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Sobolev Spaces on Quasi-Kähler Complex Varieties

Haisheng LIU[1]

Abstract If V is an irreducible quasi-Kähler complex variety and E is a vector bundle over $\operatorname{reg}(V)$, the author proves that $W_0^{1,2}(\operatorname{reg}(V), E) = W^{1,2}(\operatorname{reg}(V), E)$, and that for $\dim_{\mathbb{C}} \operatorname{reg}(V) > 1$, the natural inclusion $W^{1,2}(\operatorname{reg}(V), E) \hookrightarrow L^2(\operatorname{reg}(V), E)$ is compact, the natural inclusion $W^{1,2}(\operatorname{reg}(V), E) \hookrightarrow L^{\frac{2v}{v-1}}(\operatorname{reg}(V), E)$ is continuous.

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1 Introduction

In [4], Bei proved several results on Sobolev spaces of irreducible complex projective varieties. The one we concern here in this paper is the following theorem.

Theorem 1.1 (cf. [4, Theorem 4.1]) *Let $V \subset \mathbb{CP}^n$ be an irreducible complex projective variety of complex dimension v. Let E be a vector bundle over* $\operatorname{reg}(V)$ *and let h be a metric on E, Riemannian if E is a real vector bundle. Hermitian if E is a complex vector bundle. Let g be the Kähler metric on* $\operatorname{reg}(V)$ *induced by the Fubini-study metric of $\mathbb{CP}^n$. Finally, let $\nabla : C^\infty(\operatorname{reg}(V), E) \to C^\infty(\operatorname{reg}(V), T^*\operatorname{reg}(V) \otimes E)$ be a metric connection. We have the following properties*:

(1) $W^{1,2}(\operatorname{reg}(V), E) = W_0^{1,2}(\operatorname{reg}(V), E)$.

(2) *Assume that $v > 1$. Then there exists a continuous inclusion $W^{1,2}(\operatorname{reg}(V), E) \hookrightarrow L^{\frac{2v}{v-1}}(\operatorname{reg}(V), E)$.*

(3) *Assume that $v > 1$. Then the inclusion $W^{1,2}(\operatorname{reg}(V), E) \hookrightarrow L^2(\operatorname{reg}(V), E)$ is a compact operator.*

These results have many applications, for example, on the L^2-theory of Kähler manifolds with singularities. The situation of non-Kähler complex spaces also interests us a lot. The geometry on non-Kähler manifolds has long been studied in many papers.

In this paper, we generalize the results on Sobolev space of projective varieties in [4] and get the corresponding results on quasi-Kähler complex subvarieties, using basically the same method that Bei [4] used. Let M be a differential manifold of dimension n with a Hermitian metric h. The metric h is often identified with the associated positive Hermitian $(1,1)$-form ω_h on X, called the Kähler form (or the fundamental form) of h. The metric ω_h is said to be

Manuscript received December 2, 2015. Revised February 23, 2017.
[1]Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China.
E-mail: haishengliu2014@163.com

quasi-Kähler if $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$. Quasi-Kähler manifolds are important because they include the classes $\mathscr{AK}$ and $\mathscr{NK}$ of almost Kähler and nearly Kähler manifolds. A large part of the theory of the geometry and topology of Kähler manifolds can be carried over to the class $\mathscr{NK}$ (cf. [10]). As a generalization of Kähler manifolds, we are interested in quasi-Kähler manifolds naturally.

In Section 2, we recall some basic concepts and notations that will be used in this paper, such as almost complex manifold, Hermitian manifold (or complex manifold), metric connection, etc. We also recall the fundamental knowledge of Sobolev space and operator extension theory of differential operators which are densely defined on a Hilbert space.

In Section 3, we prove several results analogous to the corresponding results in [4]. Firstly, we prove a proposition which states that there exists a certain sequence of cut-off functions on an irreducible quasi-Kähler complex variety.

Proposition 1.1 *Let V be an irreducible quasi-Kähler complex variety of M and let g be the metric on* $\mathrm{reg}(V)$ *induced by the Hermitian metric h of M. Then there exists a sequence of Lipschsitz functions with compact support $\{\phi_j\}_{j\in\mathbb{N}}$ such that*

(1) $0 \le \phi_j \le 1$ *for each j;*

(2) $\phi_j \to 1$ *point-wise*;

(3) $\phi_j \in \mathcal{D}(d_{0,\min})$ *for each $j \in \mathbb{N}$ and* $\lim\limits_{j\to\infty} \|d_{0,\min}\phi_j\|_{L^2\Omega^1(\mathrm{reg}(V),g)} = 0$.

In particular, $1 \in \mathcal{D}(d_{0,\min})$.

Secondly, we show some results on the Sobolev space of an irreducible quasi-Kähler complex variety.

Theorem 1.2 *Let M be a compact quasi-Kähler complex manifold with a Hermitian metric h and V be an irreducible complex analytic subvariety of dimension v of M. Let E be a vector bundle over* $\mathrm{reg}(V)$ *and let h be a metric on E, Riemannian if E is a real vector bundle, Hermitian if E is a complex vector bundle. Let g be the metric on* $\mathrm{reg}(V)$ *induced by the metric of M. Finally let* $\nabla : C^\infty(\mathrm{reg}(V), E) \to C^\infty(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E)$ *be a metric connection. We have the following properties*:

(1) $W^{1,2}(\mathrm{reg}(V), E) = W_0^{1,2}(\mathrm{reg}(V), E)$.

(2) *Assume that $v > 1$. Then there exists a continuous inclusion* $W^{1,2}(\mathrm{reg}(V), E) \hookrightarrow L^{\frac{2v}{v-1}}(\mathrm{reg}(V), E)$.

(3) *Assume that $v > 1$, then the inclusion* $W^{1,2}(\mathrm{reg}(V), E) \hookrightarrow L^2(\mathrm{reg}(V), E)$ *is a compact operator.*

As in [4], the proof of this theorem depends on Kato's inequality, Sobolev inequality and the existence of a suitable sequence of cut-off functions. When V is an irreducible complex projective variety, the regular part of V, $\mathrm{reg}(V)$, is a Kähler manifold endowed with the incomplete Kähler metric, i.e., the Fubini-study metric. The fact that any Kähler submanifold is minimal yields the mean curvature equals 0, i.e., $H = 0$. This enables us to use the results in [14, 16]. Meanwhile, we would like to point out that some of those results only require the condition that the mean curvature is bounded on $\mathrm{reg}(V)$.

Thirdly, we get the following useful proposition.

Proposition 1.2 *Let* $(\mathrm{reg}(V), g)$ *be as in Theorem* 3.1. *Let* E *and* F *be two vector bundles over* $\mathrm{reg}(V)$ *endowed respectively with metrics* h *and* ρ, *Riemannian if* E *and* F *are real vector bundles, Hermitian if* E *and* F *are complex vector bundles. Finally, let* $\nabla : C^\infty(\mathrm{V}, E) \to C^\infty(M, T^*\mathrm{reg}(V) \otimes E)$ *be a metric connection. Consider a first order differential operator of this type*:

$$D := \theta_0 \circ \nabla : C_c^\infty(\mathrm{reg}(V), E) \to C_c^\infty(\mathrm{reg}(V), F), \tag{1.1}$$

where $\theta_0 \in C^\infty(\mathrm{reg}(V), \mathrm{Hom}(T^*\mathrm{reg}(V) \otimes E, F))$. *Assume that* θ_0 *extends as a bounded operator*

$$\theta : L^2(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E) \to L^2(\mathrm{reg}(V), F).$$

Then we have the following inclusion:

$$\mathrm{dom}(D_{\max}) \cap L^\infty(\mathrm{V}, E) \subset \mathrm{dom}(D_{\min}). \tag{1.2}$$

In particular, (1.2) holds when D *is the de Rham differential* $\mathrm{d}_k : \Omega_c^k(\mathrm{reg}(V)) \to \Omega_c^{k+1}(\mathrm{reg}(V))$, *a Dirac operator* $D : C_c^\infty(\mathrm{reg}(V), E) \to C_c^\infty(\mathrm{reg}(V), E)$, *or the Dolbeault operator* $\bar{\partial}_{p,q} : \Omega_c^{p,q}(\mathrm{reg}(V)) \to \Omega_c^{p,q+1}(\mathrm{reg}(V))$.

And after this proposition, we have several remarks and corollaries.

In Section 4, we show that Theorem 3.1 can be applied on Schrödinger operators and heat operators on the irreducible quasi-Kähler complex varieties.

2 Preliminary

In this section, we recall some basic concepts and notations that will be used throughout this paper.

2.1 Basic concepts, notations and results on almost complex manifolds and quasi-Kähler manifolds

An almost complex manifold M is a differentiable manifold on which there exists a $(1,1)$-tensor J, which we may consider as an isomorphism $J : \mathscr{X}(M) \to \mathscr{X}(M)$, satisfying the condition $J^2 = -1$. An almost complex manifold is orientable and of even dimension. Denote $\langle \cdot, \cdot \rangle$ a Riemannian metric on M and $\mathscr{X}(M)$ the real vector fields over M. M is an almost Hermitian manifold (or an almost complex manifold) if J is compatible with the inner product in such a way that $\langle JX, JY \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathscr{X}(M)$. There are tow special tensors defined in terms of the almost complex structure J that are very important. Firstly, the Kähler form (or the fundamental form) associated to $(M, \langle \cdot, \cdot \rangle, J)$ is defined as

$$\omega(X, Y) := -\langle (X), J(Y) \rangle = \langle J(X), (Y) \rangle, \quad \forall X, Y \in \mathscr{X}(M). \tag{2.1}$$

Since $\omega(X, Y) = \langle JX, Y \rangle = \langle JJX, JY \rangle = \langle -X, JY \rangle = -\langle JY, X \rangle = -\omega(Y, X)$, we get that ω is skew-symmetric, thus a differential form. Secondly, the so-called Nijenhuis tensor, is a

(1, 2)-tensor S defined by

$$S(X,Y)=[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY],\quad \forall X,Y\in\mathscr{X}(M). \tag{2.2}$$

By easy calculations, one can see that

$$S(X,Y)=-S(Y,X),\quad S(JX,Y)=S(X,JY)=-JS(X,Y).$$

A well-known theorem in [7, 17] states that M is a complex manifold if and only if S vanishes identically. If we extend the Riemannian connection ∇_X of M to be a derivation on the tensor algebra of M, then we have the formulae

$$\nabla_X(J)(Y)=\nabla_X(JY)-J\nabla_X(Y), \tag{2.3}$$

$$\nabla_X(\omega)(Y,Z)=\langle\nabla_X(J)(Y),Z\rangle. \tag{2.4}$$

Let $X,Y\in\mathscr{X}(M)$, M is called

almost Kähler, if $\mathrm{d}\omega=0$,

nearly Kähler, if $\nabla_X(J)(Y)+\nabla_Y(J)(X)=0$,

quasi-Kähler, if $\nabla_X(J)(Y)+\nabla_{JX}(J)(JY)=0$.

An almost complex structure is said to be integrable if it is induced by a complex structure. Actually, if M is a complex manifold of dimension n. Denote TM the real tangent space, $\mathrm{TM}^{\mathbb{C}}$ the complex tangent space. Let $(z_1,\cdots,z_n)$ be a local holomorphic coordinate in a neighborhood U of $p\in M$. Write $z_j=x_j+\mathrm{i}y_j$. Then $(x_1,\cdots,x_n,y_1,\cdots,y_n)$ is a smooth real coordinate in U and $\{\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial y_n}\}$ gives a local frame of the tangent bundle TM. Denote by

$$\frac{\partial}{\partial z_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_j}-\mathrm{i}\frac{\partial}{\partial y_j}\Big),\quad \frac{\partial}{\partial \overline{z}_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_j}+\mathrm{i}\frac{\partial}{\partial y_j}\Big)$$

for each $1\le j\le n$. Then $T^{1,0}M$ is the complex subbundle of $\mathrm{TM}^{\mathbb{C}}$ spanned by $\{\frac{\partial}{\partial z_1},\cdots,\frac{\partial}{\partial z_n}\}$, while $T^{0,1}M$ is spanned by $\{\frac{\partial}{\partial \overline{z}_n},\cdots,\frac{\partial}{\partial \overline{z}_n}\}$. Let us consider the isomorphism $J:\mathrm{TM}\to\mathrm{TM}$ defined by

$$J\frac{\partial}{\partial x_j}=\frac{\partial}{\partial y_j},\quad J\frac{\partial}{\partial y_j}=-\frac{\partial}{\partial x_j}$$

for each $1\le j\le n$. Then the map J defined above is the almost complex structure induced by the complex structure of M. The following results are very useful.

Proposition 2.1 (cf. [9, Proposition 5.2] *If M is Kähler, almost Kähler, nearly Kähler, quasi-Kähler, or Hermitian, then any Hermitian submanifold of M has the same property.*

Theorem 2.1 (cf. [9, Theorem 5.7]) *A quasi-Kähler submanifold is a minimal variety.*

From the above two results, we know that every complex submanifold of a quasi-Kähler complex manifold is quasi-Kähler and minimal. Since the regular part of a complex variety is a complex submanifold, we could use the words "quasi-Kähler variety" without any ambiguity. As we have mentioned above, the fact that the projective variety is minimal plays an important role in the proofs of some theorems in [4, 14]. Thus, we could expect that similar results could be proved on irreducible quasi-Kähler complex varieties.

2.2 L^p-space, Sobolev space and operator extension

We will recall some basic notations on L^p-spaces, Sobolev spaces and differential operators, all of which will be used in this paper. We refer to [3, 5, 12], or the appendix in [20] for a thorough discussion on the relative materials. Let (M, g) be an open and possibly incomplete Riemannian manifold of dimension m. Let E be a vector bundle over M of rank r with a metric h, Hermitian if E is a complex vector bundle, Riemannian if E is a real vector bundle. Let dv_g denote the volume element of g. We consider M endowed with Riemannian measure as in [12, p. 59] or [5, p. 29]. A section is called measurable if, for any local trivialization (U, ϕ) of E, every ϕ^r of $\phi(s|_U) = (\phi^1, \cdots, \phi^r)$ is a measurable function. For a fixed measurable section s, define its local norm(or point-wise norm) respect to h as $|s|_h := (h(s, s))^{\frac{1}{2}}$. Then for every p, $1 \le p < \infty$, we can define the L^p-norm of a measurable section s,

$$\|s\|_{L^p(M,E)} := \Big(\int_M |s|_h^p \mathrm{dv}_g\Big)^{\frac{1}{p}},$$

and thus the L^p-space of measurable sections over M,

$$L^p(M, E) := \{\|s\|_{L^p(M,E)} < \infty\}.$$

When $1 \le p < \infty$, $L^p(M, E)$ is a Banach space; furthermore if $1 < p < \infty$, $L^p(M, E)$ is a reflexive Banach space, i.e., $L^p(M, E) \cong (L^p(M, E))'$. When $p = 2$, $L^2(M, E)$ is a Hilbert space with the natural inner product

$$\langle s, t\rangle_{L^2(M,E)} := \int_M h(s, t)\mathrm{dv}_g.$$

Another important and useful conclusion is that when $1 \le p < \infty$, $L^p(M, E)$ has a natural dense subset $C_c^\infty(M, E)$, the space of smooth sections with compact support. We can also define $L^\infty(M, E)$ to be the space of measurable sections whose essential supremum is bounded, i.e., $L^\infty(M, E) := \{s \mid \operatorname{ess\,sup}|s|_h < \infty\}$. Notice that $L^\infty(M, E)$ is also a Banach space. We would like to clarify the notations a little more. The spaces $L^p(M, E)$ clearly depend on M, E, h, g, whereas we still denote them as $L^p(M, E)$ instead of $L^p(M, h, E, g)$ when there is no danger of confusion. Particularly, if E is the trivial bundle $M \times \mathbb{R}$, we will write $L^p(M, g)$ instead of $L^p(M, \mathbb{R})$ while for the k-th exterior power of the cotangent bundle $\Lambda^k T^*M$, we will write $L^p\Omega^k(M, g)$ instead of $L^p(M, \Lambda^k T^*M)$. Suppose F is another vector bundle over M endowed with a metric ρ and $P : C_c^\infty(M, E) \to C_c^\infty(M, F)$ is a differential operator of order $d \in \mathbb{N}$. The formal adjoint of P,

$$P^t : C_c^\infty(M, F) \to C_c^\infty(M, E)$$

is the differential operator defined by the following property: For each $u \in C_c^\infty(M, E)$ and for each $v \in C_c^\infty(M, F)$, we have the identity

$$\int_M h(u, P^t v)\mathrm{dv}_g = \int_M \rho(Pu, v)\mathrm{dv}_g.$$

On the basis of the above, we can now recall some knowledge of the extensions of an operator. One can check that the operator P defined above is an unbounded, densely defined and closable operator from $L^p(M,E)$ to $L^p(M,F)$. In general cases, P may have several different closed extensions between the minimal and maximal extensions. For completeness, we recall the definitions of the minimal and maximal extensions below. The domain of the maximal extension of $P : L^p(M,E) \to L^p(M,F)$ is defined, in the distributional sense, as

$$\begin{aligned}\mathrm{dom}(P_{\max}) :=\Big\{ & s \in L^2(M,E) : \exists v \in L^2(M,F) \\ & \text{s.t.} \int_M h(s, P^t\phi)\mathrm{d}v_g = \int_M \rho(v,\phi)\mathrm{d}v_g \forall \phi \in C_c^\infty(M,F)\Big\},\end{aligned} \tag{2.5}$$

and the minimal extension as

$$\begin{aligned}\mathrm{dom}(P_{\min}) := \{ & s \in L^2(M,E) : \exists \{s_i\} \subset C_c^\infty(M,E), s_i \xrightarrow{L^2} s, \\ & Ps_i \xrightarrow{L^2} w, \quad \text{where } w \in L^2(M,F)\}.\end{aligned} \tag{2.6}$$

We put $P_{\max}s = v$, $P_{\min}s = w$ by the definition above. On the other hand, the minimal extension of P is the closure of $C_c^\infty(M,E)$ under the graph norm $\|s\|_{L^2(M,E)} + \|Ps\|_{L^2(M,F)}$. One can check the following two important identity:

$$P^*_{\max} = P^t_{\min}, \quad P^*_{\min} = P^t_{\max}, \tag{2.7}$$

which means that $P^t_{\min}$ is the adjoint of $P_{\max}$ respect to the Hilbert space $L^2(M,E)$ and $L^2(M,F)$, and similarly $P^t_{\max}$ with respect to $P_{\min}$. Another useful fact is the orthogonal decomposition of the L^2-space:

$$L^2(M,E) = \ker(P_{\min}) \oplus \overline{\mathrm{im}(P^*_{\min})} = \ker(P_{\min}) \oplus \overline{\mathrm{im}(P^t_{\max})}. \tag{2.8}$$

Next, we recall the concepts and notations of Sobolev space associated to a metric connection. Let E and h be defined as above. Let $\nabla : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$ be a metric connection, i.e., a connection that is compatible with the differential and the metric in such a way: for each $s, u \in C^\infty(M)$ we have $\mathrm{d}(h(s,u)) = h(\nabla s, u) + h(s, \nabla u)$. Let $\nabla^t : C_c^\infty(M, T^*M \otimes E) \to C_c^\infty(M,E)$ be the formal adjoint of ∇ with respect to $\widetilde{h}$ and g. Then we can define the Sobolev space $W^{1,2}(M,E)$ as

$$\begin{aligned}W^{1,2}(M,E) := \Big\{ & s \in L^2(M,E) : \exists\ v \in L^2(M, T^*M \otimes E) \\ & \text{s.t.} \int_M h(s, \nabla^t\phi)\mathrm{d}v_g = \int_M \widetilde{h}(v,\phi)\mathrm{d}v_g,\ \forall \phi \in C_c^\infty(M, T^*M \otimes E)\Big\}.\end{aligned} \tag{2.9}$$

By the definition of $\mathrm{dom}(P_{\max})$, we can see that $W^{1,2}(M,E) = \mathrm{dom}(\nabla_{\max})$. We can also define the Sobolev space $W_0^{1,2}(M,E)$ as follows:

$$\begin{aligned}W_0^{1,2}(M,E) := \{ & s \in L^2(M,E) : \exists\ \{s_i\} \subset C_c^\infty(M,E), s_i \xrightarrow{L^2} s, \\ & \nabla s_i \xrightarrow{L^2} w, \quad \text{where } w \in L^2(M, T^*M \otimes E)\}.\end{aligned} \tag{2.10}$$

Analogously to the case of maximal extension, by the definition of $\mathrm{dom}(P_{\min})$, we can also conclude that $W_0^{1,2}(M,E) = \mathrm{dom}(\nabla_{\min})$. If E is the trivial bundle $M \times \mathbb{R}$, we will write

$W^{1,2}(M,g)$ and $W_0^{1,2}(M,g)$ when there is no danger of confusion. We refer the reader to [1] for more information on Sobolev spaces. We would like to recall the following result for the reader's convenience.

Proposition 2.2 *Let (M,g) be an open and possibly incomplete Riemannian manifold of dimension m. Let E be a vector bundle over M endowed with a metric h. Let $U \subset M$ be an open subset with compact closure. Consider the spaces $L^2(U, E|_U)$ and $W_0^{1,2}(M,E)$ where U is endowed with the metric $g|_U$. Then the natural inclusion*

$$W_0^{1,2}(U, E|_U) \hookrightarrow L^2(U, E|_U) \tag{2.11}$$

is a compact operator. Therefore the map

$$i_0 : W_0^{1,2}(U, E|_U) \to L^2(M, E)$$

given by

$$i_0(f) = \begin{cases} f & on\, U, \\ 0 & on\, M \backslash U \end{cases} \tag{2.12}$$

is an injective and compact operator.

For the proof of the above proposition, we refer the reader to [15, p. 349] and [20, p. 179].

The de Rham differential operator acting on the space of smooth k-forms with compact support, $\mathrm{d}_k : \Omega_c^k(M) \to \Omega_c^{k+1}(M)$. Given a Riemannian metric g on M, we denote by $\langle \cdot, \cdot \rangle_{g_k}$ and by $|\cdot|_{g_k}$ respectively the metric and the point-wise norm induced by g on $\Lambda^k T^*M$ for each $k = 0, \cdots, m$, where $m = \dim M$. When $k = 1$, we will simply denote the corresponding term by $\langle \cdot, \cdot \rangle_g$ and $|\cdot|_g$ instead of $\langle \cdot, \cdot \rangle_{g_1}$ and $|\cdot|_{g_1}$. We will denote by $\widetilde{g}_k$ the metric that g induces on $T^*M \otimes \Lambda^k T^*M$. Following the definitions, we denote by $\mathrm{d}_{k,\max/\min} : L^2\Omega^k(M,g) \to L^2\Omega^{k+1}(M,g)$ respectively the maximal and minimal extension of d_k acting on the space of L^2 k-forms.

2.3 Basics on quadratic forms and the Friedrich extension

We will recall some basic results on quadratic forms and the Friedrich extension of a positive and symmetric operator. The reader can find a general description in [15, C.1]. One can also find a more thorough discussion on this topic in [18–19]. Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. A quadratic form is a sesquilinear map $Q : \mathrm{dom}(Q) \times \mathrm{dom}(Q) \to \mathbb{C}$, where $\mathrm{dom}(Q)$ is a dense linear subspace of H. Q is called positive if $Q(u,u) \geq 0$ for any $u \in \mathrm{dom}(Q)$. A positive quadratic form Q is called closed if $(\mathrm{dom}(Q), \|\cdot\|_Q)$ is complete, where

$$\|u\|_Q := (Q(u,u) + \|u\|^2)^{\frac{1}{2}}, \quad \text{for } u \in \mathrm{dom}(Q). \tag{2.13}$$

Let $B : H \to H$ be a linear unbounded densely defined operator. B is called self-adjoint if $B = B^*$, symmetric if $B \subset B^*$, and positive if $\langle Bu, u \rangle \geq 0$ for any $u \in \mathrm{dom}(B)$. Now if we assume that B is symmetric and positive, we can define the quadratic form associated to B as $Q_B(u,v) := \langle Bu, v \rangle$. Denote by $\langle \cdot, \cdot \rangle_B$ the inner product given by $\langle \cdot, \cdot \rangle + Q_B(,)$ and by $\mathrm{dom}(Q_B)$ the completion $\mathrm{dom}(B)$ through $\langle\cdot,\cdot\rangle_B$. One can check that the identity map $\mathrm{Id} : \mathrm{dom}(B) \to \mathrm{dom}(B)$ extends as a bounded injective map $i_{Q_B} : \mathrm{dom}(Q_B) \to H$. By this injective map, $\mathrm{dom}(Q_B)$ can always be identified with its image in H, that is

$$\begin{aligned}&\{u \in H : \exists\, \{u_n\}_{n\in\mathbb{N}} \subset \mathrm{dom}(B) \text{ s.t. } \langle u_n - u, u_n - u\rangle \to 0,\\ &\langle u_n - u_m, u_n - u_m\rangle_B \to 0 \text{ as } m, n \to \infty\}.\end{aligned} \tag{2.14}$$

Following the above discussion, we can define the Friedrich extension of a positive self-adjoint operator B, denoted by $B^{\mathcal{F}}$. The domain of Friedrich extension is given by

$$\{u \in \mathrm{dom}(Q_B) \mid \exists\, v \in H \text{ s.t. } Q_B(u, w) = \langle v, w\rangle\ , \forall\, w \in \mathrm{dom}(Q_B)\},$$

and we put $B^{\mathcal{F}}u := v$. One can check that $B^{\mathcal{F}}$ is a positive and self-adjoint operator. What's more, the above definition is equivalent to

$$\begin{aligned}\mathrm{dom}(B^{\mathcal{F}}) = \{u \in \mathrm{dom}(B^*) \mid \exists\, \{u_n\} \subset \mathrm{dom}(B) \text{ s.t. } &\langle u - u_n, u - u_n\rangle \to 0,\\ &\langle B(u_n - u_m), u_n - u_m\rangle \to 0 \text{ as } n, m \to \infty\}\end{aligned} \tag{2.15}$$

and $B^{\mathcal{F}} = B^*(u)$, that is

$$\mathrm{dom}(B^{\mathcal{F}}) := \mathrm{Q_B} \cap \mathrm{B}^*, \quad B^{\mathcal{F}}u := B^*u$$

for $u \in \mathrm{dom}(B^{\mathcal{F}})$. One can also find a more complete discussion of Friedrich extension in [2].

3 Sobolev Spaces on Irreducible Quasi-Kähler Complex Varieties

In this section, we generalize the results of [4] and we use just the same method used in [4]. Actually, we just use Bei's proof and notations with some changes to adapt it to our situation. We work on irreducible quasi-Kähler complex varieties V. This mean that V is locally the zero set of a (finite) family of holomorphic functions of a compact quasi-Kähler complex manifold such that it is impossible to decompose V as a (finite) union of complex varieties which are not equal to V. To be more precise, if $V = V_1 \cup V_2$ where V_1, V_2 are complex varieties, then we have $V_i = \varnothing$ or $V_i = V$ for $i = 1, 2$. We refer the reader to [11] for more details on this topic. Given an irreducible quasi-Kähler complex variety, we denote the singular subset of V by $\mathrm{sing}(V)$ and the regular part by $\mathrm{reg}(V) := V \setminus \mathrm{sing}(V)$. The regular part $\mathrm{reg}(V)$ then becomes a quasi-Kähler complex manifold as we can see by [9]. Usually, if $\mathrm{sing}(V) \neq \varnothing$, $\mathrm{reg}(V)$ is an open and incomplete quasi-Kähler complex manifold with the induced metric from M. Now we prove a proposition which claims the existence of a certain sequence of cut-off functions. This result is similar to that contained in [14, p. 871], [21, Theorems 3.1–3.2] and [4, Theorem 4.2].

Proposition 3.1 (cf. [4, Proposition 4.1]) *Let M be a complex manifold and let h and g be two Hermitian metrics on M such that $g \geq h$. Then for each $\eta \in \Omega^1_c(M)$, we have $\|\eta\|_{L^2\Omega^1(M,g)} \leq \|\eta\|_{L^2\Omega^1(M,h)}$. Therefore the identity map $Id : \Omega^1_c(M) \to \Omega^1_c(M)$ extends as a continuous inclusion $L^2\Omega^1(M, g) \hookrightarrow L^2\Omega^1(M, h)$, so that for each $\phi \in L^2\Omega^1(M, g)$ we have $\|\phi\|_{L^2\Omega^1(M,h)} \leq \|\phi\|_{L^2\Omega^1(M,g)}$.*

Proof The proof is essentially a calculation of linear algebra. One can see more details in [8, p. 146] for example.

Proposition 3.2 *Let M be a compact quasi-Kähler complex manifold with a Hermitian metric h and V be an irreducible quasi-Kähler complex variety of M, and let g be the metric on* $\mathrm{reg}(V)$ *induced by the metric of M. Then there exists a sequence of Lipschitz functions with compact support $\{\phi_j\}_{j\in\mathbb{N}}$ such that*

(1) $0 \le \phi_j \le 1$ *for each j;*

(2) $\phi_j \to 1$ *point-wise*;

(3) $\phi_j \in \mathcal{D}(d_{0,\min})$ *for each $j \in \mathbb{N}$ and* $\lim\limits_{j\to\infty} \|d_{0,\min}\phi_j\|_{L^2\Omega^1(\mathrm{reg}(V),g)} = 0$.

In Particular, $1 \in \mathcal{D}(d_{0,\min})$.

Proof Let $\pi : \widetilde{V} \to V$ be a resolution of singularities (which exists thank to the fundamental work in [13]). We recall that $\pi : \widetilde{V} \to V$ is a holomorphic and surjective map such that

$$\pi|_{\widetilde{V}\backslash E} : \widetilde{V}\backslash E \to V\backslash \mathrm{sing}(V)$$

is a biholomorphism where $E = \pi^{-1}(\mathrm{sing}(V))$ is the exceptional set. Moreover, we can assume that E is a divisor with only normal crossings, that is, the irreducible components of E are regular and meet complex transversely. In particular, $\widetilde{V}\backslash\pi^{-1}(\mathrm{reg}(V))$ is a union of finite number compact complex submanifolds, which means $\widetilde{V}\backslash\pi^{-1}(\mathrm{reg}(V)) = \bigcup\limits_{i=1}^{m} S_i$ for some $m \in \mathbb{N}_+$. Therefore, we have $\mathrm{codim}_{\mathbb{R}}(S_i) \ge 2$ for each $i = 1, \cdots, m$. Suppose that h is a Hermitian metric on $\widetilde{V}$. Let us define $V' := \pi^{-1}(\mathrm{reg}(V))$ and $h' = h|_{V'}$. Firstly, we will show that there is a sequence of Lipschitz functions $\{\psi_j\}_{j\in\mathbb{N}}$ with compact support on (V', h'), which satisfies the three properties stated in the proposition. We use the method used in [6, 14]. Define $M_i := \widetilde{V}\backslash S_i$. Let r_i be the distance function to S_i induced by h. let $\varepsilon_j := \frac{1}{n^2}$ and $\varepsilon'_j := \mathrm{e}^{-\varepsilon^{-2}} = \mathrm{e}^{-j^4}$. Then we define ψ_{j,M_i} as

$$\psi_{j,M_i} := \begin{cases} 1, & r_i \ge \varepsilon_j, \\ \Big(\dfrac{r_i}{\varepsilon_j}\Big)^{\varepsilon_j}, & 2\varepsilon'_j \le r_i \le \varepsilon_j, \\ \Big(\dfrac{2\varepsilon'_j}{\varepsilon_j}\Big)^{\varepsilon_j}\Big(\dfrac{r_i}{\varepsilon'} - 1\Big), & \varepsilon'_j \le r_i \le 2\varepsilon'_j, \\ 0, & 0 \le r_i \le \varepsilon'_j. \end{cases} \tag{3.1}$$

We can easily check that each ψ_{j,M_i} defined in (3.1) is a Lipschitz function with compact support, partially by the fact that M is compact. Then by [12, Theorem 11.3], [4, Proposition 1.2] and the fact that M_i has finite volume, we get that $\{\psi_{j,M_i}\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathrm{d}_{0,\min})$ on $(M_i, h|_{M_i})$. We can also easily check that $0 \le \psi_{j,M_i} \le 1$ and $\lim\limits_{j\to\infty} \psi_{j,M_i} = 1$ point-wise. Moreover, by [6], we can get

$$\lim_{j\to\infty} \|\mathrm{d}_{0,\min}\psi_{j,M_i}\|_{L^2\Omega^1(M_i,h|_{M_i})} = 0. \tag{3.2}$$

We recall that the previous limit is based on an estimate of the volume of a tubular neighborhood of S_i. In this estimate, the lower bound of the real codimension of S_i is a key factor. Now we define

$$\psi_j := \prod_{i=1}^{m} \psi_{j,M_i}.$$

For each $j \in \mathbb{N}$, ψ_j is defined as a product of a finite number of non negative Lipschitz functions with compact support and bounded above by 1. We can easily check that ψ_j is a nonnegative Lipschitz function with compact support and bounded above by 1 itself. Thus, arguing as above, we can conclude that ${\psi_j}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathrm{d}_{0,\min})$ on (V', h'). Clearly for each ψ_j we have $0 \leq \psi_j \leq 1$ and $\lim_{j\to\infty} \psi_j = 1$ point-wise. Now we have to show that

$$\lim_{j\to\infty} \langle \mathrm{d}_{0,\min}\psi_j, \mathrm{d}_{0,\min}\psi_j \rangle_{L^2\Omega^1(V',h')} = 0. \tag{3.3}$$

We have $\mathrm{d}_{0,\min}\psi_j = \sum_{i=1}^{m} \gamma_i \mathrm{d}_{0,\min}\psi_{j,M_i}$, where γ_i is given by the product

$$\psi_{j,M_1} \cdots \psi_{j,M_{i-1}} \psi_{j,M_{i+1}} \cdots \psi_{j,M_m}.$$

By the fact that $0 \leq \gamma_i \leq 1$ to establish (3.3), it is enough to show that

$$\lim_{j\to\infty} \langle \mathrm{d}_{0,\min}\psi_{j,M_p}, \mathrm{d}_{0,\min}\psi_{j,M_q} \rangle_{L^2\Omega^1(V',h')} = 0$$

for each $p, q \in \{1, \cdots, m\}$.

This follows because

$$\langle \mathrm{d}_{0,\min}\psi_{j,M_p}, \mathrm{d}_{0,\min}\psi_{j,M_q} \rangle_{L^2\Omega^1(V',h')} \leq \|\mathrm{d}_{0,\min}\psi_{j,M_p}\|_{L^2\Omega^1(V',h')} \|\mathrm{d}_{0,\min}\psi_{j,M_q}\|_{L^2\Omega^1(V',h')},$$

and by (3.2), we have

$$\lim_{j\to\infty} \|\mathrm{d}_{0,\min}\psi_{j,M_p}\|_{L^2\Omega^1(V',h')} = 0$$

and

$$\lim_{j\to\infty} \|\mathrm{d}_{0,\min}\psi_{j,M_q}\|_{L^2\Omega^1(V',h')} = 0.$$

These relations allow us to conclude that on (V', h') there is a sequence of Lipschitz functions with compact support $\{\psi_j\}_{j\in\mathbb{N}}$, which satisfies the three properties stated in this proposition. Now let $\widetilde{g}$ be the Kähler metric on V' defined as $\pi^* g$. We can see $\widetilde{g}$ as the pullback of the metric on M through the map $\pi : \widetilde{V} \to V \subset M$. By the fact that $\mathrm{d}\pi$, the differential of π, degenerates on $\widetilde{V} \backslash V'$, we get that $\widetilde{g} \leq Ch'$, for some positive real constant $C > 0$. Now as an immediate application of Proposition 3.1, we can conclude that the sequence $\{\psi_j\}_{j\in\mathbb{N}}$ satisfies the three properties stated in this proposition also with respect to the balanced manifold $(V', \widetilde{g})$. Finally, by the fact that $\pi|_{V'} : (V', \widetilde{g}) \to (\mathrm{reg}(V), g)$ is an isometry, defining $\phi_j := \psi_j \circ (\pi|_{V'})^{-1}$, we obtain our desired sequence on $(\mathrm{reg}(V), g)$.

We would like to point out that in the proof of Proposition 3.2, we use the resolution theorem of Hironaka. Thus we should assume that the background manifold should be both quasi-Kähler and complex, namely, we only consider quasi-Kähler complex varieties. Next we give our main theorem in this paper.

Theorem 3.1 *Let M be a compact quasi-Kähler complex manifold with a Hermitian metric h and V be an irreducible quasi-Kähler complex subvariety of M. Let E be a vector bundle over* $\mathrm{reg}(V)$ *and let h' be a metric on E, Riemannian if E is a real vector bundle, Hermitian if E is a complex vector bundle. Let g be the metric on* $\mathrm{reg}(V)$ *induced by the metric h. Finally, let $\nabla : C^\infty(\mathrm{reg}(V), E) \to C^\infty(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E)$ be a metric connection. We have the following properties*:

(1) $W^{1,2}(\mathrm{reg}(V), E) = W_0^{1,2}(\mathrm{reg}(V), E)$.

(2) *Assume that $v > 1$, then there exists a continuous inclusion*

$$W^{1,2}(\mathrm{reg}(V), E) \hookrightarrow L^{\frac{2v}{v-1}}(\mathrm{reg}(V), E).$$

(3) *Assume that $v > 1$, then the inclusion $W^{1,2}(\mathrm{reg}(V), E) \hookrightarrow L^2(\mathrm{reg}(V), E)$ is a compact operator.*

Proof The first point follows by Proposition 3.2 and [4, Proposition 3.1]. The continuous inclusion $W_0^{1,2}(\mathrm{reg}(V), g) \hookrightarrow L^{\mathrm{reg}(V),g}$ is established in [14, p. 874] or [21, p. 113]. Now, by the first point of this theorem (or by [14, Theorem 4.1] or by [21, Corollary 3.1]), we know that $W^{1,2}(\mathrm{reg}(V), g) = W_0^{1,2}(\mathrm{reg}(V), g)$ and therefore we have the continuous inclusion

$$W^{1,2}(\mathrm{reg}(V), g) \hookrightarrow L^{\frac{2v}{v-1}}(\mathrm{reg}(V), g).$$

By [4, Proposition 2.1], we get the continuous inclusion

$$C^\infty(\mathrm{reg}(V), E) \cap W^{1,2}(\mathrm{reg}(V), E) \hookrightarrow L^{\frac{2v}{v-1}}(\mathrm{reg}(V), E).$$

Finally, by the density of $C^\infty(\mathrm{reg}(V), E) \cap W^{1,2}(\mathrm{reg}(V), E)$ in $W^{1,2}(\mathrm{reg}(V), E)$ (cf. [4, Proposition 1.2]), the continuous inclusion $W^{1,2}(\mathrm{reg}(V), E) \hookrightarrow L^{\frac{2v}{v-1}}(\mathrm{reg}(V), E)$ is established. Finally the third point is a consequence of the second point and [4, Proposition 3.3].

Remark 3.1 The statement of Theorem 3.1 can be reformulated as follows. $\mathrm{dom}(\nabla_{\max}) = \mathrm{dom}(\nabla_{\min})$, there is a continuous inclusion $\mathrm{dom}(\nabla_{\max}) \hookrightarrow L^{\frac{2v}{v-1}}(\mathrm{reg}(V), E)$ and the natural inclusion $\mathrm{dom}(\nabla_{\max}) \hookrightarrow L^2(\mathrm{reg}(V), E)$ is a compact operator where $\mathrm{dom}(\nabla_{\max})$ is endowed with the corresponding graph norm.

Remark 3.2 We would like to mention that the key point which enables us to generalize the results of [4] from the irreducible complex projective varieties to the quasi-Kähler complex subvarieties is that every quasi-Kähler complex submanifold is minimal.

Corollary 3.1 *Under the assumptions of Theorem* 3.1, $\mathrm{im}(\nabla_{\min}) = \mathrm{im}(\nabla_{\max})$ *is a closed subspace of* $L^2(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E)$.

Proof By Theorem 3.1, we know that $\nabla_{\min} = \nabla_{\max}$ and therefore $\mathrm{im}(\nabla_{\min}) = \mathrm{im}(\nabla_{\max})$. Then the corollary follows by [4, Corollary 3.1].

Proposition 3.3 *Let* $(\mathrm{reg}(V), g)$ *be as in Theorem* 3.1. *Let E and F be two vector bundles over* $\mathrm{reg}(V)$ *endowed respectively with metrics h and ρ, Riemannian if E and F are real vector bundles, Hermitian if E and F are complex vector bundles. Finally, let* $\nabla : C^\infty(\mathrm{V}, E) \to$

$C^\infty(M, T^*\mathrm{reg}(V) \otimes E)$ be a metric connection. Consider a first order differential operator of this type:

$$D := \theta_0 \circ \nabla : C_c^\infty(\mathrm{reg}(V), E) \to C_c^\infty(\mathrm{reg}(V), F), \tag{3.4}$$

where $\theta_0 \in C^\infty(\mathrm{reg}(V), \mathrm{Hom}(T^*\mathrm{reg}(V) \otimes E, F))$. Assume that θ_0 extends as a bounded operator,

$$\theta : L^2(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E) \to L^2(\mathrm{reg}(V), F).$$

Then we have the following inclusion:

$$\mathrm{dom}(D_{\max} \cap L^\infty(\mathrm{V}, E)) \subset \mathrm{dom}(D_{\min}). \tag{3.5}$$

In particular (3.5) *holds when D is the de Rham differential $\mathrm{d}_k : \Omega_c^k(\mathrm{reg}(V)) \to \Omega_c^{k+1}(\mathrm{reg}(V))$, a Dirac operator $D : C_c^\infty(\mathrm{reg}(V), E) \to C_c^\infty(\mathrm{reg}(V), E)$, or the Dolbeault operator $\bar{\partial}_{p,q} : \Omega_c^{p,q}(\mathrm{reg}(V)) \to \Omega_c^{p,q+1}(\mathrm{reg}(V))$.*

Proof This follows from Theorem 3.1 and [4, Proposition 3.2].

4 Schrödinger Operators on Irreducible Quasi-Kähler Complex Varieties

Let V be a quasi-Kähler complex subvariety, $\mathrm{reg}(V)$ be its regular part and E be a vector bundle over $\mathrm{reg}(V)$ endowed with a metric h, Riemannian if E is a real vector bundle, Hermitian if E is a complex vector bundle. Let h' be the induced metric of h on $\mathrm{reg}(V)$ and let $\nabla : C^\infty(\mathrm{reg}(V), E) \to C^\infty(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E)$ be a metric connection. We consider some Schrödinger type operators

$$\nabla^t \circ \nabla + L, \tag{4.1}$$

where $\nabla^t : C_c^\infty(\mathrm{reg}(V), T^*\mathrm{reg}(V) \otimes E) \to C_c^\infty(\mathrm{reg}(V), E)$ is the formal adjoint of ∇ and $L \in C^\infty(\mathrm{reg}(V), \mathrm{End}(E))$.

Theorem 4.1 *Let V, E, g, h, and ∇ be as described above. Let*

$$P := \nabla^t \circ \nabla + L, P : C_c^\infty(\mathrm{reg}(V), E) \to C_c^\infty(\mathrm{reg}(V), E)$$

be a Schrödinger type operator with $L \in C^\infty(\mathrm{reg}(V), \mathrm{End}(E))$. Assume that:

(1) *P is symmetric and positive.*

(2) *There is a constant $c \in \mathbb{R}$ such that for each $s \in C^\infty(\mathrm{reg}(V), E)$, we have*

$$h(Ls, s) \geq ch(s, s).$$

Let $P^{\mathcal{F}} : L^2(\mathrm{reg}(V), E) \to L^2(\mathrm{reg}(V), E)$ be the Friedrich extension of P and let

$$\delta_0^{\mathcal{F}} : L^2(\mathrm{reg}(V), g) \to L^2(\mathrm{reg}(V), g)$$

be the Friedrich extension of $\Delta_0 : C_c^\infty(\mathrm{reg}(V)) \to C_c^\infty(\mathrm{reg}(V))$. *Then the heat operator associated to* $P^{\mathcal{F}}$,

$$\mathrm{e}^{-tP^{\mathcal{F}}} : L^2(\mathrm{reg}(V), E) \to L^2(\mathrm{reg}(V), E)$$

is a trace class operator and its trace satisfies the following inequality:

$$\mathrm{Tr}(\mathrm{e}^{-tP^{\mathcal{F}}}) \leq m\mathrm{e}^{-tc}\mathrm{Tr}(\mathrm{e}^{-t\Delta_0^{\mathcal{F}}}), \tag{4.2}$$

where m *is the rank of the vector bundle* E.

Proof This follows by [4, Proposition 3.5] and Theorem 3.1.

Let (M, g) be as above. Let $k_P(t, x, y)$ be the smooth kernel of the hear operator $\mathrm{e}^{-tP^{\mathcal{F}}}$. Denote the pointwise operator norm of the heat operator as $\|k_P(t, x, y)\|_{h,\mathrm{op}}$.

Proposition 4.1 *Under the assumptions of Theorem* 4.1. *Assume that* $\dim_{\mathbb{C}} V > 1$. *Then the following inequality holds for* $0 < t < 1$:

$$\|k_P(t, x, y)\|_{h,\mathrm{op}} \leq C\mathrm{e}^{-tc}t^{-v}. \tag{4.3}$$

This implies that:

(1) $\mathrm{e}^{-tP^{\mathcal{F}}}$ *is a ultracontractive operator for each* $0 < t < 1$. *This means that for each* $0 < t < 1$, *there exists* $C_t > 0$ *such that*

$$\|\mathrm{e}^{-tP^{\mathcal{F}}} s\|_{L^\infty(\mathrm{reg}(V),E)} \leq C_t \|s\|_{L^1(\mathrm{reg}(V),E)} \tag{4.4}$$

for each $s \in L^1(\mathrm{reg}(V), E)$. *In particular, for each* $0 < t < 1$, $\mathrm{e}^{-tP^{\mathcal{F}}} : L^1(\mathrm{reg}(V), E) \to L^\infty(\mathrm{reg}(V), E)$ *is continuous.*

(2) *If* s *is an eigensection of* $P^{\mathcal{F}} : L^2(\mathrm{reg}(V), E) \to L^2(\mathrm{reg}(V), E)$ *then* $s \in L^\infty(\mathrm{reg}(V), E)$.

Proof The proof follows by [4, Proposition 3.6] and Theorem 3.1.

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