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# Ricci Positive Metrics on the Moment-Angle Manifolds*

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**Abstract** In this paper, the authors consider the problem of which (generalized) moment-angle manifolds admit Ricci positive metrics. For a simple polytope $P$, the authors can cut off one vertex $v$ of $P$ to get another simple polytope $P_v$, and prove that if the generalized moment-angle manifold corresponding to $P$ admits a Ricci positive metric, the generalized moment-angle manifold corresponding to $P_v$ also admits a Ricci positive metric. For a special class of polytope called Fano polytopes, the authors prove that the moment-angle manifolds corresponding to Fano polytopes admit Ricci positive metrics. Finally some conjectures on this problem are given.

**Keywords** Moment-Angle manifolds, Simple polytope, Cutting off face, Positive Ricci curvature, Fano polytope
**2000 MR Subject Classification** 22E46, 53C30

## 1 Introduction

The moment-angle manifold $Z$ comes from two different ways:

(1) The transverse intersections in $\mathbb{C}^n$ of real quadrics of the form $\sum\limits_{i=1}^{n} a_i|z_i|^2 = 0$ with the unit Euclidean sphere of $\mathbb{C}^n$.

(2) An abstract construction from a simple polytope $P^n$ with $m$ facets.

The study of the first one led to the discovery of a new special class of compact non-kähler complex manifolds in the work of López, Verjovsky and Meersseman [10–12], now known as the LV-M manifolds, which helps us understand the topology of non-kähler complex manifolds.

The study of the second one is related to the quasitoric manifolds in the following way: For every quasitoric manifold $\pi : M^{2n} \to P^n$ there is a principal $T^{m-n}$-bundle $Z \to M^{2n}$ whose composite map with $\pi$ makes $Z$ a $T^m$-manifold with orbit space $P^n$. The topology of the manifolds $Z$ provides an effective tool for understanding inter-relations between algebraic and combinatorial aspects such as the Stanley-Reisner rings, the subspace arrangements and the cubical complexes, etc.

Nowadays, studies of moment-angle manifolds are mainly focused on the following two aspects:

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(1) The cohomology of moment-angle manifolds and topology of some special moment-angle manifolds.

(2) The geometry of moment-angle manifolds in its convex, complex-analytic, symplectic and Lagrangian aspects.

In this paper, we pay attention to the Riemannian metric property of moment-angle manifolds in aspect of Ricci curvature. In Riemannian geometry, one of the most important themes is to study the relationship between the curvature and globally topological or geometrical property of Riemannian manifolds. In three types of curvature, scalar curvature has the weakest relationship with the geometrical property of the manifolds, but it is the best understood one according to the work of Gromov-Lawson [8–9] and Schoen-Yau [13–14]. Sectional curvature has the closest relationship with the topological and geometrical property. In some sense, sectional curvature controls nearly all aspects of Riemannian geometry. In order to get some geometric properties of manifolds, usually we should give some restriction of sectional curvature. Besides, one of the most important problems in geometry is the classification of the Riemannian manifolds with sectional positive (or non-negative) metrics and sectional negative (or non-positive) metrics, which as known is far from totally understanding. As a second order symmetric tensor, Ricci curvature is closely related to many elliptic, parabolic and nonlinear differential equations in geometry. Ricci curvature also plays an important role in the general relativity theory in physics and the existence problem of Ricci positive metric (or Einstein metric, Kähler-Einstein metric) on a given manifold is also important. However, we have few methods to determine whether a given manifold can admit a Ricci positive metric. Until now, we have not known many examples of manifolds with positive Ricci curvature (Biquotients, connected sums of $S^{n_i} \times S^{m_i}$ (see [15, 17]), Fano varieties, some principal $G$ bundles on Ricci positive manifolds (see [6], etc).

For the moment-angle manifolds, the existence of scalar positive metric can be easily proved (see [16]). As far as we know, [1] is the only paper concerned with the Ricci positive metrics on moment-angle manifolds. The authors constructed Riemannian metrics of positive Ricci curvature on 3 special moment-angle manifolds. In this paper, we also study the problem of which moment-angle manifolds admit Ricci positive metrics. We prove the following two theorems.

**Theorem 1.1** *Let $P$ be a simple polytope, $P_v$ be a simple polytope which is obtained by cutting off one vertex $v$ on $P$. If the generalized moment-angle manifold corresponding to $P$ admits a Ricci positive metric, then the generalized moment-angle manifold corresponding to $P_v$ also admits a Ricci positive metric.*

**Theorem 1.2** *If $P$ is a Fano polytope, then the moment-angle manifold corresponding to $P$ admits a Ricci positive metric.*

In the next two sections, we respectively prove these two theorems and finally give two conjectures concerned with the existence of positive Ricci curvature on moment-angle manifolds.

## 2 Cutting off One Vertex on a Simple Polytope

**Definition 2.1** *A convex polytope $P$ is the convex hull of a finite set of points in some*

*$\mathbb{R}^n$. 0-dimensional faces are called vertices, codimension one faces are called facets. If there are exactly $n$ facets meeting at each vertex of $n$-dimensional convex polytope $P^n$, this polytope are called simple.*

Given a simple polytope $P^n$, let $\mathcal{F} = \{F_1, \cdots, F_m\}$ be the set of facets of $P$. For each facet $F_i \in \mathcal{F}$, we use $T_{F_i}$ to denote the 1-dimensional coordinate subgroup of $T^{\mathcal{F}} \cong T^m$ corresponding to $F_i$. Then assign to every face $G$ the coordinate subtorus $T_G = \prod\limits_{F_i \supset G} T_{F_i} \subset T^{\mathcal{F}}$. For every point $q \in P$ we denote by $G(q)$ the unique face containing $q$ in the relative interior. We can define the moment-angle manifold corresponding to $P$ by the following way.

**Definition 2.2** *For a simple polytope $P$ introduce the moment-angle manifold $Z_P = (T^{\mathcal{F}} \times P)/\sim$, where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T_{G(q)}$.*

Alternatively, we can define moment-angle manifold in another way: Since $P$ is a simple polytope, the dual of the boundary of $P$ is a simplicial $(n-1)$-sphere, which we denote by $K$. Let $[m] = \{1, \cdots, m\}$ represent the vertices of $K$, $\sigma$ be a simplex in the complex $K$. Define

$$(D^2)_\sigma \times T_{\widehat{\sigma}} = \{(z_1, z_2, \cdots, z_m) \in (D^2)^m : |z_j| = 1 \text{ for } j \notin \sigma\}$$

and define the moment-angle complex $Z_K$ corresponding to $K$ as

$$Z_K = \bigcup_{\sigma \in K} (D^2)_\sigma \times T_{\widehat{\sigma}} \subset (D^2)^m.$$

From [2], $Z_K$ is homeomorphic to $Z_P$. When we study the topology of moment-angle manifold corresponding to simple polytopes (or simplicial complexes), we consider the behavior of moment-angle manifolds under some surgeries on the simple polytopes (or simplicial complexes). One important surgery is cutting off faces of polytopes (or bistellar moves on simplicial complexes).

**Definition 2.3** *Let $P$ be a simple polytope of dimension $n$ with $m$ facets, which is the convex hull of finitely many vertices in $\mathbb{R}^n$. For any face $G$ in $P$, we can find a hyperplane $H(x) = \sum\limits_{i=1}^{n} a_i x_i = b$ satisfying that $H(v) > b$ and $H(w) < b$ for any vertex $v \in G$ and $w \notin G$. The set $P \cap \{x \mid H(x) \leq b\}$ is a new simple polytope $P_G$, which is called to be obtained from $P$ by cutting off the face $G$.*

Let $K$ be the dual simplicial complex of the boundary of $P$ on the vertex set $[m]$, and $\sigma \in K$ be the simplex dual to the face $G$ of $P$. Then the dual simplicial complex $K_G$ of the boundary of $P_G$ can be expressed as

$$K_G := (K - \sigma * \mathrm{link}_K \sigma) \cup (\partial\sigma * \mathrm{link}_K \sigma * \{*\}),$$

where $\mathrm{link}_K \sigma = \{\tau \in K : \tau * \sigma \in K\}$ and $\{*\}$ is an additional point.

We consider a simple case that $\mathrm{link}_K \sigma$ is the boundary of $k$-simplex $\tau$. In this case,

$$K_G := (K - \sigma * \partial\tau) \cup (\partial\sigma * \partial\tau * \{*\}).$$

By the definition, the moment-angle complex corresponding to $K_G$ is

$$Z_{K_G} = (Z_K \times S^1 - T^{m-n-1} \times D_\sigma^{2(n-k)} \times S_\tau^{2k+1} \times S^1) \cup T^{m-n-1} \times S_\sigma^{2(n-k)-1} \times S_\tau^{2k+1} \times D^2,$$

where $T^{m-n-1}\times S_\sigma^{2(n-k)-1}\times S_\tau^{2k+1}\times D^2$ is attached along its boundary

$$T^{m-n-1}\times S_\sigma^{2(n-k)-1}\times S_\tau^{2k+1}\times S^1.$$

Obviously, $Z_{K_G}$ is diffeomorphic to $\partial[(Z_K - T^{m-n-1}\times D_\sigma^{2(n-k)}\times S_\tau^{2k+1})\times D^2]$, which can be interpreted as performing an "equivariant surgery operation" on $Z_K\times S^1$.

It may be very complicated if we consider the topology of the moment-angle manifold corresponding to $K_G$ when $G$ is a high dimensional face. However, we have known the change of topology of the moment-angle manifold after cutting off a vertex $v$ on a polytope $P$.

After cutting off one vertex $v$ of the simple polytope $P$, we obtain a new simple polytope $P_v$. Let $K_P$ and $K_{P_v}$ be the duals of the boundary of $P$ and $P_v$, $\sigma$ be the maximal simplex in $K_P$ dual to the vertex $v$ of the simple polytope $P$. Then we have $K_{P_v}=K_P\#_\sigma\partial\triangle^n$ ($\triangle^n$ is the standard $n$-dimensional simplex, and the choice of a maximal simplex in $\partial\triangle^n$ is irrelevant). By the definition, the moment-angle complex corresponding to $P$ (or $K_P$) is

$$Z_P=\bigcup_{\sigma\in K_P}(D^2)_\sigma\times T_{\widehat{\sigma}}\subset (D^2)^m.$$

Then we can express the moment-angle complex corresponding to $P_v$ (or $K_{P_v}$) as follows (see [2, 6.4]):

$$\begin{aligned}Z_{P_v}&\cong (Z_P\times S^1 - T^{m-n}\times D_\sigma^{2n}\times S^1)\bigcup_{T^{m-n}\times S_\sigma^{2n-1}\times S^1} T^{m-n}\times S_\sigma^{2n-1}\times D^2\\ &\cong \partial[(Z-T_{\widehat{\sigma}}^{m-n}\times D_\sigma^{2n})\times D^2].\end{aligned}$$

In [7], Gitler and López conjectured that $Z_{P_v}$ is diffeomorphic to

$$\partial[(Z_P-D^{n+m})\times D^2]\#\mathop{\#}_{j=1}^{m-n}\binom{m-n}{j}(S^{j+2}\times S^{m+n-j-1}).$$

In [3], we proved this conjecture by the following way.

First, we construct an isotopy of $T_{\widehat{\sigma}}^{m-n}$ in $Z$ to move it to the regular embedding (see Remark 2.1) $T^{m-n}\subseteq D^{m-n+1}\subseteq D^{m+n}\subseteq Z$. The key lemma in the construction is as follows.

**Lemma 2.1** *We have two embeddings of $T^k$ in $D^{k+2}$ :*
*(1) $T^k=T^{k-1}\times S^1\subseteq D^k\times D^2$, where $T^{k-1}$ is the regular embedding in $D^k$.*
*(2) $T^k\subseteq D^{k+1}\subseteq D^{k+1}\times D^1$, where $T^k$ is the regular embedding in $D^{k+1}$.*
*These two embeddings are isotopic.*

**Proof** The normal bundle of the regular embedding $T^{k-1}$ in $D^k$ is trivial, so we can choose a neighborhood of $T^{k-1}$ which is diffeomorphic to $T^{k-1}\times\mathbb{R}^1$. We can construct an isotopy of $T^k$ in $D^{k+2}$ :

$$\begin{aligned}&H: T^{k-1}\times S^1\times I\to T^{k-1}\times\mathbb{R}\times D^2,\\ &H(x,\mathrm{e}^{\mathrm{i}\theta},t)=(x,t\cos\theta,(1-t)\cos\theta,\sin\theta).\end{aligned}$$

An examination of this isotopy proves the lemma.

Using this lemma, we can inductively construct an isotopy of $T_{\widehat{\sigma}}^{m-n}$ in $Z$ to move it to the regular embedding $T^{m-n}\subseteq D^{m-n+1}\subseteq D^{m+n}\subseteq Z$, thus prove the following proposition.

**Proposition 2.1** *$Z_{P_v}$ is diffeomorphic to*

$$\partial[(Z_P - D^{n+m}) \times D^2] \,\#\, \partial[(S^{m+n} - T^{m-n} \times D^{2n}) \times D^2],$$

*where $T^{m-n} \times D^{2n}$ is the regular embedding in $S^{m+n}$.*

**Remark 2.1** We construct the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$ as follows: $S^1 \subseteq D^2 \subseteq \mathbb{R}^2$. Assume that we have constructed the embedding of $T^{i-1}$ into $D^i \subseteq \mathbb{R}^i$. Represent $(i+1)$-sphere as $S^{i+1} = D^i \times S^1 \cup S^{i-1} \times D^2$. By the assumption, the torus $T^i = T^{i-1} \times S^1$ can be embedded into $D^i \times S^1$ and therefore into $S^{i+1}$. Since $T^i$ is compact and $S^{i+1}$ is the one-point compactification of $\mathbb{R}^{i+1}$, we have $T^i \subseteq \mathbb{R}^{i+1}$. Inductively, we can construct the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$ (or $D^{k+1}$). The regular embedding of $T^k$ into $\mathbb{R}^n$ is $T^k \subseteq \mathbb{R}^{k+1} \times \{0\} \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}$, where $T^k \subseteq \mathbb{R}^{k+1} \times \{0\}$ is the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$.

Similarly, we can construct the regular embedding of $(S^n)^k$ into $\mathbb{R}^{nk+1}$.

Then we prove the following by induction.

**Proposition 2.2** *$\partial[(S^{m+n} - T^{m-n} \times D^{2n}) \times D^2]$ is diffeomorphic to*

$$\mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1}),$$

*where $T^{m-n} \times D^{2n}$ is the regular embedding in $S^{m+n}$.*

Combining these two propositions, the conjecture is proved. However, we can replace the pair $(D^2, S^1)$ with $(D^{k+1}, S^k)$ $(k \geq 2)$ in the definition of moment-angle manifolds to obtain generalized moment-angle manifolds. We use $Z_{P,k}$ to denote the generalized moment-angle manifold corresponding to $P$, then the generalized moment-angle manifold $Z_{P_v,k}$ corresponding to $P_v$ is diffeomorphic to $Z_{P_v,k} \cong \partial[(Z - (S^k)^{m-n}_{\widehat{\sigma}} \times (D^{k+1})^n_{\sigma}) \times D^{k+1}]$. In a way similar to the case of $k = 1$, we construct an isotopy of $(S^k)^{m-n}_{\widehat{\sigma}}$ in $Z_{P,k}$ to move it to the regular embedding $(S^k)^{m-n} \subseteq D^{k(m-n)+1} \subseteq D^{km+n} \subseteq Z_{P,k}$ and we have a lemma similar to Lemma 2.1.

**Lemma 2.2** *There are two embedding of $(S^n)^k$ into $D^{nk+2}$ :*

*(1) $(S^n)^k \subseteq D^{nk+1} \times \{0\} \subseteq D^{nk+2}$, where $(S^n)^k \subseteq D^{nk+1} \times \{0\}$ is the regular embedding.*

*(2) $(S^n)^k = (S^n)^{k-1} \times S^n \subseteq D^{nk-n+1} \times D^{n+1} = D^{nk+2}$, where $(S^n)^{k-1} \subseteq D^{nk-n+1}$ and $S^n \subseteq D^{n+1}$ are regular embeddings.*

*These two embeddings are isotopic to each other in $D^{nk+2}$.*

**Proof** The normal bundle of the regular embedding $(S^n)^{k-1}$ in $D^{n(k-1)+1}$ is trivial, so we can choose a neighborhood of $(S^n)^{k-1}$ which is diffeomorphic to $(S^n)^{k-1} \times \mathbb{R}^1$. We can construct an isotopy of $(S^n)^k$ in $D^{nk+2}$ :

$$\begin{aligned} &H : (S^n)^{k-1} \times S^n \times I \to (S^n)^{k-1} \times \mathbb{R}^1 \times D^{n+1}, \\ &H(x, (y_1, y_2, \cdots, y_{n+1}), t) = (x, ty_{n+1}, (y_1, y_2, \cdots, y_n, (1-t)y_{n+1})), \end{aligned}$$

where we use $(y_1, y_2, \cdots, y_{n+1})$ to express the unit sphere $S^n(1)$ in $\mathbb{R}^{n+1}$. An examination of this isotopy proves the lemma.

By this lemma, we can construct an isotopy of $(S^k)^{m-n}_{\widehat{\sigma}}$ in $Z_{P,k}$ to move it to the regular embedding $(S^k)^{m-n} \subseteq D^{k(m-n)+1} \subseteq D^{km+n} \subseteq Z_{P,k}$, thus prove the following proposition.

**Proposition 2.3** *$Z_{P_v,k}$ is diffeomorphic to*

$$\partial[(Z_{P,k} - D^{km+n}) \times D^{k+1}] \# \partial[(S^{km+n} - (S^k)^{m-n} \times D^{(k+1)n}) \times D^{k+1}],$$

*where $(S^k)^{m-n} \times D^{(k+1)n}$ is the regular embedding in $S^{km+n}$.*

Then using the same method of proving Proposition 1.2 in [3], we can prove the following by induction (see Section 4).

**Proposition 2.4** *$\partial[(S^{km+n} - (S^k)^{m-n} \times D^{(k+1)n}) \times D^{k+1}]$ is diffeomorphic to*

$$\mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{k(j+1)+1} \times S^{k(m-j)+n-1}),$$

*where $(S^k)^{m-n} \times D^{(k+1)n}$ is the regular embedding in $S^{km+n}$.*

Combining these two propositions, we can prove the following theorem.

**Theorem 2.1** *If $P$ is a simple polytope, the generalized moment-angle manifold $Z_{P_v,k}$ corresponding to $P_v$ is diffeomorphic to*

$$\partial[(Z_{P,k} - D^{n+km}) \times D^{k+1}] \# \mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{k(j+1)+1} \times S^{k(m-j)+n-1}).$$

In order to prove Theorem 1.1, we firstly recall some notations and theorems.

Suppose that we are given a Riemannian manifold $M^{p+d}$ having positive Ricci curvature and an isometric embedding: $\iota : S^p(\rho) \times D^d(R, N) \to M$ where $S^p(\rho)$ is the $p$-sphere with the round metric of radius $\rho$, $D^d(R, N)$ denotes a geodesic ball of radius $R$ in the $d$-sphere with the round metric of radius $N$. We can regard $\iota$ as a trivialization of the normal bundle of $\iota(S^p \times \{0\})$. A corollary of the main Lemma 1 in [15] is the following result.

**Theorem 2.2** (see [17, Section 4, Theorem ]) *Let $\overline{M} \cong (M - S^p \times D^d) \cup D^{p+1} \times S^{d-1}$ be the result of performing surgery on $\iota(S^p \times \{0\})$ using the trivialization $\iota$, and assume $p \geq 1$, $d \geq 3$. Then there exists $\kappa(p, d, RN^{-1}) > 0$ such that if $\frac{\rho}{N} < \kappa$ then $\overline{M}$ can be equipped with a Ricci positive curvature, the metric on a neighborhood $S^{d-1} \times D^{p+1}$ of $S^{d-1}$ is the product of the metric on a round sphere $S^{d-1}$ and the metric on a geodesic ball $D^{p+1}$ in the $(p+1)$-sphere.*

**Remark 2.2** In [15], the authors used the warped product to construct a Ricci positive metric on $D^d \times S^p$ such that the metric on a submanifold $(S^{d-1} \times I) \times S^p \subseteq D^d \times S^p$ ($S^{d-1} \times \{0\} \times S^p$ is the boundary $\partial D^d \times S^p$) is Ricci positive satisfying that

(1) the metric on the submanifold $S^{d-1} \times [0, \epsilon] \times S^p$ is isomeric to a neighborhood of the boundary of the product of a geodesic ball $D^d$ in the $d$-sphere and a round sphere $S^p$,

(2) the metric on the submanifold $S^{d-1} \times [1-\epsilon, 1] \times S^p$ is isometric to a neighborhood of the boundary of the product of a round sphere $S^{d-1}$ and a geodesic ball $D^{p+1}$ in the $(p+1)$-sphere.

So there exists a Ricci positive metric on

$$\overline{M} \cong (M - S^p \times D^d) \cup D^{p+1} \times S^{d-1} = (M - S^p \times D^d) \cup S^{d-1} \times I \times S^p \cup D^{p+1} \times S^{d-1}$$

such that

(1) the metric on $M - S^p \times D^d$ inherits from the Ricci positive metric on $M$,

(2) the metric on $S^{d-1} \times I \times S^p$ is the Ricci positive metric constructed above,

(3) the metric on $D^{p+1} \times S^{d-1}$ is isometric to the product of the metric on a geodesic ball $D^{p+1}$ in the $(p+1)$-sphere and the metric on a round sphere $S^{d-1}$.

The proof of Theorem $A$ in [17] shows the following theorem.

**Theorem 2.3** *Let $S^m \times D^n$ $(m > n \geq 3)$ be a neighborhood of an embedded sphere $S^m$ in $M$. If manifold $M$ admits a Ricci positive metric such that the restricted metric on $S^m \times D^n$ is the product metric of the round metric of sphere $S^m$ and a geodesic ball $D^n$ in the $n$-sphere, then any connected sum $M\#S^{m_1} \times S^{n_1}\# \cdots \#S^{m_k} \times S^{n_k}$ admits a metric of positive Ricci curvature for $m_i, n_i \geq 3$ and $m_i + n_i = m + n$ for all $i$.*

**Remark 2.3** Consider the Ricci positive metrics on $D^n \times S^{p+q+1}$, where $n \geq 3,\ p \geq 2,\ q \geq 1$. Let $D^{n+q+1} \times S^p = D^n \times (D^{q+1} \times S^p) \subseteq D^n \times S^{p+q+1}$ be the product of embedding $D^n \xrightarrow{\text{Id}} D^n$ and $D^{q+1} \times S^p \subseteq S^{p+q+1}$. Then there is a Ricci nonnegative metric on $D^n \times S^{p+q+1}$ such that

(1) a neighborhood of $\partial D^n \times S^{p+q+1}$ is isomeric to a neighborhood of the boundary of the product of a geodesic ball $D^n$ in the $n$-sphere and a round sphere $S^{p+q+1}$,

(2) the submanifold $D^{n+q+1} \times S^p$ is isometric to the product metric of a geodesic ball $D^{n+q+1}$ in the $(n+q+1)$-sphere and a round sphere $S^p$.

Without loss of generality, assume that $m_i \geq n_i$, so $m - n_i \geq 1$. $M\#S^{m_i} \times S^{n_i}$ can be expressed by $(M - S^{n_i-1} \times D^{m_i+1}) \cup D^{n_i} \times S^{m_i}$, where $S^{n_i-1} \times D^{m_i+1} \subseteq S^m \times D^n \subseteq M$ is the product of embedding $(S^{n_i-1} \times D^{m+1-n_i}) \subseteq S^m$ and $D^n \xrightarrow{\text{Id}} D^n$. So there exists a Ricci nonnegative metric on $(M - S^{n_i-1} \times D^{m_i+1}) \cup D^{n_i} \times S^{m_i}$ such that

(1) the metric on $M - S^m \times D^n$ inherits from the Ricci positive metric on $M$,

(2) the metric on $S^m \times D^n - S^{n_i-1} \times D^{m_i+1}$ inherits from the Ricci nonnegative metric on $S^m \times D^n$ constructed by the method above,

(3) the metric on $D^{n_i} \times S^{m_i}$ is isometric to the product of the metric on a geodesic ball $D^{n_i}$ in the $n_i$-sphere and the metric on a round sphere $S^{m_i}$.

By choosing several small geodesic sub-balls $D^n$ of $D^n$ and constructing a Ricci nonnegative metric on each $S^m \times D^n$ by the method above, we obtain a metric of nonnegative Ricci curvature on $M\#S^{m_1} \times S^{n_1}\# \cdots \#S^{m_k} \times S^{n_k}$. As the metric is Ricci positive at many points, by [5] this metric can be deformed to one with everywhere strictly positive Ricci curvature.

Now we come to the proof of Theorem 1.1.

**Proof of Theorem 1.1** As noted in [15, p. 134], if manifold $M^m$ admits a Ricci positive metric, then the metric can be deformed to be a Ricci positive one containing a geodesic ball $D^m$ in the $m$-sphere. So we can always assume that the manifold $M$ with a Ricci positive metric contains a geodesic ball $D^m$ in the $m$-sphere. If the generalized moment-angle manifold $Z_{P,k}$ admits a Ricci positive metric, the product of the metric on $Z_{P,k}$ and a round metric on $S^k$ is Ricci positive containing $D^{n+km} \times S^k$ the metric of which is the product of the metric on a geodesic ball $D^{n+km}$ in the $(n+km)$-sphere and the metric on a round sphere $S^k$. With Theorem 2.2, we can prove that

$$\partial[(Z_{P,k} - D^{n+km}) \times D^{k+1}] \cong (Z_{P,k} \times S^k - D^{n+km} \times S^k) \cup S^{n+km-1} \times D^{k+1}$$

admits a Ricci positive metric, the restricted metric on a neighborhood $S^{n+km-1} \times D^{k+1}$ of $S^{n+km-1}$ is the product of the metric on a round sphere $S^{n+km-1}$ and the metric on a geodesic ball $D^{k+1}$ in the $(k+1)$-sphere, when $m,n,k \geq 2$, $n+km-1 > k+1$. By Theorems 2.2–2.3, we can prove that the generalized moment-angle manifold

$$Z_{P_v,k} \cong \partial[(Z_{P,k} - D^{n+km}) \times D^{k+1}] \# \mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{k(j+1)+1} \times S^{k(m-j)+n-1})$$

admits a Ricci positive metric if generalized moment-angle manifold $Z_{P,k}$ admits a Ricci positive metric.

## 3 Fano Polytope

In this section, we will prove that the moment-angle manifolds corresponding to Fano polytopes admit Ricci positive metrics. Now we come to the definition of Fano polytope.

**Definition 3.1** *Let $Q$ be a simplicial convex polytope in $\mathbb{R}^n$ whose vertices are primitive lattice vectors $\{l_i\}$ ($l_i \in \mathbb{Z}^n$), and which contains* 0 *in the interior. If $a_1, \cdots, a_n$ are the vertices of a facet of $Q$, we suppose* $\det(a_1, \cdots, a_n) = \pm 1$ *for every facet. Then we call $Q$ a Fano polytope.*

The boundary of $Q$ is a simplicial sphere $K$, from which we can construct a moment-angle manifold $Z_K$. Alternatively, we can define the moment-angle manifold in another way: The dual of $Q$: $P = \{u \in \mathbb{R}^n \mid \langle u, v\rangle \leq 1, \forall v \in Q\}$ is a simple polytope. The normal vector of each facet can be chosen as one of the lattice vectors $\{l_i\}$, we assume that the lattice vector corresponding to facet $F_i$ is $l_i$. We can construct the moment-angle manifold $Z_P$ corresponding to $P$ which is homeomorphic to $Z_K$.

In order to prove Theorem 1.2, we firstly recall a theorem in [6].

**Theorem 3.1** *Let $Y$ be a compact connected Riemannian manifold with a metric of positive Ricci curvature. Let $\pi : P \to Y$ be a principal bundle over $Y$ with compact connected structure group $G$. If the fundamental group of $P$ is finite, then $P$ admits a $G$ invariant metric with positive Ricci curvature so that $\pi$ is a Riemannian submersion.*

Now we come to the proof of Theorem 1.2.

**Proof of Theorem 1.2** Given a Fano polytope $Q$, we can define the complete fan $\Sigma(Q)$ whose cones are generated by those sets of vertices $l_{i_1}, \cdots, l_{i_k}$ which are in one face of $Q$. From this fan, we can construct a toric variety $M_P$. This toric variety is smooth and Fano (see [4]) (Fano means that the anticanonical divisor is ample). By Calabi-Yau's theorem (see [18]), the Fano variety $M_P$ admits a Ricci positive metric.

Topologically, toric Fano variety can be constructed from the polytope $P$ and the lattice vectors $\{l_i\}$ by the following way (see [2]): We identify the torus $T^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$. For each point $q \in P$, define $G(q)$ as the smallest face that contains $q$ in its relative interior. The normal subspace to $G(q)$ is spanned by the primitive vectors $l_i$ corresponding to those facets $F_i$ which contain $G(q)$. Since $N$ is a rational space, it projects to a subtorus of $T^n$, which we denote by $T(q)$. Then as a topological space, the toric Fano variety

$$M_P = T^n \times P/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T(q)$.

From [2], the moment-angle manifold $Z_P$ is a principal $T^{m-n}$ bundle $Z_P \to M_P$. Since $Z_P$ is simply connected and $M_P$ admits a Ricci positive metric, by Theorem 3.1, $Z_P$ admits a $T^{m-n}$ invariant metric with positive Ricci curvature.

Now we give a conjecture.

**Conjecture 3.1** $P$ is a simple polytope.

(1) $k \geq 1$. If a generalized moment-angle manifold $Z_{P,k}$ admits a Ricci positive metric, so does $Z_{P,k+1}$.

(2) For $k \geq 2$, $Z_{P,k}$ admits a Ricci positive metric for every simple polytope $P$. Moment-angle manifold $Z_P$ admits a Ricci positive metric for every irreducible simple polytope $P$.

If the conjecture (1) is true, by Theorem 1.2, the generalized moment-angle manifolds corresponding to Fano polytopes admit Ricci positive metrics; by Theorem 1.1, we can prove that the generalized moment-angle manifolds corresponding to polytopes obtained by cutting off vertices of Fano polytopes admit Ricci positive metrics. So we can obtain a class of polytopes that the corresponding generalized moment-angle manifolds admit Ricci positive metrics. Besides, in [1], the authors constructed a Ricci positive metric on the moment-angle manifold corresponding to the polytope $P_v$ obtained by cutting off one vertex $v$ on the 3-cube $P^3$. However, the dual of $P_v$ is a Fano polytope. So by Theorem 1.2, we can prove that the corresponding moment-angle manifold admits a Ricci positive metric.

From [8], we know that any manifold obtained from a manifold which admits scalar positive curvature by performing surgeries in codimension $\geq 3$ also admits a scalar positive curvature. For the Ricci curvature, when we perform surgery on manifolds with Ricci positive metrics, whether the manifold $(M - S^p \times D^{n-p}) \cup D^{p+1} \times S^{n-p-1}$ obtained by surgery can admit Ricci positive curvature may depend on the restricted metric of $S^p \times D^{n-p}$ in $M$ (see [15, 17]). Similarly, suppose that the generalized moment-angle manifold $Z_{P,k}$ corresponding to $P$ admits a Ricci positive metric. After cutting off a face $G$ of $P$, the dual simplicial complex $K_G$ of the boundary of $P_G$ can be expressed as

$$K_G := (K - \sigma * \mathrm{link}_K\sigma) \cup (\partial\sigma * \mathrm{link}_K\sigma * \{*\}),$$

where $\mathrm{link}_K\sigma = \{\tau \in K : \tau * \sigma \in K\}$ and $\{*\}$ is an additional point. We hope that the restricted metric of the submanifold in $Z_{P,k}$ corresponding to $\sigma * \mathrm{link}_K\sigma$ can be "good" enough that we can extend the Ricci positive metric to $Z_{P_G,k}$.

## 4 Appendix

In this appendix, we will prove Proposition 2.4 by induction.

While $m - n = 1$, the manifold

$$\begin{aligned}
&\partial[(S^{km+n} - (S^k)^{m-n} \times D^{(k+1)n}) \times D^{k+1}] \\
&= \partial[(S^{(k+1)n+k} - S^k \times D^{(k+1)n}) \times D^{k+1}] \\
&\cong \partial(S^{(k+1)n-1} \times D^{2k+2}) \\
&\cong S^{(k+1)n-1} \times S^{2k+1}.
\end{aligned}$$

Inductively suppose that we have proved that $\partial[(S^{(k+1)n+ki} - (S^k)^i \times D^{(k+1)n}) \times D^{k+1}]$ is diffeomorphic to

$$\mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}).$$

We proceed to prove that $\partial[(S^{(k+1)n+k(i+1)} - (S^k)^{i+1} \times D^{(k+1)n}) \times D^{k+1}]$ is diffeomorphic to

$$\mathop{\#}_{j=1}^{i+1} \binom{i+1}{j} (S^{k(j+1)+1} \times S^{k(n+i+1-j)+n-1}).$$

Since $(S^k)^{i+1} \times D^{(k+1)n} \subseteq D^{(k+1)n+k(i+1)} \subseteq S^{(k+1)n+k(i+1)}$ is the regular embedding,

$$\begin{aligned}
&(S^k)^{i+1} \times D^{(k+1)n} \\
&= S^k \times ((S^k)^i \times D^{(k+1)n}) \\
&\subseteq S^k \times D^{(k+1)n+ki} \\
&\subseteq S^k \times D^{(k+1)n+ki} \cup D^{k+1} \times S^{(k+1)n+ki-1} \\
&= S^{(k+1)n+k(i+1)},
\end{aligned}$$

where $(S^k)^i \times D^{(k+1)n} \subseteq D^{(k+1)n+ki}$ is the regular embedding. So the manifold $\partial[(S^{(k+1)n+k(i+1)} - (S^k)^{i+1} \times D^{(k+1)n}) \times D^{k+1}]$ is diffeomorphic to

$$\begin{aligned}
&\partial[(S^k \times D^{(k+1)n+ki} \cup D^{k+1} \times S^{(k+1)n+ki-1} - (S^k)^{i+1} \times D^{(k+1)n}) \times D^{k+1}] \\
&\cong \partial[((S^k \times S^{(k+1)n+ki} - S^k \times D^{(k+1)n+ki}) \cup D^{k+1} \times S^{(k+1)n+ki-1} \\
&\quad - S^k \times (S^k)^i \times D^{(k+1)n}) \times D^{k+1}] \\
&\cong \partial[((S^k \times (S^{(k+1)n+ki} - (S^k)^i \times D^{(k+1)n}) \\
&\quad - S^k \times D^{(k+1)n+ki}) \cup D^{k+1} \times S^{(k+1)n+ki-1}) \times D^{k+1}] \\
&\cong (\partial[S^k \times (S^{(k+1)n+ki} - (S^k)^i \times D^{(k+1)n}) \times D^{k+1}] \\
&\quad - S^k \times D^{(k+1)n+ki} \times S^k) \cup D^{k+1} \times S^{(k+1)n+ki-1} \times S^k \\
&\cong \Big(S^k \times \Big(\mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1})\Big) \\
&\quad - S^k \times D^{(k+1)n+ki} \times S^k\Big) \cup D^{k+1} \times S^{(k+1)n+ki-1} \times S^k.
\end{aligned}$$

By induction, it is diffeomorphic to

$$\partial\Big[\Big(\mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}) - D^{(k+1)n+ki} \times S^k\Big) \times D^{k+1}\Big].$$

As

$$D^{(k+1)n+ki} \times S^k \subseteq D^{(n+i+1)k+n} \subseteq \mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}),$$

$$\partial\Big[\Big(\mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}) - D^{(k+1)n+ki} \times S^k\Big) \times D^{k+1}\Big]$$

is diffeomorphic to

$$\begin{aligned}
&\partial\Big[\Big(\mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}) \# (S^{k(n+i+1)+n} - D^{(k+1)n+ki} \times S^k)\Big) \times D^{k+1}\Big] \\
&\cong \partial\Big[\Big(\mathop{\#}_{j=1}^{i} \binom{i}{j} (S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}) \# S^{(k+1)n+ki-1} \times D^{k+1}\Big) \times D^{k+1}\Big],
\end{aligned}$$

Recalling Lemmas 1–2 in [7], we can generalize these two lemmas as the following.

**Lemma 4.1** *Assume $k \geq 2$,*

(1) *Let $M$ and $N$ be connected and closed $n$-manifolds. Then $\partial[(M\#N - D^n) \times D^k]$ is diffeomorphic to $\partial[(M - D^n) \times D^k]\#\partial[(N - D^n) \times D^k]$.*

(2) *Let $M, N$ be connected $n$-manifolds. If $M$ is closed but $N$ has non-empty boundary, then $\partial[(M\#N) \times D^k]$ is diffeomorphic to $\partial[(M - D^n) \times D^k]\#\partial(N \times D^k)$.*

(3) *$\partial[(S^p \times S^q - D^{p+q}) \times D^k] = S^p \times S^{q+k-1}\#S^{p+k-1} \times S^q$.*

The proof of the lemma is the same as that of Lemma 1 and Lemma 2 in [7].

Using this lemma, $\partial\big[\big(\mathop{\#}\limits_{j=1}^{i} \binom{i}{j}(S^{k(j+1)+1} \times S^{k(n+i-j)+n-1})\#S^{(k+1)n+ki-1} \times D^{k+1}\big) \times D^{k+1}\big]$ is diffeomorphic to

$$
\begin{aligned}
&\partial\Big[\Big(\mathop{\#}\limits_{j=1}^{i}\binom{i}{j}(S^{k(j+1)+1} \times S^{k(n+i-j)+n-1}) - D^{(k+1)n+(i+1)k}\Big) \times D^{k+1}\Big]\\
&\#\partial[S^{(k+1)n+ki-1} \times D^{k+1} \times D^{k+1}]\\
\cong\ &\mathop{\#}\limits_{j=1}^{i}\binom{i}{j}\partial[(S^{k(j+1)+1} \times S^{k(n+i-j)+n-1} - D^{(k+1)n+(i+1)k}) \times D^{k+1}]\#S^{(k+1)n+ki-1} \times S^{2k+1}\\
\cong\ &\mathop{\#}\limits_{j=1}^{i}\binom{i}{j}(S^{k(j+1)+1} \times S^{k(n+i+1-j)+n-1}\#S^{k(j+1)+1+k} \times S^{k(n+i-j)+n-1})\\
&\#S^{(k+1)n+ki-1} \times S^{2k+1}\\
\cong\ &\mathop{\#}\limits_{j=1}^{i+1}\binom{i+1}{j}(S^{k(j+1)+1} \times S^{k(n+i+1-j)+n-1})
\end{aligned}
$$

By induction, we can prove Proposition 2.4.

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