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Forward and Backward Mean-Field Stochastic Partial Differential Equation and Optimal Control*

Maoning TANG[1] Qingxin MENG[2] Meijiao WANG[3]

Abstract This paper is mainly concerned with the solutions to both forward and backward mean-field stochastic partial differential equation and the corresponding optimal control problem for mean-field stochastic partial differential equation. The authors first prove the continuous dependence theorems of forward and backward mean-field stochastic partial differential equations and show the existence and uniqueness of solutions to them. Then they establish necessary and sufficient optimality conditions of the control problem in the form of Pontryagin's maximum principles. To illustrate the theoretical results, the authors apply stochastic maximum principles to study the infinite-dimensional linear-quadratic control problem of mean-field type. Further, an application to a Cauchy problem for a controlled stochastic linear PDE of mean-field type is studied.

Keywords Mean-field, Stochastic partial differential equation, Backward stochastic partial differential equation, Optimal control, Maximum principle, Adjoint equation
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1 Introduction

In recent years, due to many practical and theory applications, in the finite dimensional cases, the stochastic differential equation of mean-field type, also called mean-field stochastic differential equation (MF-SDE for short), and the corresponding optimal control problem and financial applications have been studied expensively. For more details on these topics, the interested reader is referred to [1, 4, 6, 11, 13–17, 19–22] and therein. On the other hand, intuitively speaking, the adjoint equation of a controlled state process driven by the MF-SDE is a mean-field backward stochastic differential equation (MF-BSDE for short). So it is not until Buckdahn et al. [3, 5] established the theory of the MF-BSDEs that the optimal control problem of mean-field type has become a popular topic where the adjoint equation associated with the stochastic maximum principle is a MF-BSDE.

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[1]Department of Mathematical Sciences, Huzhou University, Huzhou 313000, Zhejiang, China. E-mail: tmorning@zjhu.edu.cn
[2]Corresponding Author. Department of Mathematical Sciences, Huzhou University, Huzhou 313000, Zhejiang, China. E-mail: mqx@zjhu.edu.cn
[3]Business School, University of Shanghai for Science and Technology, Shanghai 200093, China. E-mail: mjiao_wang@163.com
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The purpose of this paper is to extend the finite dimensional MF-SDE and MF-BSDE and the corresponding optimal control problem to infinite dimensional case, i.e., to mean-field stochastic partial differential equations (MF-SPDE for short) and backward mean-field stochastic partial differential equations (MF-BSPDE for short). We will establish the basic theory of MF-SPDE and MF-BSPDE and the basic optimal control theory for MF-SPDE. Precisely speaking, by Itô's formula in the Gelfand triple and under some proper assumptions, we firstly prove continuous dependence property of the solution to both MF-SPDE and MF-BSPDE on the parameter. Then the existence and uniqueness of solutions to MF-SPDE and MF-BSPDE is proved by the continuous dependence theorem and the classic parameter extension approach. The second main result established in this paper is the corresponding sufficient and necessary stochastic maximum principle for the optimal control problem of MF-BSPDE, which are obtained by establishing a convex variation formula under the convexity assumption of the control domain. Finally, to illustrate our results, we apply the stochastic maximum principles to a mean-field linear-quadratic (LQ for short) control problem of MF-SPDE. Using the necessary and sufficient maximum principles, the optimal control strategy is given explicitly in a dual representation. As an application, a LQ problem for a concrete cauchy problem of controlled mean-field stochastic partial equation is solved.

The rest of this paper is organized as follows. Section 2 gives notations and framework. In Section 3, we prove the continuous dependence theory and the existence and uniqueness of solutions to MF-SPDE in the abstract form. In Section 4, we prove the continuous dependence theory and the existence and uniqueness of solutions to MF-BSPDE in the abstract form. In Section 5, the optimal control problem of MF-SPDE is studied in detail where we establish the stochastic sufficient and necessary maximum principles under convex control domain assumption. Sections 6 applies the stochastic maximum principles to solve linear-quadratic optimal control problems of MF-SPDE. The final section concludes the paper.

Moreover, we refer to [7–9, 12] on the existence, uniqueness and regularity of solutions to infinite dimensional BSEEs as well as backward stochastic partial differential equations.

2 Notations

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space on which one-dimensional real-valued Brownian motion $\{W(t), 0 \leq t \leq T\}$ is defined with $\mathbb{F} \triangleq \{\mathscr{F}_t, 0 \leq t \leq T\}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. Denote by $\mathbb{E}[\cdot]$ the expectation with respect to the probability $\mathbb{P}$. We denote by $\mathscr{P}$ the predictable σ-algebra associated with $\mathbb{F}$. For any topological space Λ, we denote by $\mathscr{B}(\Lambda)$ its Borel σ-algebra. Let X be any Hilbert space in which the norm is denoted by$\|\cdot\|_X$. Next we introduce the following spaces:

- $\mathcal{M}^2_{\mathscr{F}}(0,T;X)$: The Space of all X-valued $\mathscr{F}$-adapted processes $f \triangleq \{f(t,\omega), (t,\omega) \in [0,T]\times\Omega\}$ endowed with the norm $\|f\|_{\mathcal{M}^2_{\mathscr{F}}(0,T;X)} \triangleq \sqrt{\mathbb{E}\big[\int_0^T \|f(t)\|^2_X \mathrm{d}t\big]} < \infty$;

• $\mathcal{S}^2_{\mathscr{F}}(0,T;X)$: The space of all X-valued $\mathscr{F}$-adapted càdlàg processes $f \triangleq \{f(t,\omega),(t,\omega)\in[0,T]\times\Omega\}$ endowed with the norm $\|f\|_{\mathcal{S}^2_{\mathscr{F}}(0,T;X)} \triangleq \sqrt{\mathbb{E}\big[\sup_{0\le t\le T}\|f(t)\|^2_X\big]}<\infty$;

• $\mathcal{L}^p(\Omega,\mathscr{F},\mathbb{P};X)$: The space of all X-valued $\mathscr{F}$-measurable random variables ξ endowed with the norm $\|\xi\|_{\mathcal{L}^p(\Omega,\mathscr{F},\mathbb{P};X)} \triangleq \sqrt{\mathbb{E}[\|\xi\|^p_X]}<\infty$, where $p\ge 1$ are given real number.

3 Mean-Field Stochastic Partial Differential Equation

This section is devoted to the study of the MF-SPDE in an abstract form. Let $(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}}) = (\Omega\times\Omega,\mathscr{F}\times\mathscr{F},\mathbb{P}\times\mathbb{P})$ be the product of $(\Omega,\mathscr{F},\mathbb{P})$ with itself. We endow this product space with the filtration $\{\overline{\mathscr{F}}_t\}_{0\le t\le T}=\{\mathscr{F}_t\times\mathscr{F}_t\}_{0\le t\le T}$. By $\overline{\mathscr{P}}$ we denote the product $\mathscr{P}\times\mathscr{P}$. Let $\overline{\mathbb{E}}$ denote the expectation with respect to the product probability space $\overline{\Omega}$. Denote by $\mathcal{M}^2_{\overline{\mathscr{F}}}(0,T;X)$ the set of all X-valued $\overline{\mathscr{F}}$-adapted processes $f\triangleq\{f(t,\omega',\omega),(t,\omega',\omega)\in[0,T]\times\overline{\Omega}\}$ such that $\|f\|_{\mathcal{M}^2_{\overline{\mathscr{F}}}(0,T;X)}\triangleq\sqrt{\overline{\mathbb{E}}\big[\int_0^T\|f(t)\|^2_X\mathrm{d}t\big]}<\infty$. For $p\ge 1$, a random variable $\xi\in L^p(\Omega,\mathscr{F},\mathbb{P};X)$ originally defined on Ω can be extended canonically to $\overline{\Omega}$: $\xi'(\omega',\omega)=\xi(\omega'),(\omega',\omega)\in\overline{\Omega}$. For any $\theta\in\mathcal{L}^p(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}};X)$, the variable $\theta(\cdot,\omega):\Omega\to X$ belongs to $\mathcal{L}^p(\Omega,\mathscr{F},\mathbb{P};X)$, $\mathbb{P}(\mathrm{d}\omega)$-a.s., we denote its expectation by

$$\mathbb{E}'[\theta(\cdot,\omega)]=\int_\Omega\theta(\omega',\omega)\mathbb{P}(\mathrm{d}\omega').$$

Notice that $\mathbb{E}'[\theta]=\mathbb{E}'[\theta(\cdot,\omega)]\in L^p(\Omega,\mathscr{F},\mathbb{P};X)$ and

$$\overline{\mathbb{E}}[\theta](=\int_{\overline{\Omega}}\theta\mathrm{d}\overline{\mathbb{P}}=\int_\Omega\mathbb{E}'[\theta(\cdot,\omega)]\mathbb{P}\mathrm{d}(\omega))=\mathbb{E}[\mathbb{E}'[\theta]].$$

Let

$$V\subset H=H^*\subset V^*$$

be Gelfand triple, i.e., $(H,(\cdot,\cdot)_H)$ is a separable Hilbert spaces and V is a reflextive Banach space such that H is identified with its dual space H^* by the Riesz isomorphism and V is densely embedded in H. We denote by $\langle\cdot,\cdot\rangle$ the duality product between V and V^*. Moreover, we denote by $\mathscr{L}(V,V^*)$ the set of all bounded linear operators from V into V^*. In the Gelfand triple (V,H,V^*), consider the following operators

$$\begin{aligned}&A=A(t,\omega):[0,T]\times\Omega\to\mathscr{L}(V,V^*),\\&b=b(t,\omega',\omega,x',x):[0,T]\times\overline{\Omega}\times H\times H\to H,\\&g=g(t,\omega',\omega,x',x):[0,T]\times\overline{\Omega}\times H\times H\to H,\end{aligned}\tag{3.1}$$

which satisfy the following standard assumption.

Assumption 3.1 Suppose that there exist constant $\alpha>0,\lambda$, and C such that the following conditions holds for all $x,x',\overline{x},\overline{x}'$ and a.e. $(t,\omega',\omega)\in[0,T]\times\overline{\Omega}$.

(i) (Measurability) The operator A is $\mathscr{P}/\mathscr{B}(\mathscr{L}(V,V^*))$ measurable; b and g are $\overline{\mathscr{P}}\otimes\mathscr{B}(H)\otimes\mathscr{B}(H)/\mathscr{B}(H)$ measurable;

(ii) (Integrality) $b(\cdot,0,0),g(\cdot,0,0)\in\mathcal{M}^2_{\overline{\mathscr{F}}}(0,T;H)$;

(iii) (Coercivity)

$$\langle A(t)x, x\rangle + \lambda\|x\|_H^2 \geq \alpha\|x\|_V^2; \tag{3.2}$$

(iv) (Boundedness)

$$\sup_{(t,\omega)\in[0,T]\times\Omega} \|A(t,\omega)\|_{\mathscr{L}(V,V^*)} \leq C; \tag{3.3}$$

(v) (Lipschitz Continuity)

$$\|b(t,x',x) - b(t,\overline{x}',\overline{x})\|_H + \|g(t,x',x) - g(t,\overline{x}',\overline{x})\|_H \leq C[\|x-\overline{x}\|_H + \|x'-\overline{x}'\|_H]. \tag{3.4}$$

Using the above notations, in the Gelfand triple (V, H, V^*), we consider the MF-SPDE in the following abstract form with the coefficients (A, b, g) defined by (3.1) and the initial value $x \in H$:

$$\begin{cases} \mathrm{d}X(t) = \{-A(t)X(t) + \mathbb{E}'[b(t,X'(t),X(t)]\}\mathrm{d}t + \mathbb{E}'[g(t,X'(t),X(t))]\mathrm{d}W(t), \\ \qquad\qquad t\in[0,T], \\ X(t) = \ x \in H, \end{cases} \tag{3.5}$$

where we have used the following notation defined by

$$\mathbb{E}'[b(t,X'(t),X(t)] = \int_\Omega b(t,\omega',\omega,X(t,\omega'),X(t,\omega))\mathbb{P}(\mathrm{d}\omega') \tag{3.6}$$

and

$$\mathbb{E}'[g(t,X'(t),X(t)] = \int_\Omega g(t,\omega',\omega,X(t,\omega'),X(t,\omega))\mathbb{P}(\mathrm{d}\omega'). \tag{3.7}$$

Now we give the definition of the solution to the MF-SPDE (3.5).

Definition 3.1 *An V-valued, $\mathbb{F}$-adapted process $X(\cdot)$ is said to be a solution to MF-SPDE* (3.5)*, if $X(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V)$ such that for a.e. $(t,\omega)\in[0,T]\times\Omega$ and every $\phi\in V$, we have*

$$\begin{aligned}(X(t),\phi)_H = (x,\phi)_H &- \int_0^t \langle A(s)X(s),\phi\rangle\,\mathrm{d}s + \int_0^t (\mathbb{E}'[b(s,X'(s),X(s))],\phi)_H\mathrm{d}s \\ &+ \int_0^t (\mathbb{E}'[g(s,X'(s),X(s))],\phi)_H\mathrm{d}W(s), \quad t\in[0,T],\end{aligned} \tag{3.8}$$

or alternatively, in the sense of V^*, $X(\cdot)$ have the following Itô form:

$$\begin{aligned}X(t) = x &- \int_0^t A(s)X(s)\mathrm{d}s + \int_0^t \mathbb{E}'[b(s,X'(s),X(s))]\mathrm{d}s \\ &+ \int_0^t \mathbb{E}'[g(s,X'(s),X(s))]\mathrm{d}W(s).\end{aligned} \tag{3.9}$$

The following result is the continuous dependence theorem of the solution to the MF-SPDE (3.5) on the coefficients (A, b, g) and the initial value x which is also called a priori estimate for the solution.

Theorem 3.1 (Continuous Dependence Theorem of MF-SPDE) *Suppose that $X(\cdot)$ is a solution to MF-SPDE* (3.5) *with the initial value x and the coefficients (A,b,g) satisfying Assumptions* 3.1. *Then we have the following estimate:*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le T}\|X(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|X(t)\|_V^2\mathrm{d}t\Big]\\&\le K\Big\{\overline{\mathbb{E}}[\|x\|_H^2]+\overline{\mathbb{E}}\Big[\int_0^T\|b(t,0,0)\|_H^2\mathrm{d}t\Big]+\overline{\mathbb{E}}\Big[\int_0^T\|g(t,0,0)\|_H^2\mathrm{d}t\Big]\Big\},\end{aligned}\tag{3.10}$$

where K is a positive constant which only depend on the constants C,T,α and λ. Further, suppose that $\overline{X}(\cdot)$ is the solution to MF-SPDE (3.5) *with the initial value $\overline{x}$ and the coefficients $(A,\overline{b},\overline{g})$ satisfying Assumption* 3.1. *Then we have*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le T}\|X(t)-\overline{X}(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|X(t)-\overline{X}(t)\|_V^2\mathrm{d}t\Big]\\&\le K\Big\{\|x-\overline{x}\|_H^2+\overline{\mathbb{E}}\Big[\int_0^T\|b(t,\overline{X}'(t),\overline{X}(t))-\overline{b}(t,\overline{X}'(t),\overline{X}(t))\|_H^2\mathrm{d}t\Big]\\&\quad+\overline{\mathbb{E}}\Big[\int_0^T\|g(t,\overline{X}'(t),\overline{X}(t))-\overline{g}(t,\overline{X}'(t),\overline{X}(t))\|_H^2\mathrm{d}t\Big]\Big\}.\end{aligned}\tag{3.11}$$

Proof It suffices to prove (3.11) since the estimate (3.10) can be obtained as a direct consequence of (3.11) by taking the coefficient $(A,\overline{b},\overline{g})=(A,0,0)$ with which the solution to MF-SPDE (3.5) is $\overline{x}(\cdot)=0$. In order to simplify our notation, we denote by

$$\begin{aligned}&\widehat{X}(t)\triangleq X(t)-\overline{X}(t),\\&\widehat{b}(t)\triangleq b(t,\overline{X}'(t),\overline{X}(t))-\overline{b}(t,\overline{X}'(t),\overline{X}(t)),\\&\widehat{g}(t)\triangleq g(t,\overline{X}'(t),\overline{X}(t))-\overline{g}(t,\overline{X}'(t),\overline{X}(t)).\end{aligned}$$

Using Itô's formula to $\|\widehat{X}(t)\|_H^2$, we get that

$$\begin{aligned}\|\widehat{X}(t)\|_H^2=&\|\widehat{x}\|_H^2-2\int_0^t\Big\langle A(s)\widehat{X}(s),\widehat{X}(s)\Big\rangle\mathrm{d}s+2\int_0^t(\mathbb{E}'[b(s,X'(s),X(s))\\&-\overline{b}(s,\overline{X}'(s),\overline{X}(s))],\widehat{X}(s))_H\mathrm{d}s+2\int_0^t(\mathbb{E}'[g(s,X'(s),X(s))\\&-\overline{g}(s,\overline{X}'(s),\overline{X}(s))],\widehat{X}(s)))_H\mathrm{d}W(s)\\&+\int_0^t\|\mathbb{E}'[g(s,X'(s),X(s)-\overline{g}(s,\overline{X}'(s),\overline{X}(s))]\|_H^2\mathrm{d}s.\end{aligned}\tag{3.12}$$

In view of Assumption 3.1 and the elementary inequality $2ab\le a^2+b^2$, $\forall a,b>0$, we obtain

$$\begin{aligned}&\|\widehat{X}(t)\|_H^2+2\alpha\int_0^t\|\widehat{X}(s)\|_V^2\mathrm{d}s\\&\le\|\widehat{x}\|_H^2+K(C,\lambda)\int_0^t\|\widehat{X}(s)\|_H^2\mathrm{d}s+K(C)\int_0^t\mathbb{E}\|\widehat{X}(s)\|_H^2\mathrm{d}s+\int_0^t\mathbb{E}'\|\widehat{b}(s)\|_H^2\mathrm{d}s\\&\quad+2\int_0^T\mathbb{E}'\|\widehat{g}(s)\|_H^2\mathrm{d}s+2\int_0^t(\mathbb{E}'[g(s,X'(s),X(s))\end{aligned}$$

$$- \overline{g}(s,\overline{X}'(s),\overline{X}(s))],\widehat{X}(s)))_H \mathrm{d}W(s). \tag{3.13}$$

Taking expectations on both sides of (3.13) leads to

$$\begin{aligned}&\mathbb{E}[\|\widehat{X}(t)\|_H^2] + 2\alpha\mathbb{E}\Big[\int_0^t \|\widehat{X}(s)\|_V^2 \mathrm{d}s\Big]\\ &\leq \|x\|_H^2 + K(\lambda,C)\mathbb{E}\Big[\int_0^t \|\widehat{X}(s)\|_H^2 \mathrm{d}s\Big] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{b}(s)\|_H^2 \mathrm{d}s\Big] + 2\overline{\mathbb{E}}\Big[\int_0^T \|\widehat{g}(s)\|_H^2 \mathrm{d}s\Big]. \end{aligned}\tag{3.14}$$

Then applying Grönwall's inequality to (3.14) yields

$$\begin{aligned}&\sup_{0\leq t\leq T}\mathbb{E}[\|\widehat{X}(t)\|_H^2] + \mathbb{E}\Big[\int_0^T \|\widehat{X}(t)\|_V^2 \mathrm{d}t\Big]\\ &\leq K\Big\{\|\widehat{x}\|_H^2 + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{b}(t)\|_H^2 \mathrm{d}t\Big] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{g}(t)\|_H^2 \mathrm{d}t\Big]\Big\},\end{aligned}\tag{3.15}$$

where K is a positive constant depending only on T, C, α and λ.

Furthermore, in view of (3.13) and (3.15), the Lipschitz continuity condition (see(3.24)) and the Burkholder-Davis-Gundy, we get that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leq t\leq T}\|\widehat{x}(t)\|_H^2\Big]\\ &\leq K\Big\{\|\widehat{x}\|_H^2 + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{b}(t)\|_H^2 \mathrm{d}t\Big] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{g}(t)\|_H^2 \mathrm{d}t\Big]\Big\}\\ &\quad + 2\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t (\mathbb{E}'[g(s,X'(s),X(s)) - \overline{g}(s,\overline{X}'(s),\overline{X}(s))],\widehat{X}(s)))_H \mathrm{d}W(s)\Big|\Big]\\ &\leq K\Big\{\|x\|_H^2 + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{b}(t)\|_H^2 \mathrm{d}t\Big] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{g}(t)\|_H^2 \mathrm{d}t\Big]\Big\} + \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq t\leq T}\|\widehat{X}(t)\|_H^2\Big].\end{aligned}\tag{3.16}$$

Therefore, (3.11) can be obtained by combining (3.15)–(3.16). The proof is complete.

Theorem 3.2 (Existence and Uniqueness Theorem of MF-SPDE) *Let Assumption* 3.1 *be satisfied. Then for any given initial value x, the MF-SPDE* (3.5) *admits a unique solution $X(\cdot) \in S^2_{\mathscr{F}}(0,T;H)$.*

Proof The uniqueness of the solution of MF-SPDE (3.5) is implied by the a priori estimate (3.11). Consider a family of MF-SPDE parameterized by $\rho \in [0,1]$ as follows:

$$\begin{aligned}X(t) = x &- \int_0^t A(s)X(s)\mathrm{d}s + \int_0^t [\rho\mathbb{E}'[b(s,X'(s),X(s))] + b_0(s)]\mathrm{d}s\\ &+ \int_0^t [\rho\mathbb{E}'[g(s,X'(s),X(s))] + g_0(s)]\mathrm{d}W(s),\end{aligned}\tag{3.17}$$

where $b_0(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ and $g_0(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ are two any given stochastic process. It is easily seen that the original MF-SPDE (3.5) is "embedded" in the MF-SPDE (3.17) when we take the parameter $\rho = 1$ and $b_0(\cdot) \equiv 0, g_0(\cdot) \equiv 0$. Obviously, the MF-SPDE (3.17) have coefficients $(A, \rho b + b_0, \rho g + g_0)$ satisfying Assumption 3.1 with the same Lipschitz constant

C. Suppose for any $b_0(\cdot), g_0(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ and some parameter $\rho = \rho_0$, the MF-SPDE (3.17) admits a unique solution $X(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V)$. For any parameter ρ, we can rewrite the MF-SPDE (3.17) as

$$\begin{aligned} X(t) = x - \int_0^t A(s)X(s)\mathrm{d}s + \int_0^t [\rho_0\mathbb{E}'[b(s,X'(s),X(s))] + b_0(s) \\ + (\rho-\rho_0)\mathbb{E}'[b(s,X'(s),X(s))]]\mathrm{d}s + \int_0^t [\rho_0\mathbb{E}'[g(s,X'(s),X(s))] + g_0(s) \\ + (\rho-\rho_0)\mathbb{E}'[g(s,X'(s),X(s))]]\mathrm{d}W(s). \end{aligned} \tag{3.18}$$

Therefore, by our above supposition, for any $x(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V)$, the following MF-SPDE

$$\begin{aligned} X(t) = x - \int_0^t A(s)X(s)\mathrm{d}s + \int_0^t [\rho_0\mathbb{E}'[b(s,X'(s),X(s))] + b_0(s) \\ + (\rho-\rho_0)\mathbb{E}'[b(s,x'(s),x(s))]]\mathrm{d}s + \int_0^t [\rho_0\mathbb{E}'[g(s,X'(s),X(s))] + g_0(s) \\ + (\rho-\rho_0)\mathbb{E}'[g(s,x'(s),x(s))]]\mathrm{d}W(s) \end{aligned} \tag{3.19}$$

admits a unique solution $X(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V)$. Consequently, now we can define a mapping from $\mathcal{M}^2_{\mathscr{F}}(0,T;V)$ onto itself and denote by $X(\cdot) = \Gamma(x(\cdot))$.

In view of the Lipschitz continuity of b and g and a priori estimate (3.11), for any $x_i(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V)$, $i = 1, 2$, we obtain

$$\begin{aligned} \|\Gamma(x_1(\cdot)) - \Gamma(x_2(\cdot))\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)} &= \|X_1(\cdot) - X_2(\cdot)\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)} \\ &\le K|\rho-\rho_0|^2 \cdot \|x_1(\cdot) - x_2(\cdot)\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)}. \end{aligned}$$

Here $K \triangleq K(T, C, \lambda, \alpha)$ is a positive constant independent of ρ. Set $\theta = \frac{1}{2K}$. Then we conclude that as long as $|\rho-\rho_0|^2 \le \theta$, the mapping Γ is a contraction in $\mathcal{M}^2_{\mathscr{F}}(0,T;V)$ which implies that MF-SPDE (3.17) is solvable. It is well-known that the MF-SPDE (3.17) with $\rho_0 = 0$ admits a unique solution by the classic existence and uniqueness theory of SPDE (see [18]). Starting from $\rho = 0$, one can reach $\rho = 1$ in finite steps and this finishes the proof of solvability of the MF-SPDE (3.5). Moreover, from Lemma 3.1 and the a priori estimate (3.10), we obtain $X(\cdot) \in \mathcal{S}^2_{\mathscr{F}}(0,T;V)$. This completes the proof.

We conclude this section by studying another type of MF-SPDE in the following abstract stochastic evolution form:

$$\begin{cases} \mathrm{d}X(t) = \{-A(t)X(t) + b(t,\mathbb{E}[X(t)],X(t))\}\mathrm{d}t + g(t,\mathbb{E}[X(t)],X(t))\mathrm{d}W(t), & t \in [0,T], \\ X(0) = \ x \in H, \end{cases} \tag{3.20}$$

where the coefficients

$$\begin{aligned} A &= A(t,\omega) : [0,T] \times \Omega \to \mathscr{L}(V,V^*), \\ b &= b(t,\omega,x',x) : [0,T] \times \Omega \times H \times H \to H, \\ g &= g(t,\omega,x',x) : [0,T] \times \Omega \times H \times H \to H \end{aligned} \tag{3.21}$$

are given random mappings.

We make the following standard assumptions on the coefficients (A, f, ξ).

Assumption 3.2 Suppose that there exist constant $\alpha > 0, \lambda$ and C such that the following conditions holds for all $x, x', \overline{x}, \overline{x}' \in H$ and a.e. $(t,\omega) \in [0,T] \times \Omega$.

(i) (Measurability) The operator A is $\mathscr{P}/\mathscr{B}(\mathscr{L}(V,V^*))$ measurable; b and g are $\mathscr{P}\otimes\mathscr{B}(H)\otimes\mathscr{B}(H)/\mathscr{B}(H)$ measurable;

(ii) (Integrality) $b(\cdot,0,0), g(\cdot,0,0) \in \mathcal{M}^2_{\mathscr{F}}(0,T;H)$;

(iii) (Coercivity)

$$\langle A(t)x, x\rangle + \lambda\|x\|_H^2 \geq \alpha\|x\|_V^2; \tag{3.22}$$

(iv) (Boundedness)

$$\sup_{(t,\omega)\in[0,T]\times\Omega} \|A(t,\omega)\|_{\mathscr{L}(V,V^*)} \leq C; \tag{3.23}$$

(v) (Lipschitz Continuity)

$$\begin{aligned} &\|b(t,x',x) - b(t,\overline{x}',\overline{x})\|_H + \|g(t,x',x) - g(t,\overline{x}',\overline{x})\|_H \\ &\leq C[\|x-\overline{x}\|_H + \|x'-\overline{x}'\|_H]. \end{aligned} \tag{3.24}$$

Similar to Theorems 3.1–3.2, we have the following two important results on the solution to MF-SPDE (3.20).

Theorem 3.3 *Let Assumption* 3.2 *be satisfied. Then for any given initial value x, the MF-SPDE* (3.20) *has a unique solution $X(\cdot) \in S^2_{\mathscr{F}}(0,T;H)$.*

Theorem 3.4 *Let Assumption* 3.2 *be satisfied. Suppose that $X(\cdot)$ be the solution to MF-SPDE* (3.20) *with initial value $x \in H$. Then the following estimate holds*:

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq t\leq T}\|X(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|X(t)\|_V^2 \mathrm{d}t\Big] \\ &\leq K\Big\{\mathbb{E}[\|x\|_H^2] + \overline{\mathbb{E}}\Big[\int_0^T \|b(t,0,0)\|_H^2 \mathrm{d}t\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \|g(t,0,0)\|_H^2 \mathrm{d}t\Big]\Big\}, \end{aligned} \tag{3.25}$$

where K is a positive constant depending only on T, C, α and λ. Further, suppose that $\overline{X}(\cdot)$ is the solution to MF-SPDE (3.20) *with the coefficients $(A, \overline{b}, \overline{g})$ satisfying Assumption* 3.2 *and the initial value $\overline{x} \in H$. Then we have*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq t\leq T}\|X(t)-\overline{X}(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|X(t)-\overline{X}(t)\|_V^2 \mathrm{d}t\Big] \\ &\leq K\Big\{\|x-\overline{x}\|_H^2 + \mathbb{E}\Big[\int_0^T \|b(t,\mathbb{E}\overline{X}(t),\overline{X}(t)) - \overline{b}(t,\mathbb{E}\overline{X}(t),\overline{X}(t))\|_H^2 \mathrm{d}t\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \|g(t,\mathbb{E}\overline{X}(t),\overline{X}(t)) - \overline{g}(t,\mathbb{E}\overline{X}(t),\overline{X}(t))\|_H^2 \mathrm{d}t\Big]\Big\}. \end{aligned} \tag{3.26}$$

4 Mean-Field Backward Stochastic Partial Differential Equation

In this section, in Gelfand triple (V, H, V^*), we begin to investigate the MF-BSPDE in the following abstract stochastic evolution form:

$$\begin{cases} \mathrm{d}Y(t) = [A(t)Y(t) + \mathbb{E}'[f(t, Y'(t), Z'(t), Y(t), Z(t))]\mathrm{d}t + Z(t)\mathrm{d}W(t), \quad t \in [0, T], \\ Y(T) = \xi, \end{cases} \tag{4.1}$$

where the coefficients (A, f, ξ) are the following mappings

$$\begin{aligned} &A = A(t, \omega) : [0, T] \times \Omega \to \mathscr{L}(V, V^*), \\ &f = f(t, \omega', \omega, y', z', y, z) : [0, T] \times \overline{\Omega} \times V \times H \times V \times H \to H, \\ &\xi = \xi(\omega) : \Omega \to H. \end{aligned} \tag{4.2}$$

In the above, we have used the following notation defined by

$$\begin{aligned} &\mathbb{E}'[f(t, Y'(t), Z'(t), Y(t), Z(t)] \\ = &\int_\Omega f(t, \omega', \omega, Y'(t, \omega'), Z'(t, \omega'), Y(t, \omega), Z(t, \omega'))\mathbb{P}(\mathrm{d}\omega'). \end{aligned} \tag{4.3}$$

Furthermore, we make the following standard assumption on the coefficients (A, f, ξ).

Assumption 4.1 Suppose that there exist constant $\alpha > 0, \lambda$ and C such that the following conditions holds for all $(y', z', y, z), (\overline{y}', \overline{z}', \overline{y}, \overline{z}) \in V \times H \times V \times H$ and a.e. $(t, \omega', \omega) \in [0, T] \times \overline{\Omega}$.

(i) (Measurability) The operator A is $\mathscr{P}/\mathscr{B}(\mathscr{L}(V, V^*))$-measurable; f is $\overline{\mathscr{P}} \otimes \mathscr{B}(V) \otimes \mathscr{B}(H) \otimes \mathscr{B}(V) \otimes \mathscr{B}(H)/\mathscr{B}(H)$-measurable; ξ is $\mathscr{F}_T$-measurable;

(ii) (Integrality) $f(\cdot, 0, 0, 0, 0) \in \mathcal{M}^2_{\mathscr{F}}(0, T; H)$ and $\xi \in \mathcal{L}^2(\mathscr{F}_T; H)$;

(iii) (Coercivity)

$$\langle A(t)x, x\rangle + \lambda\|x\|^2_H \geq \alpha\|x\|^2_V; \tag{4.4}$$

(iv) (Boundedness)

$$\sup_{(t,\omega)\in[0,T]\times\Omega} \|A(t, \omega)\|_{\mathscr{L}(V,V^*)} \leq C; \tag{4.5}$$

(v) (Lipschitz Continuity)

$$\begin{aligned} &\|f(t, y', z', y, z) - f(t, \overline{y}', \overline{z}', \overline{y}, \overline{z})\|_H \\ \leq &C(\|y' - \overline{y}'\|_V + \|z' - \overline{z}'\|_H + \|y - \overline{y}\|_V + \|z - \overline{z}\|_H). \end{aligned} \tag{4.6}$$

Now we give the definition of the solutions to MF-BSPDE (4.1).

Definition 4.1 *A $(V \times H)$-valued, $\mathbb{F}$-adapted process pair $(Y(\cdot), Z(\cdot))$ is said to be a solution to the MF-BSPDE* (4.1)*, if $Y(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0, T; V)$ and $Z(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0, T; H)$ such that*

$$\begin{aligned} (Y(t), \phi)_H = (\xi, \phi)_H &- \int_t^T \mathbb{E}'[f(s, Y'(s), Z'(s), Y(s), Z(s)), \phi)_H]\mathrm{d}s \\ &- \int_t^T \langle A(s)Y(s), \phi\rangle\, \mathrm{d}s - \int_t^T (Z(s), \phi)_H \mathrm{d}W(s), \quad t \in [0, T] \end{aligned} \tag{4.7}$$

holds for every $\phi \in V$ *and a.e.* $(t,\omega) \in [0,T]\times\Omega$*, or alternatively, in the sense of* V^*, $(Y(\cdot), Z(\cdot))$ *satisfies the following Itô form:*

$$\begin{aligned} Y(t) = \xi - \int_t^T \mathbb{E}'[f(s, Y'(s), Z'(s), Y(s), Z(s))]\mathrm{d}s \\ - \int_t^T A(s)Y(s)\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s), \quad t \in [0,T]. \end{aligned} \tag{4.8}$$

The following result gives the continuous dependence theorem for the solution to the MF-BSPDE (4.1) with respect to the coefficients (A, f, ξ), which also is referred to as a priori estimate for the solution.

Theorem 4.1 (Continuous Dependence Theorem of MF-BSPDE) *Suppose that* $(Y(\cdot), Z(\cdot))$ *is a solution to the MF-BSPDE* (4.1) *with the coefficients* (A, f, ξ) *satisfying Assumption* 4.1. *Then we have the following a priori estimate*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T} \|Y(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|Y(t)\|_V^2 \mathrm{d}t\Big] + \mathbb{E}\Big[\int_0^T \|Z(t)\|_H^2 \mathrm{d}t\Big] \\ &\le K\Big\{\mathbb{E}[\|\xi\|_H^2] + \overline{\mathbb{E}}\Big[\int_0^T \|f(t,0,0,0,0)\|_H^2 \mathrm{d}t\Big]\Big\}. \end{aligned} \tag{4.9}$$

Here $K \triangleq K(T, C, \alpha, \lambda)$ *is a positive constant depending only on* T, C, α *and* λ. *Assume that* $(\overline{Y}(\cdot), \overline{Z}(\cdot))$ *is a solution to the MF-BSPDE* (4.1) *with the coefficients* $(A, \overline{f}, \overline{\xi})$ *satisfying Assumption* 4.1. *Then it holds that*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T} \|Y(t) - \overline{Y}(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|Y(t) - \overline{Y}(t)\|_V^2 \mathrm{d}t\Big] + \mathbb{E}\Big[\int_0^T \|Z(t) - \overline{Z}(t)\|_H^2 \mathrm{d}t\Big] \\ &\le K\Big\{ + \overline{\mathbb{E}}\Big[\int_0^T \|f(t, \overline{Y}'(t), \overline{Z}'(t), \overline{Y}(t), \overline{Z}(t)) - \overline{f}(t, \overline{Y}'(t), \overline{Z}'(t), \overline{Y}(t), \overline{Z}(t))\|_H^2 \mathrm{d}t\Big] \\ &\quad + \mathbb{E}[\|\xi - \overline{\xi}\|_H^2]\Big\}. \end{aligned} \tag{4.10}$$

Proof If we take the coefficients $(A, \overline{f}, \overline{\xi}) = (A, 0, 0)$, then the corresponding solution to the MF-BSPDE (4.1) is $(\overline{Y}(\cdot), \overline{Z}(\cdot)) = (0, 0)$ and the estimate (4.9) follows from the estimate (4.10) immediately. Therefore, it suffices to prove that (4.10) holds. To simplify our notation, we define

$$\begin{aligned} &\widehat{Y}(t) \triangleq Y(t) - \overline{Y}(t), \quad \widehat{Z}(t) \triangleq Z(t) - \overline{Z}(t), \quad \widehat{\xi} \triangleq \xi - \overline{\xi}, \\ &\widehat{f}(t) \triangleq f(t, \overline{Y}'(t), \overline{Z}'(t), \overline{Y}(t), \overline{Z}(t)) - \overline{f}(t, \overline{Y}'(t), \overline{Z}'(t), \overline{Y}(t), \overline{Z}(t)). \end{aligned}$$

Using Itô's formula to $\|\widehat{Y}(t)\|_H^2$ and Assumption 4.1 and the classic inequality $2ab \le \frac{1}{\varepsilon}a^2 + \varepsilon b^2$, $\forall a, b > 0$, $\varepsilon > 0$, we have

$$\begin{aligned} &\|\widehat{Y}(t)\|_H^2 + 2\alpha \int_t^T \|\widehat{Y}(s)\|_V^2 \mathrm{d}s + \int_t^T \|\widehat{Z}(s)\|_H^2 \mathrm{d}s \\ &\le \|\widehat{\xi}\|_H^2 + K(C, \lambda, \varepsilon) \int_t^T \|\widehat{Y}(s)\|_H^2 \mathrm{d}s + \varepsilon \int_t^T \|\widehat{Y}(s)\|_V^2 \mathrm{d}s + \varepsilon \int_t^T \|\widehat{Z}(s)\|_H^2 \mathrm{d}s \end{aligned}$$

$$+ K(C,\varepsilon)\mathbb{E}\Big[\int_t^T \|\widehat{Y}(s)\|_H^2 \mathrm{d}s\Big] + \varepsilon\mathbb{E}\Big[\int_t^T \|\widehat{Y}(s)\|_V^2 \mathrm{d}s\Big] + \varepsilon\mathbb{E}\Big[\int_t^T \|\widehat{Z}(s)\|_H^2 \mathrm{d}s\Big]$$
$$+ \mathbb{E}'\int_t^T \|\widehat{f}(s)\|_H^2 \mathrm{d}s - 2\int_t^T (\widehat{Y}(s),\widehat{Z}(s))_H \mathrm{d}W(s). \tag{4.11}$$

Taking expectations on both sides of (4.11) and taking ε small enough such that $2\alpha - 2\varepsilon > 0$ and $1-2\varepsilon > 0$, we get

$$\mathbb{E}[\|\widehat{Y}(t)\|_H^2] + \mathbb{E}\Big[\int_t^T \|\widehat{Y}(s)\|_V^2 \mathrm{d}s\Big] + \mathbb{E}\Big[\int_t^T \|\widehat{Z}(s)\|_H^2 \mathrm{d}s\Big]$$
$$\le K(T,C,\alpha,\lambda)\Big\{\mathbb{E}\|\widehat{\xi}\|_H^2 + \overline{\mathbb{E}}\int_t^T \|\widehat{f}(s)\|_H^2 \mathrm{d}s + \mathbb{E}\int_t^T \|\widehat{Y}(s)\|_H^2 \mathrm{d}s\Big\}. \tag{4.12}$$

Here $K(T,C,\alpha,\lambda)$ is a general positive constant depending on α, T, C, and λ.

Then applying Grönwall's inequality to (4.12), we obtain

$$\sup_{0\le t\le T}\mathbb{E}[\|\widehat{Y}(t)\|_H^2] + \mathbb{E}\Big[\int_0^T \|\widehat{Y}(t)\|_V^2 \mathrm{d}t\Big] + \mathbb{E}\Big[\int_0^T \|\widehat{Z}(t)\|_H^2 \mathrm{d}t\Big]$$
$$\le K(T,C,\alpha,\lambda)\Big\{\mathbb{E}[\|\widehat{\xi}\|_H^2] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{f}(t)\|_H^2 \mathrm{d}t\Big]\Big\}. \tag{4.13}$$

In view of (4.11), (4.13) and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}\|\widehat{Y}(t)\|_H^2\Big] &\le K(T,C,\alpha,\lambda)\Big\{\mathbb{E}[\|\widehat{\xi}\|_H^2] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{f}(t)\|_H^2 \mathrm{d}t\Big]\Big\}\\ &\quad + 2\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\int_t^T (\widehat{Y}(s),\widehat{Z}(s))_H \mathrm{d}W(s)\Big|\Big]\\ &\le K(T,C,\alpha,\lambda)\Big\{\mathbb{E}[\|\widehat{\xi}\|_H^2] + \mathbb{E}\Big[\int_0^T \|\widehat{f}(t)\|_H^2 \mathrm{d}t\Big]\Big\}\\ &\quad + \frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T}\|\widehat{Y}(t)\|_H^2\Big],\end{aligned} \tag{4.14}$$

which implies that

$$\mathbb{E}\Big[\sup_{0\le t\le T}\|\widehat{Y}(t)\|_H^2\Big] \le K(T,C,\alpha,\lambda)\Big\{\mathbb{E}[\|\widehat{\xi}\|_H^2] + \overline{\mathbb{E}}\Big[\int_0^T \|\widehat{f}(t)\|_H^2 \mathrm{d}t\Big]\Big\}. \tag{4.15}$$

There we conclude that (4.10) holds by (4.15) with (4.13). The proof is complete.

Theorem 4.2 (Existence and Uniqueness Theorem of MF-BSPDE) *Let the coefficients* (A,f,ξ) *satisfy Assumption* 4.1. *Then MF-BSPDE* (4.1) *admits a unique solution* $(Y(\cdot),Z(\cdot)) \in \mathcal{S}_{\mathscr{F}}^2(0,T;V)\times\mathcal{M}_{\mathscr{F}}^2(0,T;H)$.

Proof The uniqueness of the solution of MF-BSPDE (4.1) is implied by the a priori estimate (4.10). Consider a family of MF-BSPDE parameterized by $\rho\in[0,1]$ as follows:

$$\begin{aligned}Y(t) = \xi - \int_t^T \{&A(s)Y(s) + \rho\mathbb{E}'[f(s,Y'(s),Z'(s),Y(s),Z(s))]\\ &+ f_0(s)\}\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s),\end{aligned} \tag{4.16}$$

where $f_0(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ is an arbitrary stochastic process.

It is easily seen that the original MF-BSPDE (4.1) is "embedded" in the MF-SPDE (4.16) when we take the parameter $\rho = 1$ and $f_0(\cdot) \equiv 0$. Obviously, the MF-BSPDE (4.16) has coefficients $(A, \rho f + b_0, \xi)$ satisfying Assumption 3.1. Suppose for some $\rho = \rho_0$ and any $f_0 \in \mathcal{M}^2_{\mathscr{F}}(0,T;H)$, the MF-BSPDE (4.16) admits a unique solution $(Y(\cdot), Z(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$. Then for any ρ, we can rewrite the MF-BSPDE(4.16) as follows:

$$\begin{aligned} Y(t) = \xi - \int_t^T &\{A(s)Y(s) + \rho_0 \mathbb{E}'[f(s, Y'(s), Z'(s), Y(s), Z(s))] \\ &+ f_0(s) + (\rho - \rho_0)\mathbb{E}'[f(s, Y'(s), Z'(s), Y(s), Z(s))]\}\mathrm{d}s \\ &- \int_t^T Z(s)\mathrm{d}W(s). \end{aligned} \tag{4.17}$$

Thus by our above assumption, for any stochastic process pair $(y(\cdot), z(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$, the following MF-BSPDE

$$\begin{aligned} Y(t) = \xi - \int_t^T &\{A(s)Y(s) + \rho_0 \mathbb{E}'[f(s, Y'(s), Z'(s), Y(s), Z(s))] \\ &+ f_0(s) + (\rho - \rho_0)\mathbb{E}'[f(s, y'(s), z'(s), y(s), z(s))]\mathrm{d}s \\ &- \int_t^T Z(s)\mathrm{d}W(s) \end{aligned} \tag{4.18}$$

admits a unique solution $(Y(\cdot), Z(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$, which implies that we can define a mapping from $\mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ onto itself denoted by $\mathcal{I}(y(\cdot), z(\cdot)) = (Y(\cdot), Z(\cdot))$.

In view of the a priori estimate (4.10) and the Lipschitz continuity of f, for any $(y_i(\cdot), z_i(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ $(i = 1, 2)$, it holds that

$$\begin{aligned} &\|\mathcal{I}(y_1(\cdot), z_1(\cdot)) - \mathcal{I}(y_2(\cdot), z_2(\cdot))\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)\times\mathcal{M}^2_{\mathscr{F}}(0,T;H)} \\ &= \|(Y_1(\cdot), Z_1(\cdot)) - (Y_2(\cdot), Z_2(\cdot))\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)\times\mathcal{M}^2_{\mathscr{F}}(0,T;H)} \\ &\leq K\overline{\mathbb{E}}\Big[\int_0^T \|\rho_0 f(s, Y_2'(s), Z_2'(s), Y_2(s), Z_2(s)) + (\rho - \rho_0) f(s, y_1'(s), z_1'(s), y_1(s), z_1(s)) \\ &\quad - \rho_0 f(s, Y_2'(s), Z_2'(s), Y_2(s), Z_2(s)) - (\rho - \rho_0) f(s, y_2'(s), z_2'(s), y_2(s), z_2(s))\|^2_H \mathrm{d}s\Big] \\ &\leq K|\rho - \rho_0|^2 \times \|(y_1(\cdot), z_1(\cdot)) - (y_2(\cdot), z_2(\cdot))\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)\times\mathcal{M}^2_{\mathscr{F}}(0,T;H)}. \end{aligned} \tag{4.19}$$

Here we note that $K \triangleq K(T, C, \lambda, \alpha)$ is a constant independent of ρ and

$$\begin{aligned} \|(Y_1(\cdot), Z_1(\cdot)) - (Y_2(\cdot), Z_2(\cdot))\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)\times\mathcal{M}^2_{\mathscr{F}}(0,T;H)} &\triangleq \|Y_1(\cdot) - Y_2(\cdot)\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;V)} \\ &\quad + \|Z_1(\cdot) - Z_2(\cdot)\|^2_{\mathcal{M}^2_{\mathscr{F}}(0,T;H)}. \end{aligned}$$

Set $\theta = \frac{1}{2K}$. Then we conclude that as long as $|\rho - \rho_0|^2 \leq \theta$, the mapping $\mathcal{I}$ is a contraction in $\mathcal{M}^2_{\mathcal{F}}(0,T;V) \times \mathcal{M}^2_{\mathcal{F}}(0,T;H)$ which implies that MF-BSPDE (4.16) is solvable. In view of [7, Proposition 3.2], we know that the MF-BSPDE (4.16) with $\rho_0 = 0$ admits a unique solution. Now we can start from $\rho = 0$ and then reach $\rho = 1$ in finite steps which finishes the proof of solvability of the MF-BSPDE (4.16). Moreover, from the a priori estimate (4.9), we obtain $Y(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0,T;H)$. This completes the proof.

5 Optimal Control of Mean-Field Stochastic Partial Differential Equation

5.1 Formulation of the optimal control problem

In this subsection, we present our optimal control problem studied in this paper. Firstly, in the Gelfand triple (V, H, V^*), consider the following controlled system:

$$\begin{cases} \mathrm{d}X(t) = [-A(t)X(t) + h(t, X(t), \mathbb{E}[X(t)], u(t))]\mathrm{d}t \\ \qquad\qquad + g(t, X(t), \mathbb{E}[X(t)], u(t))\mathrm{d}W(t), \quad t \in [0, T], \\ X(0) = x \in H \end{cases} \tag{5.1}$$

with the cost functional

$$J(u(\cdot)) = \mathbb{E}\Big[\int_0^T l(s, X(s), \mathbb{E}[X(s)], u(s))\mathrm{d}t + \Phi(X(T), \mathbb{E}[X(T)])\Big]. \tag{5.2}$$

In the above, $A : [0, T] \times \Omega \to \mathscr{L}(V, V^*)$, $h, g : [0, T] \times \Omega \times H \times H \times \mathscr{U} \to H$, $l : [0, T] \times \Omega \times H \times H \times \mathscr{U} \to \mathbb{R}$, $\Phi : \Omega \times H \times H \to \mathbb{R}$.

Let us make the following assumption.

Assumption 5.1 (i) $\mathscr{U}$ is a nonempty convex closed subset of a real separable Hilbert space U.

(ii) The operator A is $\mathscr{P}/\mathscr{B}(\mathscr{L}(V, V^*))$-measurable and satisfies the conditions (iii) and (iv) in Assumption 3.2.

(iii) The mappings h and g are $\mathscr{P} \otimes \mathscr{B}(H) \otimes \mathscr{B}(H) \otimes \mathscr{B}(\mathscr{U})/\mathscr{B}(H)$-measurable such that $h(\cdot, 0, 0, 0), g(\cdot, 0, 0, 0) \in \mathcal{M}^2_{\mathscr{F}}(0, T; H)$. Moreover, for almost all $(t, \omega) \in [0, T] \times \Omega$, h and g have continuous and uniformly bounded Gâteaux derivatives $h_x, h_{x'}, g_x, g_{x'}, h_u$ and g_u.

(iv) The mappings l is $\mathscr{P} \otimes \mathscr{B}(H) \otimes \mathscr{B}(H) \otimes \mathscr{B}(\mathscr{U})/\mathscr{B}(\mathbb{R})$-measurable and Φ is $\mathscr{F}_T \otimes \mathscr{B}(H) \otimes \mathscr{B}(H)/\mathscr{B}(\mathbb{R})$-measurable. For almost all $(t, \omega) \in [0, T] \times \Omega$, l has continuous Gâteaux derivatives $l_x, l_{x'}$ and l_u, $\Phi(\omega, x)$ has continuous Gâteaux derivative Φ_x. Moreover, for all $(x, x', u) \in H \times H \times \mathscr{U}$ and almost all $(t, \omega) \in [0, T] \times \Omega$, there is a constant $C > 0$ such that

$$\begin{aligned} &|l(t, x, x', u)| \le C(1 + \|x\|_H^2 + \|x'\|_H^2 + \|u\|_U^2), \\ &\|l_x(t, x, x', u)\|_H + \|l_{x'}(t, x, x', u)\|_H + \|l_u(t, x, x', u)\|_U \\ \le\ &C(1 + \|x\|_H + \|x'\|_H + \|u\|_U) \end{aligned}$$

and

$$\begin{aligned} &|\Phi(x, x')| \le C(1 + \|x\|_H^2 + \|x'\|_H^2), \\ &\|\Phi_x(x, x')\|_H \le C(1 + \|x\|_H + \|x'\|_H). \end{aligned}$$

Now we define as follows.

Definition 5.1 *A predictable control process $u(\cdot)$ is said to be admissible if $u(\cdot) \in \mathcal{M}^2(0, T; U)$ and $u(t) \in \mathscr{U}$, a.e. $t \in [0, T]$, $\mathbb{P}$-a.s. Denote by $\mathcal{A}$ the set of all admissible control processes.*

Given $u(\cdot) \in \mathcal{A}$, (5.1) is a MF-SPDE with random coefficients. From Theorem 3.3, it is easily seen that under Assumption 5.1, (5.1) admits a unique solution $X(\cdot) \equiv X^u(\cdot) \in \mathcal{S}^2_{\mathscr{F}}(0,T;H)$ and the cost functional is well-defined. In the case that $X(\cdot)$ is the solution of (5.1) corresponding to $u(\cdot) \in \mathcal{A}$, we call $(u(\cdot); X(\cdot))$ an admissible pair, and $X(\cdot)$ an admissible state process.

Our optimal control problem can be stated as follows.

Problem 5.1 Minimizes (5.2) over $\mathcal{A}$.

Any $\overline{u}(\cdot) \in \mathcal{A}$ satisfying

$$J(\overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot)) \tag{5.3}$$

is called an optimal control process of Problem 5.1. The corresponding state process $\overline{X}(\cdot)$ and the admissible pair $(\overline{u}(\cdot); \overline{X}(\cdot))$ is called an optimal state process and an optimal pair of Problem 5.1, respectively.

For any admissible pair $(u(\cdot); X(\cdot))$, the adjoint equation of the state equation (5.1) is defined as the following BSDE whose unknown variables is a pair of $\mathbb{F}$-adapted processes $(p(\cdot), q(\cdot))$,

$$\begin{cases} \mathrm{d}p(t) = -\{-A^*(t)p(t) + h_x^*(t, X(t), \mathbb{E}[X(t)], u(t))p(t) \\ \qquad\qquad + \mathbb{E}[h_x^*(t, X(t), \mathbb{E}[X(t)], u(t))p(t)] \\ \qquad\qquad + g_x^*(t, X(t), \mathbb{E}[X(t)], u(t))q(t) + \mathbb{E}[g^*(t, X(t), \mathbb{E}[X(t)], u(t))q(t)] \\ \qquad\qquad + l_x(t, X(t), \mathbb{E}[X(t)], u(t)) + \mathbb{E}[l_x(t, X(t), \mathbb{E}[X(t)], u(t))]\}\mathrm{d}t \\ \qquad\qquad + q(t)\mathrm{d}W(t), \quad t \in [0,T], \\ p(T) = \Phi_x(X(T), \mathbb{E}[X(T)]) + \mathbb{E}[\Phi_{x'}(X(T), \mathbb{E}[X(T)])]. \end{cases} \tag{5.4}$$

Indeed, the above equation is a linear MF-BSPDE, where A^* is the adjoint operator of A. Further, we can easily see that A^* also satisfies the boundedness and coercivity conditions. In view of Theorem 4.2, the linear MF-BSPDE (5.4) has a unique solution $(p(\cdot), q(\cdot)) \in \mathcal{S}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$.

Define the Hamiltonian $\mathcal{H} : [0,T] \times \Omega \times H \times H \times \mathscr{U} \times V \times H \to \mathbb{R}$ by

$$\mathcal{H}(t, x, x', u, p, q) := (h(t, x, x', u), p)_H + (g(t, x, x', u), q)_H + l(t, x, x', u). \tag{5.5}$$

Under Assumption 5.1, we can see that the Hamiltonian $\mathcal{H}$ is also continuously Gâteaux differentiable in (x, x', u). Denote by $\mathcal{H}_x$, $\mathcal{H}_{x'}$ and $\mathcal{H}_u$ the corresponding Gâteaux derivatives.

Therefore, using the notation of Hamiltonian $\mathcal{H}$, the adjoint equation (5.4) can be written as

$$\begin{cases} \mathrm{d}p(t) = -\{-A^*(t)p(t) + \mathcal{H}_x(t) + \mathbb{E}[\mathcal{H}_{x'}(t)]\mathrm{d}t + q(t)\mathrm{d}W(t), \quad t \in [0,T], \\ p(T) = \Phi_x(X(T), \mathbb{E}[X(T)]) + \mathbb{E}\Phi_{x'}(X(T), \mathbb{E}[X(T)]). \end{cases} \tag{5.6}$$

Here we have used the following shorthand notation:

$$\mathcal{H}(t) \triangleq \mathcal{H}(t, X(t), \mathbb{E}X(t), u(t), p(t), q(t)). \tag{5.7}$$

5.2 A variation formula for the cost functional

Suppose that $(u(\cdot); X(\cdot))$ and $(\overline{u}(\cdot); \overline{X}(\cdot))$ are any two given admissible control pairs. And let $(\overline{p}(\cdot), \overline{q}(\cdot))$ be the solution to the corresponding adjoint equation (5.4) associated with the admissible control pair $(\overline{u}(\cdot); \overline{X}(\cdot))$. In order to simplify our notation, in the rest of the paper we shall use the following shorthand notation

$$\begin{aligned}
&\rho(t) \triangleq \rho(t, X(t), \mathbb{E}[X(t)], u(t)), \quad \rho \triangleq h, g,\\
&\overline{\rho}(t) \triangleq \rho(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t)), \quad \rho \triangleq h, g,\\
&\mathcal{H}(t) \triangleq \mathcal{H}(t, X(t), \mathbb{E}[X(t)], u(t), \overline{p}(t), \overline{q}(t)),\\
&\overline{\mathcal{H}}(t) \triangleq \mathcal{H}(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t), \overline{p}(t), \overline{q}(t)).
\end{aligned} \tag{5.8}$$

To obtain the variation formula for the cost functional, we need the following basic result.

Lemma 5.1 *Let Assumption* 5.1 *be satisfied. Then difference* $J(u(\cdot)) - J(\overline{u}(\cdot))$ *of the cost functionals associated with the two admissible pairs* $(u(\cdot); X(\cdot))$ *and* $(\overline{u}(\cdot); \overline{X}(\cdot))$ *has the following representation*:

$$\begin{aligned}
&J(u(\cdot)) - J(\overline{u}(\cdot))\\
&= \mathbb{E}\Big[\int_0^T \Big\{\mathcal{H}(t) - \overline{\mathcal{H}}(t) - (\overline{\mathcal{H}}_x(t) + \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)], X(t) - \overline{X}(t))_H\Big\}\mathrm{d}t\Big]\\
&\quad + \mathbb{E}\Big[\Phi(X(T), \mathbb{E}[X(T)]) - \Phi(\overline{x}(T), \mathbb{E}[\overline{X}(T)])\\
&\quad - (\Phi_x(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) + \mathbb{E}[\Phi_{x'}(\overline{x}(T), \mathbb{E}[\overline{X}(T)])], X(T) - \overline{X}(T))_H\Big].
\end{aligned} \tag{5.9}$$

Proof Suppose that $(u(\cdot); X(\cdot))$ and $(\overline{u}(\cdot); \overline{X}(\cdot))$ are any two given admissible control pairs. By the state equation (5.1), it is easy to check that the difference $X(t) - \overline{X}(t)$ satisfies the following MF-SPDE:

$$\begin{cases}
\mathrm{d}(X(t) - \overline{X}(t)) = [-A(t)(X(t) - \overline{X}(t)) + h(t) - \overline{h}(t)]\mathrm{d}t\\
\qquad\qquad\qquad\qquad + [g(t) - \overline{g}(t)]\mathrm{d}W(t), \quad t \in [0, T],\\
X(0) - \overline{X}(0) = 0.
\end{cases} \tag{5.10}$$

And by the definition of the adjoint equation (see (5.6)), we can get that $(\overline{p}(\cdot), \overline{q}(\cdot))$ satisfies the following MF-BSPDE

$$\begin{cases}
\mathrm{d}\overline{p}(t) = -\{-A^*(t)\overline{p}(t) + \overline{\mathcal{H}}_x(t) + \mathbb{E}[\overline{\mathcal{H}}_{x'}]\mathrm{d}t + \overline{q}(t)\mathrm{d}W(t), \quad t \in [0, T],\\
p(T) = \ \Phi_x(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) + \mathbb{E}[\Phi_{x'}(\overline{X}(T), \mathbb{E}[X(T)])].
\end{cases} \tag{5.11}$$

Then using Itô's formula to $(\overline{p}(t), X(t) - \overline{X}(t))_H$, we get that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \{(\overline{p}(t), h(t) - \overline{h}(t))_H + (\overline{q}(t), g(t) - \overline{g}(t))_H\}\mathrm{d}t\Big]\\
&= \mathbb{E}\Big[\int_0^T (\overline{\mathcal{H}}_x(t) + \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)], X(t) - \overline{X}(t))_H \mathrm{d}t\Big]\\
&\quad + \mathbb{E}\Big[(\Phi_x(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) + \mathbb{E}\Phi_{x'}(\overline{X}(T), \mathbb{E}[\overline{X}(T)]), X(T) - \overline{X}(T))_H\Big].
\end{aligned} \tag{5.12}$$

In view of the definitions of the cost functional and the Hamiltonian $\mathcal{H}$ (see (5.5) and (5.2)), we can see that

$$\begin{aligned}J(u(\cdot))-J(\overline{u}(\cdot))=\mathbb{E}\Big[\int_0^T\{\mathcal{H}(t)-\overline{\mathcal{H}}(t)-(\overline{p}(t),h(t)-\overline{h}(t))_H-(\overline{q}(t),g(t)-\overline{g}(t))_H\}\mathrm{d}t\Big]\\+\mathbb{E}[\Phi(X(T),\mathbb{E}[X(T)])-\Phi(\overline{X}(T),\mathbb{E}[\overline{X}(T)])].\end{aligned}\tag{5.13}$$

Then (5.9) can be immediately obtained by substituting (5.12) into (5.13). The proof is complete.

Next we derive a variational formula for the cost functional (5.2).

Lemma 5.2 *Let Assumption* 5.1 *be satisfied. Then we have the following variational formula*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}J(\overline{u}(\cdot)+\varepsilon(v(\cdot)-\overline{u}(\cdot)))|_{\varepsilon=0}&=\lim_{\varepsilon\to0^+}\frac{J(\overline{u}(\cdot)+\varepsilon(v(\cdot)-\overline{u}(\cdot)))-J(\overline{u}(\cdot))}{\varepsilon}\\&=\mathbb{E}\Big[\int_0^T(\overline{\mathcal{H}}_u(t),v(t)-\overline{u}(t))_U\mathrm{d}t\Big],\end{aligned}\tag{5.14}$$

where $\overline{u}(\cdot)$ *and* $v(\cdot)$ *are any two given admissible controls, and* $0\le\varepsilon\le1$.

Proof Suppose that $(\overline{u}(\cdot);\overline{X}(\cdot))$ is a given admissible pair and $(\overline{p}(\cdot),\overline{q}(\cdot))$ is the corresponding adjoint process. Define a perturbed control process of $\overline{u}(\cdot)$ as follows:

$$u^\varepsilon(\cdot)\triangleq\overline{u}(\cdot)+\varepsilon(v(\cdot)-\overline{u}(\cdot)),\quad 0\le\varepsilon\le1,\tag{5.15}$$

where $v(\cdot)$ is any given admissible control. Due to the convexity of the control domain $\mathscr{U}$, $u^\varepsilon(\cdot)$ belongs to $\mathcal{A}$. Let $X^\varepsilon(\cdot)$ be the state process corresponding to the control $u^\varepsilon(\cdot)$. We will use the following shorthand notation:

$$\mathcal{H}^\varepsilon(t)\triangleq\mathcal{H}(t,X^\varepsilon(t),\mathbb{E}X^\varepsilon(t),u^\varepsilon(t),\overline{p}(t),\overline{q}(t)).\tag{5.16}$$

Using the shorthand notations (5.8) and (5.16), from Lemma 5.1, we get that

$$\begin{aligned}&J(u^\varepsilon(\cdot))-J(\overline{u}(\cdot))\\=&\mathbb{E}\Big[\int_0^T\{\mathcal{H}^\varepsilon(t)-\overline{\mathcal{H}}(t)-(\overline{\mathcal{H}}_x(t)+\mathbb{E}[\overline{\mathcal{H}}_{x'}(t)],X^\varepsilon(t)-\overline{X}(t))_H-(\overline{\mathcal{H}}_u(t),u^\varepsilon(t)-\overline{u}(t))_U\}\mathrm{d}t\Big]\\&+\mathbb{E}[\Phi(X^\varepsilon(T),\mathbb{E}[X^\varepsilon(T)])]-\Phi(\overline{X}(T),\mathbb{E}[\overline{X}(T)])\\&-(\Phi_x(\overline{X}(T),\mathbb{E}[\overline{X}(T)])+\mathbb{E}[\Phi_{x'}(\overline{X}(T),\mathbb{E}[\overline{X}(T)])],X^\varepsilon(T)-\overline{X}(T))_H]\\&+\mathbb{E}\Big[\int_0^T(\overline{\mathcal{H}}_u(t),u^\varepsilon(t)-\overline{u}(t))_U\mathrm{d}t\Big].\end{aligned}\tag{5.17}$$

In view of Taylor series expansion, it follows that

$$\begin{aligned}\mathbb{E}\Big[\int_0^T\{\mathcal{H}^\varepsilon(t)-\overline{\mathcal{H}}(t)\}\mathrm{d}t\Big]&=\mathbb{E}\Big[\int_0^T\int_0^1\{(\mathcal{H}_x^{\varepsilon,\lambda}(t),X^\varepsilon(t)-\overline{X}(t))_H\\&\quad+(\mathcal{H}_{x'}^{\varepsilon,\lambda}(t),\mathbb{E}[X^\varepsilon(t)]-\mathbb{E}[\overline{X}(t)])_H+(\mathcal{H}_u^{\varepsilon,\lambda}(t),u^\varepsilon(t)-\overline{u}(t))_U\}\mathrm{d}\lambda\mathrm{d}t\Big]\\&=\mathbb{E}\Big[\int_0^T\int_0^1\{(\mathcal{H}_x^{\varepsilon,\lambda}(t)+\mathbb{E}[\mathcal{H}_{x'}^{\varepsilon,\lambda}(t)],X^\varepsilon(t)-\overline{X}(t))_H\end{aligned}$$

$$+ (\mathcal{H}_u^{\varepsilon,\lambda}(t), u^\varepsilon(t) - \overline{u}(t))_U\}\mathrm{d}\lambda\mathrm{d}t\Big], \tag{5.18}$$

where

$$\mathcal{H}^{\varepsilon,\lambda}(t) \triangleq \mathcal{H}(t, X^{\varepsilon,\lambda}(t), \mathbb{E}[X^{\varepsilon,\lambda}(t)], u^{\varepsilon,\lambda}(t), \overline{p}(t), \overline{q}(t))$$

and

$$X^{\varepsilon,\lambda}(t) \triangleq \overline{X}(t) + \lambda(X^\varepsilon(t) - \overline{X}(t)),$$
$$u^{\varepsilon,\lambda}(t) \triangleq \overline{u}(t) + \lambda(u^\varepsilon(t) - \overline{u}(t)).$$

On the other hand, it follows from the definition of u^ε (see (5.15)) that

$$\mathbb{E}\Big[\int_0^T \|u^\varepsilon(t) - \overline{u}(t)\|_U^2 \mathrm{d}t\Big] = \varepsilon^2 \mathbb{E}\Big[\int_0^T \|v(t) - \overline{u}(t)\|_U^2 \mathrm{d}t\Big]. \tag{5.19}$$

Further, in view of the continuous dependence theorem of MF-SPDE (see Theorem 3.4), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} \|X^\varepsilon(t) - \overline{X}(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|X^\varepsilon(t) - \overline{X}(t)\|_V^2 \mathrm{d}t\Big] \\
&\le K\mathbb{E}\Big[\int_0^T \|u^\varepsilon(t) - \overline{u}(t)\|_U^2 \mathrm{d}t\Big] \\
&= K\varepsilon^2 \mathbb{E}\Big[\int_0^T \|v(t) - \overline{u}(t)\|_U^2 \mathrm{d}t\Big].
\end{aligned} \tag{5.20}$$

Therefore, combining (5.18)–(5.20) yields

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \{\mathcal{H}^\varepsilon(t) - \overline{\mathcal{H}}(t) - (\overline{\mathcal{H}}_x(t) + \mathbb{E}\overline{\mathcal{H}}_{x'}(t), x^\varepsilon(t) - \overline{x}(t))_H - (\overline{\mathcal{H}}_u(t), u^\varepsilon(t) - \overline{u}(t))_U\}\mathrm{d}t\Big] \\
&= \mathbb{E}\Big[\int_0^T \int_0^1 \{(\mathcal{H}_x^{\varepsilon,\lambda}(t) + \mathbb{E}[\mathcal{H}_{x'}^{\varepsilon,\lambda}(t)] - \overline{\mathcal{H}}_x(t) - \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)], X^\varepsilon(t) - \overline{X}(t))_H \\
&\quad + (\mathcal{H}_u^{\varepsilon,\lambda}(t) - \overline{\mathcal{H}}_u(t), u^\varepsilon(t) - \overline{u}(t))_U\}\mathrm{d}\lambda\mathrm{d}t\Big] \\
&\le \Big\{\mathbb{E}\Big[\int_0^T \int_0^1 \|(\mathcal{H}_x^{\varepsilon,\lambda}(t) + \mathbb{E}[\mathcal{H}_{x'}^{\varepsilon,\lambda}(t)] - \overline{\mathcal{H}}_x(t) - \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)]\|_H^2 \mathrm{d}t\mathrm{d}\lambda\Big]\Big\}^{\frac{1}{2}} \\
&\quad \cdot \Big\{\mathbb{E}\Big[\int_0^T \|X^\varepsilon(t) - \overline{X}(t)\|_H^2\Big]\Big\}^{\frac{1}{2}} + \Big\{\mathbb{E}\Big[\int_0^T \int_0^1 \|(\mathcal{H}_u^{\varepsilon,\lambda}(t) - \overline{\mathcal{H}}_u(t)\|_H^2 \mathrm{d}t\mathrm{d}\lambda\Big]\Big\}^{\frac{1}{2}} \\
&\quad \cdot \Big\{\mathbb{E}\Big[\int_0^T \|u^\varepsilon(t) - \overline{u}(t)\|_H^2\Big]\Big\}^{\frac{1}{2}} \\
&\le K\varepsilon\Big\{\mathbb{E}\Big[\int_0^T \int_0^1 \|(\mathcal{H}_x^{\varepsilon,\lambda}(t) + \mathbb{E}[\mathcal{H}_{x'}^{\varepsilon,\lambda}(t)] - \overline{\mathcal{H}}_x(t) - \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)]\|_H^2 \mathrm{d}t\mathrm{d}\lambda\Big]\Big\}^{\frac{1}{2}} \\
&\quad + K\varepsilon\Big\{\mathbb{E}\Big[\int_0^T \int_0^1 \|(\mathcal{H}_u^{\varepsilon,\lambda}(t) - \overline{\mathcal{H}}_u(t)\|_U^2 \mathrm{d}t\mathrm{d}\lambda\Big]\Big\}^{\frac{1}{2}} \\
&= o(\varepsilon),
\end{aligned} \tag{5.21}$$

where the last equality can be obtained by the fact that

$$\lim_{\varepsilon\to 0} \Big\{\mathbb{E}\Big[\int_0^T \int_0^1 \|(\mathcal{H}_x^{\varepsilon,\lambda}(t) + \mathbb{E}[\mathcal{H}_{x'}^{\varepsilon,\lambda}(t)] - \overline{\mathcal{H}}_x(t) - \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)]\|_H^2 \mathrm{d}t\mathrm{d}\lambda\Big]\Big\}^{\frac{1}{2}}$$

$$+\lim_{\varepsilon\to 0}\Big\{\mathbb{E}\Big[\int_0^T\int_0^1\|(\mathcal{H}_u^{\varepsilon,\lambda}(t)-\overline{\mathcal{H}}_u(t)\|_U^2\mathrm{d}t\mathrm{d}\lambda\Big]\Big\}^{\frac{1}{2}}=0, \tag{5.22}$$

which can be got by combining Assumption 5.1, (5.19)–(5.20) and the dominated convergence theorem.

We can similarly get that

$$\begin{aligned}&\mathbb{E}\Big[\Phi(X^\varepsilon(T),\mathbb{E}[X^\varepsilon(T)])-\Phi(\overline{X}(T),\mathbb{E}[\overline{X}(T)])\\&-\Big(\Phi_x(\overline{X}(T),\mathbb{E}[\overline{X}(T)])+\mathbb{E}[\Phi_{x'}(\overline{X}(T),\mathbb{E}[\overline{X}(T)])],X^\varepsilon(T)-\overline{X}(T)\Big)_H\Big]=o(\varepsilon).\end{aligned} \tag{5.23}$$

Hence, by substituting (5.21) and (5.23) into (5.17), we get that

$$\lim_{\varepsilon\to 0}\frac{J(u^\varepsilon(\cdot))-J(\overline{u}(\cdot))}{\varepsilon}=\mathbb{E}\Big[\int_0^T(\overline{\mathcal{H}}_u(t),v(t)-\overline{u}(t))_U\mathrm{d}t\Big].$$

The proof is complete.

5.3 Stochastic maximum principle

In this subsection, we will establish the necessary and sufficient maximum principle for the optimal control of Problem 5.1.

Theorem 5.1 (Necessary Stochastic Maximum Principle) *Let Assumption* 5.1 *be satisfied. Let* $(\overline{u}(\cdot);\overline{x}(\cdot))$ *be an optimal pair of Problem* 5.1 *associated with the adjoint process* $(\overline{p}(\cdot),\overline{q}(\cdot))$. *Then the following minimum condition holds*:

$$(\overline{\mathcal{H}}_u(t),v-\overline{u}(t))_U\geq 0, \tag{5.24}$$

$\forall v\in\mathscr{U}$, *for a.e.* $t\in[0,T]$, $\mathbb{P}$-*a.s.*

Proof For any admissible control $v(\cdot)\in\mathcal{A}$, it follows from Lemma 5.2 that

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T(\overline{\mathcal{H}}_u(t),v(t)-\overline{u}(t))_U\mathrm{d}t\Big]\\&=\lim_{\varepsilon\to 0}\frac{J(u^\varepsilon(\cdot))-J(\overline{u}(\cdot))}{\varepsilon}\geq 0,\end{aligned} \tag{5.25}$$

where the last inequality can be get directly since $(\overline{u}(\cdot);\overline{X}(\cdot))$ is an optimal pair of Problem 5.1. Then minimum condition (5.24) can be obtained by the classic argument following [2]. For the similar proof, we refer to [16]. The proof is complete.

Next we will give the verification theorem of optimality, namely, the sufficient maximum principle for the optimal control of Problem 5.1. Besides Assumption 5.1, the verification theorem relies on some convexity assumptions of the Hamiltonian and the terminal cost.

Theorem 5.2 (Sufficient Maximum Principle) *Let Assumption* 5.1 *be satisfied. Let* $(\overline{u}(\cdot);\overline{X}(\cdot))$ *be an admissible pair associated with the adjoint process* $(\overline{p}(\cdot),\overline{q}(\cdot))$. *Suppose that for almost all* $(t,\omega)\in[0,T]\times\Omega$,

(1) $\mathcal{H}(t,x,x',u,\overline{p}(t),\overline{q}(t))$ *is convex in* (x,x',u);

(2) $\Phi(x,x')$ *is convex in* (x,x');

(3) $\overline{\mathcal{H}}(t)=\min_{u\in\mathscr{U}}\mathcal{H}(t,\overline{X}(t),\mathbb{E}[\overline{X}(t)],u,\overline{p}(t),\overline{q}(t))$,

then $(\overline{u}(\cdot);\overline{X}(\cdot))$ *is an optimal pair of Problem* 5.1.

Proof Given an arbitrary admissible pair $(u(\cdot); X(\cdot))$. By Lemma 5.1, we get

$$\begin{aligned}
&J(u(\cdot)) - J(\overline{u}(\cdot)) \\
= &\mathbb{E}\Big[\int_0^T \Big\{\mathcal{H}(t) - \overline{\mathcal{H}}(t) - (\overline{\mathcal{H}}_x(t) + \mathbb{E}[\overline{\mathcal{H}}_{x'}(t)], X(t) - \overline{X}(t))_H\Big\}\mathrm{d}t\Big] \\
&+ \mathbb{E}\Big[\Phi(X(T), \mathbb{E}[X(T)]) - \Phi(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) \\
&- \Big(\Phi_x(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) + \mathbb{E}[\Phi_{x'}(\overline{X}(T), \mathbb{E}[\overline{X}(T)])], X(T) - \overline{X}(T)\Big)_H\Big].
\end{aligned} \tag{5.26}$$

By the convexity of $\mathcal{H}(t, x, x', u, , \overline{p}(t), \overline{q}(t))$ and $\Phi(x', x)$, in view of [10, Proposition 1.54], we have

$$\begin{aligned}
\mathcal{H}(t) - \overline{\mathcal{H}}(t) \geq (\overline{\mathcal{H}}_x(t), X(t) - \overline{X}(t))_H + (\overline{\mathcal{H}}_{x'}(t), \mathbb{E}[X(t)] - \mathbb{E}[\overline{X}(t)])_H \\
+ (\overline{\mathcal{H}}_u(t), u(t) - \overline{u}(t))_U
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
&\Phi(X(T), \mathbb{E}[X(T)]) - \Phi(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) \\
\geq &(\Phi_x(\overline{X}(T), \mathbb{E}[\overline{X}(T)]), X(T) - \overline{X}(T))_H \\
&+ (\Phi_{x'}(\overline{X}(T), \mathbb{E}[\overline{X}(T)]), \mathbb{E}[X(T)] - \mathbb{E}[\overline{X}(T)])_H.
\end{aligned} \tag{5.28}$$

In addition, in view of the convex optimization principle (see [10, Proposition 2.21]), the optimality condition 3 implies that for almost all $(t, \omega) \in [0, T] \times \Omega$,

$$(\overline{\mathcal{H}}_u(t), u(t) - \overline{u}(t))_U \geq 0. \tag{5.29}$$

Substituting (5.27)–(5.29) into (5.26) yields

$$J(u(\cdot)) - J(\overline{u}(\cdot)) \geq 0.$$

Therefore, since $u(\cdot)$ is arbitrary, $\overline{u}(\cdot)$ is an optimal control process and $(\overline{u}(\cdot); \overline{X}(\cdot))$ is an optimal pair. The proof is complete.

5.4 Optimality system of mean-field stochastic partial differential equation

For any admissible pair $(\overline{u}(\cdot), \overline{X}(t))$, consider the following stochastic system:

$$\begin{cases}
\mathrm{d}\overline{X}(t) = [-A(t)\overline{X}(t) + h(s, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t))]\mathrm{d}t + g(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t))\mathrm{d}W(t), \\
\mathrm{d}\overline{p}(t) = -[-A^*(t)\overline{p}(t) + h_x^*(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t))\overline{p}(t) \\
\qquad + \mathbb{E}[h_x^*(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t))\overline{p}(t)] \\
\qquad + g_x^*(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t))\overline{q}(t) + \mathbb{E}[g_x^*(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], \overline{u}(t))\overline{q}(t)]] \\
\qquad + l_x(t, \overline{X}(t), \mathbb{E}\overline{X}(t), \overline{u}(t)) + \mathbb{E}[l_x(t, \overline{X}(t), \mathbb{E}\overline{X}(t), \overline{u}(t))]]\mathrm{d}t \\
\qquad + \overline{q}(t)\mathrm{d}W(t), \quad t \in [0, T], \\
\overline{X}(0) = x, \\
p(T) = \Phi_x(\overline{X}(T), \mathbb{E}[\overline{X}(T)]) + \mathbb{E}\Phi_{x'}(\overline{X}(T), \mathbb{E}[\overline{X}(T)]), \\
(\overline{\mathcal{H}}_u(t), v - \overline{u}(t))_U \geq 0, \quad \forall v \in \mathscr{U}.
\end{cases} \tag{5.30}$$

Note that this is a mean-field fully-coupled forward-backward stochastic partial differential equation consisting of the state equation (5.1), the adjoint equation (5.4) and the minimum condition of (5.24). The forward-backward equation (5.30) is referred to as the stochastic Hamiltonian system or the optimality system of Problem 5.1. The 4-tuple stochastic process $(\overline{u}(\cdot), \overline{X}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;U) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$ satisfying the above is called the solution of (5.30). Under proper assumptions, we can claim that the existence of the optimal control of Problem 5.1 is equivalent to the solvability of the stochastic Hamiltonian system (5.30).

Corollary 5.1 *Let Assumption* 5.1 *and Conditions* 1-2 *in Theorem* 5.2 *be satisfied. Then the existence of the optimal control of Problem* 5.1 *is equivalent to the existence of a solution to the stochastic Hamiltonian system.* (5.30)*.*

Proof For the sufficient part, suppose that the stochastic Hamiltonian system (5.30) admits an adapted solution $(\overline{u}(\cdot), \overline{X}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;U) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$, then we begin to prove the existence of the optimal control of Problem 5.1. In fact, from the minimum condition in the stochastic Hamiltonian system (5.30) and the convexity of $\mathcal{H}(t, \overline{X}(t), \mathbb{E}[\overline{X}(t)], u, \overline{p}(t), \overline{q}(t))$ with u, we know that

$$\mathcal{H}(t, \overline{X}(t), \mathbb{E}\overline{X}(t), \overline{u}(t), \overline{p}(t), \overline{q}(t)) = \min_{u \in \mathscr{U}} \mathcal{H}(t, \overline{X}(t), \mathbb{E}\overline{X}(t), u, \overline{p}(t), \overline{q}(t)).$$

Therefore, in view of the sufficient stochastic maximum principle (see Theorem 5.2), we get that $(\overline{u}(\cdot); \overline{X}(\cdot))$ is an optimal pair.

For the necessary part, suppose that $(\overline{u}(\cdot); \overline{X}(\cdot))$ is an optimal pair associated with the corresponding adjoint process $(\overline{p}(\cdot), \overline{q}(\cdot))$, then in view of the necessary stochastic maximum principle, we get that the stochastic Hamiltonian system (5.30) has an adapted solution

$$(\overline{u}(\cdot), \overline{X}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;U) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H).$$

The proof is complete.

6 An Application: Linear-Quadratic Optimal Control Problems for Mean-Field Stochastic Partial Differential Equation

The case where the system dynamics are described by a set of linear differential equations and the cost functional is described by a quadratic function is called the LQ problem which is one of the most important optimal control problems. The reader is referred to [23, Chapter 6] for a complete survey on this topic. In this section, an infinite-dimensional LQ problem of mean-field type will be discussed. As an application, we will solve an LQ problem for a Cauchy problem of a stochastic linear parabolic PDE of mean field type.

6.1 LQ optimal control of mean- field stochastic partial differential equation

This subsection is devoted to applying the stochastic maximum principles to study an infinite-dimensional linear-quadratic optimal control problem of mean field type, and establish the explicit dual characterization of the optimal control with stochastic Hamiltonian system of mean field type.

Consider the following linear quadratic optimal control problem. Minimize over $\mathcal{A} = \mathcal{M}^2_{\mathscr{F}}(0,T;U)$ the following quadratic cost functional

$$\begin{aligned} J(u(\cdot)) = &\mathbb{E}[(\Phi_1 X(T), X(T))_H] + \mathbb{E}[(\Phi_2 \mathbb{E}[X(T)], \mathbb{E}[X(T)])_H] \\ &+ \mathbb{E}\Big[\int_0^T (G_1(s)X(s), X(s))_H \mathrm{d}s\Big] + \mathbb{E}\Big[\int_0^T (G_2(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)])_H \mathrm{d}s\Big] \\ &+ \mathbb{E}\Big[\int_0^T (N(s)u(s), u(s))_U \mathrm{d}s\Big], \end{aligned} \tag{6.1}$$

where $X(\cdot)$ is the solution of the controlled linear MF-SPDE in the Gelfand triple (V, H, V^*):

$$\begin{cases} \mathrm{d}X(t) = [-A(t)X(t) + B_1(t)X(t) + B_2(t)\mathbb{E}[X(t)] + C(t)u(t)]\mathrm{d}t \\ \qquad\quad + [D_1(t)X(t) + D_2(t)\mathbb{E}[X(t)] + F(t)u(t)]\mathrm{d}W(t), \\ X(0) = x, \quad t \in [0,T]. \end{cases} \tag{6.2}$$

Here $A, B_1, B_2, C, D_1, D_2, F, G_1, G_2, N, \Phi_1$ and Φ_2 are given random mappings such that $A : [0,T]\times\Omega \to \mathscr{L}(V, V^*)$, $B_1, B_2, D_1, D_2, G_1, G_2 : [0,T]\times\Omega \to \mathscr{L}(H,H)$, $C, F : [0,T]\times\Omega \to \mathscr{L}(U,H)$, $N : [0,T]\times\Omega \to \mathscr{L}(U,U)$ and $\Phi_1, \Phi_2 : \Omega \to \mathscr{L}(H,H)$, satisfying the following assumptions.

Assumption 6.1 The operator A satisfies the coercivity and boundedness conditions, i.e., (iii) and (iv) in Assumption 3.1. The mappings $A, B_1, B_2, C, D_1, D_2, F, G_1, G_2, N$, G_1, G_2 and N are uniformly bounded $\mathbb{F}$-predictable processes, Φ_1 and Φ_2 are uniformly bounded $\mathcal{F}_T$-measurable random variables.

Assumption 6.2 The stochastic processes G_1, G_2, N and the random variables Φ_1 and Φ_2 are nonnegative operators, a.e. $t \in [0,T]$, $\mathbb{P}$-a.s. Moreover, N is uniformly positive a.e. $t \in [0,T]$, $\mathbb{P}$-a.s., i.e., for $\forall u \in U$, $(Nu, u)_U \geq k(u,u)_U$, for some positive constant k, a.e. $t \in [0,T]$, $\mathbb{P}$-a.s.

In the general control Problem 5.1, we specify the coefficients h, g, l and Φ with

$$\begin{aligned} &h(t,x,x',u) = B_1(t)x + B_2(t)x' + C(t)u, \\ &g(t,x,x',u) = D_1(t)x + D_2(t)x' + F(t)u, \\ &l(t,x,x',u) = (G_1(t)x, x)_H + (G_2(t)x', x')_H + (N(t)u,u)_U, \\ &\Phi(x,x') = (\Phi_1 x, x)_H + (\Phi_2 x', x')_H. \end{aligned}$$

By Assumptions 6.1–6.2, it is easily to check that Assumption 5.1 on the coefficients (A, h, g, l, Φ) holds. So our LQ problem can be embedded in Problem 5.1. In this case, the Hamiltonian $\mathcal{H}$ has the following form:

$$\begin{aligned} \mathcal{H}(t,x,x',u,p,q) = &(B_1(t)x + B_2(t)x' + C(t)u, p)_H + (D_1(t)x + D_2(t)x + F(t)u, q)_H \\ &+ (G_1(t)x, x)_H + (G_2(t)x, x)_H + (N(t)u,u)_U. \end{aligned} \tag{6.3}$$

Here we denote the adjoint operators of B_1, B_2, C_1, C_2, D, and F by B_1^*, B_2^*, C_1^*, C_2^*, D^* and F_1^*, respectively. Associated with an admissible pair $(u(\cdot); X(\cdot))$, the adjoint equation (5.4) has the following form:

$$\begin{cases} \mathrm{d}p(t) = -\{-A^*(t)p(t) + B_1^*(t)X(t) + +\mathbb{E}[B_2^*(t)p(t)] + D_1^*(t)q(t) + \mathbb{E}[D_2^*(t)q_2(t)] \\ \qquad\qquad + 2G_1(t)X(t) + 2\mathbb{E}[G_2(t)X(t)]\}\mathrm{d}t + q(t)\mathrm{d}W(t), \quad t \in [0,T], \\ p(T) = 2\Phi_1 X(T) + 2\mathbb{E}[\Phi_2 X(T)]. \end{cases} \tag{6.4}$$

Because in this case, there is no constraint on the control, the minimum condition (5.24) of the optimal control is

$$\mathcal{H}_u(t, X(t), \mathbb{E}[X(t)], p(t), q(t), u(t)) = 0. \tag{6.5}$$

Therefore the stochastic Hamiltonian system is the following fully-coupled linear forward-backward stochastic partial differential equation

$$\begin{cases} \mathrm{d}X(t) = [-A(t)x(t) + B_1(t)X(t) + B_2(t)\mathbb{E}[X(t)] + C(t)u(t)]\mathrm{d}t \\ \qquad\qquad + [D_1(t)X(t) + D_2(t)\mathbb{E}[X(t)] + F(t)u(t)]\mathrm{d}W(t), \\ \mathrm{d}p(t) = -\{-A^*(t)p(t) + B_1^*(t)X(t) + \mathbb{E}[B_2^*(t)p(t)] + D_1^*(t)q(t) + \mathbb{E}[D_2^*(t)q_2(t)] \\ \qquad\qquad + 2G_1(t)X(t) + 2\mathbb{E}[G_2(t)X(t)]\}\mathrm{d}t + q(t)\mathrm{d}W(t), \quad t \in [0,T], \\ x(0) = x, \\ p(T) = 2\Phi_1 X(T) + 2\mathbb{E}[\Phi_2 X(T)], \\ \mathcal{H}_u(t, X(t), \mathbb{E}[X(t)], p(t), q(t), u(t)) = 0. \end{cases} \tag{6.6}$$

Now we give the dual characterization of the optimal control.

Theorem 6.1 *Let Assumptions* 6.1–6.2 *be satisfied. Then our LQ problem has a unique optimal control, which implies that the stochastic Hamiltonian system* (6.6) *has a unique adapted solution* $(\overline{u}(\cdot), \overline{X}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot)) \in \mathcal{M}_{\mathscr{F}}^2(0,T;U) \times \mathcal{M}_{\mathscr{F}}^2(0,T;V) \times \mathcal{M}_{\mathscr{F}}^2(0,T;V) \times \mathcal{M}_{\mathscr{F}}^2(0,T;H)$. *Moreover the optimal control is given by*

$$\overline{u}(t) = -\frac{1}{2}N^{-1}(t)[C^*(t)\overline{p}(t) + F^*(t)\overline{q}(t)]. \tag{6.7}$$

Proof Let $(u(\cdot), X(\cdot))$ and $(\overline{u}(\cdot), \overline{X}(\cdot))$ be any two admissible control pairs. In view of the continuous dependence theorem of MF-SPDE (see Theorem 4.1), we have

$$\begin{aligned} &|J(u(\cdot)) - J(\overline{u}(\cdot))|^2 \\ &\leq K\Big\{\mathbb{E}\Big[\int_0^T |u(t) - \overline{u}(t)|^2 \mathrm{d}t\Big]\Big\} \times \Big\{\mathbb{E}\Big[\int_0^T |u(t)|^2 \mathrm{d}t\Big] + \mathbb{E}\Big[\int_0^T |\overline{u}(t)|^2 \mathrm{d}t\Big] + x\Big\}. \end{aligned} \tag{6.8}$$

Thus, it follows that

$$J(u(\cdot)) - J(\overline{u}(\cdot)) \to 0, \quad \text{as } u(\cdot) \to \overline{u}(\cdot) \text{ in } \mathcal{A}, \tag{6.9}$$

which implies that the cost functional $J(u(\cdot))$ is continuous over $\mathcal{M}_{\mathscr{F}}^2(0,T;U)$.

From the uniformly strictly positivity of the process N, we conclude that the cost functional $J(u(\cdot))$ is strictly convex and

$$J(u(\cdot)) \geq k\mathbb{E}\Big[\int_0^T \|u(t)\|_U^2 \mathrm{d}t\Big] = k\|u(\cdot)\|^2_{\mathcal{M}_{\mathscr{F}}^2(0,T;U)}.$$

Therefore, the cost functional $J(u(\cdot))$ is coercive, i.e.,

$$\lim_{\|u(\cdot)\|_{\mathcal{M}^2_{\mathscr{F}}(0,T;U)}\to\infty} J(u(\cdot)) = \infty.$$

In the end, we get the uniqueness and existence of the optimal control $\overline{u}(\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;U)$ of our LQ problem by [10, Proposition 2.12].

Now we begin to prove that the stochastic Hamiltonian system (6.6) has a unique adapted solution. Indeed, in view of Corollary 5.1, the existence of the optimal control $\overline{u}(\cdot)$ of our LQ problem 5.1 implies that the stochastic Hamiltonian system (6.6) has a solution $(\overline{u}(\cdot), \overline{x}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot)) \in \mathcal{M}^2_{\mathscr{F}}(0,T;U) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;V) \times \mathcal{M}^2_{\mathscr{F}}(0,T;H)$. Here $\overline{x}(\cdot)$ is the optimal state and $(\overline{p}(\cdot), \overline{q}(\cdot))$ is the adjoint process corresponding the optimal control $\overline{u}(\cdot)$. If the stochastic Hamiltonian system (6.6) has another adapted solution $(\overline{u}'(\cdot), \overline{x}'(\cdot), \overline{p}'(\cdot), \overline{q}'(\cdot))$, then view of Corollary 5.1, $(\overline{u}'(\cdot); \overline{X}'(\cdot))$ have to be an optimal pair of our LQ problem. So $\overline{u}(\cdot) = \overline{u}'(\cdot)$ due to the uniqueness of the optimal control. Moreover, from the uniqueness of solutions to MF-SPDE (see Theorem 3.2) and MF-BSPDE (see Theorem 4.2), we get $(\overline{x}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot)) = (\overline{x}'(\cdot), \overline{p}'(\cdot), \overline{q}'(\cdot))$. Therefore, the stochastic Hamiltonian system (6.6) admits a unique solution. In the end, the dual characterization (6.7) of the unique optimal can be directly obtained by solving the minimum condition (6.5).

6.2 LQ control of the Cauchy problem for stochastic linear PDE of mean field type

In this subsection, in terms of the results in the previous subsection, we solve a LQ problem of a Cauchy problem for a controlled stochastic linear PDE of mean-field type.

Now we give some preliminaries of Sobolev spaces. For $m = 0, 1$, introduce the space $H^m \triangleq \{\phi : \partial_z^\alpha \phi \in L^2(\mathbb{R}^d),\ \text{for any } \alpha := (\alpha_1, \cdots, \alpha_d) \text{ with } |\alpha| := |\alpha_1| + \cdots + |\alpha_d| \le m\}$ with the norm

$$\|\phi\|_m \triangleq \Big\{ \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} |\partial_z^\alpha \phi(z)|^2 \mathrm{d}z \Big\}^{\frac{1}{2}}.$$

The dual space of H^1 is denoted by H^{-1}. Put $V = H^1$, $H = H^0$, $V^* = H^{-1}$. Then we claim that (V, H, V^*) is a Gelfand triple.

Suppose that the control domain is $\mathscr{U} = U = H$. For any admissible control $u(\cdot,\cdot) \in \mathcal{M}^2_{\mathscr{F}}(0,T;U)$, we introduce the controlled Cauchy problem, where the state process is the following stochastic partial differential equation of mean-field type in divergence form:

$$\begin{cases} \mathrm{d}y(t,z) = \{\partial_{z^i}[a^{ij}(t,z)\partial_{z^j} y(t,z)] + b^i(t,z)\partial_{z^i} y(t,z) + c(t,z)y(t,z) + \eta(t,z)\mathbb{E}[y(t,z)] \\ \qquad\qquad + u(t,z)\}\mathrm{d}t + [\rho(t,z)y(t,z) + \sigma(t,z)\mathbb{E}[y(t,z)] + u(t,z)]\mathrm{d}W(t), \\ \qquad\qquad (t,z) \in [0,T]\times\mathbb{R}^d, \\ y(0,z) = x \in \mathbb{R}^d \end{cases} \tag{6.10}$$

and the cost functional is

$$\inf_{u(\cdot,\cdot)\in\mathcal{M}^2_{\mathscr{F}}(0,T;U)} \Big\{ \mathbb{E}\Big[\int_{\mathbb{R}^d} y^2(T,z)\mathrm{d}z + \mathbb{E}\Big[\int_{\mathbb{R}^d} |\mathbb{E}y(T,z)|^2 \mathrm{d}z + \iint_{[0,T]\times\mathbb{R}^d} y^2(s,z)\mathrm{d}z\mathrm{d}s$$

$$+\iint_{[0,T]\times\mathbb{R}^d}|\mathbb{E}y(s,z)|^2\mathrm{d}z\mathrm{d}s+\iint_{[0,T]\times\mathbb{R}^d}u^2(s,z)\mathrm{d}z\mathrm{d}s\Big]\Big\}. \tag{6.11}$$

Here the coefficients a^{ij}, b^i, c, η, ρ, σ are given random functions satisfying the following assumptions, for some fixed constants $K\in(1,\infty)$ and $\kappa\in(0,1)$.

Assumption 6.3 a^{ij}, b^i, c, η, ρ and σ are $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)$-measurable taking values in the space of real symmetric $d\times d$ matrices, $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}$, respectively, and are bounded by K.

Assumption 6.4 a^{ij} satisfies the following super-parabolic condition

$$\kappa I\le 2a^{ij}(t,\omega,z)\le KI,\quad \forall(t,\omega,z)\in[0,T]\times\Omega\times\mathbb{R}^d,$$

where I is the $(d\times d)$-identity matrix.

In this case, in the Gelfand triple (V,H,V^*), the state equation (6.10) can be written as the abstract MF-SPDE:

$$\begin{cases}\mathrm{d}y(t)=[-A(t)y(t)+B_2(t)\mathbb{E}[y(t)]+u(t)]\mathrm{d}t\\ \qquad\qquad +[D_1(t)y(t)+D_2(t)\mathbb{E}[y(t)]+u(t)]\mathrm{d}W(t),\quad t\in[0,T],\\ y(0)=x,\end{cases} \tag{6.12}$$

where the operators A, B_2, D_1, D_2 are denoted by

$$\begin{aligned}&A(t)\phi(z)\triangleq-\partial_{z^i}[a^{ij}(t,z)\partial_{z^j}\phi(z)]-b^i(t,z)\partial_{z^i}\phi(z)-c(t,z)\phi(z),\quad\forall\phi\in V,\\ &B_2(t)\phi(z)\triangleq\eta(t,z),\quad D_1(t)\phi(z)\triangleq\rho(t,z)\phi(z),\quad D_2(t)\phi(z)\triangleq\sigma(t,z)\phi(z),\quad\forall\phi\in H.\end{aligned}$$

Then we write the optimal control problem as

$$\begin{aligned}\inf_{u(\cdot)\in\mathcal{M}^2_{\mathscr{F}}(0,T;U)}&\Big\{\mathbb{E}\Big[(y(T),y(T))_H+\int_0^T(y(s),y(s))_H\mathrm{d}s+(\mathbb{E}[y(T)],\mathbb{E}[y(T)])_H\\ &+\int_0^T(\mathbb{E}[y(s)],\mathbb{E}[y(s)])_H\mathrm{d}s+\int_0^T(u(s),u(s))_H\mathrm{d}s\Big]\Big\}.\end{aligned} \tag{6.13}$$

Thus this optimal control problem becomes a special case of our LQ problem in the previous subsection, where C, F, N and G are identity operators and $B_1=0$. From Assumptions 6.3–6.4, it is easy to check that the optimal control problem (6.13) satisfies Assumptions 6.1–6.2. So in view of Theorem 6.1, we claim that the optimal control $\overline{u}(\cdot)$ has the following explicit characterization:

$$\overline{u}(t)=-\frac{1}{2}\{\overline{p}(t)+\overline{q}(t)\},$$

where $(\overline{p}(\cdot),\overline{q}(\cdot))$ is the unique solution of the adjoint equation

$$\begin{cases}\mathrm{d}p(t)=-\{-A^*(t)p(t)+\mathbb{E}[B_2^*(t)p(t)]+D_1^*(t)q(t)\\ \qquad\qquad +\mathbb{E}[D_2^*(t)q(t)]+2y(t)\}\mathrm{d}t+q(t)\mathrm{d}W(t),\quad t\in[0,T],\\ p(T)=2y(T).\end{cases} \tag{6.14}$$

Here A^*, B_2^*, D_1^*, D_2^* denote the adjoint operators of A, B, D_1, D_2. More specifically,

$$A^*(t)\phi(z) = -\partial_{z^i}[a^{ij}(t,z)\partial_{z^j}\phi(z)] + b^i(t,z)\partial_{z^i}\phi(z) - [c(t,z) - \partial_{z^i}b^i(t,z)]\phi(z), \quad \forall\phi \in V$$

and

$$B_2^* = B_2, \quad D_1^* = D_1, \quad D_2^* = D_2.$$

7 Conclusion

In this paper, the MF-SPDE and MF-BSPDE and the corresponding optimal control problem for MF-SPDE have been investigated. We have established the existence, uniqueness and continuous dependence theorems of solutions to MF-SPDE and MF-BSPDE, respectively. For the optimal control problem of MF-SPDE, we have obtained necessary and sufficient conditions for optimal controls in the form of maximum principles. An an application, the LQ problem for MF-SPDE was investigated to illustrate our optimal control theory result established. As a result, the existence, uniqueness and explicit duality presentation of the optimal control have been obtained.

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