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Fast Growth Entire Functions Whose Escaping Set Has Hausdorff Dimension Two*

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Abstract The authors study a family of transcendental entire functions which lie outside the Eremenko-Lyubich class in general and are of infinity growth order. Most importantly, the authors show that the intersection of Julia set and escaping set of these entire functions has full Hausdorff dimension. As a by-product of the result, the authors also obtain the Hausdorff measure of their escaping set is infinity.

Keywords Dynamic systems, Entire function, Julia set, Escaping set, Hausdorff dimension
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1 Introduction

Let f be a transcendental entire function and denote by f^n the n-th iteration of f, $n \in \mathbb{N}$. The Fatou set $F(f)$ is defined as the set consisting of all $z \in \mathbb{C}$, where $\{f^n\}$ forms a normal family in the sense of Montel (or, equivalently, is equicontinuous). The complement $J(f)$ of $F(f)$ is called the Julia set of f. Both sets are completely invariant. Maximal connected subset of $F(f)$ is called connected Fatou component. For an introduction to the basic properties of these sets, we refer to the survey [3] and the books [2, 15].

Besides Julia set and Fatou set, there is another important and interesting set in the studies of dynamics of transcendental entire functions, which is the escaping set $I(f) := \{z;\ f^n(z) \to \infty \text{ as } n \to \infty\}$. Eremenko [9] proved that $I(f)$ is not empty and $J(f) = \partial I(f)$ for entire functions.

In 1987, McMullen [14] proved a result on size of Julia set, that is $\dim J(E_\lambda) = 2$ for $\lambda \neq 0$, where $E_\lambda = \lambda \exp(z)$. In his proofs, he first showed that these results hold for the escaping set $I(f) := \{z;\ f^n(z) \to \infty \text{ as } n \to \infty\}$, and then $I(f) \subset J(f)$ for the functions E_λ.

We denote by $\operatorname{sing}(f^{-1})$ the set of all values in which some branch of f^{-1} cannot be defined; that is, the set of all critical and finite asymptotic values. A class $\mathcal{B}$ of entire functions, introduced by Eremenko and Lyubich [10] as follows, has particular interests to many mathe-

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maticians.

$$\mathcal{B} = \{f : f \text{ is transcendental entire function and } \operatorname{sing}(f^{-1}) \text{ is bounded}\}.$$

The importance of this class lies in the fact that functions in $\mathcal{B}$ have some expanding properties near ∞, it is similar to the way in which the derivative of the exponential map is large when the exponential itself is large. Using the expanding property, Eremenko and Lyubich showed that $I(f) \subset J(f)$ for $f \in \mathcal{B}$. It is easy to see that $E_\lambda \in \mathcal{B}$, thus $I(E_\lambda) \subset J(E_\lambda)$.

McMullen's result initiated a large body of research on Hausdorff dimension for Julia set and escaping set (see [1, 8, 22]). In [1, 19], Barański and Schubert independently proved that $\dim J(f) = 2$ if $f \in \mathcal{B}$ has finite order of growth. For more results, we refer to the publication [21].

In 2010, Bergweiler and Karpińska [7] studied entire functions f which could be outside of the class $\mathcal{B}$ and proved the following theorem.

Theorem A *Supposed that f is an entire function and that there exist $A, B, C, r_0 > 1$ such that*

$$A \log M(r, f) \le \log M(Cr, f) \le B \log M(r, f) \quad \text{for all } r > r_0.$$

Then

$$\dim(I(f) \cap J(f)) = 2.$$

The fast escaping set, $I^o(f) \subset I(f)$,which was introduced in [6] and has been studied systematically in [18], is defined by

$$I^o(f) = \{z : \text{There exists } l \in \mathbb{N} \text{ such that } |f^{n+l}(z)| \ge M^n(r, f) \text{ for } n \in \mathbb{N}\},$$

where $M^n(r, f)$ denotes repeated iteration of $M(r, f)$.

In 2015, Sixsmith [20] researched Hausdorff dimension of the Julia set and fasting escaping set for a class entire functions of genus zero, which also lie outside the Eremenko-Lyubich class in general.

Theorem B *Suppose that f is an entire function of the form*

$$f(z) = cz^q \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right), \quad \text{where } c \neq 0,\ q \in \mathbb{N} \text{ and } 0 < |a_1| \le |a_2| \le \cdots .$$

Suppose that there exist positive constants θ_1 and θ_2 such that $0 \le \theta_2 - \theta_1 < \pi$, and also $N \in \mathbb{N}$ such that $\arg(a_n) \in [\theta_1, \theta_2]$ in the sense of modulus 2π, for $n \ge N$. Then

$$\dim I^o(f) = 2.$$

Suppose also that f has only simply connected Fatou components. Then

$$\dim(J(f) \cap I^o(f)) = 2.$$

Recently, Bergweiler and Chyzhykov [5] studied a class of entire functions whose zeros are in the neighborhood of certain rays and that have completely regular growth with a certain growth error term (also see [4]). They proved the following theorem.

Theorem C *Let* $\theta_1 < \theta_2 < \cdots < \theta_m < \theta_{m+1} = \theta_1 + 2\pi$ *and let* $h : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* $h(\theta) > 0$ *for* $\theta_j < \theta < \theta_{j+1}$ *and* $j = 1, \cdots, m$*. Let* f *be an entire function,* $\rho(r)$ *be a proximate order and* $\varepsilon(r) = \frac{1}{\log^N(r)}$ *for some* $N \in \mathbb{N}$*, where* $\log^N$ *denotes the* N*-th iterate of the logarithm. Suppose that*

$$\log |f(re^{i\theta})| = h(\theta)r^{\rho} + O(r^{\rho(r)}\varepsilon(r)),$$

whenever $|\theta - \theta_j| > \varepsilon(r)$ *for* $j = 1, \cdots, m$*. Then* $I(f) \cap J(f)$ *has positive measure.*

In this paper, as a complement of Theorems A, B and C, we study a family of entire functions which are of infinitely order rather than finite order and whose zeros are equally distributed on circles rather than in a sectors or rays. Also the family of entire functions are outside of Eremenko-Lyubich class $\mathcal{B}$, and most importantly, $\dim(J(f) \cap I^o(f)) = 2$. However, it seems difficult to prove $J(f) \cap I(f)$ has positive Lebesgue measure. In Section 5, we also obtain some theorems on Hausdorff measures of the intersection of Julia set and escaping set.

Let $\alpha_1(r)$ be any positive differentiable increasing function on the interval $(1, \infty)$ with $\alpha_1'(r) > \alpha_1(r)$ (e.g. e^{2r}). Set

$$\alpha(x) = \int_1^x \frac{\alpha_1(t)}{t}\,dt. \tag{1.1}$$

Let $\{r_j\}$ be the sequence defined by

$$\alpha_1(r_j) = 2^{j+1}, \quad j = 1, 2, 3, \cdots. \tag{1.2}$$

This sequence $\{r_j\}$ is uniquely determined, strictly increasing and unbounded. Let

$$f(z) = \prod_{j=1}^{\infty} \left(1 + \left(\frac{z}{r_j}\right)^{n_j}\right), \tag{1.3}$$

where

$$n_j = 2^j. \tag{1.4}$$

The entire functions in (1.3) have been studied in [23].

Theorem 1.1 *Let* f *be defined as in* (1.3)*. Then*

$$\dim(J(f) \cap I^o(f)) = 2.$$

Obviously, the following corollary is a straightforward consequence of the above theorem.

Corollary 1.1 *Let* f *be defined as in* (1.3)*. Then*

$$\dim(J(f) \cap I(f)) = 2.$$

2 Notation and Definitions

First, we need some notations in Nevanlinna theory. We denote the maximum modulus of f by $M(r,f)=\max_{|z|=r}|f(z)|$; the number of zeros of $f-a$ in the disc $\{z;|z|<r\}$ by $n(r,a)$; or simply, by $M(r),\ n(r)$, respectively. The growth order of entire function f is defined by

$$\rho(f)=\limsup_{r\to\infty}=\frac{\log\log M(r,f)}{\log r}.$$

For more introduction, we refer the reader to books [11, 13].

To state McMullen's technique, we first give what it means for a family of sets to satisfy the 'nesting condition'.

Definition 2.1 (Nesting Condition) *Let $\mathcal{A}_l$ be a finite collection of compact, disjoint and connected subsets of $\mathbb{C}$ with positive Lebesgue measure. Let $\mathbb{A}_l$ be the union of the elements of $\mathcal{A}_l$, we say $\{\mathcal{A}_l\}$ satisfy the Nesting conditions if it has the following conditions*:

(a) *For every element $l\in\mathbb{N}$ and $G\in\mathcal{A}_{l+1}$, there exists unique $F\in\mathcal{A}_l$ such that $G\subset F$.*

(b) *There exists a decreasing sequence (d_l) tend to 0 such that*

$$\max_{F\in\mathcal{A}_l}\{\operatorname{diam}F\}\le d_l \quad \textit{for all } l\in\mathbb{N}.$$

(c) *There exists a sequence $\{\Delta_l\}$ of positive real such that*

$$\operatorname{dens}(\mathbb{A}_{l+1},F)=\frac{\operatorname{area}(\mathbb{A}_{l+1}\cap F)}{\operatorname{area}(F)}\ge\Delta_l \quad \textit{for all } l\in\mathbb{N}, F\in\mathcal{A}_l.$$

Thus the intersection $\mathbb{A}=\bigcap_{l=1}^{\infty}\mathbb{A}_l$ is a non-empty and compact set.

A gauge function is a monotonically increasing function $h:[0,\varepsilon)\to[0,+\infty)$ which is continuous from the right and satisfies $h(0)=0$.

Definition 2.2 (Hausdorff Dimension and Hausdorff Measure) *Let $A\subset\mathbb{R}^n$ is a set, $\delta>0$ and h is a gauge function. Then we call*

$$H^h(A):=\liminf_{\delta\to0}\Big\{\sum_{j=1}^{\infty}h(\operatorname{diam}(A_j));\ A\subset\bigcup_{j=1}^{\infty}A_j \ \textit{and}\ \operatorname{diam}(A_j)<\delta\Big\}$$

the Hausdorff measure with respect to h, where $\operatorname{diam}(A_j)=\sup_{x,y\in A_j}|x-y|$ is the diameter of A_j.

The Hausdorff measure is an outer measure for Borel sets which is measurable. In particular, when $h^s(r)=r^s$ $(s>0)$ then $H^{h^s}(A)$ is the s-dimension Hausdorff measure of A. If $H^{h^s}(A)<\infty$ and $t>s$, then $H^{h^t}(A)=0$; if $H^{h^s}(A)>0$ and $t<s$, then $H^{h^t}(A)=\infty$.

Moreover, there exists a constant s such that $H^{h^t}(A)=0$ for all $t>s$ and $H^{h^t}(A)=\infty$ for all $t<s$. The above s is called Hausdorff dimension of A and denote $s=\dim(A)$.

As we known, for given $\lambda\in\big(0,\frac{1}{\mathrm{e}}\big)$, the function E_λ has two fixed points α_λ and β_λ, where α_λ is attracting and $\beta_\lambda>1$ is repelling. Recall that a classical result of Koenigs says that there exists a function Φ_λ holomorphic in a neighborhood $D(\lambda)$ of β_λ which satisfies $\Phi_\lambda(\beta_\lambda)=0,\ \Phi'_\lambda(\beta_\lambda)=1$ and

$$\Phi_\lambda(E_\lambda(z))=\beta_\lambda\Phi_\lambda(z),\quad z,\ E_\lambda(z)\in U. \tag{2.1}$$

It is easy to see from $\Phi_\lambda(x) \in \mathbb{R}$ for $x \in \mathbb{R} \cap U$ and (2.1) that Φ_λ admits a real value continuation to $[\beta_\lambda, \infty)$. Moreover, $\Phi_\lambda(x)$ tends to ∞ as $x \to \infty$.

3 Properties of Our Function f and Lemmas

In the sequel, we will replace $N(r,\ 0)$ and $n(r,\ 0)$ by the simpler notations $N(r)$ and $n(r)$, if there are no any confusions.

Lemma 3.1 *Let r_j be as in* (1.2). *Then*

$$\lim_{j\to\infty}\left(\frac{r_{j+1}}{r_j}\right) = 1.$$

Proof By (1.2), there is a $\xi \in (r_j,\ r_{j+1})$ such that

$$\alpha_1(r_j) = \alpha_1(r_{j+1}) - \alpha_1(r_j) = \alpha_1'(\xi)(r_{j+1} - r_j).$$

Also

$$r_{j+1} - r_j = \frac{\alpha_1(r_j)}{\alpha_1'(\xi)} \le \frac{\alpha_1(r_j)}{\alpha_1(r_j)} \le 1.$$

Since r_j is increasing to ∞, we now have that

$$0 \le \liminf\left(\frac{r_{j+1}}{r_j} - 1\right) \le \limsup\left(\frac{r_{j+1}}{r_j} - 1\right) = 0.$$

This proves the lemma.

Lemma 3.2 *Let f be defined as in* (1.3). *Then f is an entire function and satisfies* ($z = r\mathrm{e}^{\mathrm{i}\theta}$)

$$\log|f(z)| = N(r) + O(1), \tag{3.1}$$
$$N(r) \le T_f(r) \le \log M(r) = N(r) + O(1), \tag{3.2}$$
$$\left|\frac{zf'(z)}{f(z)}\right| = n(r) - \log r + O(1) \tag{3.3}$$

for all large $r \in \bigcup_{j=j_0}^{\infty}(r_j + a^j,\ r_{j+1} - a^{j+1})$, *where j_0 is a large positive integer and $a \in \mathbb{R}$ with* $\frac{1}{2} < |a| < 1$.

Proof Let $r > 0$, with $r \in [r_{k-1},\ r_k)$. Then $n(r) = 2^k - 2 \le \alpha_1(r) - 2$ and

$$n(r) = \frac{1}{2}\alpha_1(r_k) - 2 \ge \frac{1}{2}\alpha_1(r) - 2. \tag{3.4}$$

Our upper bound for $n(r)$ and (1.1) yield that

$$N(r) \le \alpha(r) - 2\log r, \quad r > r_0. \tag{3.5}$$

Choose $a \in \left(\frac{1}{2}, 1\right)$, and set

$$\begin{aligned} s_j &= r_j - a^j, \quad S_j = r_j + a^j, \\ E_j &= [s_j,\ S_j], \quad E = \cup E_j, \end{aligned} \tag{3.6}$$

and observe that $\mu(E) \le \sum_{j=1}^{\infty} 2a^j = A$, where A depends only on a. Clearly, for all sufficiently large j, we can assume that $E_j \cap E_{j+1} = \emptyset$. Therefore, there is a j_0 such that for all large r with $r \notin \bigcup_{j=j_0} E_j$ if and only if $r \in \bigcup_{j=j_0}^{\infty} (r_j + a^j,\ r_{j+1} - a^{j+1})$. Since $\alpha_1(r) > 2^r$, so we have for large j,

$$r_j < j + 1. \tag{3.7}$$

We first prove that (1.3) defines an entire function.

If $r > S_k$, we have from (3.6)–(3.7) that for all large k,

$$\log \frac{r}{r_k} \ge \log\Big(1 + \frac{a^k}{r_k}\Big) \ge \frac{a^k}{2r_k} \ge \frac{a^k}{2(1+k)},$$

so that by (1.4),

$$\Big(\frac{r}{r_k}\Big)^{n_k} \ge \exp\Big(\frac{(2a)^k}{2(k+1)}\Big). \tag{3.8}$$

If $r < s_k$, we see from (3.6)–(3.7) that for all large k,

$$\log \frac{r}{r_k} \le \log\Big(1 - \frac{a^k}{r_k}\Big) \le -\frac{a^k}{r_k} \le -\frac{a^k}{(1+k)}$$

and so

$$\Big(\frac{r}{r_k}\Big)^{n_k} \le \exp\Big(-\frac{(2a)^k}{1+k}\Big). \tag{3.9}$$

In general, the $\{E_j\}$ need not be disjoint. But for any $r \notin E$, there exists a unique r_k such that $r \in (r_k,\ r_{k+1}) \setminus E$, $r > S_j$ $(j = 1, \cdots, k)$ and $r < s_j$ $(j = k+1, \cdots)$. Thus we have with $z = re^{i\theta}$,

$$\begin{aligned}
\log|f(z)| &= \sum_{j=1}^{k} n_j \log\Big|\frac{z}{r_j}\Big| + \sum_{j=1}^{k} \log\Big|\Big(\frac{r_j}{z}\Big)^{n_j} + 1\Big| + \sum_{j=k+1}^{\infty} \log\Big|1 + \Big(\frac{z}{r_j}\Big)^{n_j}\Big| \\
&= N(r) + \sum_{j=1}^{k} \log\Big|\Big(\frac{r_j}{z}\Big)^{n_j} + 1\Big| + \sum_{j=k+1}^{\infty} \log\Big|1 + \Big(\frac{z}{r_j}\Big)^{n_j}\Big| \\
&= N(r) + \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned} \tag{3.10}$$

Since $2a > 1$, we see from (3.8)–(3.9) respectively that if r is large enough, then

$$\begin{aligned}
|\mathrm{I}_1| &\le \sum_{j=1}^{k} \Big(\frac{r_j}{r}\Big)^{n_j} \le \sum_{j=1}^{\infty} \exp\Big(-\frac{(2a)^j}{2(j+1)}\Big) + O(1) = O(1), \\
|\mathrm{I}_2| &\le \sum_{j=k+1}^{\infty} \Big(\frac{r}{r_j}\Big)^{n_j} \le \sum_{j=1}^{\infty} \exp\Big(-\frac{(2a)^j}{1+j}\Big) + O(1) = O(1).
\end{aligned}$$

Thus f is entire, and (3.1)–(3.2) hold. Furthermore, it follows from (3.5) and (3.2) that

$$T_f(r) = N(r) + O(1) \le \alpha(r), \quad r > r_*,\ r \notin E.$$

As above, we assume that $r \in (r_k,\ r_{k+1})\backslash E$, with k large. Thus, as in (3.10),

$$\frac{zf'(z)}{f(z)} = \sum_{j=1}^{k} n_j + \sum_{j=1}^{k} \frac{-n_j}{1+\left(\dfrac{z}{r_j}\right)^{n_j}} + \sum_{j=k+1}^{\infty} \frac{n_j\left(\dfrac{z}{r_j}\right)^{n_j}}{1+\left(\dfrac{z}{r_j}\right)^{n_j}}$$
$$= n(r) + \mathrm{J}_1 + \mathrm{J}_2.$$

Since $2a > 1$, we see from (1.4) and (3.8)–(3.9) that

$$|\mathrm{J}_1| \le 2\sum_{j=1}^{k} \frac{n_j}{\left(\dfrac{r}{r_j}\right)^{n_j}} \le 2\sum_{j=1}^{\infty} 2^j \exp\Big(-\frac{(2a)^j}{2(1+j)}\Big) + O(1) = O(1),$$

$$|\mathrm{J}_2| \le \sum_{j=k+1}^{\infty} \frac{n_j\left(\dfrac{r}{r_j}\right)^{n_j}}{1-\left(\dfrac{r}{r_j}\right)^{n_j}} \le 3\sum_{j=1}^{\infty} 2^j \exp\Big(-\frac{(2a)^j}{1+j}\Big) + O(1) = O(1).$$

Consequently, if $z = r\mathrm{e}^{\mathrm{i}\theta}$,

$$\frac{zf'(z)}{f(z)} = n(r) + O(1), \quad r > r_*,\ r \notin E,$$

and so (3.3) holds. Thus, the lemma is completely proved.

Lemma 3.3 (see [12]) *Let $g(r) : (0,\infty) \to \mathbb{R}, h(r) : (0,\infty) \to \mathbb{R}$ be non-decreasing functions. If $g(r) \le h(r)$, $r \notin (0,1] \cup H$, where $H \subset (1,\infty)$ is a set of finite logarithmic measure, then for any $\alpha > 1$, there exists r_0 such that $g(r) \le h(\alpha r)$ for all $r \ge r_0$.*

Lemma 3.4 *Let f be defined as in* (1.3)*. Then, for all large r,*

$$\log M(3r) \ge \frac{3}{2}\log M(r).$$

Proof We know that, for $r \in [r_k, r_{k+1})$ and $s \in [r_{k+1}, r_{k+2})$, $n(r) = 2^{k+1} - 2$ and $n(s) = 2^{k+2} - 2$. Therefore (later, we take $s = r_{k+1}$)

$$n(s) = 2n(r) - 6.$$

Further, when $r \in [r_k, r_{k+1})$,

$$2r \ge 2r_k \ge \frac{3}{2}r_{k+1}$$

for all large r, or, all large k since $\frac{r_{k+1}}{r_k} \to 1 < \frac{4}{3}$. Thus, for any large r, there is a k such that

$$r \in [r_k, r_{k+1}), \quad 2r \in [r_{k+1}, \infty).$$

Hence, $n(2r) \ge n(r_{k+1}) = 2n(r) - 6$ for all large r. Consequently,

$$\int_{r_0}^{r} \frac{n(2t)}{t}\mathrm{d}t \ge 2\int_{r_0}^{r} \frac{n(t)}{t}\mathrm{d}t - C\log r,$$

which implies

$$N(2r) \ge 2N(r) - c\log r \ge \frac{7}{4}N(r).$$

By Lemma 3.2, we have $\log M(r) = N(r) + O(1)$ for $r \in E$ with $mE < \infty$. Set

$$F = \{r, \quad r \in E \text{ or } 2r \in E\}.$$

Then $mF < \infty$ and when $r \in \mathbb{R} \setminus F$, we get

$$\log M(2r) \geq \frac{3}{2} \log M(r).$$

It follows from Lemma 3.3 that $\log M(3r) \geq \frac{3}{2} \log M(r)$ for all large r. Thus, the lemma is proved.

Remark 3.1 It seems $\log M(3r)$ cannot be bounded from above by $C \log M(r)$ for any positive constant C.

We also need the following lemmas. Firstly, we introduce the Koebe distortion theorem.

Lemma 3.5 (Koebe Distortion Theorem) *Let $z_0 \in \mathbb{C}, r > 0$ and f be a univalent function in this disk $D(z_0, r)$. If $z \in D(z_0, r)$, then*

$$r^2 |f'(z_0)| \frac{r - |z - z_0|}{(r + |z - z_0|)^3} \leq |f'(z)| \leq r^2 |f'(z_0)| \frac{r + |z - z_0|}{(r - |z - z_0|)^3}$$

and

$$r^2 |f'(z_0)| \frac{|z - z_0|}{(r + |z - z_0|)^2} \leq |f(z) - f(z_0)| \leq r^2 |f'(z_0)| \frac{|z - z_0|}{(r - |z - z_0|)^2}.$$

For our use, we need the following version.

Lemma 3.6 *Let G be a domain and K be a compact subset of G. Then there exists a positive constant C such that if f is univalent in G and $z, \xi \in K$, then $|f'(\xi)| \leq C|f'(z)|$.*

Remark 3.2 The constant C in above lemma only depends on the relative location of G and K.

The following lemma developed by McMullen [16, Proposition 2.2], plays an important role in calculating that Hausdorff dimension.

Lemma 3.7 (see [14, Proposition 2.2]) *Let $\mathbb{A}, \Delta_l$ and d_l be as in Definition* 2.1. *Then*

$$\limsup_{l \to \infty} \frac{\sum_{n=1}^{l} |\log \Delta_n|}{|\log d_l|} \geq 2 - \dim \mathbb{A}.$$

In order to construct the intersection nesting sets $\mathcal{A}$, we shall use the Ahlfors islands theorem.

Lemma 3.8 (see [13, Theorem 6.2]) *Let Ω_v, $v \in \{1, 2, 3\}$ are Jordan domains with pairwise disjoint closures, let $a \in \mathbb{C}, r > 0$ and f be a analytic function from the disk $D(a, r)$ to the complex plane $\mathbb{C}$. If there exists $\mu > 0$ such that*

$$\frac{|f'(a)|}{1 + |f(a)|^2} \geq \frac{\mu}{r}.$$

Then $D(a, r)$ has a subdomain U which is mapped bijectively onto one of the domain Ω_v.

Lemma 3.9 (see [24, Corollary 5]) *Let f be a transcendental meromorphic function with at most finitely many poles and let $d > 1$ be a constant. If for all sufficiently large $R > 0$, we have*

$$\log M(CR, f) > d \log M(R, f) \quad \textit{for some } C > 1,$$

then $J(f)$ has an unbounded component and all components of $F(f)$ are simply connected.

4 Proof of Theorem 1.1

First, we construct a family $\{\mathcal{A}_n\}_{n=1}^{\infty}$, which is of the nesting condition defined in (2.1). As before, we denote the disk with radius r and center a by $D(a, r)$ and the annulus $A_j = \{z;\ S_j < |z| < s_{j+1}\}$ for $j \geq j_0$, as in Lemma 3.2.

For any large integer k and $a \in A_k$, recalling that $f(z)$ has no zeros in the domain A_k, we can define a holomorphic function

$$h(z) = \log f(z) - \log f(a)$$

in a neighborhood of a.

Thus $h(a) = 0$ and Lemma 3.2 implies

$$\frac{h'(a)}{1 + |h(a)|^2} = |h'(a)| = \frac{|f'(a)|}{|f(a)|} = \frac{n(|a|)}{|a|} + o(1) \geq \frac{n(S_k)}{s_{k+1}},$$

when k is large.

Since $\frac{r_{j+1}}{r_j}$ tends to 1, $\frac{s_{j+1}}{S_j}$ also holds. There exists a constant $\mu < 1$ satisfying

$$\frac{n(S_k)}{s_{j+1}} \geq \mu \frac{n(S_j)}{S_j}$$

for all large S_j.

Let $t_k = \frac{S_k}{n(S_k)}$ and $\Omega_v = \{z;\ 0 \leq \mathrm{Re}(z - \log f(a)) \leq \log 2,\ \mathrm{Im}(z - \log f(a) - 8\pi v) \leq 2\pi\}$, where $v = 1, 2, 3$. It follows form Ahlfors islands theorem, i.e., Lemma 3.8, that there is a set $U \subset D(a, t_k) \subset A_k$ such that $h(z)$ is univalent from U onto one of Ω_v. Noting that $f(z) = \mathrm{e}^{\log f(z)}$, we obtain that the set $U \subset D(a, t_k) \subset A_k$ is mapped bijectively by f onto the annulus

$$\{z:\ |f(a)| < |z| < 2|f(a)|\}.$$

Because of $|f(a)|$ is much bigger than 1 for all large k, there must be some positive integer k_1 greater than k and satisfying

$$|f(a)| < S_{k_1} < s_{k_1+1} < 2|f(a)|. \tag{4.1}$$

Therefore, there are finitely pair-disjoint disks $D(a, t_{k_1})$, say, $\{D(a_{(k_1,i)},\ t_{k_1})\}_{i=1}^{p_1}$, that are contained in A_{k_1}. Consequently, there are finitely many pair-disjoint subdomains $V_{(k_1,i)} \subset D(a_{(k_1,i)}, t_{k_1})$ such that

$$\mathrm{dens}\Big(\bigcup_{i=1}^{p_1} V_{(k_1,i)},\ A_{k_1}\Big) > \frac{1}{2},$$

and further, for each $i = 1, 2, \cdots, p_1$, f bijectively maps $V_{(k_1,i)}$ onto an annulus A_{k_2}, for some integer $k_2 > k_1$, with

$$|f(a_{(k_1,i)})| < S_{k_2} < s_{k_2+1} < 2|f(a_{(k_1,i)})|.$$

Now we use the above argument repeatedly to construct our $\mathcal{A}_n$ with the nesting condition property. Set

$$\mathcal{A}_0 = \{A_{j_0}\}, \quad \mathcal{A}_1 = \{V_{(j_0,i)} : i = 1, 2, \cdots, m_0\},$$

where for any $1 \le i \le m_0$, $V_{(j_0,i)}$ is contained in A_{j_0} and f bijectively maps $V_{(j_0,i)}$ onto an annulus A_{j_1} for some integer $j_1 > j_0$ while

$$\operatorname{dens}\Big(\bigcup_{i=1}^{m_0} V_{(j_0,i)},\ A_{j_0}\Big) > \frac{1}{2},$$

where $\mathbb{A}_1 = \bigcup_{i=1}^{m_0} V_{(j_0,i)}$, which is defined as the union of all elements in $\mathcal{A}_1$.

Now we construct $\mathcal{A}_2$. By using the above argument, we have finitely many pair-disjoint subdomains $V_{(j_1,i)} \subset D(a_{(j_1,i)},\ t_{j_1}) \subset A_{j_1}$, say $i = 1, \cdots, m_1$, such that, for any $1 \le i \le m_1$, f bijectively maps $V_{(j_1,i)}$ onto an annulus A_{j_2} for some integer $j_2 > j_1$ and

$$\operatorname{dens}\Big(\bigcup_{i=1}^{m_0} V_{(j_1,i)},\ A_{j_1}\Big) > \frac{1}{2}.$$

Therefore, we define

$$\mathcal{A}_2 = \{f^{-1}_{V_{(j_0,k)}}(V_{(j_1,i)}),\ i = 1, 2, \cdots, m_1, k = 1, \cdots, m_0\},$$

where $f^{-1}_{V_{(j_0,k)}}$ is the inverse function f restricted on $V_{(j_0,k)}$. Clearly, for any fixed k with $1 \le k \le m_0$, $f^{-1}_{V_{(j_0,k)}}(V_{(j_1,i)}) \subset V_{(j_0,k)}$ for $i = 1, \cdots, m_1$. Moreover, for any $V_{(j_0,k)} \in \mathcal{A}_1$, we have from Lemma 3.5 that

$$\begin{aligned}\operatorname{dens}(\mathbb{A}_2,\ V_{(j_0,k)}) &= \operatorname{dens}\Big(f^{-1}_{V_{(j_0,k)}}\Big(\bigcup_{i=1}^{m_0} V_{(j_1,i)}\Big),\ f^{-1}_{V_{(j_0,k)}}(A_{j_1})\Big)\\ &\ge \frac{1}{C^2}\operatorname{dens}\Big(\bigcup_{i=1}^{m_0} V_{(j_1,i)},\ A_{j_1}\Big) \ge \frac{1}{2C^2} \overset{\text{def}}{=} \Delta_1,\end{aligned}$$

where, again, $\mathbb{A}_2$ is defined as the union of all elements in $\mathcal{A}_2$.

Repeating this process, we obtain

$$\mathcal{A}_{n+1} = \{f_q^{-n}(V_{(j_n,i)}) :\ i = 1, 2, \cdots, m_n,\ q = 1, \cdots, m_{n-1}\},$$

where f_q^{-n} is the n-th inverse of f restricted on related previous $V_{(j_t,s)}$ for a suitable j_t, s which depend on q; $V_{(j_n,i)}$ is contained in the annulus A_{j_n} with

$$\operatorname{dens}\Big(\bigcup_{i=1}^{m_n} V_{(j_n,i)},\ A_{j_n}\Big) > \frac{1}{2},$$

and f is a univalent map from $V_{(j_n,i)}$ to an annulus $A_{j_{n+1}}$. Thus, for any $F \in \mathcal{A}_n$, applying Lemma 3.5 to f^n gives

$$\begin{aligned}\operatorname{dens}(\mathbb{A}_{n+1}, F) &= \operatorname{dens}\Big(f_q^{-n}\Big(\bigcup_{i=1}^{m_{n-1}} V_{j_n,i}\Big), F\Big)\\ &\geq \frac{1}{C^2}\operatorname{dens}\Big(\bigcup_{i=1}^{m_{n-1}} V_{(j_n,i)},\ A_{j_n}\Big)\\ &\geq \frac{1}{2C^2} \stackrel{\text{def}}{=} \Delta_n.\end{aligned}$$

For any $F \in \mathcal{A}_n$, say, F_n, by our construction, there is $F_{n-1} \in \mathcal{A}_{n-1}$ such that $F_n \subset F_{n-1}$ and $f^{n-1}(F_n) = V_{(j_{n-1},i)} \subset A_{j_n}$ for some i. In the following, we will drop i from the notation $a_{(j_n,i)}$, $V_{(j_n,i)}$ and simply use a_{j_n}, V_{j_n}, respectively.

Noting, $V_{j_n} \subset D(a_{j_n}, t_{j_n})$, we get from Lemma 3.5 that

$$\begin{aligned}\operatorname{diam} F_n &\leq C_1 \,|(f^{-(n-1)})'(a_{j_n})|\operatorname{diam}(V_{j_{n-1}})\\ &\leq C_1 \frac{1}{|(f^{n-1})'(z_0)|}\frac{S_{j_{n-1}}}{n(S_{j_{n-1}})},\end{aligned}$$

where $z_0 \in F_n$ with $f^{(n-1)}(z_0) = a_{j_n}$ and C_1 is a constant.

Moreover, by Lemma 3.2,

$$\begin{aligned}|(f^{n-1})'(z_0)| &= \prod_{i=0}^{n-2} |f'(f^i(z_0))| \geq \frac{1}{2}\prod_{i=0}^{n-2} |f^{i+1}(z_0)|\,\frac{n(|f^i(z_0)|)}{|f^i(z_0)|}\\ &\geq \frac{1}{2}\prod_{i=1}^{n-1} \mu\, S_{j_{i+1}}\,\frac{n(S_{j_i})}{S_{j_i}} = \frac{\mu\, S_{j_n}}{2S_{j_1}}\prod_{i=1}^{n-1} n(S_{j_i}).\end{aligned}$$

So

$$\operatorname{diam} F_n \leq \frac{S_{j_1}}{C_1\,\mu}\prod_{i=1}^{n}\frac{1}{n(S_{j_i})}. \tag{4.2}$$

Recalling our definition of f, we have $n(S_{j_i}) = 2^{j_i+1} - 2 \geq 2^{j_i}$ for all large j_i. Thus

$$\operatorname{diam} F_n \leq C2^{-(j_1+j_2+\cdots+j_n)} \stackrel{\text{def}}{=} d_n.$$

Hence,

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} |\log \Delta_i|}{|\log d_n|} = 0,$$

and consequently, Lemma 3.7 implies that

$$\dim\mathbb{A} = \dim\Big(\bigcap_{i=1}^{\infty} \mathbb{A}_i\Big) = 2.$$

Furthermore, applying Lemmas 3.4 and 3.9 to our f, we have $A_{j_n} \cap J(f)$ is not empty for all large j_n. The completely invariance of $J(f)$ implies that $\mathbb{A}_n \cap J(f)$ is also not empty for all n. So $\mathbb{A} \subset J(f)$ from the definition of $\mathbb{A}$.

Lemma 3.2 also implies $\mathbb{A} \subset I^o(f)$. Therefore, we have $\mathbb{A} \subset J(f) \cap I^o(f)$. The theorem is completely proved.

5 Results on Hausdorff Measure

In [14], McMullen remarked that $H^h(J(E_\lambda)) = \infty$ when $h(t) = t^2 \log^m \left(\frac{1}{t}\right)$, $m \in \mathbb{N}$ for exponential maps E_λ.

Recently, Peter [16–17] studied the Hausdorff measure on Julia set of exponential functions and entire functions in class $\mathcal{B}$ by introducing gauge function Φ. He obtained the following theorems.

Theorem D *Define* $\lambda \in \left(0, \frac{1}{\mathrm{e}}\right), \beta_\lambda, \Phi_\lambda$ *as above, let* $K_\lambda = \frac{\log 2}{\log \beta_\lambda}$ *and* $h(t) = t^2 g(t)$ *be a gauge function. If*

$$\liminf_{t\to 0} \frac{\log g(t)}{\log \Phi_{\lambda\left(\frac{1}{t}\right)}} > K_\lambda,$$

then $H^h(J(E_\lambda)) = \infty$ *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.

Theorem E *Let* $\lambda \in \left(0, \frac{1}{\mathrm{e}}\right)$. *There exists* $K > 0$ *with the following property: If* $f \in \mathcal{B}$ *and* $\rho(f) = \rho > \frac{1}{2}$, *then* $H^h(J(f)) = \infty$, *where* $h(t) = t^2\left(\Phi_\lambda\left(\frac{1}{t}\right)\right)^\kappa$ *and* $\kappa > \frac{\log \rho + K}{\log \beta_\lambda}$.

Naturally, we want to know that, for which gauge functions h, the Hausdorff measure of $J(f)$ and $I^o(f)$ is ∞, where f is defined as (1.3). We obtain the following results.

Theorem 5.1 *Let* $m \in \mathbb{N}$ *and* $h(t) = t^2 \log^m \left(\frac{1}{t}\right)$. *Then* $H^h(J(f) \cap I^o(f)) = \infty$.

Theorem 5.2 *Suppose* $\lambda \in \left(0, \frac{1}{\mathrm{e}}\right)$, $\kappa > \frac{\log(\frac{1}{\Delta})}{\log \beta_\lambda}$ *and* Φ_λ *is as above. Let* $h(t) = t^2 g(t)$ *be a gauge function. If*

$$\liminf_{t\to 0} \frac{\log g(t)}{\log \Phi_\lambda\left(\frac{1}{t}\right)} > \kappa.$$

Then $H^h(J(f) \cap I^o(f)) = \infty$.

Peter in [16] developed McMullen's technique and proved the following lemma which is a main tool in this kind of studies.

Lemma 5.1 (see [16, Lemma 3.3]) *Let* $\mathbb{A}, \{d_n\}$ *and* $\{\Delta_n\}$ *be as above. Let* $\varepsilon > 0$ *and* $\varphi : (0, \varepsilon) \to \mathbb{R}_{\geq 0}$ *be a decreasing continuous function such that* $t^2\varphi(t)$ *is increasing. Further, suppose that* $\lim_{t\to 0} t^2\varphi(t) = 0$ *and*

$$\lim_{n\to\infty} \varphi(d_n) \prod_{j=1}^{n} \Delta_j = \infty. \tag{5.1}$$

Define

$$h : [0, \varepsilon) \to \mathbb{R}, \quad t \mapsto \begin{cases} t^2\varphi(t), & t > 0, \\ 0, & t = 0. \end{cases}$$

Then $h(t)$ *is a continuous gauge function and* $H^h(\mathbb{A}) = \infty$.

Proof of Theorem 5.1 All notations in the following proof are defined in the proof of Theorem 1.1. Thanks to Lemma 5.1, it is sufficient to check that φ satisfies the condition (5.1). To accomplish that, we need to estimate $\operatorname{diam} F$, which is different what we have in the proof of Theorem 1.1.

Since $\alpha_1'(r) > \alpha_1(r)$, we can get $\alpha_1(r) > 3r$ for sufficiently large r. Recalling (3.4), (4.2) and the fact that $n(r)$ is constant when $r \in [S_{j_n},\ s_{j_n+1}]$, we get

$$\operatorname{diam} F_n \le \frac{S_{j_1}}{C_1\,\mu}\prod_{i=1}^{n}\frac{1}{n(S_{j_i})} \le \frac{1}{n(S_{j_n})} = \frac{1}{n(s_{j_n+1})} \le \frac{1}{s_{j_n+1}}.$$

Lemma 3.2 implies that $|f(z)| \ge K\mathrm{e}^{N(|z|)} \ge \mathrm{e}^{|z|}$, where K is a constant. Moreover, by the augment used in (4.1) and the fact that $|a_{(j_{n-1},i)}| \in (S_{j_{n-1}},\ s_{j_n})$, we get

$$s_{j_n+1} > |f(a_{(j_{n-1},i)})| \ge \mathrm{e}^{S_{j_{n-1}}}$$

for all $n \in \mathbb{N}$. Thus $s_{j_n+1} > (\mathrm{e}^x)^n\big|_{x=S_{j_0}}$, where $(\mathrm{e}^x)^n$ denote the n-th iteration of e^x.

Thus (5.1) is verified for the decreasing continuous function $\varphi(t) = \log^m\left(\frac{1}{t}\right)$, $\Delta_n = \frac{1}{2C^2}$ and $d_n = \frac{1}{(\mathrm{e}^x)^n}\big|_{x=s_{j_0}}$. It follows that the theorem is proved.

Proof of Theorem 5.2 As the same reason we need to check (5.1) holds for $\varphi(t) = \left(\Phi_\lambda\left(\frac{1}{t}\right)\right)^\kappa$. Noting that the functional equation (2.1), we can deduce that

$$\Phi_\lambda(E_\lambda(z)) = \beta_\lambda\Phi_\lambda(z), \quad z, E_\lambda(z) \in U.$$

Combining with $d_n = \frac{1}{(\lambda \mathrm{e}^x)^n|_{x=r_0}}$, we get that

$$\Phi_\lambda\Big(\frac{1}{d_n}\Big)^\kappa \prod_{j=1}^{n}\Delta_j = \Phi_\lambda((\lambda\mathrm{e}^x)^n|_{x=r_0})^\kappa\Delta^n = (\beta_\lambda^n\Phi_\lambda(r_0))^\kappa\Delta^n = (\beta_\lambda^\kappa\Delta)^n\,\Phi_\lambda(r_0)^\kappa.$$

Thus the assumption $\kappa > \frac{(\log(\frac{1}{\Delta}))}{\log\beta_\lambda}$ implies (5.1) is true, and consequently, the theorem is proved.

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References

[1] Barański, K., Hausdorff dimension of hairs and ends for entire maps of finite order, *Math. Proc. Camb. Phil. Soc.*, **145**, 2008, 719–737.

[2] Beardon, A. F., Iteration of Rational Functions, Spinger-Verlag, Berlin, 1991.

[3] Bergweiler, W., Iteration of meromorphic functions, *Bull. Amer. Math. Soc. (N. S.)*, **29**, 1993, 151–188.

[4] Bergweiler, W., Lebesgue measure of Julia sets and escaping sets of certain entire functions, *Fund. Math.*, **242**, 2018, 281–301.

[5] Bergweiler, W. and Chyzhykov, I, Lebesgue measure of escaping sets of entire functions of completely regular growth, *J. London Math. Soc.*, **94**(2), 2016, 639–661.

[6] Bergweiler, W. and Hinkkanen, A., On semiconjugation of entire functions, *Math. Pro. Camb. Phil. Soc.*, **126**(3), 1999, 565–574.

[7] Bergweiler, W. and Karpińska, B., On the Hausdorff dimension of the Julia set of a regularly growing entire function, *Math. Pro. Camb. Phil. Soc.*, **148**, 2010, 531–551.

[8] Bergweiler, W., Karpińska, B. and Stallard, G. M., The growth rate of an entire function and the Hausdorff dimension of its Julia set, *J. London Math. Soc.*, **80**(2), 2009, 680–698.

[9] Eremenko, A. E., On the Iteration of Entire Functions, Dynamical Systems and Ergodic Theory, **23**, Banach Center Publ., Warsaw, 1989, 339–345.

[10] Eremenko, A. E. and Lyubich, M. Yu., Dynamical properties of some classes of entire functions, *Ann. Inst. Fourier,* **42**, 1992, 989–1020.

[11] Goldberg, A. A. and Ostrovsikii, I. V., Value Distribution of Meromorphic Functions, Translation of Mathematical Monographs, **236**, *Amer. Math. Soc.*, Province, Rhode Island, 2008.

[12] Gundersen, G., Finite order solutions of second order linear differential equations, *Trans. Amer. Math. Soc.*, **305**, 1988, 415–429.

[13] Hayman, W. K., Meromorphic Functions, Clarendon Press, Oxford, UK, 1964.

[14] McMullen, C., Area and Hausdorff dimension of Julia sets of entire functions, *Trans. Amer. Math. Soc.*, **300**, 1987, 329–342.

[15] Milnor, J., Dynamics in One Complex Variable, Friedr. Vieweg & Sohn, Braunschweig, 1999.

[16] Peter, J., Hausdorff measure of Julia sets in the exponential family, *J. London Math. Soc.*, **82**(1), 2010, 229–255.

[17] Peter, J., Hausdorff measure of escaping and Julia sets for bounded-type functions of finite order, *Ergod. Th. & Dynam. Sys.*, **33**(1), 2013, 284–302.

[18] Rippon, P. J. and Stallard, G. M., Fast escaping points of entire functions, *Pro. London Math. Soc.*, **105**(4), 2012, 787–820.

[19] Schubert, H., Über die Hausdorff-Dimension der Juliamenge von Funktionen endlicher Ordnung, Dissertation, University of Kiel, 2007, http://eldiss.uni-kiel.de/macau/receive/dissertation_diss_00002124.

[20] Sixsmith, D., Functions of genus zero for which the fast escaping set has Hausdorff dimension two, *Pro. Amer. Math. Soc.,* **143**(6), 2015, 2597–2612.

[21] Stallard, G. M., Dimensions of Julia sets of transcendental meromorphic functions, Transcendental Dynamics and Complex Analysis, London Math. Soc. Lect. Note **348**, Rippon, P. J. and Stallard, G. M.(eds.), Cambridge University Press, Cambridge, 2008, 425–446.

[22] Taniguchi, M., Size of the Julia set of structurally finite transcendental entire function, *Math. Proc. Camb. Phil. Soc.*, **135**, 2003, 181–192.

[23] Ye, Z., On Nevnalinna's error terms, *Duke Math. J.*, **64**(2), 1991, 243–260.

[24] Zheng, J. H., On multiply-connected Fatou components in iteration of meromorphic functions, *J. Math. Anal. Appl.*, **313**, 2006, 24–37.