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Noncommutative Constrained KP Hierarchy and Multi-component Noncommutative Constrained KP Hierarchy*

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Abstract In this paper, the authors define the noncommutative constrained Kadomtsev-Petviashvili (KP) hierarchy and multi-component noncommutative constrained KP hierarchy. Then they give the recursion operators for the noncommutative constrained KP (NcKP) hierarchy and multi-component noncommutative constrained KP (NmcKP) hierarchy. The authors hope these studies might be useful in the study of D-brane dynamics whose noncommutative coordinates emerge from limits of the M theory and string theory.

Keywords Recursion operator, Noncommutative constrained KP hierarchy, Multi-component noncommutative constrained KP hierarchy
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1 Introduction

It is well known that possessing infinite number of symmetries is a common property of the classical integrable systems. There are many results on concrete forms of symmetries (see [1–3]). The recursion operator is one kind of effective tools to generate symmetries of integrable systems (see [4–5]). On the other hand, the recursion operator is also used to establish the Hamiltonian structure of integrable systems (see [1, 6–7]) and integrable flows of curves (see [8]). So it is vital to construct the recursion operator for integrable systems. The noncommutative theory gives rise to various new physics such as the canonical commutation relation $[q, p] = \mathrm{i}\hbar$ in quantum mechanics which leads to the so-called space-space uncertainty relation. In the context of the effective theory of D-branes, the noncommutative gauge theories are found to be equivalent to ordinary gauge theories in the presence of background magnetic fields. Noncommutative solitons play important roles in the study of D-brane dynamics in which noncommutative coordinates are known to emerge from limits of M theory and string theory as shown in [9].

The KP hierarchy (see [10–11]) is one of the most important integrable systems in mathematics and physics as Toda system. However, there are many noncommutative equations

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deserving to be studied, such as the noncommutative constrained KP (NcKP) hierarchy, the noncommutative multi-component KP (NmKP) hierarchy, the multi-component noncommutative constrained KP hierarchy defined in this paper and so on. They have extensive effect on systems in the noncommutative space. In the papers [7, 12–13], several different methods were used to construct recursion operators. Furthermore, it was highly non-trivial to reduce some results from the noncommutative KP hierarchy (NcKP) to the noncommutative constrained KP hierarchy and the multi-component noncommutative constrained KP hierarchy. In the papers [14–15], recursion operators for the commutative 1-constrained BKP and CKP hierarchies were given. In addition, recursion operators for the noncommutative KP hierarchy were given in [16], and recursion operator for the multi-component constrained KP (mcKP) hierarchy was researched. The purpose of this paper is to give recursion operators of the noncommutative constrained KP hierarchy and the multi-component noncommutative constrained KP hierarchy.

The organization of this paper is as follows. We recall some basic facts for the noncommutative KP systems and constraints on the basis constrained KP (cKP) hierarchy in Section 2. In Section 3, the recursion operator for the noncommutative constrained KP hierarchy is discussed and used to generate the flow equations, which are consistent with results given by eigenfunction equations of this sub-hierarchy. Based on some basic knowledge of the multi-component KP (mKP) hierarchy and the multi-component constrained KP (mcKP) hierarchy, we give the Lax equation of the multi-component noncommutative constrained KP hierarchy and its constraints in Section 4. In Section 5, the recursion operator for the two-component noncommutative constrained KP hierarchy is discussed.

2 The Noncommutative KP Systems and Constraints

As we all know, the noncommutative KP hierarchy is one of the most important topics in the area of classical integrable systems (see [17]). In the noncommutative system, $\star$ is defined by

$$f(x)\star g(x)=\exp\Big(\frac{\mathrm{i}}{2}\theta^{uv}\partial_{a^u}\partial_{b^v}\Big)f(a)g(b)\mid_{a=b=x}=f(x)g(x)+\frac{\mathrm{i}}{2}\theta^{uv}\partial_u f(x)\partial_v g(x)+\vartheta(\theta^2),$$

where $\vartheta(\theta^2)$ means the higher order terms of θ. We also get that $[x^u,x^v]_\star=x^u\star x^v-x^v\star x^u=\mathrm{i}\theta^{uv}$, and when $\theta^{uv}\to 0$, the noncommutative system can be reduced to the commutative ones. The noncommutative KP hierarchy is constructed by the pseudo-differential Lax operator $L=\partial+u_2\partial^{-1}+u_3\partial^{-2}+\cdots$ as

$$L_{t_n}=[B_n,L]_\star:=B_n\star L-L\star B_n,$$

where $B_n=(L^n)_+$ and "+" means the nonnegative projection on powers of ∂. So we can get the noncommutative KP equation by the t_2 (denoted by y) flows and t_3 (denoted by t) flows (see [16, 18]):

$$(4u_t-u_{xxx}-6u\star u_x-6u_x\star u)_x-3u_{yy}+6[u_x,u_y]_\star=0, \tag{2.1}$$

where $u = u_2$. To calculate the noncommutative quasi-differential operators, we need the following definition.

Definition 2.1 *If the operator A is a differential operator and has form $A := \sum\limits_{n=0}^{\infty} \partial^n a_n$, then we define $A^* \star g(x) = \sum\limits_{m=0}^{\infty} (-1)^m (\partial^m g(x)) \star a_m$.*

The eigenfunction q and conjugate eigenfunction r of noncommutative KP hierarchy are defined by

$$q_{t_m} = B_m \star q, \quad r_{t_m} = -B_m^* \star r. \tag{2.2}$$

However, we should use a formal adjoint operation $*$ for an arbitrary pseudo-differential operator $P = \sum\limits_i p_i \star \partial^i$, $P^* = \sum\limits_i (-1)^i \partial^i p_i$ to define the noncommutative KP hierarchy, such as, $\partial^* = -\partial$, $(\partial^{-1})^* = -\partial^{-1}$, and $(A \star B)^* = B^* \star A^*$ for two operators A, B.

What is more, in view of symmetry constraint, the so called "noncommutative constrained KP hierarchy" (NcKP) is a very interesting sub-hierarchy, and the Lax operator for noncommutative 1-constrained KP is given by

$$L = \partial + \sum_{i=1}^{n} q_i \star \partial^{-1} r_i, \tag{2.3}$$

where q_i (r_i) is the eigenfunction(adjoint eigenfunction) of L in (2.3). In the following context, we take $n = 1$ for simplicity, i.e.,

$$L = \partial + q \star \partial^{-1} r. \tag{2.4}$$

Note that q and r satisfy the eigenfunction equations (2.2) associated with L in (2.4). The 1-constrained KP hierarchy whose evolution equations are as follows:

$$\frac{\partial L}{\partial t_n} = [B_n, L]_\star, \quad n = 1, 2, \cdots. \tag{2.5}$$

In order to get the explicit form of the flow equations, we need B_n,

$$\begin{aligned}
B_1 &= \partial, \\
B_2 &= \partial^2 + 2(q \star r), \\
B_3 &= \partial^3 + 3q \star r \star \partial + 3q_x \star r, \\
&\ \vdots
\end{aligned}$$

After computing from eigenfunction equations (2.2) directly, we can get the first few flows of the noncommutative cKP hierarchy

$$\begin{cases} q_{t_1} = q_x, \\ r_{t_1} = r_x, \end{cases} \tag{2.6}$$

$$\begin{cases} q_{t_2} = q_{xx} + 2(q \star r \star q), \\ r_{t_2} = -r_{xx} - 2(r \star q \star r), \end{cases} \tag{2.7}$$

$$\begin{cases} q_{t_3} = q_{xxx} + 3q_x \star r \star q + 3q \star r \star q_x + q \star r_x \star q + r \star q_x \star q, \\ r_{t_3} = r_{xxx} + 3r_x \star q \star r + 3r \star q \star r_x, \end{cases} \tag{2.8}$$

$$\vdots$$

It is very difficult to observe recursion operator from equations on t_1 flows, t_2 flows and t_3 flows above. We shall find it in next section from eigenfunction equations on q and r, and may use it to generate any higher order flows. To illustrate the validity of recursion operator, we can use it to generate t_4 flows from trivial flows, i.e., t_3 flows, and further generate t_5 flows from t_4 flows which will be omitted here.

3 The Recursion Operator for a Noncommutative Constrained KP Hierarchy

In this section, the form of recursion operator R will be given, where R_q denotes to be right multiplied by q.

In order to calculate recursion operator conveniently, we give the following lemma.

Lemma 3.1 *The following four noncommutative identities hold true*:

$$(B_n f \star \partial^{-1} g)_- = B_n(f) \star \partial^{-1} g, \tag{3.1}$$

$$(f \star \partial^{-1} g \star B_n)_- = f \star \partial^{-1} B_n^*(g), \tag{3.2}$$

$$f_1 \star \partial^{-1} g_1 \star f_2 \star \partial^{-1} g_2 = f_1 \star \Big(\int g_1 \star f_2 \Big) \star \partial^{-1} g_2 - f_1 \star \partial^{-1} \Big(\int g_1 \star f_2 \Big) \star g_2. \tag{3.3}$$

Now, we define the following four operators:

$$\begin{aligned} \mathbf{R}_{11} &= \mathcal{L} + R_q \star \partial^{-1} R_r, \\ \mathbf{R}_{12} &= q \star \partial^{-1} R_q + R_q \star \partial^{-1} q, \\ \mathbf{R}_{21} &= R_r \star \partial^{-1} R_r, \\ \mathbf{R}_{22} &= \mathcal{L} + q \star \partial^{-1} R_r + R_r \star \partial^{-1} q. \end{aligned}$$

Theorem 3.1 *The recursion relation of flows for the noncommutative* 1*-cKP hierarchy* (2.5) *is as follows*:

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_{m+1}} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix}_{t_m}. \tag{3.4}$$

Proof Denote A_n as $(\mathcal{L}^n)_-$, $n = 1, 2, \cdots$. By considering the noncommutative KP condition and (2.2), q and r should satisfy the same equation, i.e.,

$$B_m(q) = q_{t_m}, \quad B_m^*(r) = -r_{t_m}, \tag{3.5}$$

then

$$q_{t_{m+1}} = (\mathcal{L} \star \mathcal{L}^m)_+ \star q = B_1 \star B_m \star q + (B_1 \star A_m)_+ \star q + (A_1 \star B_m)_+ \star q, \tag{3.6}$$

$$r_{t_{m+1}} = (\mathcal{L} \star \mathcal{L}^m)_+ \star r = B_1 \star B_m \star r + (B_1 \star A_m)_+ \star r + (A_1 \star B_m)_+ \star r. \tag{3.7}$$

Let us firstly calculate $(B_1 \star A_m)_+$. We set $A_m = \partial^{-1} a_1 + \partial^{-2} a_2 + \cdots$. So $(B_1 \star A_m)_+ = a_1$. The identity $\mathrm{Res}_\partial[\mathcal{L}^m, \mathcal{L}]_\star = 0$ yields

$$\mathrm{Res}_\partial[B_m, \mathcal{L}]_\star = \mathrm{Res}_\partial[-A_m, \mathcal{L}]_\star = \mathrm{Res}_\partial[-A_m, B_1]_\star. \tag{3.8}$$

The first residue of (3.8) equals $\mathrm{Res}_\partial \mathcal{L}_{t_m} = q_{t_m} \star r + q \star r_{t_m}$ and the last residue of (3.8) yields $\mathrm{Res}_\partial[\partial, \partial^{-1} a_1 + \partial^{-2} a_2 + \cdots]_\star = a_{1x}$. So a_1 can be expressed as

$$a_1 = \int q_{t_m} \star r + q \star r_{t_m}. \tag{3.9}$$

Hence, we can directly calculate $(B_1 \star A_m)_+$ as

$$(B_1 \star A_m)_+ = \int q_{t_m} \star r + q \star r_{t_m}.$$

Considering the term $(A_1 \star B_m)_+$, we write it as $A_1 \star B_m - (A_1 \star B_m)_-$. The first term is relevant to t_m flow. By using the Lemma 3.1, we can compute the second term

$$(A_1 \star B_m)_- = (q\partial^{-1} r \star B_m)_- = q\partial^{-1} B_m^*(r) = -q \star \partial^{-1} r_{t_m}.$$

After bringing these results into (3.6), we get the recursion flow of q,

$$q_{t_{m+1}} = [\mathcal{L} + R_q \star \partial^{-1} R_r] \star q_{t_m} + [q \star \partial^{-1} R_q + R_q \star \partial^{-1} q] \star r_{t_m}.$$

Similarly after bringing these results into (3.7), we get the recursion flow of r,

$$r_{t_{m+1}} = [R_r \star \partial^{-1} R_r] \star q_{t_m} + [\mathcal{L} + q \star \partial^{-1} R_r + R_r \star \partial^{-1} q] \star r_{t_m}.$$

Therefore, we get the recursion operator written in (3.4).

Let us inspect whether the results from this recursion operator are consistent with what from the eigenfunction (2.2). After tedious calculating, we can show that they are consistent which contains the t_2 flows and t_3 flows. Of course we can generate any higher order flows.

4 Lax Equations of the Multi-component Noncommutative Constrained KP Hierarchy

The multi-component noncommutative constrained KP hierarchy can be defined by the following Lax operator

$$L = \partial + \sum_{i=1}^{n} q_i \star \partial^{-1} r_i, \tag{4.1}$$

where q_i and r_i are $N \times N$ matrix functions take values in noncommutative coordinate space. In the following context, we take $n = 1$ for simplicity. We consider the Lax operator of the multi-component commutative constrained KP hierarchy as

$$L = \partial + q \star \partial^{-1} r. \tag{4.2}$$

In order to get $q_{t_n^j}$ and $r_{t_n^j}$, we need the following definition.

Definition 4.1 *If the matrix operator B is a differential operator and has form $B := \sum_{n=0}^{\infty} \partial^n a_n$, then we define $B^* \star g(x) = \sum_{m=0}^{\infty} (-1)^m (\partial^m g(x)) \star a_m$.*

The eigenfunction q and the adjoint eigenfunction r of the multi-component noncommutative constrained KP hierarchy satisfy the following Sato equations

$$q_{t_n^j} = (B_n^j)_+ \star q, \quad r_{t_n^j} = -(B_n^j)_+^* \star r, \tag{4.3}$$

where B_n^j will be defined later. For an arbitrary matrix-valued pseudo-differential operator $P = \Sigma p_i \partial^i$, we can denote $*$ as a formal adjoint operation which is defined by $P^* = \Sigma(-1)^i \partial^i p_i^{\mathrm{T}}$, and we have $(f \star g)^* = g^* \star f^*$ for two operators f and g. Here, we list some identities, which will be used in the following sections: $A^* = A^{\mathrm{T}}$, $(A \star B)^* = B^{\mathrm{T}} \star A^{\mathrm{T}}$, $(A \star \partial \star B)^* = -B^{\mathrm{T}} \star \partial \star A^{\mathrm{T}}$, where A and B are $N \times N$ matrix functions. We can rewrite the operator L in a dressing form as

$$L = P \star \partial \star P^{-1}, \tag{4.4}$$

where

$$P = E + \sum_{i \geq 1} p_i \star \partial^{-i}. \tag{4.5}$$

So the equations of the multi-component noncommutative constrained KP hierarchy is

$$\partial_{t_n^j} L = [(B_n^j)_+, L]_\star, \quad n = 1, 2, 3, \cdots, \tag{4.6}$$

where

$$B_n^j = L^n \star C_j, C_j = P \star E_{jj} \star P^{-1}. \tag{4.7}$$

In the following part, we give the Lax equations of the two-component noncommutative constrained KP hierarchy. The Lax operator has form

$$L = \partial + q \star \partial^{-1} \star r, \tag{4.8}$$

where q and r are 2×2 matrix functions. The Lax equations of the two-component noncommutative constrained KP hierarchy are defined by

$$\partial_{t_n^j} L = [(B_n^j)_+, L]_\star, \quad n = 1, 2, 3, \cdots, \ j = 1, 2, \tag{4.9}$$

$$B_n^j = L^n \star C_j, \tag{4.10}$$

and we give the following constraint on C_1 and C_2,

$$C_1 = E_{11} + \alpha \star \partial^{-1} \star \beta, \quad C_2 = E_{22} - \alpha \star \partial^{-1} \star \beta, \tag{4.11}$$

where α and β are 2×2 matrix functions. We have $C_1 + C_2 = E$. The eigenfunction α and the adjoint eigenfunction β are defined respectively by

$$\frac{\partial \alpha}{\partial t_n^j} = (B_n^j)_+ \star \alpha, \quad \frac{\partial \beta}{\partial t_n^j} = -(B_n^j)_+^* \star \beta, \quad j = 1, 2. \tag{4.12}$$

5 Recursion Operator for Two-component Noncommutative Constrained KP Hierarchy

In this section, we shall give the recursion operator for the two-component noncommutative constrained KP hierarchy.

Theorem 5.1 *The recursion operators of* t_m^j $(j = 1, 2)$ *flows for* (4.9) *are as follows*:

$$q_{t_{m+1}^j} = A_{j1} \star q_{t_m^j} + A_{j2} \star r_{t_m^j} + A_{j3} \star \alpha_{t_m^j} + A_{j4} \star \beta_{t_m^j}, \tag{5.1}$$
$$r_{t_{m+1}^j} = B_{j1} \star q_{t_m^j} + B_{j2} \star r_{t_m^j} + B_{j3} \star \alpha_{t_m^j} + B_{j4} \star \beta_{t_m^j}, \tag{5.2}$$
$$\alpha_{t_{m+1}^j} = C_{j1} \star q_{t_m^j} + C_{j2} \star r_{t_m^j} + C_{j3} \star \alpha_{t_m^j} + C_{j4} \star \beta_{t_m^j}, \tag{5.3}$$
$$\beta_{t_{m+1}^j} = D_{j1} \star q_{t_m^j} + D_{j2} \star r_{t_m^j} + D_{j3} \star \alpha_{t_m^j} + D_{j4} \star \beta_{t_m^j}. \tag{5.4}$$

The specific form of A_{jn}, B_{jn}, C_{jn}, D_{jn} $(j = 1, 2,\ n = 1, 2, 3, 4)$ *will be shown in the following context.*

Proof The equations of the multi-component noncommutative constrained KP hierarchy are defined by

$$\partial_{t_n^j} L = [(B_n^j)_+, L]_\star, \quad n = 1, 2, 3, \cdots, \tag{5.5}$$

where

$$B_n^j = L^n \star C_j, C_j = P \star E_{jj} \star P^{-1}. \tag{5.6}$$

So we have

$$B_1^j = L \star C_j, C_j = E_{jj} + (-1)^{j-1} \alpha \star \partial^{-1} \star \beta,$$

therefore

$$\begin{aligned} B_1^j = {} & E_{jj}\partial + (-1)^{j-1}\alpha \star \beta + q\partial^{-1}\Big[r \star E_{jj} - (-1)^{j-1}\Big(\int r \star \alpha\Big) \star \beta\Big] \\ & + (-1)^{j-1}\Big[\alpha_x + q \star \Big(\int r \star \alpha\Big)\Big]\partial^{-1}\beta. \end{aligned}$$

Denote A_n^j as $(B_n^j)_-$, $n=1,2,\cdots$. By considering (4.3) and (4.12), we have the following results:

$$(B_m^j)_+^*(q)=-q_{t_m^j},\quad (B_m^j)_+^*(r)=-r_{t_m^j},\quad (B_m^j)_+^*(\alpha)=-\alpha_{t_m^j},\quad (B_m^j)_+^*(\beta)=-\beta_{t_m^j}. \tag{5.7}$$

Then we have

$$\begin{aligned} q_{t_{m+1}^j} &= (B_1^j\star B_m^j)_+\star q \\ &= (B_1^j)_+\star(B_m^j)_+\star q+((B_1^j)_+\star A_m^j)_+\star q+(A_1^j\star(B_m^j)_+)_+\star q, \end{aligned}\tag{5.8}$$

$$\begin{aligned} r_{t_{m+1}^j} &= (B_1^j\star B_m^j)_+\star r \\ &= (B_1^j)_+\star(B_m^j)_+\star r+((B_1^j)_+\star A_m^j)_+\star r+(A_1^j\star(B_m^j)_+)_+\star r, \end{aligned}\tag{5.9}$$

$$\begin{aligned} \alpha_{t_{m+1}^j} &= (B_1^j\star B_m^j)_+\star \alpha \\ &= (B_1^j)_+\star(B_m^j)_+\star \alpha+((B_1^j)_+\star A_m^j)_+\star \alpha+(A_1^j\star(B_m^j)_+)_+\star \alpha, \end{aligned}\tag{5.10}$$

$$\begin{aligned} \beta_{t_{m+1}^j} &= (B_1^j\star B_m^j)_+\star \beta \\ &= (B_1^j)_+\star(B_m^j)_+\star \beta+((B_1^j)_+\star A_m^j)_+\star \beta+(A_1^j\star(B_m^j)_+)_+\star \beta. \end{aligned}\tag{5.11}$$

Firstly, we shall calculate $((B_1^j)_+\star A_m^j)_+$. Set $A_m^j=\partial^{-1}a_1+\partial^{-2}a_2+\cdots$. Then we have $((B_1^j)_+\star A_m^j)_+=E_{jj}\star a_1$. The identity $\mathrm{Res}_\partial[B_m^j,B_1^j]_\star=0$ yields

$$\mathrm{Res}_\partial[(B_m^j)_+,B_1^1+B_1^2]_\star=\mathrm{Res}_\partial[-A_m^j,(B_1^1+B_1^2)_+]_\star=\mathrm{Res}_\partial[(B_1^1+B_1^2)_+,A_m^j]_\star. \tag{5.12}$$

The first residue of (5.12) equals

$$\mathrm{Res}_\partial L_{t_m^j}=(q\star r)_{t_m^j},$$

and the last residue of (5.12) yields

$$\mathrm{Res}_\partial[\partial,\partial^{-1}a_1+\partial^{-2}a_2+\cdots]_\star=a_{1x}.$$

So we get

$$a_1=\int(q\star r)_{t_m^j}. \tag{5.13}$$

Hence

$$((B_1^j)_+\star A_m^j)_+=E_{jj}\star\int(q\star r)_{t_m^j}. \tag{5.14}$$

Next, we consider the term $(A_1^j\star(B_m^j)_+)_+$. We write it as $A_1^j\star(B_m^j)_+-(A_1^j\star(B_m^j)_+)_-$. Using the identity (3.1), we can compute the second term

$$\begin{aligned}(A_1^j\star(B_m^j)_+)_- = \Big[&q\partial^{-1}\Big(r\star E_{jj}-(-1)^{j-1}\Big(\int r\star\alpha\Big)\star\beta\Big) \\ &+(-1)^{j-1}\Big(\alpha_x+q\star\Big(\int r\star\alpha\Big)\Big)\partial^{-1}\beta\star(B_m^j)_+\Big]_-\end{aligned}$$

$$= q\partial^{-1}(B_m^j)_+^* \star C_j^*(r) - (-1)^{j-1}\Big(\alpha_x + q\star\Big(\int r\star\alpha\Big)\Big)\partial^{-1}\beta_{t_m^j}.$$

Hence, we should calculate the following term

$$\begin{aligned}
(B_m^j)_+^* \star C_j^*(r) &= C_j^* \star (B_m^j)_+^*(r) + [(B_m^j)_+^*, C_j^*]_\star(r) \\
&= -C_j^*(r_{t_m^j}) - (C_j^*)_{t_m^j}(r) \\
&= -r_{t_m^j} \star E_{jj} + (-1)^{j-1}\Big(\int r_{t_m^j}\star\alpha\Big)\star\beta + (-1)^{j-1}\Big(\int r\star\alpha\Big)\star\beta_{t_m^j} \\
&\quad + (-1)^{j-1}\Big(\int r\star\alpha_{t_m^j}\Big)\star\beta.
\end{aligned}$$

Therefore, we can get

$$\begin{aligned}
(A_1^j \star (B_m^j)_+)_- &= -q\partial^{-1}r_{t_m^j}\star E_{jj} + (-1)^{j-1}q\partial^{-1}\Big(\int r_{t_m^j}\star\alpha\Big)\star\beta \\
&\quad + (-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta_{t_m^j} + (-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha_{t_m^j}\Big)\star\beta \\
&\quad - (-1)^{j-1}\alpha_x\partial^{-1}\beta_{t_m^j} - (-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta_{t_m^j}.
\end{aligned}$$

After bringing these results into (5.8), we get the recursion flow of q as follows:

$$\begin{aligned}
q_{t_{m+1}^j} &= (B_1^j)_+\star(B_m^j)_+\star q + ((B_1^j)_+\star A_m^j)_+\star q + (A_1^j\star(B_m^j)_+)_+\star q \\
&= B_1^j\star q_{t_m^j} + ((B_1^j)_+\star A_m^j)_+\star q - (A_1^j\star(B_m^j)_+)_-\star q \\
&= \Big[E_{jj}\partial + (-1)^{j-1}\alpha\star\beta + q\partial^{-1}r\star E_{jj} - (-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta \\
&\quad + (-1)^{j-1}\alpha_x\partial^{-1}\beta + (-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta + E_{jj}\star R_q\partial^{-1}R_r\Big]\star q_{t_m^j} \\
&\quad + [E_{jj}\star R_q\partial^{-1}q + q\partial^{-1}R_q\star R_{E_{jj}} - (-1)^{j-1}q\partial^{-1}R_q\star R_\beta\partial^{-1}R_\alpha]\star r_{t_m^j} \\
&\quad + [-(-1)^{j-1}q\partial^{-1}R_q\star R_\beta\partial^{-1}r]\star\alpha_{t_m^j} + \Big[-(-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\partial^{-1}R_q\Big]\star\beta_{t_m^j}.
\end{aligned}$$

Similarly, we can get the other recursion flows:

$$\begin{aligned}
r_{t_{m+1}^j} &= (B_1^j)_+\star(B_m^j)_+\star r + ((B_1^j)_+\star A_m^j)_+\star r + (A_1^j\star(B_m^j)_+)_+\star r \\
&= B_1^j\star r_{t_m^j} + ((B_1^j)_+\star A_m^j)_+\star r - (A_1^j\star(B_m^j)_+)_-\star r \\
&= E_{jj}\star R_r\partial^{-1}R_r\star q_{t_m^j} + \Big[E_{jj}\partial + (-1)^{j-1}\alpha\star\beta + q\partial^{-1}r\star E_{jj} \\
&\quad - (-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta + (-1)^{j-1}\alpha_x\partial^{-1}\beta + (-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta \\
&\quad + E_{jj}\star R_r\partial^{-1}q + q\partial^{-1}R_r\star R_{E_{jj}} - (-1)^{j-1}q\partial^{-1}R_r\star R_\beta\partial^{-1}R_\alpha\Big]\star r_{t_m^j} \\
&\quad + [-(-1)^{j-1}q\partial^{-1}R_r\star R_\beta\partial^{-1}r]\star\alpha_{t_m^j} + \Big[-(-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star R_r \\
&\quad + (-1)^{j-1}\alpha_x\partial^{-1}R_r + (-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}R_r\Big]\star\beta_{t_m^j}, \\
\alpha_{t_{m+1}^j} &= (B_1^j)_+\star(B_m^j)_+\star\alpha + ((B_1^j)_+\star A_m^j)_+\star\alpha + (A_1^j\star(B_m^j)_+)_+\star\alpha
\end{aligned}$$

$$
\begin{aligned}
&= B_1^j \star \alpha_{t_m^j} + ((B_1^j)_+ \star A_m^j)_+ \star \alpha - (A_1^j \star (B_m^j)_+)_- \star \alpha \\
&= E_{jj} \star R_\alpha \partial^{-1} R_r \star q_{t_m^j} + [E_{jj} \star R_\alpha \partial^{-1} q + q\partial^{-1} R_\alpha \star R_{E_{jj}} \\
&\quad - (-1)^{j-1} q\partial^{-1} R_\alpha \star R_\beta \partial^{-1} R_\alpha] \star r_{t_m^j} + \Big[E_{jj}\partial + (-1)^{j-1}\alpha \star \beta + q\partial^{-1} r \star E_{jj} \\
&\quad - (-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta + (-1)^{j-1}\alpha_x\partial^{-1}\beta + (-1)^{j-1} q \star \Big(\int r\star\alpha\Big)\partial^{-1}\beta \\
&\quad - (-1)^{j-1} q\partial^{-1} R_\alpha \star R_\beta \partial^{-1} r\Big] \star \alpha_{t_m^j} + \Big[-(-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star R_\alpha \\
&\quad + (-1)^{j-1}\alpha_x\partial^{-1} R_\alpha + (-1)^{j-1} q\star\Big(\int r\star\alpha\Big)\partial^{-1} R_\alpha\Big]\star\beta_{t_m^j},
\end{aligned}
$$

$$
\begin{aligned}
\beta_{t_{m+1}^j} &= (B_1^j)_+ \star (B_m^j)_+ \star \beta + ((B_1^j)_+ \star A_m^j)_+ \star \beta + (A_1^j \star (B_m^j)_+)_+ \star \beta \\
&= B_1^j \star \beta_{t_m^j} + ((B_1^j)_+ \star A_m^j)_+ \star \beta - (A_1^j \star (B_m^j)_+)_- \star \beta \\
&= E_{jj} \star R_\beta \partial^{-1} R_r \star q_{t_m^j} + [E_{jj} \star R_\beta \partial^{-1} q + q\partial^{-1} R_\beta \star R_{E_{jj}} \\
&\quad - (-1)^{j-1} q\partial^{-1} R_\beta \star R_\beta \partial^{-1} R_\alpha] \star r_{t_m^j} + [-(-1)^{j-1} q\partial^{-1} R_\beta \star R_\beta \partial^{-1} r] \star \alpha_{t_m^j} \\
&\quad + \Big[E_{jj}\partial + (-1)^{j-1}\alpha\star\beta + q\partial^{-1} r\star E_{jj} - (-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta \\
&\quad + (-1)^{j-1}\alpha_x\partial^{-1}\beta + (-1)^{j-1} q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta - (-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star R_\beta \\
&\quad + (-1)^{j-1}\alpha_x\partial^{-1} R_\beta + (-1)^{j-1} q\star\Big(\int r\star\alpha\Big)\partial^{-1} R_\beta\Big]\star\beta_{t_m^j}.
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
A_{j1} &= E_{jj}\partial + (-1)^{j-1}\alpha\star\beta + q\partial^{-1} r\star E_{jj} - (-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta + (-1)^{j-1}\alpha_x\partial^{-1}\beta \\
&\quad + (-1)^{j-1} q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta + E_{jj}\star R_q\partial^{-1} R_r, \\
A_{j2} &= E_{jj}\star R_q\partial^{-1} q + q\partial^{-1} R_q\star R_{E_{jj}} - (-1)^{j-1} q\partial^{-1} R_q\star R_\beta\partial^{-1} R_\alpha, \\
A_{j3} &= -(-1)^{j-1} q\partial^{-1} R_q\star R_\beta\partial^{-1} r, \\
A_{j4} &= -(-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\partial^{-1} R_q, \\
B_{j1} &= E_{jj}\star R_r\partial^{-1} R_r, \\
B_{j2} &= E_{jj}\partial + (-1)^{j-1}\alpha\star\beta + q\partial^{-1} r\star E_{jj} - (-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta \\
&\quad + (-1)^{j-1}\alpha_x\partial^{-1}\beta + (-1)^{j-1} q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta + E_{jj}\star R_r\partial^{-1} q \\
&\quad + q\partial^{-1} R_r\star R_{E_{jj}} - (-1)^{j-1} q\partial^{-1} R_r\star R_\beta\partial^{-1} R_\alpha, \\
B_{j3} &= -(-1)^{j-1} q\partial^{-1} R_r\star R_\beta\partial^{-1} r, \\
B_{j4} &= -(-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star R_r + (-1)^{j-1}\alpha_x\partial^{-1} R_r + (-1)^{j-1} q\star\Big(\int r\star\alpha\Big)\partial^{-1} R_r, \\
C_{j1} &= E_{jj}\star R_\alpha\partial^{-1} R_r, \\
C_{j2} &= E_{jj}\star R_\alpha\partial^{-1} q + q\partial^{-1} R_\alpha\star R_{E_{jj}} - (-1)^{j-1} q\partial^{-1} R_\alpha\star R_\beta\partial^{-1} R_\alpha, \\
C_{j3} &= E_{jj}\partial + (-1)^{j-1}\alpha\star\beta + q\partial^{-1} r\star E_{jj} - (-1)^{j-1} q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta
\end{aligned}
$$

$$+(-1)^{j-1}\alpha_x\partial^{-1}\beta+(-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta-(-1)^{j-1}q\partial^{-1}R_\alpha\star R_\beta\partial^{-1}r,$$

$$C_{j4}=-(-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star R_\alpha+(-1)^{j-1}\alpha_x\partial^{-1}R_\alpha+(-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}R_\alpha,$$

$$D_{j1}=E_{jj}\star R_\beta\partial^{-1}R_r,$$

$$D_{j2}=E_{jj}\star R_\beta\partial^{-1}q+q\partial^{-1}R_\beta\star R_{E_{jj}}-(-1)^{j-1}q\partial^{-1}R_\beta\star R_\beta\partial^{-1}R_\alpha,$$

$$D_{j3}=-(-1)^{j-1}q\partial^{-1}R_\beta\star R_\beta\partial^{-1}r,$$

$$\begin{aligned}D_{j4}={}&E_{jj}\partial+(-1)^{j-1}\alpha\star\beta+q\partial^{-1}r\star E_{jj}-(-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star\beta+(-1)^{j-1}\alpha_x\partial^{-1}\beta\\&+(-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}\beta-(-1)^{j-1}q\partial^{-1}\Big(\int r\star\alpha\Big)\star R_\beta+(-1)^{j-1}\alpha_x\partial^{-1}R_\beta\\&+(-1)^{j-1}q\star\Big(\int r\star\alpha\Big)\partial^{-1}R_\beta.\end{aligned}$$

6 Conclusions and Discussions

The recursion operators in (3.4) for the noncommutative cKP system are found from the eigenfunction equations on q and r. These operators are used to generate t_2 flows (see (2.8)) from the t_1 flows of this special hierarchy, which are consistent with flows from eigenfunction (3.5). The validity of the recursion operators are demonstrated. Of course one can also use it to generate higher order flows. Moreover, the Lax equation of multi-component noncommutative constrained KP hierarchy is discussed and we construct recursion operators for two-component noncommutative constrained hierarchy. Moreover, we can also get the following reduction chain:

$$\text{NmKP hierarchy}\xrightarrow{L=\partial+q\star\partial^{-1}r}\text{NmcKP hierarchy}\xrightarrow{N=1}\text{NcKP hierarchy}.\tag{6.1}$$

Also, we can get the following reduction chain after taking $\theta=0$:

$$\text{mKP hierarchy}\xrightarrow{L=\partial+q\partial^{-1}r}\text{mcKP hierarchy}\xrightarrow{N=1}\text{cKP hierarchy}.\tag{6.2}$$

In our future works, we shall try to discuss the recursion operators for the multi-component noncommutative constrained BKP and CKP hierarchies which might be useful in multi-orthogonal polynomials in a noncommutative space.

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References

[1] Fokas, A. S. and Santini, P. M., The recursion operator of the Kadomtsev-Petviashvili equation and the squared eigenfunctions of the Schrödinger operator, *Stud. Appl. Math.*, **75**, 1986, 179–185.

[2] Fokas, A. S. and Santini, P. M., Recursion operators and bi-Hamiltonian structures in multidimensions, II, *Commun. Math. Phys.*, **116**, 1988, 449–474.

[3] Matsukidaira, J., Satsuma, J. and Strampp, W., Conserved quantities and symmetries of KP hierarchy, *J. Math. Phys.*, **31**, 1990, 1426–1434.

[4] Fuchssteiner, B. and Fokas, A. S., Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Phys. D.*, **4**, 1981, 47–66.

[5] Olver, P. J., Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986.

[6] Santini, P. M. and Fokas, A. S., Recursion operators and bi-Hamiltonian structures in multidimensions, I, *Commun. Math. Phys.*, **115**, 1988, 375–419.

[7] Strampp, W. and Oevel, W., Recursion operators and Hamitonian structures in Sato theory, *Lett. Math. Phys.*, **20**, 1990, 195–210.

[8] Stephen, C. A., Bi-Hamiltonian operators, integrable flows of curves using moving frames and geometric map equations, *J. Phys. A: Math. Gen.*, **39**, 2006, 2043–2072.

[9] Douglas, M. R. and Nekrasov, N. A., Noncommutative field theory, *Rev. Mod. Phys.*, **73**, 2002, p.977.

[10] Dickey, L. A., Soliton Equations and Hamiltonian Systems, 2nd ed., World Scintific, Singapore, 2003.

[11] Date, E., Kashiwara, M., Jimbo, M., et al., Transformation groups for soliton equations, Nonlinear Integrable Systems-Classical and Quantum Theory, Jimbo, M. and Miwa, T. (eds.), World Scientific, Singapore, 1983, 39–119.

[12] Cheng, Y., Constraints of the Kadomtsev-Petviashvili hierarchy, *J. Math. Phys.*, **33**, 1992, 3774–3782.

[13] Gurses, M., Karasu, A. and Sokolov, V. V., On construction of recursion operators from Lax representation, *J. Math. Phys.*, **40**, 1999, 6473–6490.

[14] Loris, I., Recursion operator for a constrained BKP system, Proceedings of the Workshop on Nonlinearity, Integrability and All That Twenty years after NEEDS'79, Boiti, M., Martina, L., Pempinelli, F., et al., (eds.), World Scientific, Singapore, 1999, 325–330.

[15] Li, C. Z., Tian, K. L., He, J. S. and Cheng, Y., Recursion operator for a constrained CKP hierarchy, *Acta Mathematica Scientia, Ser. B*, **31**, 2011, 1295–1302.

[16] He, J. S., Tu, J. Y., Li, X. D. and Wang, L. H., Explicit flow equations and recursion operator of the ncKP hierarchy, *Nonlinearity*, **24**, 2011, 2875–2890.

[17] Hamanaka, M., Noncommutative integrable systems and quasidetermiants, Nonlinear and Modern Mathematical Physics, AIP Conf. Proc., 1212, Amer. Inst. Phys., Melville, NY, 2010, 122–135.

[18] Sooman, G. M., Soliton Solutions of Noncommutative Integrable Systems, University of Glasgow, 2009.