kl} \alpha_{ik}\beta_{jl}(\mu + \widetilde{\mu})\mathrm{d}V_g. \tag{3.1}$$

Since $\overline{h} = h + \widetilde{h}$ is positive definite, we can view $\overline{h} = \overline{h}_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ as a Riemannian metric defined on $\mathcal{M}$. Then the cotangent bundle is endowed with the metric

$$\langle \mathrm{d}x^i, \mathrm{d}x^j \rangle = \overline{h}^{ij}, \tag{3.2}$$

and the metric induces a metric on the tensor bundle $T^*\mathcal{M} \otimes T^*\mathcal{M}$,

$$\langle \mathrm{d}x^i \otimes \mathrm{d}x^k, \mathrm{d}x^j \otimes \mathrm{d}x^l \rangle = \overline{h}^{ij}\overline{h}^{kl}. \tag{3.3}$$

Note that $\det(\overline{h})(\mu + \widetilde{\mu}) > 0$ on $\mathcal{M}$. The integral defined by (3.1) is an inner product.

In what follows we will show the tensor $W = 0$ by $(W, W) = 0$, where $W = W_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the solution to (2.14)–(2.15), hence prove Theorem 1.1 and Theorem 1.3.

A direct computation shows

$$\begin{aligned}
(W, W) &= \int_{\mathcal{M}} \frac{\det(\overline{h})}{\det(g)} \overline{h}^{ij}\overline{h}^{kl} W_{ik}W_{jl}(\mu + \widetilde{\mu}) \\
&= \int_{\mathcal{M}} \frac{\det(\overline{h})}{\det(g)} \overline{h}^{ij}\overline{h}^{kl} (2\Phi_{i,k} + \overline{h}_{ik}(\mu - \widetilde{\mu}))W_{jl} \\
&= \int_{\mathcal{M}} \frac{\det(\overline{h})}{\det(g)} \overline{h}^{ij}\overline{h}^{kl} 2\Phi_{i,k}W_{jl} \\
&= 2\int_{\partial\mathcal{M}} X \cdot \vec{\nu}\mathrm{d}V_{\partial\mathcal{M}} - 2\int_{\mathcal{M}} \Phi_i\Big(\frac{\det(\overline{h})}{\det(g)} \overline{h}^{ij}\overline{h}^{kl} W_{jl}\Big)_{,k},
\end{aligned} \tag{3.4}$$

where $X = \frac{\det(\overline{h})}{\det(g)} \overline{h}^{ij}\overline{h}^{kl} \varphi_i W_{jl} \frac{\partial}{\partial x^k}$ and $\vec{\nu}$ is outward normal along the $\partial\mathcal{M}$. In the third equality, we use $\overline{h}^{ij}W_{ij} = 0$, and the fourth equality is an application of divergence theorem.

For $i = 1$,

$$\begin{aligned}
&(\det(\overline{h})\overline{h}^{ij}\overline{h}^{kl}W_{jl})_{,k} \\
&= (\overline{A}_{11}\overline{h}^{1l}W_{1l} + \overline{A}_{12}\overline{h}^{1l}W_{2l})_{,1} + (\overline{A}_{11}\overline{h}^{2l}W_{1l} + \overline{A}_{12}\overline{h}^{2l}W_{2l})_{,2} \\
&= (-\overline{A}_{11}\overline{h}^{2l}W_{2l} + \overline{A}_{12}\overline{h}^{1l}W_{2l})_{,1} + (\overline{A}_{11}\overline{h}^{2l}W_{1l} - \overline{A}_{12}\overline{h}^{1l}W_{1l})_{,2} \\
&= (-\overline{h}_{22}\overline{h}^{2l}W_{2l} - \overline{h}_{12}\overline{h}^{1l}W_{2l})_{,1} + (\overline{h}_{22}\overline{h}^{2l}W_{1l} + \overline{h}_{12}\overline{h}^{1l}W_{1l})_{,2} \\
&= -(\delta_2^l W_{2l})_{,1} + (\delta_2^l W_{1l})_{,2} \\
&= W_{21,2} - W_{22,1} \\
&= 0,
\end{aligned} \tag{3.5}$$

where $\overline{A}_{ij} = \det(\overline{h})\overline{h}^{ij}$ is the cofactor of $\overline{h}$. In the second equality and the last equality, we have used $\overline{h}^{ij}W_{ij} = 0, W_{ij,k} = W_{ik,j}$. Similarly, for $i = 2$, we also have

$$(\det(\overline{h})\overline{h}^{ij}\overline{h}^{kl}W_{jl})_{,k} = 0.$$

If $\mathcal{M} = \mathbb{S}^2$, in the integral by parts the boundary term vanishes; if $\mathcal{M}$ is Alexandrov's annuli, on the boundary $W = 0$ by Lemma 2.1 hence the boundary term vanishes too. Both of the two terms in (3.4) vanish, $(W, W) = 0$, $W \equiv 0$.

To prove Theorem 1.2 and Theorem 1.4, we introduce the following inner product: For any two $(0, 2)$-symmetric tensors $\alpha = \alpha_{ik}\mathrm{d}x^i\mathrm{d}x^k, \beta = \beta_{jl}\mathrm{d}x^j\mathrm{d}x^l$,

$$(\alpha, \beta) = \int_{\mathbb{S}^2} \frac{\det(h)}{\det(g)} h^{ij}h^{kl}\alpha_{ik}\beta_{jl}\mu\mathrm{d}V_g.$$

In what follows we will show the tensor $w = 0$ by $(w, w) = 0$, where $w = w_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the solution to (2.21)–(2.22), hence prove Theorem 1.2 and Theorem 1.4.

A direct computation shows

$$\begin{aligned}(w, w) &= \int_{\mathcal{M}} \frac{\det(h)}{\det(g)} h^{ij}h^{kl}w_{ik}w_{jl}\mu \\ &= \int_{\mathcal{M}} \frac{\det(h)}{\det(g)} h^{ij}h^{kl}\Big(\varphi_{i,k} + \frac{h_{ik}\nu}{\mu}\Big)w_{jl} \\ &= \int_{\mathcal{M}} \frac{\det(h)}{\det(g)} h^{ij}h^{kl}\varphi_{i,k}w_{jl} \\ &= \int_{\partial\mathcal{M}} X \cdot \vec{\nu}\mathrm{d}V_{\partial\mathcal{M}} - \int_{\mathcal{M}} \varphi_i\Big(\frac{\det(h)}{\det(g)} h^{ij}h^{kl}w_{jl}\Big)_{,k},\end{aligned} \tag{3.6}$$

where $X = \frac{\det(h)}{\det(g)} h^{ij}h^{kl}\varphi_i w_{jl}\frac{\partial}{\partial x^k}$ and $\vec{\nu}$ is outward normal along the $\partial\mathcal{M}$.

If $\mathcal{M} = \mathbb{S}^2$, a similar argument in (3.5) yields $(w, w) = 0$, $w \equiv 0$.

If $\mathcal{M}$ is Alexandrov's annuli, we have

$$(w, w) = \int_{\partial\mathcal{M}} X \cdot \vec{\nu}\mathrm{d}V_{\partial\mathcal{M}}. \tag{3.7}$$

Note the right-hand side of (3.7) is invariant under coordinate change. So we use geodesic coordinates based on $\partial\mathcal{M}$. Without loss of generality, we merely consider the case that $\mathcal{M}$ is a disk, and then $\partial\mathcal{M}$ is a planar curve denoted by σ. On the boundary, we have $h_{11} = h_{12} = 0$,

$w_{11} = 0$ and μ is constant.

$$
\begin{aligned}
&\int_{\partial\mathcal{M}} X \cdot \vec{\nu} \mathrm{d}V_{\partial\mathcal{M}} \\
&= \int_\sigma \frac{\det(h)}{\det(g)} h^{ij} h^{2l} \varphi_i w_{jl} \mathrm{d}s \\
&= \int_\sigma \frac{\det(h)}{\det(g)} \varphi_1 (h^{11}h^{22}w_{12} + h^{12}h^{21}w_{21} + h^{11}h^{21}w_{11} + h^{12}h^{22}w_{22}) \\
&\quad + \int_\sigma \frac{\det(h)}{\det(g)} \varphi_2 (h^{21}h^{22}w_{12} + h^{22}h^{21}w_{21} + h^{21}h^{21}w_{11} + h^{22}h^{22}w_{22}) \\
&= \int_\sigma \frac{\det(h)}{\det(g)} \varphi_1 (h^{11}h^{22}w_{12} + h^{12}(h^{21}w_{21} + h^{11}w_{11} + h^{22}w_{22})) \\
&= \int_\sigma \frac{\det(h)}{\det(g)} \varphi_1 (h^{11}h^{22}w_{12} - h^{12}h^{12}w_{21}) \\
&= \int_\sigma \varphi_1 w_{21},
\end{aligned} \tag{3.8}
$$

where in the third equality we use the fact $h_{11} = h_{12} = 0$, $w_{11} = 0$ and in the fourth equality we use $h^{ij}w_{ij} = 0$.

In what follows we will show

$$
\frac{1}{\mu} \oint_\sigma \varphi_s F \mathrm{d}s \le 0,
$$

where $F = w_{12}\mu$.

Recall on the boundary σ, $h_{11} = h_{12} = 0$, $w_{11} = 0$ and $\Gamma_{11}^2 = -\Gamma_{12}^1 = k_g, \Gamma_{11}^1 = \Gamma_{12}^2 = 0$. By (2.26), we have on the boundary

$$
\begin{cases}
\varphi_{ss} = k_g \varphi_t, \\
\varphi_{ts} = -k_g \varphi_s + F,
\end{cases} \tag{3.9}
$$

which is nothing else but an ODE of φ_s and φ_t. We can rewrite (3.9) in complex form

$$
\frac{\mathrm{d}}{\mathrm{d}s}(\varphi_s + \sqrt{-1}\varphi_t) + \sqrt{-1}k_g(\varphi_s + \sqrt{-1}\varphi_t) = \sqrt{-1}F.
$$

For convenience, we introduce a new variable $\theta = \int_0^s k_g \in [0, 2\pi]$ and let $c_1 = \varphi_s(0),\ c_2 = \varphi_t(0)$. Then the solution to (3.9) is

$$
\varphi_s(\theta) = -\cos\theta(u(\theta) - c_1) + \sin\theta(v(\theta) + c_2), \tag{3.10}
$$

where $f = \frac{F}{k_g}$ and

$$
u(\theta) = \int_0^\theta f(x) \sin x \mathrm{d}x, \quad v(\theta) = \int_0^\theta f(x) \cos x \mathrm{d}x.
$$

Suppose that the boundary lies on the plane $z = 0$. By the motion of moving frame we have on the boundary

$$
\begin{cases}
\vec{r}_{ss} = k_g r_t, \\
\vec{r}_{ts} = -k_g \vec{r}_s.
\end{cases} \tag{3.11}
$$

It is easy to check

$$\begin{cases} \vec{r}_s(\theta) = (\cos(\theta+\alpha),\, \sin(\theta+\alpha),\, 0), \\ \vec{r}_t(\theta) = (-\sin(\theta+\alpha),\, \cos(\theta+\alpha),\, 0), \end{cases} \tag{3.12}$$

where α is a fixed constant.

In fact (2.4) follows from $\vec{r}_s(2\pi) = \vec{r}_s(0)$. Note that on σ, $\vec{Y}_s = -w_{12}\vec{r}_s$ and μ is constant. By $\int_\sigma \vec{Y}_s = 0$, we have $u(2\pi) = v(2\pi) = 0$.

Since $\oint_\sigma \varphi_s \mathrm{d}s = 0$,

$$\begin{aligned} \oint_\sigma \varphi_s \mathrm{d}s &= \int_0^{2\pi} \varphi_s(\theta)\frac{1}{k_g}\mathrm{d}\theta \\ &= \int_0^{2\pi} (-\cos\theta u(\theta) + \sin\theta v(\theta))\frac{1}{k_g}\mathrm{d}\theta \\ &= 0, \end{aligned} \tag{3.13}$$

where we use (2.3)–(2.4).

Hence

$$\begin{aligned} \oint_\sigma \varphi_s F \mathrm{d}s &= \int_0^{2\pi} (-f\cos\theta(u(\theta)-c_1) + f\sin\theta(v(\theta)+c_2))\mathrm{d}\theta \\ &= \int_0^{2\pi} (-f\cos\theta u(\theta) + f\sin\theta v(\theta))\mathrm{d}\theta \\ &= \int_0^{2\pi} (-v'(\theta)u(\theta) + v(\theta)u'(\theta))\mathrm{d}\theta \\ &= 2\int_0^{2\pi} -v'(\theta)u(\theta)\mathrm{d}\theta. \end{aligned} \tag{3.14}$$

We define a new closed planar curve Γ by parameter equations

$$\begin{cases} x_1(\theta) = \displaystyle\int_0^\theta \frac{\cos x}{k_g(x)}\mathrm{d}x, \\ x_2(\theta) = \displaystyle\int_0^\theta \frac{\sin x}{k_g(x)}\mathrm{d}x. \end{cases} \tag{3.15}$$

A direct computation shows that the curvature of Γ is k_g and the area bounded by the curve is

$$\begin{aligned} S &= -\oint_\Gamma x_2 \mathrm{d}x_1 \\ &= \int_0^{2\pi} \frac{\cos\theta}{k_g(\theta)} \int_0^\theta \frac{\sin x}{k_g(x)}\mathrm{d}x\mathrm{d}\theta \\ &> 0. \end{aligned}$$

And we introduce two new functions

$$\begin{aligned} U(\theta) &= u(\theta) + C\int_0^\theta \frac{\sin x}{k_g(x)}\mathrm{d}x, \\ V(\theta) &= v(\theta) + C\int_0^\theta \frac{\cos x}{k_g(x)}\mathrm{d}x, \end{aligned}$$

where

$$C = -\frac{u(\pi)}{\displaystyle\int_0^\pi \frac{\sin x}{k_g(x)}\mathrm{d}x}.$$

Then we have $U'(\theta)\cot\theta = V'(\theta)$ and $U(0) = U(\pi) = 0$. Therefore

$$\begin{aligned} 2\int_0^{2\pi} -V'(\theta)U(\theta)\mathrm{d}\theta &= 2\int_0^{2\pi} -U(\theta)U'(\theta)\cot\theta\mathrm{d}\theta \\ &= -\int_0^{2\pi} \sec^2\theta U^2(\theta)\mathrm{d}\theta \\ &\le 0 \end{aligned} \tag{3.16}$$

and integral by parts yields

$$\begin{aligned} &\int_0^{2\pi} -V'(\theta)U(\theta)\mathrm{d}\theta \\ &= \int_0^{2\pi} -\Big(v'(\theta) + C\frac{\cos\theta}{k_g(\theta)}\Big)\Big(u(\theta) + C\int_0^\theta \frac{\sin x}{k_g(x)}\mathrm{d}x\Big)\mathrm{d}\theta \\ &= \int_0^{2\pi} -v'(\theta)u(\theta)\mathrm{d}\theta + C\int_0^{2\pi} (v(\theta)\sin\theta - \cos\theta u(\theta))\frac{1}{k_g}\mathrm{d}\theta - C^2\oint_\Gamma x_2\mathrm{d}x_1 \\ &= \int_0^{2\pi} -v'(\theta)u(\theta)\mathrm{d}\theta + C^2 S, \end{aligned} \tag{3.17}$$

where in the third equality we use (3.13).

Combining (3.14)–(3.17), we have

$$\oint_\sigma \varphi_s F\mathrm{d}s \le 0, \tag{3.18}$$

and then

$$0 \le (w,w) \le \int_\sigma \varphi_1 w_{21} = \frac{1}{\mu}\oint_\sigma \varphi_s F\mathrm{d}s \le 0. \tag{3.19}$$

In what follows we give another proof of Theorem 1.2 and Theorem 1.4. The proof is more geometric than above, correspondingly for Theorem 1.4 we restrict that the component number of boundary of Alexandrov's positive annuli is 1 (disk) or 2 (annulus). We need the following lemma.

Lemma 3.1 *For any vector valued $\vec{E} : \mathcal{M} \mapsto \mathbb{R}^3$ satisfying*

$$\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{E} = 0, \tag{3.20}$$

the 1*-form defined on $\mathcal{M}$,*

$$\omega = \mathrm{d}\vec{Y}\cdot\vec{E}$$

is closed.

Proof It is obvious that ω is a 1-form. Exterior differentiation yields

$$\begin{aligned}
\mathrm{d}\omega &= \partial_j(\vec{Y}_k \cdot \vec{E})\mathrm{d}x^j \wedge \mathrm{d}x^k \\
&= (\vec{Y}_{kj} \cdot \vec{E} + \vec{Y}_k \cdot \vec{E}_j)\mathrm{d}x^j \wedge \mathrm{d}x^k \\
&= ((\vec{Y}_{21} - \vec{Y}_{12}) \cdot \vec{E} + (\vec{Y}_2 \cdot \vec{E}_1 - \vec{Y}_1 \cdot \vec{E}_2))\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&= (\vec{Y}_2 \cdot \vec{E}_1 - \vec{Y}_1 \cdot \vec{E}_2)\mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned}$$

By (3.20), we have

$$\begin{cases} \vec{r}_1 \cdot E_1 = 0, \\ \vec{r}_2 \cdot \vec{E}_2 = 0, \\ \vec{r}_1 \cdot \vec{E}_2 + \vec{r}_2 \cdot \vec{E}_1 = 0. \end{cases} \tag{3.21}$$

We can rewrite $\mathrm{d}\vec{Y}$ as

$$\begin{cases} \vec{Y}_1 = \dfrac{1}{\sqrt{\det g}}(-w_{12}\vec{r}_1 + w_{11}\vec{r}_2), \\ \vec{Y}_2 = \dfrac{1}{\sqrt{\det g}}(-w_{22}\vec{r}_1 + w_{21}\vec{r}_2). \end{cases} \tag{3.22}$$

Hence by (3.22) we get

$$\vec{Y}_2 \cdot \vec{E}_1 - \vec{Y}_1 \cdot \vec{E}_2 = \frac{w_{21}}{\sqrt{\det g}}\vec{r}_2 \cdot \vec{E}_1 + \frac{w_{12}}{\sqrt{\det g}}\vec{r}_1 \cdot \vec{E}_2 = 0.$$

ω is a closed 1- form.

Case 1 Let $\mathcal{M}$ be a disk D called Alexandrov's positive disk, $\vec{k}$ be the normal along the boundary σ, and $\vec{i}, \vec{j}$ and $\vec{k}$ form an orthogonal basis. Assume $\vec{r} \cdot \vec{k} = 0$ on the boundary σ, and $\vec{r} \cdot \vec{k} > 0$ at the interior points. We have

$$\vec{E} = \vec{k} \quad \text{or} \quad \vec{E} = \vec{i} \times \vec{r} \quad \text{or} \quad \vec{E} = \vec{j} \times \vec{r}$$

satisfy (3.20). Since $\vec{r} = g^{ij}\rho_i\vec{r}_j + \mu\vec{n} = g^{ij}\rho_i\vec{r}_j \perp \vec{n}$ on σ, and $\vec{i} \perp \vec{n}$, $\vec{j} \perp \vec{n}$, we have that $\vec{E}$ is parallel $\vec{n} = \vec{k}$ and then $\omega = \mathrm{d}\vec{Y} \cdot \vec{E} = 0$ on σ.

For convenience, we write

$$\vec{Y}_k = a_k^l \vec{r}_l,$$

where a_k^l is a $(1,1)$ tensor. The relationship between w_{ij} and a_k^l is released in (3.22). Since the first de Rham cohomology of disk is trivial, i.e., $H^1_{DR}(D) = 0$, there exists some smooth function ψ defined on the disk, such that

$$\omega = \mathrm{d}\psi = \psi_k \mathrm{d}x^k.$$

Hence we have

$$\psi_k = \vec{Y}_k \cdot \vec{E} = a_k^l \vec{r}_l \cdot \vec{E}. \tag{3.23}$$

We will show that ψ is constant hence $\omega = 0$, which is one key step to prove Theorem 1.4.

It is worth pointing out that the following idea is borrowed from [8] which proves the rigidity in prescribed curvature problem.

A simple computation shows

$$\psi_{k,j} = a^l_{k,j}\vec{r}_l \cdot \vec{E} + a^l_k h_{jl}\vec{n} \cdot \vec{E} + a^l_k \vec{r}_l \cdot \vec{E}_j.$$

Then

$$\begin{aligned}
h^{kj}\psi_{k,j} &= h^{kj}a^l_{k,j}\vec{r}_l \cdot \vec{E} + a^k_k\vec{n} \cdot \vec{E} + h^{kj}a^l_k\vec{r}_l \cdot \vec{E}_j \\
&= h^{kj}a^l_{k,j}\vec{r}_l \cdot \vec{E} + \Big(\frac{-w_{12}}{\sqrt{\det g}} + \frac{w_{21}}{\sqrt{\det g}}\Big)\vec{n} \cdot \vec{E} \\
&\quad + h^{1k}a^2_k\vec{r}_2 \cdot \vec{E}_1 + h^{2k}a^1_k\vec{r}_1 \cdot \vec{E}_2 \\
&= h^{kj}a^l_{k,j}\vec{r}_l \cdot \vec{E} + (h^{1k}a^2_k - h^{2k}a^1_k)\vec{r}_2 \cdot \vec{E}_1 \\
&= h^{kj}a^l_{k,j}\vec{r}_l \cdot \vec{E} + \frac{h^{ij}w_{ij}}{\sqrt{\det g}}\vec{r}_2 \cdot \vec{E}_1 \\
&= h^{kj}a^l_{k,j}\vec{r}_l \cdot \vec{E}.
\end{aligned} \tag{3.24}$$

By [8, Lemma 4], we have

$$(a^1_1)^2 + (a^1_2)^2 + (a^2_1)^2 + (a^2_2)^2 \le -C\det(a^j_i).$$

We conclude that

$$h^{kj}\psi_{k,j} = h^{kj}a^l_{k,j}\frac{B^m_l\psi_m}{\det a}.$$

Here B^m_l is the cofactor of a^m_l. We also have for $l = 1$,

$$\begin{aligned}
h^{ij}a^1_{i,j} &= h^{11}a^1_{1,1} + h^{12}a^1_{1,2} + h^{21}a^1_{2,1} + h^{22}a^1_{2,2} \\
&= \frac{1}{\sqrt{\det g}}(-h^{11}w_{12,1} - h^{12}w_{12,2} - h^{21}w_{22,1} - h^{22}w_{22,2}) \\
&= -\frac{1}{\sqrt{\det g}}(h^{11}w_{11,2} + h^{12}w_{12,2} + h^{21}w_{21,2} + h^{22}w_{22,2}) \\
&= -\frac{1}{\sqrt{\det g}}h^{ij}w_{ij,2} = \frac{1}{\sqrt{\det g}}h^{ij}_{,2}w_{ij}.
\end{aligned}$$

Similarly, we have

$$h^{ij}a^2_{i,j} = \frac{1}{\sqrt{\det g}}h^{ij}_{,1}w_{ij}.$$

Hopf's strong maximum principle (cf. [6, §3.2, Theorem 3.5]) tells us that ψ is a constant function on the disk since on the boundary ψ is a constant, hence

$$\omega = \mathrm{d}\psi = 0.$$

Let

$$S = \{x \mid x \in \overline{D},\ \vec{n} \cdot \vec{k} = \pm 1 \text{ or } \vec{r} \cdot \vec{k} = 0\}.$$

We have in $D \setminus S$, at least one of the following mixed products is nonzero:

$$\begin{cases} (\vec{i} \times \vec{r}, \vec{k}, \vec{n}) = -(\vec{r} \cdot \vec{k})(\vec{n} \cdot \vec{i}), \\ (\vec{j} \times \vec{r}, \vec{k}, \vec{n}) = -(\vec{r} \cdot \vec{k})(\vec{n} \cdot \vec{j}). \end{cases} \tag{3.25}$$

Recall that

$$\vec{E} = \vec{k} \quad \text{or} \quad \vec{E} = \vec{i} \times \vec{r} \quad \text{or} \quad \vec{E} = \vec{j} \times \vec{r},$$

since $\omega = \mathrm{d}\vec{Y} \cdot \vec{E}$ and $\mathrm{d}\vec{Y} \cdot \vec{n} = 0$, $\mathrm{d}\vec{Y} = 0$ in $D \setminus S$. Note that S is zero measured, by the continuity $\mathrm{d}\vec{Y} = 0$ in D.

Case 2 $\mathcal{M}$ is Alexandrov's positive annulus. Lemma 2.2 says that the boundary consists of two planar curves. We will discuss two different cases, respectively. Subcase 2.1: The two boundary planes are parallel; Subcase 2.2: The two boundary planes are not parallel.

Different from Case 1, we need some extra topology preliminary.

Lemma 3.2 *If $\vec{E} = \vec{a} \times \vec{r} + \vec{b}$ for any constant $\vec{a}$ and $\vec{b}$ which is the trivial solution to* (1.2), *we have for any component of boundary $\sigma_k, 1 \le k \le m$,*

$$\oint_{\sigma_k} \omega = \oint_{\sigma_k} \mathrm{d}\vec{Y} \cdot \vec{E} = 0, \tag{3.26}$$

hence there exists some smooth function ψ defined on the $\mathcal{M}$, such that

$$\omega = \mathrm{d}\psi.$$

Proof Integral by parts yields

$$\begin{aligned}\oint_{\sigma_k} \mathrm{d}\vec{Y} \cdot \vec{E} &= \oint_{\sigma_k} \mathrm{d}\vec{Y} \cdot (\vec{a} \times r + \vec{b}) \\ &= \vec{a} \cdot \oint_{\sigma_k} \vec{Y} \times \mathrm{d}\vec{r} + \vec{b} \cdot \oint_{\sigma_k} \mathrm{d}\vec{Y} \\ &= \vec{a} \cdot \oint_{\sigma_k} \mathrm{d}\vec{r} + \vec{b} \cdot \oint_{\sigma_k} \mathrm{d}\vec{Y} \\ &= 0,\end{aligned} \tag{3.27}$$

where we use (2.20).

For Subcase 2.1, let $\vec{k}$ be a unit vector in $\mathbb{R}^3$ which is parallel to the normals of the two boundary planes and choose $\vec{E} = \vec{k}$. Then $\omega = \mathrm{d}\vec{Y} \cdot \vec{k} = 0$ on $\partial\mathcal{M}$. In particular the normal derivative $\frac{\partial\psi}{\partial\vec{\nu}} = 0$. Similar to Case 1, by maximum principle on Neumann problem (cf. [6, §3.2, Theorem 3.6]), ψ is constant. Hence $\mathrm{d}\psi = \vec{Y}_i \cdot \vec{k}\mathrm{d}x^i = 0$,

$$\begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \cdot \vec{k} \\ \vec{r}_2 \cdot \vec{k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.28}$$

Note that on $\mathcal{M}$ at least one of $\vec{r}_1 \cdot \vec{k}, \vec{r}_2 \cdot \vec{k}$ is not zero, otherwise $\vec{k}$ is parallel to some normal on $\mathcal{M}$, but as a convex surface, its Gauss map is one-to-one and any normal on $\mathcal{M}$ differs from the normals on $\partial\mathcal{M}$ therefore is not parallel to $\vec{k}$. Hence the coefficient determinant $\det(a_j^i) = \frac{\det(w_{ij})}{\det(g)} = 0$, i.e., $\det(w_{ij}) = \det(w) = 0$. (2.21) says $\mathrm{tr}_h(w_{ij}) = \mathrm{tr}(h^{-1}w) = 0$, in addition $\det(h^{-1}w) = \frac{\det(w)}{\det(h)} = 0$, then $h^{-1}w = 0$ and $w = 0$ because h and w are symmetric, i.e., $\mathrm{d}\vec{Y} = 0$.

For Subcase 2.2, let the constant normals on σ_1, σ_2 be $\vec{n}(\sigma_1), \vec{n}(\sigma_2)$, and the constant support functions on σ_1, σ_2 be $\mu(\sigma_1), \mu(\sigma_2)$, respectively. We choose $\vec{E}$ as

$$\vec{E} = (\vec{n}(\sigma_1) \times \vec{n}(\sigma_2)) \times (\vec{r} + c_1\vec{n}(\sigma_1) + c_2\vec{n}(\sigma_2)), \tag{3.29}$$

where c_1, c_2 solves

$$\begin{pmatrix} 1 & \vec{n}(\sigma_1)\cdot\vec{n}(\sigma_2) \\ \vec{n}(\sigma_1)\cdot\vec{n}(\sigma_2) & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} \mu(\sigma_1) \\ \mu(\sigma_2) \end{pmatrix}. \tag{3.30}$$

Since $\vec{n}(\sigma_1), \vec{n}(\sigma_2)$ are not parallel, the coefficient matrix in algebraic equation (3.30) is invertible and thereby (3.30) is solvable.

Note that $\vec{r} = g^{ij}\rho_i\vec{r}_j + \mu\vec{n}$, it is easy to check that on $\partial\mathcal{M} = \bigcup\limits_{k=1}^{2}\sigma_k$, such $\vec{E}$ is parallel to normal. Then $\omega = \mathrm{d}\vec{Y}\cdot\vec{E} = 0$ on $\partial\mathcal{M}$.

Similar to Subcase 2.1, if at least one of $\vec{r}_1\cdot\vec{E}, \vec{r}_2\cdot\vec{E}$ is not zero, the tensor $w = w_{ij}\mathrm{d}x^i\mathrm{d}x^j = 0$. We will see the set

$$S_p := \{p\in\mathcal{M},\ \vec{r}_1\cdot\vec{E} = 0,\ \vec{r}_2\cdot\vec{E} = 0\} \tag{3.31}$$

is of zero measure. Then $w = 0$ everywhere on $\mathcal{M}$ by the continuity.

Let $\vec{X} = \vec{r} + c_1\vec{n}(\sigma_1) + c_2\vec{n}(\sigma_2)$, and define

$$\varphi_{\mathcal{M}}(p) = \vec{n}\cdot\vec{X}, \quad p\in\mathcal{M}. \tag{3.32}$$

We have that S_p is contained in the level set $\{p\in\mathcal{M},\ \varphi_{\mathcal{M}}(p) = 0\}$ since $\vec{n}$ is parallel to $\vec{E} = (\vec{n}(\sigma_1)\times\vec{n}(\sigma_2))\times\vec{X}$ on S_p. We will check on the level set, $\nabla\varphi_{\mathcal{M}} \neq 0$ if $\vec{X}\neq 0$.

$$\partial_i\varphi_{\mathcal{M}} = \vec{r}_i\cdot\vec{n} + \vec{X}\cdot\partial_i\vec{n} = -\vec{X}\cdot h_i^l\vec{r}_l. \tag{3.33}$$

Let $\vec{X} = a^j\vec{r}_j$ since on the level set $\vec{n}\cdot\vec{X} = 0$, if $\nabla\varphi_{\mathcal{M}} = 0$, we have

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix}\begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.34}$$

hence $a^j = 0$ and $\vec{X} = 0$. $\vec{r}$ is regular surface and the translation $\vec{X}$ is regular too, then the $\{p\in\mathcal{M}, \vec{X}(p) = 0\}$ is finite. The level set $\{p\in\mathcal{M},\ \varphi_{\mathcal{M}}(p) = 0\}$ is zero measured. As a subset of $\{p\in\mathcal{M}, \varphi_{\mathcal{M}}(p) = 0\}$, S_p is also zero measured.

Remark 3.1 If $\mathcal{M} = \mathbb{S}^2$, i.e., the case of closed convex surface, we choose

$$\vec{E} = \vec{k} \quad \text{or} \quad \vec{E} = \vec{i} \quad \text{or} \quad \vec{E} = \vec{j}.$$

Similar but simpler argument yields $\mathrm{d}\vec{Y} = 0$. Thus we complete the proof of Theorem 1.2.

As we have seen, the new proofs we give highlight the roles that the function ρ defined in (2.5) and its linearized version φ defined in (2.23) play. In fact we can extract all information from ρ which satisfies Darboux equation in isometric embedding problem as we work on the support function in Minkowski problem.

4 The Rigidity of Hypersurfaces in $\mathbb{R}^{n+1}, n\geq 3$

Similarly in the case of higher dimension, for the equation (1.2) we can assume that

$$\mathrm{d}\vec{\tau} = \vec{Y}\times\mathrm{d}\vec{r}$$

for some vector $\vec{Y} \in G_r(n-1,n+1) \cong G_r(2,n+1)$, where $G_r(r,n+1)$ is Grassmannian.

Let

$$\mathrm{d}\vec{r} = \vec{r}_j \mathrm{d}x^j, \quad \mathrm{d}\vec{Y} = \vec{Y}_i \mathrm{d}x^i, \quad 1 \le i, j \le n$$

and

$$\vec{Y}_i = W_i^{\alpha\beta} e_\alpha \wedge e_\beta, \quad 1 \le \alpha, \beta \le n+1,$$

where $\vec{r}_{n+1}$ is the normal vector, and the basis $e_\alpha \wedge e_\beta$ in $G_r(2,n+1)$ is defined by

$$e_\alpha \wedge e_\beta = \frac{1}{(n-1)!} \delta^{12\cdots(n-1)n(n+1)}_{k_1 k_2 \cdots k_{n-1} \alpha\beta} \vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}}, \tag{4.1}$$

where δ is generalized Kronecker symbol. Obviously $e_\alpha \wedge e_\beta = -e_\beta \wedge e_\alpha$, we set

$$W_i^{\alpha\beta} = -W_i^{\beta\alpha}. \tag{4.2}$$

By

$$\mathrm{d}\vec{Y} \wedge \mathrm{d}\vec{r} = 0,$$

we have

$$W_i^{\alpha\beta} e_\alpha \wedge e_\beta \wedge \vec{r}_j \mathrm{d}x^i \wedge \mathrm{d}x^j = 0, \tag{4.3}$$

i.e.,

$$\frac{1}{(n-1)!} W_i^{\alpha\beta} \delta^{12\cdots(n-1)n(n+1)}_{k_1 k_2 \cdots k_{n-1} \alpha\beta} \vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \wedge \vec{r}_j \mathrm{d}x^i \wedge \mathrm{d}x^j = 0. \tag{4.4}$$

Define a basis $E_\gamma, 1 \le \gamma \le n+1$ in $G_r(n,n+1) \cong G_r(1,n+1)$ by

$$\vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \wedge \vec{r}_j = \delta^{12\cdots(n-1)n(n+1)}_{k_1 k_2 \cdots k_{n-1} j\gamma} E_\gamma, \tag{4.5}$$

hence

$$\begin{aligned} &\frac{1}{(n-1)!} W_i^{\alpha\beta} \delta^{12\cdots(n-1)n(n+1)}_{k_1 k_2 \cdots k_{n-1} \alpha\beta} \delta^{12\cdots(n-1)n(n+1)}_{k_1 k_2 \cdots k_{n-1} j\gamma} E_\gamma \mathrm{d}x^i \wedge \mathrm{d}x^j \\ =& \frac{1}{(n-1)!} W_i^{\alpha\beta} \delta^{j\gamma}_{\alpha\beta} E_\gamma \mathrm{d}x^i \wedge \mathrm{d}x^j \\ =& 0, \end{aligned} \tag{4.6}$$

i.e., for fixed i, j and γ,

$$W_i^{\alpha\beta} \delta^{j\gamma}_{\alpha\beta} - W_j^{\alpha\beta} \delta^{i\gamma}_{\alpha\beta} = 0, \tag{4.7}$$

hence

$$W_i^{j\gamma} = W_j^{i\gamma}. \tag{4.8}$$

We claim the following lemma.

Lemma 4.1 *If* $1 \le i, j, \gamma \le n$, *then*

$$W_i^{j\gamma} = 0. \tag{4.9}$$

For the left-hand side of (4.9), by (4.2) and (4.8),

$$W_i^{j\gamma} = -W_i^{\gamma j} = -W_\gamma^{ij} = W_\gamma^{ji}. \tag{4.10}$$

And on the other hand, for the right-hand side of (4.9), by (2.2) and (4.8)

$$W_j^{i\gamma} = -W_j^{\gamma i} = -W_\gamma^{ji}, \tag{4.11}$$

so

$$W_i^{j\gamma} = -W_j^{i\gamma}.$$

Hence we can rewrite

$$\vec{Y}_i = 2W_i^{l(n+1)} e_l \wedge e_{n+1}. \tag{4.12}$$

At the same time note that for fixed i, j,

$$\begin{aligned}
\vec{Y}_i \wedge \vec{r}_j &= 2W_i^{l(n+1)} \frac{1}{(n-1)!} \delta^{12\cdots(n-1)n(n+1)}_{k_1k_2\cdots k_{n-1}l(n+1)} \vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \wedge \vec{r}_j \\
&= 2W_i^{j(n+1)} \frac{1}{(n-1)!} \delta^{12\cdots(n-1)n(n+1)}_{k_1k_2\cdots k_{n-1}j(n+1)} \vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \wedge \vec{r}_j \\
&= 2W_i^{j(n+1)} \sqrt{|g|}\vec{r}_{n+1},
\end{aligned} \tag{4.13}$$

hence letting $w_{ij} = 2W_i^{j(n+1)}\sqrt{|g|}$, the quadratic form $w_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is globally well-defined.

We rewrite (4.12) as

$$\vec{Y}_i = w_{il} \frac{1}{\sqrt{|g|}} e_l \wedge e_{n+1}. \tag{4.14}$$

In what follows we will compute the covariant derivative of $e_l \wedge e_{n+1}$.

At first we notice that

$$e_l \wedge e_{n+1} = \sum_{k_1<k_2<\cdots<k_{n-1}}^{k_1,k_2,\cdots,k_{n-1}\neq l,n+1} \delta^{12\cdots(n-1)n(n+1)}_{k_1k_2\cdots k_{n-1}l(n+1)} \vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}}, \tag{4.15}$$

therefore

$$\begin{aligned}
(e_l \wedge e_{n+1})_j = &\sum_{k_1<k_2<\cdots<k_{n-1}}^{k_1,k_2,\cdots,k_{n-1}\neq l,n+1} \delta^{12\cdots(n-1)n(n+1)}_{k_1k_2\cdots k_{n-1}l(n+1)} (h_{k_1,j}\vec{r}_{n+1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \\
&+ h_{k_2,j}\vec{r}_{k_1} \wedge \vec{r}_{n+1} \wedge \cdots \wedge \vec{r}_{k_{n-1}} + \cdots + h_{k_{n-1},j}\vec{r}_{k_1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{n+1}).
\end{aligned} \tag{4.16}$$

Since

$$\begin{aligned}
&\delta^{12\cdots(n-1)n(n+1)}_{k_1k_2\cdots k_{n-1}l(n+1)} \vec{r}_{n+1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \\
&= -\delta^{12\cdots(n-1)n(n+1)}_{(n+1)k_2\cdots k_{n-1}lk_1} \vec{r}_{n+1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \\
&= -\delta^{12\cdots(n-1)n(n+1)}_{k_2\cdots k_{n-1}(n+1)lk_1} \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} \wedge \vec{r}_{n+1},
\end{aligned}$$

and for $k_2 < k_3 < \cdots < k_{n-1} < n+1,\ k_2, k_3, k_{n-1}, n+1 \neq l, k_1$, we have

$$\delta^{12\cdots(n-1)n(n+1)}_{k_1k_2\cdots k_{n-1}l(n+1)} \vec{r}_{n+1} \wedge \vec{r}_{k_2} \wedge \cdots \wedge \vec{r}_{k_{n-1}} = -e_l \wedge e_{k_1}. \tag{4.17}$$

Similarly we have

$$(e_l \wedge e_{n+1})_j = \sum_{k \neq l, n+1} h_{kj} e_k \wedge e_l. \tag{4.18}$$

Thus

$$\begin{cases} \vec{Y}_{i,j} = \dfrac{1}{\sqrt{|g|}}\Big(w_{il,j} e_l \wedge e_{n+1} + w_{il} \displaystyle\sum_{k \neq l, n+1} h_{kj} e_k \wedge e_l\Big), \\ \vec{Y}_{j,i} = \dfrac{1}{\sqrt{|g|}}\Big(w_{jl,i} e_l \wedge e_{n+1} + w_{jl} \displaystyle\sum_{k \neq l, n+1} h_{ki} e_k \wedge e_l\Big). \end{cases} \tag{4.19}$$

By compatibility $\vec{Y}_{i,j} = \vec{Y}_{j,i}$, we have

$$w_{il,j} = w_{jl,i}, \tag{4.20}$$

$$h_{kj} w_{il} - h_{lj} w_{ik} = h_{ki} w_{jl} - h_{li} w_{jk}. \tag{4.21}$$

Remark 4.1 (4.20) shows that w_{ij} is Codazzi. In fact, (4.20)–(4.21) is a homogeneous linearized Gauss-Codazzi system.

Similar to the case of $n = 2$,

$$h^{ij} \vec{Y}_{i,j} = \frac{h^{ij}}{\sqrt{|g|}} w_{il,j} e_l \wedge e_{n+1}. \tag{4.22}$$

Hence for hypersurface in $\mathbb{R}^{n+1}$, we can use maximal principle to get the infinitesimal rigidity. But we can make use of (4.21) to reprove Theorem 1.5.

Proof of Theorem 1.5 We want to show $w_{ij} = 0$. In view that $w_{ij} \mathrm{d}x^i \mathrm{d}x^j$ is invariant under variable transformation, we consider the diagonal case, i.e., $h_{ij} = 0, i \neq j$, since at any point on the hypersurface we can diagonalize the matrix (h_{ij}) by variable transformation.

If the rank of the matrix (h_{ij}) is greater than 2, without loss of generality we can assume $h_{11}, h_{22}, h_{33} \neq 0$. By (4.21),

$$\begin{cases} h_{11} w_{22} + h_{22} w_{11} = h_{12} w_{21} + h_{21} w_{12}, \\ h_{11} w_{33} + h_{33} w_{11} = h_{13} w_{31} + h_{31} w_{13}, \\ h_{22} w_{33} + h_{33} w_{22} = h_{23} w_{32} + h_{32} w_{23}. \end{cases} \tag{4.23}$$

Since $h_{ij} = 0,\ i \neq j$, (4.23) is just a linear system of w_{11}, w_{22}, w_{33},

$$\begin{pmatrix} h_{22} & h_{11} & 0 \\ h_{33} & 0 & h_{11} \\ 0 & h_{33} & h_{22} \end{pmatrix} \begin{pmatrix} w_{11} \\ w_{22} \\ w_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.24}$$

The coefficient matrix in (4.24) is invertible, hence $w_{11} = w_{22} = w_{33}$. For other w_{ij}, by (4.21),

$$h_{11} w_{ij} + h_{ij} w_{11} = h_{1i} w_{j1} + h_{1j} w_{i1}, \tag{4.25}$$

since $i \neq 1,\ j \neq 1$ and $w_{11} = 0$, $h_{11} w_{ij} = 0$.

As for the part of global rigidity, without loss of generality we assume that the block $H_3 = (h_{ij})_{3\times 3}$ is of full rank, then its adjoint matrix H_3^* is of full rank too. By Gauss equation, every element in H_3^* is an entry of Riemannian curvature tensor which is totally determined by metric. Therefore H_3^* is intrinsic and we can recover H_3 from H_3^*. H^3 is intrinsic too, and as we proceed in the part of infinitesimal rigidity the $H = (h_{ij})_{n\times n}$ is intrinsic too.

In the proof of Theorem 1.5, we just deal with the algebraic equations, Gauss equations or its linearized equations, so we can say Theorem 1.5 is algebraic.

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