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On the Global Well-Posedness of 3-D Boussinesq System with Variable Viscosity*

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(*Dedicated to Professor Andrew J. Majda for the* 70*th birthday*)

Abstract In this paper, the authors first consider the global well-posedness of 3-D Boussinesq system, which has variable kinematic viscosity yet without thermal conductivity and buoyancy force, provided that the viscosity coefficient is sufficiently close to some positive constant in L^∞ and the initial velocity is small enough in $\dot{B}^0_{3,1}(\mathbb{R}^3)$. With some thermal conductivity in the temperature equation and with linear buoyancy force θe_3 on the velocity equation in the Boussinesq system, the authors also prove the global well-posedness of such system with initial temperature and initial velocity being sufficiently small in $L^1(\mathbb{R}^3)$ and $\dot{B}^0_{3,1}(\mathbb{R}^3)$ respectively.

Keywords Boussinesq systems, Littlewood-Paley theory, Variable viscosity, Maximal regularity of heat equation
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1 Introduction

The purpose of this paper is to investigate the global well-posedness to the following three-dimensional Boussinesq system with variable kinematic viscosity

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \nu|D|^s\theta = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t u + u\cdot\nabla u - \operatorname{div}(2\mu(\theta)d) + \nabla\Pi = \varepsilon\theta e_3,\\ \operatorname{div} u = 0,\\ (\theta,u)|_{t=0} = (\theta_0,u_0).\end{cases} \tag{1.1}$$

Here $\theta, u = (u_1,u_2,u_3)$ stand for the temperature and velocity of the fluid respectively, and $d = \big(\frac{1}{2}(\partial_i u_j + \partial_j u_i)\big)_{3\times 3}$ denotes the deformation tensor, Π is a scalar pressure function, and the kinematic viscous coefficient $\mu(\theta)$ is a smooth, positive and non-decreasing function on $[0,\infty)$. The thermal conductivity coefficient $\nu \geq 0$, and $e_3 = (0,0,1)$, $\varepsilon \geq 0$, $\varepsilon\theta e_3$ denotes buoyancy force. Furthermore, in all that follows, we shall always denote $|D|^s$ to be the Fourier multiplier with symbol $|\xi|^s$ for $s \geq 0$.

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The Boussinesq system arises from a zeroth order approximation to the coupling between Navier-Stokes equations and the thermodynamic equations. It can be used as a model to describe many geophysical phenomena (see [28]). In the Boussinesq approximation of a large class of flow problems, thermodynamic coefficients such as kinematic viscosity, specific heat and thermal conductivity may be assumed to be constants, leading to a coupled system of parabolic equations with linear second order operators.

However, there are some fluids such as lubricants or some plasma flow for which this is not an accurate assumption (see [30]), and a quasilinear parabolic system as follows has to be considered:

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta - \Delta\varphi(\theta) = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,\\ \partial_t u + u\cdot\nabla u - \mathrm{div}(2\mu(\theta)d) + \nabla\Pi = F(\theta),\\ \mathrm{div}\, u = 0. \end{cases} \tag{1.2}$$

One may check [17] and the references therein for more details about (1.2). Furthermore, under some technical assumptions, the global existence of weak solutions to (1.2) and in the case of constant viscosity, the uniqueness of such weak solutions in two space dimension was proved in [17].

Recently the System (1.2) has attracted a lot of attentions in the field of mathematical fluid dynamics. In particular, in two space dimension, with $F(\theta) = \theta e_2$ for $e_2 = (0,1)$ in (1.2), Wang and Zhang [32] proved the global existence of smooth solutions to (1.2). In this case, even with $\varphi(\theta) = 0$ and $\mu(\theta) = \mu > 0$ in (1.2), Chae [11] and Hou, Li[23] independently proved the global existence of smooth solutions to (1.2), Hmidi and Keraani [20] proved the global existence of weak solutions to (1.2) with θ_0, u_0 belonging to $L^2(\mathbb{R}^2)$ and the uniqueness of such solutions was proved for θ_0, u_0 belonging to $H^s(\mathbb{R}^2)$ for any $s > 0$, the first author of this paper and Hmidi [3] established the global well-posedness of this system with initial data satisfying $(\theta_0, u_0) \in B^0_{2,1} \times (L^2 \cap B^{-1}_{\infty,1})(\mathbb{R}^2)$. When $N \geq 3$, $e_N = (0,\cdots,1)$, and $F(\theta) = \theta e_N$, $\varphi(\theta) = 0$ and $\mu(\theta) = \mu > 0$ in (1.2), which corresponds to $\nu = \varepsilon = 0$ and $\mu(\theta) = \mu > 0$ in (1.1), Danchin and Paicu [14] proved the global well-posedness of this system with $\theta_0 \in \dot{B}^0_{N,1} \cap L^{\frac{N}{3}}(\mathbb{R}^N)$ and $u_0 \in \dot{B}^{-1+\frac{N}{p}}_{p,1} \cap L^{N,\infty}(\mathbb{R}^N)$ for $p \in [N,\infty)$ provided that

$$\|u_0\|_{L^{N,\infty}} + \mu^{-1}\|\theta_0\|_{L^{\frac{N}{3}}} \leq c\mu$$

for some sufficiently small constant c.

We should also mention that there are many studies on the so-called Boussinesq system with critical dissipation in two space dimension, which reads

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|\theta = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \partial_t u + u\cdot\nabla u + \mu|D|u + \nabla\Pi = \theta e_2,\\ \mathrm{div}\, u = 0,\\ (\theta,u)|_{t=0} = (\theta_0,u_0). \end{cases} \tag{1.3}$$

When $\nu = 0$ and $\mu > 0$, the above system is called Boussinesq-Navier-Stokes system with critical dissipation, Hmidi, Keraani and Rousset [21] proved the global well-posedness of such system. When $\nu > 0$ and $\mu = 0$, the System (1.3) is called Boussinesq-Euler system with critical dissipation, Hmidi, Keraani and Rousset [22] proved the global well-posedness of this system. Very recently even the logarithmically critical Boussinesq system was investigated by Hmidi in

[19]. There are also studies to the global well-posedness of the anisotropic Boussinesq system (with partial thermal conductivity and partial kinematic viscosity) in two space dimension (see [10, 15] for instance).

On the other hand, Abidi [2] proved the global well-posedness of (1.2) in two space dimension under the assumptions that: $\varphi(\theta) = 0$, $F(\theta) = 0$, and the initial data satisfies $\theta_0 \in \dot{B}^1_{2,1}(\mathbb{R}^2)$, $u_0 \in (L^2 \cap \dot{B}^{-1}_{\infty,1})(\mathbb{R}^2)$, moreover for some sufficiently small ε, there holds

$$\|\theta_0\|_{\dot{B}^1_{2,1}} + \|\mu(\theta_0) - 1\|_{L^\infty} \leq \varepsilon.$$

Furthermore, the authors of [6] established the global well-posedness of a 2-D Boussinesq system, which has variable kinematic viscosity and with thermal conductivity of $|D|\theta$, with general initial data provided that the viscosity coefficient is sufficiently close to some positive constant in L^∞ norm.

Motivated by [2, 6] and the recent results of the authors of [4–5] concerning the global well-posedness of inhomogeneous Navier-Stokes system with variable density (see also [16, 24]), we are going to investigate the global well-posedness to the following Boussinesq system with variable viscosity, which corresponds $\nu = \varepsilon = 0$ in (1.1):

$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(2\mu(\theta)d) + \nabla\Pi = 0, \\ \operatorname{div} u = 0, \\ (\theta, u)|_{t=0} = (\theta_0, u_0). \end{cases} \tag{1.4}$$

In all that follows, we always make the convention that, for any $\alpha > 0$, α_+ means any constant greater than α, and

$$0 < \underline{\mu} \leq \mu(\theta), \quad \mu(\cdot) \in W^{2,\infty}(\mathbb{R}^+) \quad \text{and} \quad \mu(0) = 1. \tag{1.5}$$

Theorem 1.1 *Let $\theta_0 \in (B^1_{3,1} \cap B^{(\frac{1}{2})_+}_{\infty,\infty})(\mathbb{R}^3)$ and $u_0 \in (\dot{H}^{-2\delta} \cap \dot{B}^0_{3,1})(\mathbb{R}^3)$ be a solenoidal vector filed for some $\delta \in]0, \frac{1}{2}[$. Then there exists a sufficiently small constant ε_0, and some small enough constant ε, which depends on $\|\theta_0\|_{\dot{B}^1_{3,1} \cap B^{(\frac{1}{2})_+}_{\infty,\infty}}$, such that if*

$$\|\mu(\theta_0) - 1\|_{L^\infty} \leq \varepsilon_0 \quad \textit{and} \quad \|u_0\|_{\dot{B}^0_{3,1}} \leq \varepsilon, \tag{1.6}$$

(1.4) *has a unique global solution $(\theta, u, \nabla\Pi)$ with*

$$\begin{aligned} &\theta \in \mathcal{C}([0, \infty[; B^1_{3,1}(\mathbb{R}^3)), \quad u \in \mathcal{C}([0, \infty[; \dot{B}^0_{3,1}) \cap L^1_{\text{loc}}(\mathbb{R}^+; \dot{B}^2_{3,1}) \quad \textit{and} \\ &\nabla\Pi \in L^1_{\text{loc}}(\mathbb{R}^+; \dot{B}^0_{3,1}). \end{aligned} \tag{1.7}$$

Remark 1.1 Let us give the following remarks concerning this theorem:

(1) We point out that the exact dependence of ε on $\|\theta_0\|_{\dot{B}^1_{3,1} \cap B^{(\frac{1}{2})_+}_{\infty,\infty}}$ will be given by (5.1) and (5.49). Furthermore, compared with the 2-D result in [2], here we do not require any smallness condition on the initial temperature in Theorem 1.1.

(2) The assumption that $u_0 \in \dot{H}^{-2\delta}(\mathbb{R}^3)$ is to make sure that the solution decay to 0 as time t goes to ∞, which will be essential to obtain the a priori estimate of the velocity u in the space $L^1(\mathbb{R}^+, \text{Lip}(\mathbb{R}^3))$. In fact, the exact decay rate of $u(t)$ will be given by (5.31) and (5.37). The

assumption that $\theta_0 \in B^{(\frac{1}{2})_+}_{\infty,\infty}(\mathbb{R}^3)$ is due to the variable viscosity, one may check (3) of Remark 1.2 for details.

(3) Compared with the results in [4–5] and [24] for the inhomogeneous Navier-Stokes system with variable viscosity, here ε_0 is a uniform small positive constant, which does not depend on θ_0. While in [4–5] and [24], the smallness condition for $\mu(\rho_0) - 1$ is in some sense formulated as

$$\|\mu(\rho_0) - 1\|_{L^\infty}(1 + \|\rho_0\|_{B^\delta_{\infty,\infty}}) \leq \varepsilon_0$$

for some $\delta > 0$. In general, under the assumption that

$$\|\mu(\rho_0) - 1\|_{L^\infty} \leq \varepsilon_0 \tag{1.8}$$

for some ε_0 sufficiently small, Desjardins [16] only proved the global existence of strong solutions for 2-D inhomogeneous Navier-Stokes system. Yet the uniqueness and regularities of such strong solutions are still open. Moreover, here we do not require the initial velocity $u_0 \in H^1$ as was assumed in [4–5, 16, 24].

However, with linear buoyancy force θe_3 on the right-hand side of the velocity equation in (1.4), we still do not know how to prove such global well-posedness result as Theorem 1.1 for the corresponding system. Yet with some dissipation on the temperature equation, more precisely, for the system below,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + |D|^s\theta = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u\cdot\nabla u - \operatorname{div}(2\mu(\theta)d) + \nabla\Pi = \theta e_3, \\ \operatorname{div} u = 0, \\ (\theta, u)|_{t=0} = (\theta_0, u_0), \end{cases} \tag{1.9}$$

we have the following global well-posedness result.

Theorem 1.2 *Let* $s \in \left]3 - \sqrt{6}, 1\right]$ *and* $\alpha \in]0,1[$. *Let* $\theta_0 \in (L^1 \cap \dot{B}^1_{3,1} \cap \dot{B}^\alpha_{\infty,\infty})(\mathbb{R}^3)$ *and* $u_0 \in \dot{B}^0_{3,1}(\mathbb{R}^3)$ *be a solenoidal vector field. Then there exist sufficiently small constants* η *and* η_0, *which depend on* $\|\theta_0\|_{\dot{B}^1_{3,1}\cap\dot{B}^\alpha_{\infty,\infty}}$, *such that if*

$$\|\theta_0\|_{L^1} \leq \eta_0 \quad \text{and} \quad \|u_0\|_{\dot{B}^0_{3,1}} \leq \eta, \tag{1.10}$$

(1.9) *has a unique global solution* (θ, u) *with*

$$\begin{aligned} &\theta \in \mathcal{C}([0,\infty[; \dot{B}^1_{3,1}(\mathbb{R}^3)) \cap L^1_{\text{loc}}(\mathbb{R}_+; \dot{B}^{1+s}_{3,1}(\mathbb{R}^3)) \quad \text{and} \\ &u \in \mathcal{C}([0,\infty[; \dot{B}^0_{3,1}(\mathbb{R}^3)) \cap L^1_{\text{loc}}(\mathbb{R}_+; \dot{B}^2_{3,1}(\mathbb{R}^3)), \quad \nabla\Pi \in L^1_{\text{loc}}(\mathbb{R}^+; \dot{B}^0_{3,1}(\mathbb{R}^3)). \end{aligned} \tag{1.11}$$

Remark 1.2 Let us mention the following facts about Theorem 1.2.

(1) We remark that the exact smallness conditions for η and η_0 will be given by (7.1), (7.2), (7.12) and (7.20). This result in some sense extends the global well-posedness result in [14] with constant viscosity to the case of variable kinematic viscosity. As a matter of fact, the method of the proof to Theorem 1.2 is also motivated a lot from that in [14, 27], namely, we need first to control the $L^\infty(\mathbb{R}^+; L^{3,\infty}(\mathbb{R}^3))$ norm for the velocity field u before we deal with the evolution of the Besov norm for u.

(2) We also remark that the global well-posedness of the System (1.9) is easier for larger s. However, we just choose $s \leq 1$ for simplicity. And the reason why we choose $s > 3 - \sqrt{6}$ is due

to the facts that when we manipulate the global $L^1(t_1,\infty;\dot{B}^1_{\infty,1})$ for the velocity in Subsection 7.2, we need $\theta(t_1)$ belongs to $B^{\frac{3}{2}}_{3,1}(\mathbb{R}^3)$, which is obtained by using the smooth effect of $|D|^s$ in the θ equation of (1.9), and p_s given by (5.39) satisfies $s>\frac{3}{p_s}$, which is crucial for us to work the paraproduct estimate in (7.28). We do not claim the assumption that $s\in\,]3-\sqrt{6},1]$ in Theorem 1.2 is optimal in any sense.

(3) We emphasize that the main difficulty in the proof of Theorems 1.1 and 1.2 is due to the variable viscosity. In this case, when we apply Littlewood-Paley theory and smoothing effect of heat semigroup to prove the global L^1 in time of the space Lipschize estimate for the velocity, we need some positive space derivative estimate of θ. Yet to propagate the positive space derivative estimate for θ, we require the the global L^1 in time of the space Lipschize estimate for the convection velocity u. Especially for the transport equation of (1.4), one has

$$\|\theta(t)\|_{B^\delta_{\infty,\infty}}\leq\|\theta_0\|_{B^\delta_{\infty,\infty}}\exp(C\|\nabla u\|_{L^1_t(L^\infty)})\quad\text{for }\delta\in]0,1[,$$

which makes impossible to close the a priori estimates.

Let us complete this section with the notations we are going to use in this context.

Notations Let A,B be two operators, we denote $[A;B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$. We shall denote by $(a\mid b)$ (or $(a\mid b)_{L^2}$) the $L^2(\mathbb{R}^3)$ inner product of a and b, and denote by $(d_j)_{j\in\mathbb{Z}}$ (resp. $(c_j)_{j\in\mathbb{Z}}$) a generic element of $\ell^1(\mathbb{Z})$ (resp. $\ell^2(\mathbb{Z})$) so that $\|(d_j)_{j\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})}=1$ (resp. $\|(c_j)_{j\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}=1$).

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;\,X)$ the set of continuous functions on I with values in X, and by $\mathcal{C}_b(I;\,X)$ the subset of bounded functions of $\mathcal{C}(I;\,X)$. For $q\in[1,+\infty]$, the notation $L^q(I;\,X)$ stands for the set of measurable functions on I with values in X, such that $t\mapsto\|f(t)\|_X$ belongs to $L^q(I)$. Finally for any vector field $v=(v_1,v_2,v_3)$, we denote $d(v)=\frac{1}{2}(\partial_i v_j+\partial_j v_i)_{i,j=1,2,3}$, and the Leray projection operator $\mathbb{P}\overset{\text{def}}{=}Id+\nabla(-\Delta)^{-1}\text{div}$.

2 Strategies to the Proof of Theorems 1.1 and 1.2

As the existence part of both Theorems 1.1 and 1.2 basically follows from the a priori estimates for smooth enough solutions of (1.4) and (1.9). We shall only outline the main steps in the derivation of the estimates.

2.1 Strategy to the proof of Theorem 1.1

By applying maximal regularity estimates for heat semi-group, we prove that under the assumption of (1.6), for smooth enough solution (θ,u) of (1.4) on $[0,T^*[$, there holds

$$\|u\|_{L^\infty_t(L^3)}+\|u\|_{L_t^{\frac{2p}{p-3}}(L^p)}+\|\nabla u\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})}\leq C\|u_0\|_{\dot{B}^0_{3,1}}\tag{2.1}$$

for any $p\in[6,8]$. If one assumes moreover the smallness condition (5.1), we get, by using Littlewood-Paley analysis that $T^*\geq 1$ and

$$\|u\|_{L^1_1(\dot{B}^1_{\infty,1})}\leq C\|u_0\|_{\dot{B}^0_{3,1}}(1+\|\theta_0\|_{B^{\frac{1}{2}}_{\infty,1}}).\tag{2.2}$$

With (2.2), we can prove the propagation of regularities of (θ_0, u_0) for (θ, u) on $[0,1]$, namely the estimate (5.14). This in particular ensures some $t_0 \in]0,1[$ so that there holds

$$\begin{aligned}
&\|\theta(t_0)\|_{B^1_{3,1}} \le C\|\theta_0\|_{B^1_{3,1}} \quad \text{and} \quad \|\theta(t_0)\|_{B^{(\frac12)+}_{\infty,\infty}} \le C\|\theta_0\|_{B^{(\frac12)+}_{\infty,\infty}},\\
&\|u(t_0)\|_{\dot H^{-2\delta}} \le C(\|u_0\|_{\dot H^{-2\delta}} + \|\theta_0\|_{B^1_{3,1}}\|u_0\|_{\dot B^1_{3,1}}) \quad \text{and}\\
&\|u(t_0)\|_{H^1} \le C(\|u_0\|_{\dot H^{-2\delta}} + \|\theta_0\|_{B^1_{3,1}}\|u_0\|_{\dot B^0_{3,1}}).
\end{aligned} \tag{2.3}$$

Due to (2.3), we can prove the following Desjardins type (see [16]) energy estimates for $t \in]t_0, T^*[$,

$$\begin{aligned}
&\|u\|^2_{L^\infty(t_0,t;L^2)} + \|\nabla u\|^2_{L^2(t_0,t;L^2)} \le C\|u(t_0)\|^2_{L^2},\\
&\|\nabla u\|^2_{L^\infty(t_0,t;L^2)} + \|\partial_t u\|^2_{L^2(t_0,t;L^2)} \le C\|\nabla u(t_0)\|^2_{L^2}\exp(C\|u_0\|_{\dot B^0_{3,1}}).
\end{aligned} \tag{2.4}$$

Furthermore, since $u(t_0) \in \dot H^{-2\delta}(\mathbb{R}^3)$, we can use Schonbek's approach in [31] to get

$$\|u(t)\|_{L^2} \le C\overline{C}\langle t\rangle^{-\delta} \quad \text{for any } t \in [t_0, T^*[\tag{2.5}$$

for $\overline{C}$ given by (5.31). Based on (2.5), we deduce that for $p \in [6,8]$ and satisfying (5.39),

$$\|\nabla u\|_{L^1_t(L^p)} \le C\overline{C}^{\frac{3}{p-3}}\|u_0\|^{\frac{p-6}{p-3}}_{\dot B^0_{3,1}} \quad \text{for any } t \in [0, T^*[. \tag{2.6}$$

By virtue of (2.4) and (2.6), we can prove the following key estimate by applying the smoothing effect of heat semi-group and Littlewood-Paley theory

$$\|u\|_{L^1_t(\dot B^1_{\infty,1})} \le C(\|u_0\|_{\dot B^0_{3,1}} + \|u(t_0)\|_{H^1}(1 + \|u(t_0)\|^{1-2\varepsilon}_{H^1}\|u_0\|^{2\varepsilon}_{\dot B^0_{3,1}}) + \overline{C}\|\theta_0\|_{B^{\frac12+\varepsilon}_{\infty,\infty}}) \tag{2.7}$$

for any $t \in]0, T^*[$, where $\|u(t_0)\|_{H^1}$ is determined by (2.3) and $\overline{C}$ by (5.31).

This is basically the contents of Section 5. With (2.7), we shall complete the proof to the existence part of Theorems 1.1 in Subsection 6.1 by constructing appropriate approximate solutions and passing to the limit. And finally the uniqueness part of Theorems 1.1 will be proved in Subsection 6.2 by applying Osgood lemma.

2.2 Strategy to the proof of Theorem 1.2

As in the proof of Theorem 1.1, under the smallness conditions (7.1) and (7.2), we shall prove that the corresponding local smooth solution (θ, u) of (1.9) on $[0, T^*[$ satisfies $T^* \ge 1$, and

$$\|u\|_{L^1_1(\dot B^1_{\infty,1})} \le C(\|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|^{\frac13}_{L^1}\|\theta_0\|^{\frac23}_{\dot B^1_{3,1}}). \tag{2.8}$$

Thanks to (2.8), we get, by applying the smoothing effect of $\mathrm{e}^{t\Delta}$ and $\mathrm{e}^{t|D|^s}$, that there exists some $t_1 \in]\frac12, 1[$ so that

$$\begin{aligned}
&\|u(t_1)\|_{\dot B^0_{3,1}\cap \dot B^1_{3,1}} \le C(\eta_2 + \eta_3)(1 + \|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}),\\
&\|\theta(t_1)\|_{L^1} \le \|\theta_0\|_{L^1} \quad \text{and} \quad \|\theta(t_1)\|_{\dot B^1_{3,1}\cap \dot B^{1+s}_{3,1}} \le C(\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}).
\end{aligned} \tag{2.9}$$

Starting from t_1, we write the velocity equation of (1.9) as

$$u(t) = \mathrm{e}^{(t-t_1)\Delta}u(t_1) + \int_{t_1}^{t} \mathrm{e}^{(t-t')\Delta}\mathbb{P}(-\mathrm{div}(u\otimes u) + \mathrm{div}(2(\mu(\theta)-1)d) + \theta e_3)(t')\mathrm{d}t', \tag{2.10}$$

from which, Lemma 4.2 and under the additional smallness assumption (7.20), we infer the following.

Lemma 2.1 *Under the assumptions of Proposition* 7.3 *and for* η_4 *given by* (7.20), *we have*

$$\|u\|_{L^\infty(t_1,t;L^{3,\infty})} \leq C\eta_4 \quad \textit{for any } t\in]t_1,T^*[. \tag{2.11}$$

Lemma 2.2 *Under the assumptions of Proposition* 7.3, *for* $0 < s < 1$, *one has for any* $t\in]t_1,T^*[$,

$$\begin{aligned}
&\|\nabla u\|_{L^{\frac{p_s}{p_s-3}}(t_1,t;L^{\frac{3p_s}{6-p_s}})} \leq C(\|u(t_1)\|_{\dot B^1_{3,1}} + t^{\frac{2(p_s-3)}{p_s}}\|\theta_0\|_{L^1}^{\frac{2(6-p_s)}{3p_s}}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac{5p_s-12}{3p_s}}) \quad \textit{and}\\
&\|\nabla u\|_{L^{\frac{p_s}{3}}(t_1,t;L^{\frac{3p_s}{p_s-3}})} \leq C(\|u(t_1)\|_{\dot B^0_{3,1}\cap \dot B^1_{3,1}} + t^{\frac{6-p_s}{p_s}}\eta_4 + t^{\frac{9}{2p_s}}\|\theta_0\|_{L^1}^{\frac{2(p_s-3)}{3p_s}}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac{p_s+6}{3p_s}}),
\end{aligned} \tag{2.12}$$

where p_s *is given by* (7.21).

In view of (2.10) and the above lemmas, we obtain the a priori estimate for $\|u\|_{L^1_t(\dot B^1_{\infty,1})}$ for any $t < T^*$. The detail will be presented in Proposition 7.3.

Finally the existence part of Theorem 1.2 will be proved by constructing appropriate approximate solutions and passing to the limit in Subsection 8.1. Whereas the uniqueness part of Theorems 1.2 will be proved in Subsection 8.2 through Osgood lemma type argument.

3 Littlewood-Paley Analysis and Lorentz Spaces

The proofs of Theorems 1.1 and 1.2 require Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x\in\mathbb{R}^N$ (see e.g. [7]). Let φ be a smooth function supported in the annulus $\mathcal{C} \stackrel{\mathrm{def}}{=} \{\xi\in\mathbb{R}^N, \frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ and $\chi(\xi)$ be a smooth function supported in the ball $\mathcal{B} \stackrel{\mathrm{def}}{=} \{\xi\in\mathbb{R}^N, |\,\xi|\leq\frac{4}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1 \quad \text{for } \xi\neq 0 \quad \text{and} \quad \chi(\xi) + \sum_{q\geq 0}\varphi(2^{-q}\xi) = 1 \quad \text{for all } \ \xi\in\mathbb{R}^N\,.$$

Then for $u\in\mathcal{S}'_h(\mathbb{R}^N)$ (see [7, Definition 1.26]), which means $u\in\mathcal{S}'(\mathbb{R}^N)$ and

$$\lim_{j\to-\infty}\|\chi(2^{-j}D)u\|_{L^\infty} = 0,$$

we set

$$\begin{aligned}
&\forall\, j\in\mathbb{Z},\quad \dot\Delta_j u \stackrel{\mathrm{def}}{=} \varphi(2^{-j}\mathrm{D})u \quad \text{and} \quad \dot S_j u \stackrel{\mathrm{def}}{=} \chi(2^{-j}D)u,\\
&\forall\, q\geq 0,\quad \Delta_q u \stackrel{\mathrm{def}}{=} \varphi(2^{-q}\mathrm{D})u, \quad \Delta_{-1}u \stackrel{\mathrm{def}}{=} \chi(D)u \quad \text{and} \quad S_q u \stackrel{\mathrm{def}}{=} \sum_{-1\leq q'\leq q-1}\Delta_{q'}u,
\end{aligned} \tag{3.1}$$

we have the formal decomposition

$$u = \sum_{j\in\mathbb{Z}}\dot\Delta_j\, u, \quad \forall\, u\in\mathcal{S}'_h(\mathbb{R}^N) \quad \text{and} \quad u = \sum_{q\geq -1}\Delta_q\, u, \quad \forall\, u\in\mathcal{S}(\mathbb{R}^N). \tag{3.2}$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_j\dot{\Delta}_k u \equiv 0 \quad \text{if } |j-k| \geq 2 \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{k-1}u\dot{\Delta}_k u) \equiv 0 \quad \text{if } |j-k| \geq 5. \tag{3.3}$$

We recall now the definition of homogeneous Besov spaces and Bernstein type inequalities from [7]. Similar definitions in the inhomogeneous context can be found in [7].

Definition 3.1 (see [7, Definition 2.15]) *Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'_h(\mathbb{R}^N)$, we set*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} (2^{js}\|\dot{\Delta}_j u\|_{L^p})_{\ell^r}.$$

- *For $s < \frac{N}{p}$ (or $s = \frac{N}{p}$ if $r = 1$), we define $\dot{B}^s_{p,r}(\mathbb{R}^N) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'_h(\mathbb{R}^N) \mid \|u\|_{\dot{B}^s_{p,r}} < \infty\}$.*
- *If $k \in \mathbb{N}$ and $\frac{N}{p} + k \leq s < \frac{N}{p} + k + 1$ (or $s = \frac{N}{p} + k + 1$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^N)$ is defined as the subset of distributions $u \in \mathcal{S}'_h(\mathbb{R}^N)$ such that $\partial^\beta u \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^N)$ whenever $|\beta| = k$.*

Lemma 3.1 *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ an annulus of $\mathbb{R}^N$. A constant C exists so that for any positive real number δ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers $(a,\ b)$ with $b \geq a \geq 1$, there hold*

$$\begin{aligned}
&\text{Supp } \widehat{u} \subset \delta\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1}\delta^{k+N(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\\
&\text{Supp } \widehat{u} \subset \delta\mathcal{C} \Rightarrow C^{-1-k}\delta^k\|u\|_{L^a} \leq \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a} \leq C^{1+k}\delta^k\|u\|_{L^a},\\
&\text{Supp } \widehat{u} \subset \delta\mathcal{C} \Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m}\delta^{m+N(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}.
\end{aligned} \tag{3.4}$$

Lemma 3.2 (see [7, Lemma 2.4]) *Let $\mathcal{C}$ be an annulus. A positive constant C exists so that for any $p \in]1,\infty[$ and any couple (t,λ) of positive real numbers, we have*

$$\text{Supp } \widehat{a} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}a\|_{L^p} \leq Ce^{-ct\lambda^2}\|a\|_{L^p}.$$

We also recall Bony's decomposition from [9]:

$$uv = T_u v + T'_v u = T_u v + T_v u + R(u,v), \tag{3.5}$$

where

$$\begin{aligned}
&T_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v, \quad T'_v u \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j+2}v\dot{\Delta}_j u,\\
&R(u,v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u\dot{\tilde{\Delta}}_j v \quad \text{with } \dot{\tilde{\Delta}}_j v \stackrel{\text{def}}{=} \sum_{|j'-j|\leq 1} \dot{\Delta}_{j'} v.
\end{aligned}$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r}(\mathbb{R}^N))$ from [7].

Definition 3.2 *Let $(r,\lambda,p) \in [1,\ +\infty]^3$ and $T \in]0,+\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{p\,r}(\mathbb{R}^N))$ as the completion of $\mathcal{C}([0,T];\mathcal{S}(\mathbb{R}^N))$ by the norm*

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\Big(\int_0^T \|\dot{\Delta}_j\, f(t)\|^\lambda_{L^p}dt\Big)^{\frac{r}{\lambda}}\Big)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$.

To prove Theorem 1.2, we also need to use Lorentz space $L^{p,q}(\mathbb{R}^3)$. For the convenience of the readers, we recall some basic facts on $L^{p,q}(\mathbb{R}^N)$ from [18, 25].

Definition 3.3 (see [18, Definition 1.4.6]) *For a measurable function f on $\mathbb{R}^N$, we define its non-increasing rearrangement by*

$$f^*(t) \stackrel{\text{def}}{=} \inf\{s > 0,\ \mu(\{x,\ |f(x)| > s\}) \le t\},$$

where μ denotes the usual Lebesgue measure. For $(p,q) \in [1,+\infty]^2$, the Lorentz space $L^{p,q}(\mathbb{R}^N)$ is the set of functions f such that $\|f\|_{L^{p,q}} < \infty$, with

$$\|f\|_{L^{p,q}} \stackrel{\text{def}}{=} \begin{cases} \Big(\displaystyle\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{q}} & \text{for } 1 \le q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{for } q = \infty. \end{cases}$$

We remark that Lorentz spaces can also be defined by real interpolation from Lebesgue spaces (see for instance [25, Definition 2.3]):

$$(L^{p_0}, L^{p_1})_{(\beta,q)} = L^{p,q},$$

where $1 \le p_0 < p < p_1 \le \infty$, β satisfies $\frac{1}{p} = \frac{1-\beta}{p_0} + \frac{\beta}{p_1}$ and $1 \le q \le \infty$.

Lemma 3.3 (see [25, pages 18–20]) *Let $1 < p < \infty$ and $1 \le q \le \infty$, we have*

- $$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}} \|g\|_{L^\infty}.$$
- *If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*
$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$
- *If $1 < p < \infty$, $\frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*
$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}$$

for $p = \infty$, and $\frac{1}{q_1} + \frac{1}{q_2} = 1$, then

$$\|f * g\|_{L^\infty} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

- *For $1 \le p \le \infty$ and $1 \le q_1 \le q_2 \le \infty$, we have*
$$L^{p,q_1} \hookrightarrow L^{p,q_2} \quad \text{and} \quad L^{p,p} = L^p.$$

Lemma 3.4 (see [14, Lemma 3.9]) *For $1 < p < q \le \infty$, one has*

$$L^{p,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{3(\frac{1}{q}-\frac{1}{p})}_{q,\infty}(\mathbb{R}^3).$$

As an application of the above basic facts on Littlewood-Paley theory and Lorentz spaces, we prove the following estimates.

Lemma 3.5 *Let $p \ge \frac{3}{2}$, $s \in]-1,\infty[$, $\alpha \in [0,1[$ and $a, b \in \mathcal{S}(\mathbb{R}^3)$. We have*

$$\begin{aligned} &(1) && \|ab\|_{\dot{B}^s_{p,r}} \lesssim \|a\|_{L^{3,\infty}} \|b\|_{\dot{B}^{s+1}_{p,r}} + \|b\|_{L^{3,\infty}} \|a\|_{\dot{B}^{s+1}_{p,r}} \quad \text{and} \\ &(2) && \sum_{j\in\mathbb{Z}} 2^j \|[\dot{\Delta}_j; a\cdot\nabla]b\|_{L^3} \lesssim \|\nabla a\|_{L^\infty} \|b\|_{\dot{B}^1_{3,1}} + \|a\|_{\dot{B}^{2-\alpha}_{3,1}} \|b\|_{\dot{B}^\alpha_{\infty,\infty}}. \end{aligned} \tag{3.6}$$

Proof The proof of the first inequality of (3.6) basically follows from that of (26) in [14]. For completeness, we present its proof here. Using Bony's decomposition (3.5), we write

$$ab = T_a b + T_b a + R(a,b).$$

It follows from Lemma 3.1 that

$$\begin{aligned}\|\dot\Delta_j T_a b\|_{L^p} &\lesssim \sum_{|j'-j|\le 4} \|S_{j'-1}a\|_{L^\infty}\|\dot\Delta_{j'}b\|_{L^p}\\ &\lesssim \sum_{|j'-j|\le 4} c_{j',r}2^{-j's}\|a\|_{\dot B^{-1}_{\infty,\infty}}\|b\|_{\dot B^{s+1}_{p,r}} \lesssim c_{j,r}2^{-js}\|a\|_{\dot B^{-1}_{\infty,\infty}}\|b\|_{\dot B^{s+1}_{p,r}}\end{aligned}$$

for some $(c_{j,r})_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z})$ so that $\|(c_{j,r})_{j\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})} = 1$. Similar estimate holds for $T_b a$.

Whereas applying Lemma 3.1 and Lemma 3.3 yields

$$\begin{aligned}\|\dot\Delta_j R(a,b)\|_{L^p} &\lesssim 2^j \Big\| \sum_{j'\ge j-3} \dot\Delta_{j'}a\dot{\tilde\Delta}_{j'}b\Big\|_{L^{\frac{3p}{p+3},\infty}} \lesssim 2^j \sum_{j'\ge j-3} \|\dot\Delta_{j'}a\|_{L^{3,\infty}}\|\dot{\tilde\Delta}_{j'}b\|_{L^p}\\ &\lesssim 2^j \sum_{j'\ge j-3} c_{j',r}2^{-j'(s+1)}\|a\|_{L^{3,\infty}}\|b\|_{\dot B^{s+1}_{p,r}} \lesssim c_{j,r}2^{-js}\|a\|_{L^{3,\infty}}\|b\|_{\dot B^{s+1}_{p,r}}.\end{aligned}$$

Hence by virtue of Lemma 3.4, we obtain the first inequality of (3.6).

Along the same line, by using Bony decomposition (3.5), we write

$$[\dot\Delta_j; a\cdot\nabla]b = [\dot\Delta_j; T_a]\nabla b + \dot\Delta_j T'_{\nabla b}a - T'_{\dot\Delta_j\nabla b}a.$$

It follows from the classical commutator's estimate (see [7]) that

$$\begin{aligned}\|[\dot\Delta_j; T_a]\nabla b\|_{L^3} &\lesssim \sum_{|j'-j|\le 4} 2^{-j}\|\dot S_{j'-1}\nabla a\|_{L^\infty}\|\dot\Delta_{j'}\nabla b\|_{L^3}\\ &\lesssim d_j 2^{-j}\|\nabla a\|_{L^\infty}\|b\|_{\dot B^1_{3,1}} \quad \text{for } (d_j)_{\mathbb{Z}} \in \ell^1(\mathbb{Z}).\end{aligned}$$

While it is easy to observe that

$$\begin{aligned}\|\dot\Delta_j T'_{\nabla b}a\|_{L^3} &\lesssim \sum_{j'\ge j-N_0} \|S_{j'+2}\nabla b\|_{L^\infty}\|\dot\Delta_{j'}a\|_{L^3}\\ &\lesssim \sum_{j'\ge j-N_0} d_{j'}2^{-j'}\|b\|_{\dot B^\alpha_{\infty,\infty}}\|a\|_{\dot B^{2-\alpha}_{3,1}} \lesssim d_j 2^{-j}\|b\|_{\dot B^\alpha_{\infty,\infty}}\|a\|_{\dot B^{2-\alpha}_{3,1}}.\end{aligned}$$

The same estimate holds for $T'_{\dot\Delta_j\nabla b}a$. Hence there holds (2) of (3.6). This completes the proof of the lemma.

4 Some Technical Lemmas

In this section, we shall collect some technical lemmas which will be used throughout this paper. The first one is concerning the definition of Besov spaces with negative indices through heat semi-group.

Proposition 4.1 (see [7, Theorem 2.34]) *Let s be a negative real number and $(p,r) \in [1,\infty]^2$. A constant C exists such that*

$$C^{-1}\|f\|_{\dot B^s_{p,r}} \le \big\|\|t^{-\frac{s}{2}}\mathrm{e}^{t\Delta}f\|_{L^p}\big\|_{L^r(\mathbb{R}^+;\frac{\mathrm{d}t}{t})} \le C\|f\|_{\dot B^s_{p,r}}.$$

The other key ingredient used in this paper is the maximal $L^p(L^q)$ regularity for the heat kernel.

Lemma 4.1 (see [25, Lemma 7.3]) *The operator $\mathcal{A}$ defined by*

$$F(t,x)\mapsto\int_0^t \triangle \mathrm{e}^{(t-t')\triangle}F(t',x)\mathrm{d}t'$$

is bounded from $L^p(]0,T[;L^q(R^3))$ to $L^p(]0,T[;L^q(R^3))$ for every $T\in(0,\infty]$ and $1<p,q<\infty$. Moreover, there holds

$$\|\mathcal{A}F\|_{L^p_T(L^q)}\leq C_{p,q}\|F\|_{L^p_T(L^q)}.$$

Lemma 4.2 *Let $3<p<\infty$. The operator $\mathcal{B}$ defined by $F(t,x)\mapsto\int_0^t\nabla\mathrm{e}^{(t-t')\Delta}F(t',x)\mathrm{d}t'$ is bounded from $L^{\frac{p}{p-3}}(]0,T[;L^{\frac{p}{2}}(R^3))$ to $L^{\frac{2p}{p-3}}(]0,T[;L^p(R^3))$ for every $T\in(0,\infty]$, and there holds*

$$\|\mathcal{B}(F)\|_{L_T^{\frac{2p}{p-3}}(L^p)}\leq C_p\|F\|_{L_T^{\frac{p}{p-3}}(L^{\frac{p}{2}})}. \tag{4.1}$$

If moreover, $p\in]3,6]$, one has

$$\|\mathcal{B}(F)\|_{L^\infty_T(L^3)}\leq C\|F\|_{L_T^{\frac{p}{p-3},1}(L^{\frac{p}{2}})}. \tag{4.2}$$

Proof Note that

$$\begin{aligned}\nabla\mathrm{e}^{(t-t')\triangle}F(t',x)=&\frac{\sqrt{\pi}}{(4\pi(t-t'))^2}\int_{\mathbb{R}^3}\frac{(x-y)}{2\sqrt{(t-t')}}\exp\Big\{-\frac{|x-y|^2}{4(t-t')}\Big\}F(t',y)\mathrm{d}y\\ \stackrel{\text{def}}{=}&\frac{\sqrt{\pi}}{(4\pi(t-t'))^2}K\Big(\frac{\cdot}{2\sqrt{(t-t')}}\Big)*F(t',x).\end{aligned} \tag{4.3}$$

Applying Young's inequality in the space variables yields

$$\begin{aligned}&\|\nabla\mathrm{e}^{\mu(t-t')\triangle}(1_{[0,T]}(t')F)(t',\cdot)\|_{L^p}\\ &\leq C(t-t')^{-2}\Big\|K\Big(\frac{\cdot}{2\sqrt{\pi(t-t')}}\Big)\Big\|_{L^{\frac{p}{p-1}}}\|1_{[0,T]}(t')F(t')\|_{L^{\frac{p}{2}}}\\ &\leq C(t-t')^{-\frac12+\frac{3}{2p}}\|1_{[0,T]}(t')F(t')\|_{L^{\frac{p}{2}}},\end{aligned} \tag{4.4}$$

where $1_{[0,t]}(t')$ denotes the characteristic function on $[0,t]$, from which and Hardy-Littlewood-Sobolev inequality, we conclude the proof of (4.1).

It remains to prove the limiting case, i.e., (4.2). Indeed it follows by a similar derivation of (4.4) that

$$\|\mathcal{B}(F)(t)\|_{L^3}\leq C\int_0^T(t-t')^{-\frac{3}{p}}\|F(t')\|_{L^{\frac{p}{2}}}\mathrm{d}t'.$$

Note that for $3<p\leq6$, we have

$$1_{t>0}t^{-\frac{3}{p}}\in L^{\frac{p}{3},\infty},$$

as a result, it comes out (4.2). This completes the proof of Lemma 4.2.

Lemma 4.3 *Let $s \in]0,1]$ and u be a smooth solenoidal vector filed on $[0,T]$. Let θ be a smooth enough solution of*

$$\partial_t\theta + u\cdot\nabla\theta + |D|^s\theta = 0, \quad \theta|_{t=0} = \theta_0. \tag{4.5}$$

Then one has for any $t \le T$,

(1) *for all $p \in [1,\infty]$, $\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}$;*

(2) $\|\theta(t)\|_{L^2} + \|\theta\|_{L^2_t(\dot H^{\frac{s}{2}})} \le \|\theta_0\|_{L^2}$;

(3) *for all $p \in]1,\infty[$ and $r \in [1,\infty]$,*

$$\|\theta\|_{\widetilde L^r_t(\dot B^s_{p,\infty})} \le C(\|\theta_0\|_{\dot B^{s-\frac{s}{r}}_{p,\infty}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^r_t(L^p)});$$

(4) *for any $\alpha \in]0,1[$,*

$$\|\theta\|_{\widetilde L^\infty_t(\dot B^1_{3,1})} + \|\theta\|_{L^1_t(\dot B^{1+s}_{3,1})} \le C(\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}\|u\|_{L^1_t(\dot B^{2-\alpha}_{3,1})})\exp(C\|\nabla u\|_{L^1_t(L^\infty)}).$$

Proof Part (1) follows directly from [12]. While by taking the L^2 inner product of (4.5) with θ and using $\operatorname{div} u = 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|^2_{L^2} + \||D|^{\frac{s}{2}}\theta\|^2_{L^2} = 0,$$

integrating the above inequality over $[0,t]$ yields part (2) of the lemma.

To deal with part (3), we first apply the dyadic operator $\dot\Delta_j$ to (4.5) and then use a standard commutator's process to write

$$\partial_t\dot\Delta_j\theta + (u\cdot\nabla)\dot\Delta_j\theta + |D|^s\dot\Delta_j\theta = -[\dot\Delta_j; u\cdot\nabla]\theta. \tag{4.6}$$

Taking L^2 inner product of (4.6) with $|\dot\Delta_j\theta|^{p-2}\dot\Delta_j\theta$ and using Hölder inequality, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j\theta\|^p_{L^p} + \int_{R^3}(|D|^s\dot\Delta_j\theta)|\dot\Delta_j\theta|^{p-2}\dot\Delta_j\theta\mathrm{d}x \le \|\dot\Delta_j\theta\|^{p-2}_{L^p}\|[\dot\Delta_j; u\cdot\nabla]\theta\|_{L^p}.$$

Recalling from [19, 33] the following generalized Bernstein inequality that

$$c2^{js}\|\dot\Delta_j\theta\|^p_{L^p} \le \int_{R^3}(|D|^s\dot\Delta_j\theta)|\dot\Delta_j\theta|^{p-2}\dot\Delta_j\theta\mathrm{d}x$$

for some p independent constant c. We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{ct2^{js}}\|\dot\Delta_j\theta\|_{L^p}) \le \mathrm{e}^{ct2^{js}}\|[\dot\Delta_j; u\cdot\nabla]\theta\|_{L^p},$$

from which, we infer

$$\|\dot\Delta_j\theta(t)\|_{L^p} \le \mathrm{e}^{-ct2^{js}}\|\dot\Delta_j\theta_0\|_{L^p} + \int_0^t \mathrm{e}^{-c(t-t')2^{js}}\|[\dot\Delta_j; u\cdot\nabla]\theta(t')\|_{L^p}\mathrm{d}t', \tag{4.7}$$

and hence

$$\|\theta\|_{\widetilde L^r_t(\dot B^s_{p,\infty})} \le \|\theta_0\|_{\dot B^{s-\frac{s}{r}}_{p,\infty}} + C\sup_{j\in\mathbb{Z}}\|[\dot\Delta_j, u\cdot\nabla]\theta\|_{L^r_t(L^p)}. \tag{4.8}$$

However recalling from [22, Lemma 4.3] that

$$\|[\dot{\Delta}_j; u\cdot\nabla]\theta\|_{L^p} \le C\|\theta\|_{L^\infty}\|\nabla u\|_{L^p}.$$

Resuming the above inequality into (4.8) and using part (1) of the lemma gives rise to part (3) of the lemma.

Finally we deduce from Lemma 3.5 and (4.6) that for any $\alpha \in]0,1[$,

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})} + \|\theta\|_{L^1_t(\dot{B}^{1+s}_{3,1})} &\le \|\theta_0\|_{\dot{B}^1_{3,1}} + C\int_0^t \|[\dot{\Delta}_j; u\cdot\nabla]\theta(t')\|_{\dot{B}^1_{3,1}}\mathrm{d}t' \\ &\le \|\theta_0\|_{\dot{B}^1_{3,1}} + C\int_0^t (\|\nabla u\|_{L^\infty}\|\theta\|_{\dot{B}^1_{3,1}} + \|u\|_{\dot{B}^{2-\alpha}_{3,1}}\|\theta\|_{\dot{B}^\alpha_{\infty,\infty}})(t')\mathrm{d}t'.\end{aligned}$$

Applying Gronwall's inequality gives rise to

$$\begin{aligned}&\|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})} + \|\theta\|_{L^1_t(\dot{B}^{1+s}_{3,1})} \\ &\le \Big(\|\theta_0\|_{\dot{B}^1_{3,1}} + C\int_0^t \|u\|_{\dot{B}^{2-\alpha}_{3,1}}\|\theta\|_{\dot{B}^\alpha_{\infty,\infty}}\mathrm{d}t'\Big)\exp(C\|\nabla u\|_{L^1_t(L^\infty)}).\end{aligned}\tag{4.9}$$

Whereas it follows from (4.6) and the proof of part (1) in [12] that

$$\|\dot{\Delta}_j\theta(t)\|_{L^\infty} \le \|\dot{\Delta}_j\theta_0\|_{L^\infty} + \int_0^t \|[\dot{\Delta}_j; u\cdot\nabla]\theta(t')\|_{L^\infty}\mathrm{d}t',$$

which ensures

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty(\dot{B}^\alpha_{\infty,\infty})} &\le \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}} + \int_0^t \sup_{j\in\mathbb{Z}} 2^{j\alpha}\|[\dot{\Delta}_j, u\cdot\nabla]\theta(t')\|_{L^\infty}\mathrm{d}t' \\ &\le \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}} + C\int_0^t \|\nabla u(t')\|_{L^\infty}\|\theta(t')\|_{\dot{B}^\alpha_{\infty,\infty}}\mathrm{d}t'.\end{aligned}\tag{4.10}$$

Applying Gronwall's lemma gives

$$\|\theta\|_{\widetilde{L}^\infty(\dot{B}^\alpha_{\infty,\infty})} \le \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}\exp(C\|\nabla u\|_{L^1_t(L^\infty)}).$$

Substituting the above inequality into (4.9) leads to part (4) of the lemma, and we complete the proof of Lemma 4.3.

Let us complete this section by recalling the following proposition from [1].

Proposition 4.2 (see [1, Proposition 3.3], see also [7, Theorem 3.37]) *Let $p \in]2,\infty[$ and $s \in]-\frac{3}{p}, \frac{3}{p}[$. Let u, v be two solenoidal vector field which satisfy $u \in \mathcal{C}([0,T]; \dot{B}^s_{p,r}) \cap \widetilde{L}^1_T(\dot{B}^{s+2}_{p,r})$, $\nabla v \in L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})$, and*

$$\begin{cases}\partial_t u + v\cdot\nabla u - \Delta u + \nabla\Pi = g, \quad (t,x) \in [0,T]\times\mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0.\end{cases}\tag{4.11}$$

Then there exists a constant C such that

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,r})} + \|u\|_{\widetilde{L}^1_T(\dot{B}^{s+2}_{p,r})} + \|\nabla\Pi\|_{\widetilde{L}^1_T(\dot{B}^s_{p,r})} \\ &\le (\|u_0\|_{\dot{B}^s_{p,r}} + C\|g\|_{\widetilde{L}^1_T(\dot{B}^s_{p,r})})\exp(C\|\nabla v\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}).\end{aligned}\tag{4.12}$$

Moreover, there holds

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_T(\dot{B}^{-\frac{3}{p}}_{p,\infty})}+\|u\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{3}{p}+2}_{p,\infty})}+\|\nabla\Pi\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{3}{p}}_{p,\infty})}\\&\leq(\|u_0\|_{\dot{B}^{-\frac{3}{p}}_{p,\infty}}+C\|g\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{3}{p}}_{p,\infty})})\exp(C\|\nabla v\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}).\end{aligned}\tag{4.13}$$

5 The a Priori Estimates Related to the System (1.4)

In this section, we shall establish the a priori estimates which will be used to prove the global existence part of Theorem 1.1.

5.1 The short time estimates for smooth enough solutions of (1.4)

Proposition 5.1 *Let* (θ,u) *be a smooth enough solution of* (1.4) *on* $[0,T^*[$. *Then under the assumption of* (1.6), *for all* $t\in[0,T^*[$, *we have* (2.1) *for any* $p\in[6,8]$. *If moreover we assume that*

$$\|u_0\|_{\dot{B}^0_{3,1}}\|\theta_0\|_{B^{\frac{1}{2}}_{\infty,1}}\leq\varepsilon_1\tag{5.1}$$

for some ε_1 *sufficiently small, one has* (2.2).

Proof Let $\mathbb{P}\stackrel{\text{def}}{=}Id+\nabla(-\Delta)^{-1}\text{div}$ be the Leray projection operator to the divergence free vector space. In order to prove (2.1), we first apply the operator $\mathbb{P}$ to the u equation of (1.4) to get

$$\partial_t u-\Delta u=-\mathbb{P}\text{div}(u\otimes u+(1-\mu(\theta))d),\tag{5.2}$$

or equivalently,

$$u=\mathrm{e}^{t\Delta}u_0-\int_0^t\mathrm{e}^{(t-t')\Delta}\mathbb{P}\text{div}(u\otimes u+(1-\mu(\theta))d)(t')\mathrm{d}t'.\tag{5.3}$$

Note that $3<p<\infty$, we infer from Proposition 4.1 that

$$\|\mathrm{e}^{t\Delta}u_0\|_{L_t^{\frac{2p}{p-3}}(L^p)}\leq C\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\frac{2p}{p-3}}}\leq C\|u_0\|_{\dot{B}^0_{3,1}},$$

from which and Lemma 4.2, we deduce from (5.3) that for any $p\in[6,8]$,

$$\|u\|_{L_t^{\frac{2p}{p-3}}(L^p)}\leq C(\|u_0\|_{\dot{B}^0_{3,1}}+\|u\|^2_{L_t^{\frac{2p}{p-3}}(L^p)}+\|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})}).\tag{5.4}$$

Along the same line, since $p\geq 6$, we deduce from Proposition 4.1 and Lemma 3.1 that

$$\|\mathrm{e}^{t\Delta}\nabla u_0\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})}\leq C\|\nabla u_0\|_{\dot{B}^{-2(1-\frac{3}{p})}_{\frac{p}{2},\frac{p}{p-3}}}\leq C\|u_0\|_{\dot{B}^0_{3,1}},$$

from which, (5.3) and Lemma 4.1, we infer for any $p\in[6,8]$ that

$$\|\nabla u\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})}\leq C(\|u_0\|_{\dot{B}^0_{3,1}}+\|u\|^2_{L_t^{\frac{2p}{p-3}}(L^p)}+\|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})}).\tag{5.5}$$

By summing up (5.4) and (5.5), for ε_0 being sufficiently small in (1.6), we write

$$\|u\|_{L_t^{\frac{2p}{p-3}}(L^p)} + \|\nabla u\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})} \leq C(\|u_0\|_{\dot{B}^0_{3,1}} + \|u\|^2_{L_t^{\frac{2p}{p-3}}(L^p)}) \quad \text{for any } p \in [6,8].$$

Then for ε sufficiently small in (1.6), we infer

$$\|u\|_{L_t^{\frac{2p}{p-3}}(L^p)} + \|\nabla u\|_{L_t^{\frac{p}{p-3}}(L^{\frac{p}{2}})} \leq C\|u_0\|_{\dot{B}^0_{3,1}} \quad \text{for any } p \in [6,8]. \tag{5.6}$$

While we deduce from (4.2) for $p = 6$ and (5.3) that

$$\|u\|_{L_t^\infty(L^3)} \leq C(\|u_0\|_{L^3} + \|u\|^2_{L_t^{4,2}(L^6)} + \|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L_t^{2,1}(L^3)}). \tag{5.7}$$

Note that a similar proof of (4.1) also yields

$$\|\mathcal{B}(F)\|_{L_t^{4,2}(L^6)} \lesssim \|F\|_{L^2(L^3)},$$

so that we deduce from (5.3) that

$$\|u\|_{L_t^{4,2}(L^6)} \leq C(\|S(t)u_0\|_{L_t^{4,2}(L^6)} + \|u\|^2_{L^4(L^6)} + \|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L_t^2(L^3)}). \tag{5.8}$$

On the other hand, it follows from Lemma 4.1 that

$$\|\mathcal{A}F\|_{L_T^q(L^3)} \leq C_q\|F\|_{L_T^q(L^3)},$$

which together with the fact (see [13]) that

$$L^{2,1}(L^3) = [L^{p_0}(L^3), L^{p_1}(L^3)]_{\gamma,1} \quad \text{with } 1 < p_0 < 2 < p_1 \text{ and } \frac{1}{2} = \frac{1-\gamma}{p_0} + \frac{\gamma}{p_1},$$

implies

$$\|\mathcal{A}F\|_{L_T^{2,1}(L^3)} \leq C\|F\|_{L_T^{2,1}(L^3)}.$$

Then we deduce from (5.3) that

$$\|\nabla u\|_{L_t^{2,1}(L^3)} \leq C(\|\nabla S(t)u_0\|_{L_t^{2,1}(L^3)} + \|u\|^2_{L_t^{4,2}(L^6)} + \|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L_t^{2,1}(L^3)}). \tag{5.9}$$

Thanks to the fact that

$$\|\nabla u\|_{L_t^2(L^3)} \lesssim \|\nabla u\|_{L_t^{2,1}(L^3)} \quad \text{and} \quad \|u\|_{L^4(L^6)} \lesssim \|u\|_{L_t^{4,2}(L^6)},$$

and $\|1-\mu(\theta_0)\|_{L^\infty} \ll 1$, we get, by summing up (5.8) and (5.9) that

$$\|u\|_{L_t^{4,2}(L^6)} + \|\nabla u\|_{L_t^{2,1}(L^3)} \leq C(\|S(t)u_0\|_{L_t^{4,2}(L^6)} + \|\nabla S(t)u_0\|_{L_t^{2,1}(L^3)} + \|u\|^2_{L^{4,2}(L^6)}). \tag{5.10}$$

On the other hand, it follows from Proposition 4.1 that

$$\|S(t)u_0\|_{L^{4-}(L^6)} \lesssim \|u_0\|_{\dot{B}^{-\frac{2}{4-}}_{6,4-}} \quad \text{and} \quad \|S(t)u_0\|_{L^{4+}(L^6)} \lesssim \|u_0\|_{\dot{B}^{-\frac{2}{4+}}_{6,4+}},$$
$$\|\nabla S(t)u_0\|_{L^{2-}(L^3)} \lesssim \|\nabla u_0\|_{\dot{B}^{-\frac{2}{2-}}_{3,2-}} \quad \text{and} \quad \|\nabla S(t)u_0\|_{L^{2+}(L^3)} \lesssim \|u_0\|_{\dot{B}^{-\frac{2}{2+}}_{3,2+}}.$$

Yet by virtue of the real interpolation with the pairs $\left(\frac{1}{2},2\right)$ and $\left(\frac{1}{2},1\right)$ (see [8, Theorem 6.4.5, p.160]), we get

$$L^{4,2}=[L^{4^-},L^{4^+}]_{\frac12,2}\quad\text{and}\quad L^{2,1}=[L^{2^-},L^{2^+}]_{\frac12,1},$$
$$[\dot B^{-\frac{2}{4^-}}_{6,4^-},\dot B^{-\frac{2}{4^+}}_{6,4^+}]_{\frac12,2}=\dot B^{-\frac12}_{6,2}\quad\text{and}\quad[\dot B^{-\frac{2}{2^-}}_{3,2^-},\dot B^{-\frac{2}{2^+}}_{3,2^+}]_{\frac12,1}=\dot B^{-1}_{3,1}.$$

As a result, it comes out

$$\|S(t)u_0\|_{L^{4,2}_t(L^6)}\lesssim\|u_0\|_{\dot B^{-\frac12}_{6,2}}\quad\text{and}\quad\|\nabla S(t)u_0\|_{L^{2,1}(L^3)}\lesssim\|\nabla u_0\|_{\dot B^{-1}_{3,1}},$$

and

$$\|S(t)u_0\|_{L^{4,2}_t(L^6)}+\|\nabla S(t)u_0\|_{L^{2,1}_t(L^3)}\lesssim\|u_0\|_{\dot B^{-\frac12}_{6,1}}+\|\nabla u_0\|_{\dot B^{-1}_{3,1}}\lesssim\|u_0\|_{\dot B^0_{3,1}}.$$

Then for ε sufficiently small in (1.6), we deduce from (5.10) that

$$\|u\|_{L^{4,2}_t(L^6)}+\|\nabla u\|_{L^{2,1}_t(L^3)}\lesssim\|u_0\|_{\dot B^0_{3,1}}.$$

Inserting the above inequality into (5.7) yields

$$\|u\|_{L^\infty_t(L^3)}\lesssim\|u_0\|_{\dot B^0_{3,1}}.$$

This together with (5.6) concludes the proof of (2.1).

On the other hand, by virtue of Lemma 3.2, we deduce from (5.3) that

$$\begin{aligned}\|\dot\Delta_j u(t)\|_{L^\infty}&\lesssim d_j2^j\mathrm{e}^{-c2^{2j}t}\|u_0\|_{\dot B^{-1}_{\infty,1}}\\&\quad+2^j\int_0^t\mathrm{e}^{-c2^{2j}(t-t')}(\|\Delta_j(u\otimes u)\|_{L^\infty}+\|\Delta_j((\mu(\theta)-1)d)\|_{L^\infty})(t')\mathrm{d}t'.\end{aligned}\tag{5.11}$$

Yet applying Bony's decomposition (3.5) and standard paraproduct estimates (see [7]) leads to

$$\begin{aligned}\|\dot\Delta_j(u\otimes u)(t')\|_{L^\infty}&\lesssim\sum_{|\ell-j|\le4}\|\dot S_{\ell-1}u(t')\|_{L^\infty}\|\dot\Delta_\ell u(t')\|_{L^\infty}+2^j\sum_{\ell\ge j-N_0}\|\dot\Delta_\ell u(t')\|_{L^3}\|\widetilde{\dot\Delta}_\ell u(t')\|_{L^\infty}\\&\lesssim d_j(t')\|u(t')\|_{L^3}\|u(t')\|_{\dot B^1_{\infty,1}}.\end{aligned}$$

Along the same line, we write

$$(\mu(\theta)-1)d=T_{(\mu(\theta)-1)}d+T_d(\mu(\theta)-1)+R(\mu(\theta)-1,d).$$

Applying Lemma 3.1 gives

$$\begin{aligned}\|\dot\Delta_jT_{(\mu(\theta)-1)}d(t')\|_{L^\infty}&\lesssim\sum_{|\ell-j|\le4}\|\dot S_{\ell-1}(\mu(\theta)-1)(t')\|_{L^\infty}\|\dot\Delta_\ell d(t')\|_{L^\infty}\\&\lesssim d_j(t')\|(\mu(\theta)-1)(t')\|_{L^\infty}\|u(t')\|_{\dot B^1_{\infty,1}},\end{aligned}$$

and for $p(t)\overset{\text{def}}{=}6(1+\|\nabla u\|_{L^1_t(L^\infty)})$,

$$\begin{aligned}\|\dot\Delta_jR(\mu(\theta)-1,d)(t')\|_{L^\infty}&\lesssim2^{\frac{3j}{p(t)}}\sum_{\ell\ge j-3}\|\dot\Delta_\ell\theta(t')\|_{L^\infty}\|\dot\Delta_\ell d(t')\|_{L^{p(t)}}\\&\lesssim2^{\frac{3j}{p(t)}}\sum_{\ell\ge j-3}d_\ell(t')2^{-\frac{3\ell}{p(t)}}\|\nabla u(t')\|_{L^{p(t)}}\|\theta(t')\|_{\dot B^{\frac{3}{p(t)}}_{\infty,1}}\\&\lesssim d_j(t')\|\nabla u(t')\|_{L^{p(t)}}\|\theta(t')\|_{\dot B^{\frac{3}{p(t)}}_{\infty,1}}.\end{aligned}$$

The same estimate holds for $T_d(\mu(\theta)-1)$.

Resuming the above two inequalities into (5.11) and summing up the resulting inequality over j in $\mathbb{Z}$, we arrive at

$$\begin{aligned}\|u\|_{L^1_t(\dot B^1_{\infty,1})}\le C(&\|u_0\|_{\dot B^{-1}_{\infty,1}}+\|u\|_{L^\infty_t(L^3)}\|u\|_{L^1_t(\dot B^1_{\infty,1})}+\|1-\mu(\theta_0)\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{\infty,1})}\\&+p(t)\|\nabla u\|_{L^1_t(L^{p(t)})}\|\theta\|_{L^\infty_t(\dot B^{\frac{3}{p(t)}}_{\infty,1})}).\end{aligned}\tag{5.12}$$

Without loss of generality, we may assume that $T^*>1$. We define

$$t^*\stackrel{\rm def}{=}\sup\{\ t\in]0,1]:\ p(t)=6(1+\|\nabla u\|_{L^1_t(L^\infty)})\le 8\}.\tag{5.13}$$

We claim that under the assumptions of Proposition 5.1, $t^*=1$. Otherwise, it follows from [2, Lemma 3.1] that

$$\|\theta\|_{L^\infty_t(\dot B^{\frac{3}{p(t)}}_{\infty,1})}\le C\|\theta_0\|_{\dot B^{\frac{3}{p(t)}}_{\infty,1}}(1+\|\nabla u\|_{L^1_t(L^\infty)})\le C\|\theta_0\|_{B^{\frac12}_{\infty,1}}(1+\|\nabla u\|_{L^1_t(L^\infty)}),$$

and (2.1) ensures that

$$\|\nabla u\|_{L^1_t(L^{p(t)})}\le t^{\frac{3}{2p(t)}}\|\nabla u\|_{L_t^{\frac{2p(t)}{2p(t)-3}}(L^{p(t)})}\le Ct^{\frac{3}{2p(t)}}\|u_0\|_{\dot B^0_{3,1}}\quad\text{for}\ \ t\le t^*.$$

Then taking $t=t^*$ in (5.12), for $\varepsilon_0,\varepsilon$ being sufficiently small in (1.6), we deduce from (2.1) and (5.12) that

$$\|u\|_{L^1_{t^*}(\dot B^1_{\infty,1})}\le C(\|u_0\|_{\dot B^0_{3,1}}+\|u_0\|_{\dot B^0_{3,1}}\|\theta_0\|_{B^{\frac12}_{\infty,1}}(1+\|\nabla u\|^2_{L^1_{t^*}(L^\infty)})),$$

which together with (5.1) ensures

$$\|u\|_{L^1_{t^*}(\dot B^1_{\infty,1})}\le C\|u_0\|_{\dot B^0_{3,1}}(1+\|\theta_0\|_{B^{\frac12}_{\infty,1}})\le C(\varepsilon+\varepsilon_1).$$

In particular, if we take $\varepsilon,\varepsilon_1$ so small that $C(\varepsilon+\varepsilon_1)\le\frac16$, we obtain

$$p(t^*)\le 6(1+\|u\|_{L^1_{t^*}(\dot B^1_{\infty,1})})\le 7,$$

which contradicts with (5.13) if $t^*<1$, and this in turn shows that $t^*=1$ and there holds (2.2). Hence we complete the proof of Proposition 5.1.

Proposition 5.2 *Under the assumptions of Proposition* 5.1*, one has*

$$\begin{aligned}&\|\theta\|_{\widetilde L^\infty_1(B^1_{3,1})}\le C\|\theta_0\|_{B^1_{3,1}}\quad\textit{and}\quad\|\theta\|_{\widetilde L^\infty_1(B^{(\frac12)_+}_{\infty,\infty})}\le C\|\theta_0\|_{B^{(\frac12)_+}_{\infty,\infty}},\\&\|u\|_{\widetilde L^\infty_1(\dot H^{-2\delta})}+\|u\|_{\widetilde L^1_1(\dot H^{2(1-\delta)})}\le C(\|u_0\|_{\dot H^{-2\delta}}+\|\theta_0\|_{B^1_{3,1}}\|u_0\|_{\dot B^1_{3,1}}).\end{aligned}\tag{5.14}$$

Proof We first apply the dyadic operator $\dot\Delta_j$ to (5.2) and then taking the L^2 inner product of the resulting equation with $\dot\Delta_j u$ that

$$\frac12\frac{\rm d}{{\rm d}t}\|\dot\Delta_j u(t)\|^2_{L^2}+\|\nabla\dot\Delta_j u\|^2_{L^2}=-([\dot\Delta_j\mathbb{P};u\cdot\nabla]u\mid\dot\Delta_j u)+(\dot\Delta_j\mathbb{P}{\rm div}((\mu(\theta)-1)d)\mid\dot\Delta_j u),$$

from which and Lemma 3.1, we infer

$$\|u\|_{\widetilde{L}^\infty_1(\dot H^{-2\delta})} + \|u\|_{\widetilde{L}^1_1(\dot H^{2(1-\delta)})} \lesssim \|u_0\|_{\dot H^{-2\delta}} + (2^{-2j\delta}\|[\dot\Delta_j\mathbb{P}; u\cdot\nabla]u\|_{L^1_1(L^2)})_{\ell^2} + (2^{j(1-2\delta)}\|\dot\Delta_j\mathbb{P}(\mu(\theta)-1)d\|_{L^1_1(L^2)})_{\ell^2}. \tag{5.15}$$

Whereas using Bony's decomposition (3.5) and $\operatorname{div} u = 0$, one has

$$[\dot\Delta_j\mathbb{P}; u\cdot\nabla]u = [\dot\Delta_j\mathbb{P}; T_u]\nabla u + \operatorname{div}\dot\Delta_j\mathbb{P}T'_u u - \operatorname{div} T'_{\dot\Delta_j u}u.$$

Applying commutator's estimate (see [7, 29]) gives

$$\begin{aligned}\|[\dot\Delta_j\mathbb{P}; T_u]\nabla u\|_{L^1_1(L^2)} &\lesssim \sum_{|j'-j|\le 4}\|S_{j'-1}\nabla u\|_{L^1_1(L^\infty)}\|\dot\Delta_j u\|_{L^\infty_t(L^2)}\\ &\lesssim c_j 2^{2j\delta}\|\nabla u\|_{L^1_1(L^\infty)}\|u\|_{\widetilde{L}^\infty_1(\dot H^{-2\delta})}.\end{aligned}$$

While since $\delta\in]0,\frac12[$, applying Lemma 3.1 yields

$$\begin{aligned}\|\operatorname{div}\dot\Delta_j\mathbb{P}T'_u u\|_{L^1_1(L^2)} &\lesssim 2^j\sum_{j'\ge j-N_0}\|\dot\Delta_{j'}u\|_{L^1_1(L^\infty)}\|S_{j'+2}u\|_{L^\infty_1(L^2)}\\ &\lesssim c_j 2^{2j\delta}\|\nabla u\|_{L^1_1(L^\infty)}\|u\|_{\widetilde{L}^\infty_1(\dot H^{-2\delta})}.\end{aligned}$$

The same estimate holds for $\operatorname{div} T'_{\dot\Delta_j u}u$. We thus obtain

$$\begin{aligned}\|[\dot\Delta_j\mathbb{P}; u\cdot\nabla]u\|_{L^1_1(L^2)} &\lesssim c_j 2^{2j\delta}\|\nabla u\|_{L^1_1(L^\infty)}\|u\|_{\widetilde{L}^\infty_1(\dot H^{-2\delta})}\\ &\lesssim c_j 2^{2j\delta}\|u_0\|_{\dot B^0_{3,1}}(1+\|\theta_0\|_{B^{\frac12}_{\infty,1}})\|u\|_{\widetilde{L}^\infty_1(\dot H^{-2\delta})},\end{aligned} \tag{5.16}$$

where we used (2.2) in the last step.

Along the same line, we get, by using paraproduct estimates in [7] that

$$\begin{aligned}\|\dot\Delta_j\mathbb{P}((\mu(\theta)-1)d)\|_{L^1_1(L^2)} &\lesssim \|\dot\Delta_j\mathbb{P}T_{(\mu(\theta)-1)}d\|_{L^1_1(L^2)} + \|\dot\Delta_j\mathbb{P}T'_d(\mu(\theta)-1)\|_{L^1_1(L^2)}\\ &\lesssim c_j 2^{j(2\delta-1)}\|\mu(\theta)-1\|_{L^\infty_1(L^\infty)}\|u\|_{\widetilde{L}^1_1(\dot H^{2(1-\delta)})}\\ &\quad + \|\dot\Delta_j T'_d(\mu(\theta)-1)\|_{L^1_1(L^2)}.\end{aligned}$$

However applying Lemma 3.1 gives

$$\begin{aligned}\|\dot\Delta_j T'_d(\mu(\theta)-1)\|_{L^1_1(L^2)} &\lesssim \sum_{j'\ge j-N_0}\|\dot S_{j'+2}d\|_{L^1_1(L^6)}\|\dot\Delta_{j'}(\mu(\theta)-1)\|_{L^\infty_1(L^3)}\\ &\lesssim c_j 2^{j(2\delta-1)}\|\theta\|_{\widetilde{L}^\infty_1(\dot B^1_{3,1})}\|\nabla u\|_{L^1_1(\dot B^{-2\delta}_{6,1})}\\ &\lesssim c_j 2^{j(2\delta-1)}\|\theta\|_{\widetilde{L}^\infty_1(\dot B^1_{3,1})}(\|\nabla u\|_{L^4_1(L^6)} + \|u\|_{L^\infty_1(L^3)}).\end{aligned}$$

And it follows from [7, Theorem 3.14] and (1.6), (2.2) that

$$\|\theta\|_{\widetilde{L}^\infty_1(B^1_{3,1})} \lesssim \|\theta_0\|_{B^1_{3,1}}\exp(C\|u\|_{L^1_1(\dot B^1_{\infty,1})}) \lesssim \|\theta_0\|_{B^1_{3,1}}. \tag{5.17}$$

Therefore by virtue of (2.1) for $p=12$, we obtain

$$\|\dot\Delta_j\mathbb{P}((\mu(\theta)-1)d)\|_{L^1_1(L^2)} \lesssim c_j 2^{j(2\delta-1)}(\|\mu(\theta_0)-1\|_{L^\infty}\|u\|_{\widetilde{L}^1_1(\dot H^{2(1-\delta)})} + \|\theta_0\|_{B^1_{3,1}}\|u_0\|_{\dot B^0_{3,1}}).$$

Resuming the above estimate and (5.16) into (5.15), we write

$$\begin{aligned}\|u\|_{\widetilde{L}_1^\infty(\dot{H}^{-2\delta})}+\|u\|_{\widetilde{L}_1^1(\dot{H}^{2(1-\delta)})}\leq C(&\|u_0\|_{\dot{H}^{-2\delta}}+\|\theta_0\|_{B^1_{3,1}}\|u_0\|_{\dot{B}^0_{3,1}}\\&+\|u_0\|_{\dot{B}^0_{3,1}}\|u\|_{\widetilde{L}_1^\infty(\dot{H}^{-2\delta})}(1+\|\theta_0\|_{B^{\frac12}_{\infty,1}})\\&+\|\mu(\theta_0)-1\|_{L^\infty}\|u\|_{\widetilde{L}_1^1(\dot{H}^{2(1-\delta)})}).\end{aligned}$$

Taking $\varepsilon_0,\varepsilon$ small enough in (1.6) and ε_1 small enough in (5.1), we obtain the second line of (5.14).

Finally similar to (5.17), we can prove

$$\|\theta\|_{\widetilde{L}_1^\infty(B^{(\frac12)+}_{\infty,\infty})}\leq C\|\theta_0\|_{B^{(\frac12)+}_{\infty,\infty}}.$$

This concludes the proof of the proposition.

Corollary 5.1 *Under the assumptions of Proposition* 5.1*, we can find some* $t_0\in]0,1[$ *such that* $\theta(t_0)\in(B^1_{3,1}\cap B^{(\frac12)+}_{\infty,\infty})(\mathbb{R}^3)$, $u(t_0)\in H^1(\mathbb{R}^3)$, *moreover, there holds* (2.3).

Proof Note that since $\delta\in]0,\frac12[$, we have

$$\begin{aligned}\|u\|_{L^1_1(B^1_{2,1})}&\leq\|\dot{S}_0u\|_{\widetilde{L}_1^\infty(\dot{H}^{-2\delta})}+\|(Id-\dot{S}_0)u\|_{L^1_1(\dot{B}^1_{2,1})}\\&\lesssim\|\dot{S}_0u\|_{\widetilde{L}_1^\infty(\dot{H}^{-2\delta})}+\|(Id-\dot{S}_0)u\|_{\widetilde{L}_1^1(\dot{H}^{2(1-\delta)})},\end{aligned}$$

from which and (5.14), we deduce that there exists some $t_0\in]0,1[$ so that

$$\|u(t_0)\|_{H^1}\leq\|u(t_0)\|_{B^1_{2,1}}\leq C(\|u_0\|_{\dot{H}^{-2\delta}}+\|\theta_0\|_{B^1_{3,1}}\|u_0\|_{\dot{B}^0_{3,1}}).$$

This together with (5.14) concludes the proof of (2.3).

5.2 The propagation of H^1 regularity for u

As a convention in the remaining of this section, we shall always denote t_0 to be the positive time determined by Corollary 5.1.

Proposition 5.3 *Let* (θ,u) *be a smooth enough solution of* (1.4) *on* $[0,T^*[$. *Then under the assumptions of Proposition* 5.1*, we have* (2.4) *for* $t\in]t_0,T^*[$.

Proof The proof of this proposition is motivated by that of [16, Theorem 1] for 2-D inhomogeneous Navier-Stokes system and that of [5, Proposition 2.1] for 3-D inhomogeneous Navier-Stokes system. In fact, since $\operatorname{div}u=0$, we get, by taking L^2 inner product of the velocity equation of (1.4) with u, that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|u(t)|^2\mathrm{d}x+2\int_{\mathbb{R}^3}\mu(\theta)d:d\mathrm{d}x=0,\tag{5.18}$$

integrating the above inequality over $[t_0,t]$ and using (1.5) and $\operatorname{div}u=0$, we obtain the first line of (2.4).

Whereas by taking the L^2 inner product of (5.2) with ∂_tu, we write

$$\int_{\mathbb{R}^3}|\partial_tu|^2\mathrm{d}x-\int_{\mathbb{R}^3}\operatorname{div}(2\mu(\theta)d)\mid\partial_tu\mathrm{d}x=-\int_{\mathbb{R}^3}\partial_tu\mid(u\cdot\nabla u)\mathrm{d}x.\tag{5.19}$$

Motivated by the derivation of (29) in [16], we get, by using integration by parts, that

$$\begin{aligned}-\int_{\mathbb{R}^3} \mathrm{div}(2\mu(\theta)d) \mid \partial_t u \mathrm{d}x &= \int_{\mathbb{R}^3} 2\mu(\theta)d : \partial_t d \mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mu(\theta)d : d\mathrm{d}x - \int_{\mathbb{R}^3} \partial_t(\mu(\theta))d : d\mathrm{d}x.\end{aligned}$$

Using the θ equation of (1.4) and then integration by parts, we get

$$\begin{aligned}&-\int_{\mathbb{R}^3} \partial_t(\mu(\theta))d : d\mathrm{d}x = \int_{\mathbb{R}^3} u\cdot\nabla\mu(\theta)d : d\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} u\cdot\nabla(\mu(\theta))^{-1}(\mu(\theta)d) : (\mu(\theta)d)\mathrm{d}x = \sum_{i=1}^{3}\int_{\mathbb{R}^3} u^i d : \partial_i(2\mu(\theta)d)\mathrm{d}x.\end{aligned}$$

Notice that

$$\begin{aligned}\sum_{i=1}^{3}\int_{\mathbb{R}^3} u^i d : \partial_i(2\mu(\theta)d)\mathrm{d}x &= \sum_{1\le i,k,\ell\le 3}\int_{\mathbb{R}^3} u^i\partial_k u^\ell \partial_i(2\mu(\theta)d_{k\ell})\mathrm{d}x \\ &= -\sum_{1\le i,k,\ell\le 3}\Big(\int_{\mathbb{R}^3} \partial_k u^i u^\ell\partial_i(2\mu(\theta)d_{k\ell})\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} u^i u^\ell \partial_i\partial_k(2\mu(\theta)d_{k\ell})\mathrm{d}x\Big).\end{aligned}$$

Hence due to $\mathrm{div}\, u = 0$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} |\partial_t u|^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mu(\theta)d : d\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \partial_t u \mid (u\cdot\nabla u)\mathrm{d}x - \sum_{1\le i,k,\ell\le 3}\Big(\int_{\mathbb{R}^3} u^i\,\partial_i u^\ell\,\partial_k(2\mu(\theta)d_{k\ell})\mathrm{d}x + \int_{\mathbb{R}^3} 2\mu(\theta)\partial_k u^i\,\partial_i u^\ell \mathrm{d}_{k\ell}\mathrm{d}x\Big),\end{aligned}$$

which together with the velocity equation of (1.4) implies that

$$\begin{aligned}&\int_{\mathbb{R}^3} |\partial_t u|^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mu(\theta)d : d\mathrm{d}x \\ &\le \|\partial_t u\|_{L^2}\|u\cdot\nabla u\|_{L^2} - \int_{\mathbb{R}^3} u\cdot\nabla u \mid (\partial_t u + u\cdot\nabla u + \nabla\Pi)\mathrm{d}x + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &\le C(\|u\cdot\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2) - \int_{\mathbb{R}^3} u\cdot\nabla u \mid \nabla\Pi\mathrm{d}x + \frac{1}{4}\|\partial_t u\|_{L^2}^2.\end{aligned}$$

Integrating the above inequality over $[t_0, t]$ and using again $\mathrm{div}\, u = 0$ and $\|u\|_{L^6} \le C\|\nabla u\|_{L^2}$, we infer

$$\begin{aligned}\frac{3}{4}\int_{t_0}^{t}\int_{\mathbb{R}^3} |\partial_t u|^2\mathrm{d}x\mathrm{d}t' + \|\nabla u(t)\|_{L^2}^2 \le C\Big(&\|\nabla u(t_0)\|_{L^2}^2 + \int_{t_0}^{t} \|\nabla u\|_{L^2}^2\|\nabla u\|_{L^3}^2\mathrm{d}t' \\ &+ \int_{t_0}^{t} \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2\mathrm{d}t' \\ &+ \sum_{i,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \Pi\partial_i u^k\partial_k u^i\mathrm{d}x\mathrm{d}t'\Big).\end{aligned} \tag{5.20}$$

To deal with the pressure function Π, we get, by taking space divergence to the velocity equation of (1.4), that

$$\Pi = (-\Delta)^{-1}\mathrm{div} \otimes \mathrm{div}(2\mu(\theta)d) - (-\Delta)^{-1}\mathrm{div}(u \cdot \nabla u) \tag{5.21}$$

and

$$\sum_{i,k=1}^{3} \int_{\mathbb{R}^3} (-\Delta)^{-1}\mathrm{div}(u \cdot \nabla u) \mid \partial_i u^k \partial_k u^i \mathrm{d}x = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} (-\Delta)^{-1}\partial_k \mathrm{div}(u \cdot \nabla u) \mid u \cdot \nabla u^k \mathrm{d}x,$$

from which, we deduce

$$\Big| \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} \Pi \partial_i u^k \partial_k u^i \mathrm{d}x \Big| \lesssim \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^3}^2.$$

We thus deduce from (5.20) that

$$\begin{aligned} \frac{3}{4} \int_{t_0}^{t} \int_{\mathbb{R}^3} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t' + \|\nabla u(t)\|_{L^2}^2 \leq C\Big(&\|\nabla u(t_0)\|_{L^2}^2 + \int_{t_0}^{t} \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^3}^2 \mathrm{d}t' \\ &+ \int_{t_0}^{t} \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 \mathrm{d}t' \Big). \end{aligned} \tag{5.22}$$

On the other hand, it is easy to observe that

$$\nabla u = \nabla(-\Delta)^{-1}\mathrm{div}\,\mathbb{P}(2(\mu(\theta) - 1)d) - \nabla(-\Delta)^{-1}\mathrm{div}\,\mathbb{P}(2\mu(\theta)d), \tag{5.23}$$

from which and

$$\|a\|_{L^p} \leq C\|a\|_{L^3}^{\frac{6}{p}-1} \|\nabla a\|_{L^2}^{2(1-\frac{3}{p})} \quad \text{for } p \in [3,6], \tag{5.24}$$

we infer

$$\|\nabla u\|_{L^p} \leq C(\|\mu(\theta_0) - 1\|_{L^\infty} \|\nabla u\|_{L^p} + \|\nabla u\|_{L^3}^{\frac{6}{p}-1} \|\mathbb{P}\mathrm{div}(2\mu(\theta)d)\|_{L^2}^{2(1-\frac{3}{p})}).$$

Taking ε_0 sufficiently small in (1.6), we obtain for $p \in [3,6]$,

$$\begin{aligned} \|\nabla u\|_{L^p} &\leq C\|\nabla u\|_{L^3}^{\frac{6}{p}-1} \|\partial_t u + (u \cdot \nabla)u\|_{L^2}^{2(1-\frac{3}{p})} \\ &\leq C\|\nabla u\|_{L^3}^{\frac{6}{p}-1} (\|\partial_t u\|_{L^2}^{2(1-\frac{3}{p})} + \|u\|_{L^6}^{2(1-\frac{3}{p})} \|\nabla u\|_{L^3}^{2(1-\frac{3}{p})}). \end{aligned} \tag{5.25}$$

Substituting the above inequality for $p = 4$ into (5.22), we obtain

$$\int_{t_0}^{t} \int_{\mathbb{R}^3} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t' + \|\nabla u\|_{L^\infty(t_0,t;L^2)}^2 \leq C(\|\nabla u(t_0)\|_{L^2}^2 + \int_{t_0}^{t} \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^3}^2 \mathrm{d}t'). \tag{5.26}$$

Applying Gronwall's lemma and using (2.1) for $p = 6$, we infer

$$\begin{aligned} \int_{t_0}^{t} \int_{\mathbb{R}^3} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t' + \|\nabla u\|_{L^\infty(t_0,t;L^2)}^2 &\leq C\|\nabla u(t_0)\|_{L^2}^2 \exp(C\|\nabla u\|_{L_t^2(L^3)}^2) \\ &\leq C\|\nabla u(t_0)\|_{L^2}^2 \exp(C\|u_0\|_{\dot{B}^0_{3,1}}) \quad \text{for } t \in]0, T^*[. \end{aligned} \tag{5.27}$$

This concludes the proof of (2.4).

An immediate consequence of Proposition 5.3 is the following corollary concerning the estimate of $\|\nabla u\|_{L_t^q(L^p)}$ for $3 \leq p \leq 8$.

Corollary 5.2 *Under the assumptions of Proposition* 5.1, *for any* $p \in [3,8]$ *and any* $t \in]t_0, T^*[$, *we have*

$$\begin{aligned} &\|\nabla u\|_{L^2(t_0,t;L^p)} \le C\|u_0\|_{\dot B^0_{3,1}}^{\frac{6}{p}-1}\|\nabla u(t_0)\|_{L^2}^{2(1-\frac{3}{p})}, \qquad \text{if } p\in[3,6],\\ &\|\nabla u\|_{L^{\frac{4p(p-3)}{4p^2-21p+18}}(t_0,t;L^p)} \le C\|u_0\|_{\dot B^0_{3,1}}^{\frac{p-6}{p-3}}\|\nabla u(t_0)\|_{L^2}^{\frac{3}{p-3}}, \quad \text{if } p\in[6,8]. \end{aligned} \tag{5.28}$$

Proof For $p \in [3,6]$, by virtue of (5.25), (2.1) and (2.4), we have for any $t \in]t_0, T^*[$,

$$\begin{aligned} \|\nabla u\|_{L^2(t_0,t;L^p)} &\le C(\|\nabla u\|_{L^2(t_0,t;L^3)}^{\frac{6}{p}-1}\|\partial_t u\|_{L^2(t_0,t;L^2)}^{2(1-\frac{3}{p})} + \|\nabla u\|_{L^\infty(t_0,t;L^2)}^{2(1-\frac{3}{p})}\|\nabla u\|_{L^2(t_0,t;L^3)})\\ &\le C(1+\|u_0\|_{\dot B^0_{3,1}}^{2(1-\frac{3}{p})})\|u_0\|_{\dot B^0_{3,1}}^{\frac{6}{p}-1}\|\nabla u(t_0)\|_{L^2}^{2(1-\frac{3}{p})}\exp(C\|u_0\|_{\dot B^0_{3,1}}), \end{aligned}$$

which together with (1.6) yields the first inequality of (5.28). For $p \in [6,8]$, applying Hölder's inequality gives rise to

$$\|\nabla u\|_{L^{\frac{4p(p-3)}{4p^2-21p+18}}(t_0,t;L^p)} \le C\|\nabla u\|_{L^2(t_0,t;L^6)}^{\frac{3}{p-3}}\|\nabla u\|_{L^{\frac{4p}{4p-3}}(t_0,t;L^{2p})}^{1-\frac{3}{p-3}}, \tag{5.29}$$

from which, (2.1) and the first inequality of (5.28), we conclude the proof of the second inequality of (5.28).

Corollary 5.3 *Let* $\langle t\rangle \overset{\text{def}}{=} \mathrm{e}+t$. *Then under the assumption of Proposition* 5.1, *one has*

$$\|\langle t'\rangle^{\frac12}\nabla u\|_{L^\infty(t_0,t;L^2)}^2 + \|\langle t'\rangle^{\frac12}\partial_t u\|_{L^2(t_0,t;L^2)}^2 \le C\|u(t_0)\|_{H^1}^2 \quad \text{for } t\in]t_0,T^*[. \tag{5.30}$$

Proof The proof of this proposition basically follows from that of Proposition 5.3. We first get, by multiplying (5.19) by $\langle t\rangle$ and then by using a similar derivation of (5.26), that

$$\begin{aligned} &\|\langle t'\rangle^{\frac12}\partial_t u\|_{L^2(t_0,t;L^2)}^2 + \|\langle t'\rangle^{\frac12}\nabla u\|_{L^\infty(t_0,t;L^2)}^2\\ &\le C\Big(\|\nabla u(t_0)\|_{L^2}^2 + \|\nabla u\|_{L^2(t_0,t;L^2)}^2 + \int_{t_0}^t \langle t'\rangle\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^3}^2\mathrm{d}t'\Big). \end{aligned}$$

By (2.1) and Gronwall's lemma, we thus obtain

$$\|\langle t'\rangle^{\frac12}\nabla u\|_{L^\infty(t_0,t;L^2)}^2 + \|\langle t'\rangle^{\frac12}\partial_t u\|_{L^2(t_0,t;L^2)}^2 \le C\|u(t_0)\|_{H^1}^2\exp(C\|u_0\|_{\dot B^0_{3,1}}),$$

which together with the second inequality of (1.6) leads to (5.30).

5.3 Large time decay estimate for u

Lemma 5.1 *Under the assumptions of Proposition* 5.1, *we have*

$$\begin{aligned} &\|u(t)\|_{L^2} \le C\overline{C}\langle t\rangle^{-\delta} \quad \text{for any } t\in[t_0,T^*[\quad \text{and}\\ &\overline{C} \overset{\text{def}}{=} 1 + \|u(t_0)\|_{\dot H^{-2\delta}}^2 + \|u(t_0)\|_{L^2}^2(1+\|\theta_0\|_{B^1_{3,1}}^2 + \|u(t_0)\|_{L^2}^2). \end{aligned} \tag{5.31}$$

Proof We first deduce from (5.18) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + 2\underline{\mu}\|\nabla u(t)\|_{L^2}^2 \le 0. \tag{5.32}$$

Applying Schonbek's strategy in [31], we split the frequency space $\mathbb{R}^3$ into two time-dependent domains: $\mathbb{R}^3 = S(t) \cup S(t)^c$, where $S(t) \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3 :|\,\xi| \leq \sqrt{\frac{M}{2\underline{\mu}}} g(t)\}$ for some $g(t) \lesssim \langle t\rangle^{-\frac{1}{2}}$, which will be chosen hereafter. Then we deduce from (5.32) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + g^2(t)\|u(t)\|_{L^2}^2 \leq Mg^2(t)\int_{S(t)} |\widehat{u}(t,\xi)|^2 \mathrm{d}\xi. \tag{5.33}$$

To deal with the term on the right hand side of (5.33), we write the velocity equation of (1.4) as

$$u(t) = \mathrm{e}^{(t-t_0)\Delta}u_0 + \int_{t_0}^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}(2(\mu(\theta)-1)d - u\otimes u)(t')\mathrm{d}t'.$$

Taking Fourier transform with respect to x variables gives rise to

$$|\widehat{u}(t,\xi)| \lesssim \mathrm{e}^{-(t-t_0)|\xi|^2}|\widehat{u}(t_0,\xi)| + \int_{t_0}^t \mathrm{e}^{-(t-t')|\xi|^2}|\xi||\mathcal{F}_x(2(\mu(\theta)-1)d - u\otimes u)(t')|\mathrm{d}t',$$

so that

$$\begin{aligned}\int_{S(t)} |\widehat{u}(t,\xi)|^2\mathrm{d}\xi \lesssim& \int_{S(t)} \mathrm{e}^{-(t-t_0)|\xi|^2}|\widehat{u}(t_0,\xi)|^2\mathrm{d}\xi + g^5(t)\Big(\int_0^t \|\mathcal{F}_x(u\otimes u)(t')\|_{L^\infty_\xi}\mathrm{d}t'\Big)^2\\ &+ g^4(t)\Big(\int_0^t \|\mathcal{F}_x((\mu(\theta)-1)d)(t')\|_{L^6_\xi}\mathrm{d}t'\Big)^2.\end{aligned} \tag{5.34}$$

We now estimate term by term above. It is easy to observe that

$$\int_{S(t)} \mathrm{e}^{-(t-t_0)|\xi|^2}|\widehat{u}(t_0,\xi)|^2\mathrm{d}\xi \leq \langle t\rangle^{-2\delta}\|u(t_0)\|_{\dot{H}^{-2\delta}}^2,$$

and

$$\begin{aligned}\Big(\int_{t_0}^t \|\mathcal{F}_x(u\otimes u)(t')\|_{L^\infty_\xi}\mathrm{d}t'\Big)^2 &\leq \Big(\int_{t_0}^t \|(u\otimes u)(t')\|_{L^1}\mathrm{d}t'\Big)^2\\ &\leq \Big(\int_{t_0}^t \|u(t')\|_{L^2}^2\mathrm{d}t'\Big)^2 \leq \|u\|_{L^\infty(t_0,t;L^2)}^4(t-t_0)^2,\end{aligned}$$

and

$$\begin{aligned}\Big(\int_{t_0}^t \|\mathcal{F}_x((\mu(\theta)-1)d)(t')\|_{L^6_\xi}\mathrm{d}t'\Big)^2 &\leq \Big(\int_{t_0}^t \|((\mu(\theta)-1)d)(t')\|_{L^{\frac{6}{5}}}\mathrm{d}t'\Big)^2\\ &\leq \|\mu(\theta)-1\|_{L^\infty(t_0,t;L^3)}^2\|\nabla u\|_{L^2(t_0,t;L^2)}^2(t-t_0).\end{aligned}$$

Resuming the above estimates into (5.34) and using (2.4) yields

$$\int_{S(t)} |\widehat{u}(t,\xi)|^2\mathrm{d}\xi \leq C(\langle t\rangle^{-2\delta}\|u(t_0)\|_{\dot{H}^{-2\delta}}^2 + \|u(t_0)\|_{L^2}^2(\|\mu(\theta_0)-1\|_{L^3}^2 t g^4(t) + \|u(t_0)\|_{L^2}^2 t^2 g^5(t))).$$

Inserting the resulting inequality into (5.33), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + g^2(t)\|u(t)\|_{L^2}^2 \leq C\overline{C}(g^2(t)\langle t\rangle^{-2\delta} + g^6(t)\langle t\rangle + g^7(t)\langle t\rangle^2) \tag{5.35}$$

for $\overline{C}$ given by (5.31).

As $g(t) \lesssim \langle t\rangle^{-\frac{1}{2}}$, in the case when $\delta \in]0, \frac{1}{4}[$, we deduce from (5.35) that

$$\exp\Big(\int_{t_0}^t g^2(t')\mathrm{d}t'\Big)\|u(t)\|_{L^2}^2 \le \|u(t_0)\|_{L^2}^2 + C\overline{C}\int_{t_0}^t \exp\Big(\int_{t_0}^{t'} g^2(\tau)\mathrm{d}\tau\Big) g^2(t')\langle t'\rangle^{-2\delta}\mathrm{d}t'.$$

Taking $g^2(t) = \alpha\langle t\rangle^{-1}$ (with $\alpha > 2\delta$) in the above inequality, we infer

$$\langle t\rangle^{\alpha}\|u(t)\|_{L^2}^2 \le C\overline{C}(1 + \langle t\rangle^{\alpha-2\delta}). \tag{5.36}$$

Dividing (5.36) by $\langle t\rangle^{\alpha}$ yields (5.31) for $\delta \in]0, \frac{1}{4}]$.

In the case when $\delta \in]\frac{1}{4}, \frac{1}{2}[$, we already have

$$\|u(t)\|_{L^2}^2 \le C\overline{C}\langle t\rangle^{-\frac{1}{2}} \quad \text{for } t \in [t_0, T^*[,$$

from which, we infer

$$\Big(\int_{t_0}^t \|\mathcal{F}_x(u\otimes u)(t')\|_{L^\infty_\xi}\mathrm{d}t'\Big)^2 \le \Big(\int_{t_0}^t \|u(t')\|_{L^2}^2\mathrm{d}t'\Big)^2 \le C\overline{C}^2\langle t\rangle.$$

Hence a similar derivation of (5.35) leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + g^2(t)\|u(t)\|_{L^2}^2 &\le C(\overline{C}g^2(t)\langle t\rangle^{-2\delta} + (g^6(t) + \overline{C}^2 g^7(t))\langle t\rangle)\\ &\le C\overline{C}^2 g^2(t)\langle t\rangle^{-2\delta},\end{aligned}$$

where in the last step, we used once again that $g(t) \lesssim \langle t\rangle^{-\frac{1}{2}}$. This ensures (5.31) for $\delta \in]\frac{1}{4}, \frac{1}{2}]$, and we completes the proof of Lemma 5.1.

Proposition 5.4 *Under the assumptions of Proposition* 5.1*, we have*

$$\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2(t_0,t;L^2)} + \|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}\nabla u\|_{L^\infty(t_0,t;L^2)} + \|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}\partial_t u\|_{L^2(t_0,t;L^2)} \le C\overline{C}^2 \tag{5.37}$$

for any $t \in [t_0, T^*[$ and $\overline{C}$ given by (5.31).

Proof Multiplying (5.32) by $\langle t\rangle^{2\delta_-}$ and then integrating the resulting inequality over $[t_0, t]$, we get, by applying (5.31), that

$$\begin{aligned}&\|\langle t\rangle^{\delta_-}u\|_{L^\infty(t_0,t;L^2)}^2 + 2\underline{\mu}\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2(t_0,t;L^2)}^2\\ &\lesssim \|u(t_0)\|_{L^2}^2 + \int_{t_0}^t \langle t'\rangle^{2\delta_- -1}\|u(t')\|_{L^2}^2\mathrm{d}t' \le C\overline{C}^2.\end{aligned} \tag{5.38}$$

On the other hand, by multiplying (5.19) by $\langle t\rangle^{(1+2\delta)_-}$, we deduce, by a similar derivation of (5.26), that

$$\begin{aligned}&\|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}\nabla u\|_{L^\infty(t_0,t;L^2)}^2 + \|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}\partial_t u\|_{L^2(t_0,t;L^2)}^2\\ &\lesssim \|\nabla u(t_0)\|_{L^2}^2 + \|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2(t_0,t;L^2)}^2 + \int_{t_0}^t \|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}\nabla u\|_{L^2}^2\|\nabla u\|_{L^3}^2\mathrm{d}t'.\end{aligned}$$

Applying Gronwall's lemma and using (2.1) gives rise to

$$\begin{aligned}&\|\langle t'\rangle^{(\frac12+\delta)_-}\nabla u\|^2_{L^\infty(t_0,t;L^2)}+\|\langle t'\rangle^{(\frac12+\delta)_-}\partial_t u\|^2_{L^2(t_0,t;L^2)}\\&\le C(\|\nabla u(t_0)\|^2_{L^2}+\|\langle t'\rangle^{\delta_-}\nabla u\|^2_{L^2(t_0,t;L^2)})\exp(C\|u_0\|_{\dot B^0_{3,1}}).\end{aligned}$$

This together with (2.3) and (5.38) implies (5.37), and we completes the proof of the proposition.

We now present the key estimate of this section.

Proposition 5.5 *Under the assumptions of Theorem* 1.1*, for any* $p\in[6,8]$ *satisfying*

$$\frac14-\frac{3}{2p}<\delta, \tag{5.39}$$

and $\overline{C}$ *given by* (5.31)*, we have* (2.6)*.*

Proof We first deduce from (5.25) that

$$\begin{aligned}&\|\langle t'\rangle^{(\frac12+\delta)_-}\nabla u\|_{L^2(t_0,t;L^6)}\\&\le C(\|\langle t'\rangle^{(\frac12+\delta)_-}\partial_t u\|_{L^2(t_0,t;L^2)}+\|\langle t'\rangle^{(\frac12+\delta)_-}\nabla u\|_{L^\infty(t_0,t;L^2)}\|\nabla u\|_{L^2(t_0,t;L^3)})\\&\le C\overline{C}(1+\|u_0\|_{\dot B^0_{3,1}})\le C\overline{C},\end{aligned}$$

from which, (2.1) and Hölder's inequality, we get

$$\begin{aligned}\|\langle t'\rangle^{\frac{3}{p-3}(\frac12+\delta)_-}\nabla u\|_{L^{\frac{4p(p-3)}{4p^2-21p+18}}(t_0,t;L^p)}&\le C\|\langle t'\rangle^{(\frac12+\delta)_-}\nabla u\|^{\frac{3}{p-3}}_{L^2_t(L^6)}\|\nabla u\|^{1-\frac{3}{p-3}}_{L^{\frac{4p}{4p-3}}(t_0,t;L^{2p})}\\&\le C_p\overline{C}^{\frac{3}{p-3}}\|u_0\|^{\frac{p-6}{p-3}}_{\dot B^0_{3,1}}.\end{aligned}$$

On the other hand, since p satisfies (5.39), we get, by applying Hölder's inequality, that

$$\begin{aligned}\|\nabla u\|_{L^1(t_0,t;L^p)}&\lesssim\|\langle t'\rangle^{-\frac{3}{p-3}(\frac12+\delta)_-}\|_{L^{\frac{4p(p-3)}{9(p-2)}}(t_0,t)}\|\langle t'\rangle^{\frac{3}{p-3}(\frac12+\delta)_-}\nabla u\|_{L^{\frac{4p(p-3)}{4p^2-21p+18}}(t_0,t;L^p)}\\&\le C\overline{C}^{\frac{3}{p-3}}\|u_0\|^{\frac{p-6}{p-3}}_{\dot B^0_{3,1}}\quad\text{for } t\in]t_0,T^*[.\end{aligned}$$

This together with (2.1) concludes the proof of the proposition.

5.4 The $L^1(\mathbb{R}^+;\dot B^1_{\infty,1})$ estimate for the velocity field

The goal of the this section is to present the a priori $L^1(\mathbb{R}^+;\dot B^1_{\infty,1}(\mathbb{R}^3))$ estimate for the velocity field, which is the most important ingredient used in the proof of Theorem 1.1.

Lemma 5.2 *Let* $\epsilon\in]0,\frac12[$*, let* $(\theta,u,\nabla\Pi)$ *be a smooth enough solution of* (1.4) *on* $[0,T^*[$*. Then under the assumptions of Theorem* 1.1*, we have*

$$\|u\|_{\widetilde L^1(t_0,t;\dot B^{\frac32+\epsilon}_{6,\infty})}\le C(\|u(t_0)\|_{H^1}(1+\|u(t_0)\|^{1-2\epsilon}_{H^1}\|u_0\|^{2\epsilon}_{\dot B^0_{3,1}})+\overline{C}\|\theta\|_{L^\infty_t(B^{\frac12+\epsilon}_{\infty,\infty})}) \tag{5.40}$$

for any $t\in]t_0,T^*[$ *and* $\overline{C}$ *given by* (5.31)*.*

Proof Applying $\dot\Delta_j$ to (5.2) and using a standard commutator's process, we write

$$\begin{aligned}&\partial_t\dot\Delta_j u+u\cdot\nabla\dot\Delta_j u-\Delta\dot\Delta_j u-2\mathrm{div}((\mu(\theta)-1)\mathbb{P}d(\dot\Delta_j u))\\&=-[\dot\Delta_j\mathbb{P};u\cdot\nabla]u+2\mathrm{div}[\dot\Delta_j\mathbb{P};\mu(\theta)]d.\end{aligned}\tag{5.41}$$

Throughout this paper, we always denote $d(v)\stackrel{\text{def}}{=}(\frac{1}{2}(\partial_i v_j+\partial_j v_i))_{3\times 3}$, and abbreviate $d(u)$ as d.

Taking L^2 inner product of (5.41) with $|\dot\Delta_j u|^4\dot\Delta_j u$ and using Lemma 3.1, we obtain

$$\begin{aligned}&\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\dot\Delta_j u|^6\mathrm{d}x-\int_{\mathbb{R}^3}\Delta\Delta_j u\mid|\dot\Delta_j u|^4\dot\Delta_j u\mathrm{d}x\\&\le\|\dot\Delta_j u\|_{L^6}^5\big(C2^j\|(\mu(\theta)-1)\mathbb{P}d(\dot\Delta_j u)\|_{L^6}+\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^6}+C2^j\|[\dot\Delta_j\mathbb{P},\mu(\theta)]d\|_{L^6}\big).\end{aligned}\tag{5.42}$$

Using integration by parts and [14, Lemma A.5], one has

$$\begin{aligned}-\int_{\mathbb{R}^2}\Delta\dot\Delta_j u\mid|\dot\Delta_j u|^4\dot\Delta_j u\mathrm{d}x&=\int_{\mathbb{R}^2}|\dot\Delta_j\nabla u|^2|\dot\Delta_j u|^4\mathrm{d}x\\&\quad+4\int_{\mathbb{R}^2}|\dot\Delta_j u|^4(\nabla|\dot\Delta_j u|)^2\mathrm{d}x\ge 2c2^{2j}\|\dot\Delta_j u\|_{L^6}^6\end{aligned}$$

for some uniform positive constant c.

Whereas it follows from Lemma 3.1 that

$$\|(\mu(\theta)-1)\mathbb{P}d(\dot\Delta_j u)\|_{L^6}\lesssim 2^j\|\mu(\theta)-1\|_{L^\infty}\|\dot\Delta_j u\|_{L^6}\lesssim 2^j\|\mu(\theta_0)-1\|_{L^\infty}\|\dot\Delta_j u\|_{L^6}.$$

Thus, by taking ε_0 sufficiently small in (1.6), we deduce from (5.42) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j u\|_{L^6}+c2^{2j}\|\dot\Delta_j u\|_{L^6}\le C(\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^6}+2^j\|[\dot\Delta_j\mathbb{P},\mu(\theta)]d\|_{L^6}).$$

This gives rise to

$$\begin{aligned}\|\dot\Delta_j u(t)\|_{L^6}&\le\mathrm{e}^{-c2^{2j}(t-t_0)}\|\dot\Delta_j u(t_0)\|_{L^6}+C\int_{t_0}^t\mathrm{e}^{-c2^{2j}(t-t')}(\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^6}\\&\quad+2^j\|[\dot\Delta_j\mathbb{P},\mu(\theta)]d\|_{L^6})(t')\mathrm{d}t'.\end{aligned}$$

Then by virtue of Definition 3.2, we infer

$$\begin{aligned}\|u\|_{\widetilde{L}^1(t_0,t;\dot B^{\frac{3}{2}+\epsilon}_{6,\infty})}&\le\|u(t_0)\|_{\dot B^{\frac{1}{2}+\epsilon}_{2,\infty}}+C\Big(\sup_{j\in\mathbb{Z}}2^{j(-\frac{1}{2}+\epsilon)}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1(t_0,t;L^6)}\\&\quad+\sup_{j\in\mathbb{Z}}2^{j(\frac{1}{2}+\epsilon)}\|[\dot\Delta_j\mathbb{P};\mu(\theta)]d\|_{L^1(t_0,t;L^6)}\Big).\end{aligned}\tag{5.43}$$

In what follows, we shall handle term by term the right-hand side of (5.43). Firstly applying Bony's decomposition (3.5) yields

$$[\dot\Delta_j\mathbb{P};u\cdot\nabla]u=[\dot\Delta_j\mathbb{P};T_u\cdot\nabla]u+\dot\Delta_j\mathbb{P}T'_{\nabla u}u-T'_{\nabla\dot\Delta_j u}u.$$

Applying [29, Lemma 1] gives

$$\begin{aligned}2^{j(-\frac{1}{2}+\epsilon)}\|[\dot\Delta_j\mathbb{P};T_u\cdot\nabla]u\|_{L^6}&\lesssim 2^{j(-\frac{1}{2}+\epsilon)}\sum_{|j-\ell|\le 4}\|\nabla\dot S_{\ell-1}u\|_{L^\infty}\|\dot\Delta_\ell u\|_{L^6}\\&\lesssim\|\nabla u\|_{L^6}\|u\|_{\dot B^{\frac{1}{2}+\epsilon}_{3,1}}.\end{aligned}$$

And by applying Lemma 3.1, one has

$$2^{j(-\frac{1}{2}+\epsilon)}\|\dot\Delta_j\mathbb{P}T'_{\nabla u}u\|_{L^6}\lesssim 2^{j(\frac{1}{2}+\epsilon)}\sum_{\ell\geq j-3}\|\dot\Delta_\ell u\|_{L^3}\|\dot S_{\ell+2}\nabla u\|_{L^6}$$
$$\lesssim\|\nabla u\|_{L^6}\|u\|_{\dot B^{\frac{1}{2}+\epsilon}_{3,1}}.$$

The same estimate holds for $T'_{\nabla\dot\Delta_j u}u$. Hence, thanks to the interpolation inequality

$$\|a\|_{\dot B^{\frac{1}{2}+\epsilon}_{3,1}}\lesssim\|\nabla a\|_{L^2}^{1-2\epsilon}\|\nabla a\|_{L^3}^{2\epsilon},$$

and Corollary 5.2 implies

$$\sup_{j\in\mathbb{Z}}2^{j(-\frac{1}{2}+\epsilon)}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1(t_0,t;L^6)}\leq C\|\nabla u\|_{L^2(t_0,t;L^6)}\|\nabla u\|^{1-2\epsilon}_{L^2(t_0,t;L^2)}\|\nabla u\|^{2\epsilon}_{L^2(t_0,t;L^3)}$$
$$\leq C\|u(t_0)\|_{H^1}^{2(1-\epsilon)}\|u_0\|^{2\epsilon}_{\dot B^0_{3,1}}. \tag{5.44}$$

The same process along with (2.6) for $p=6$ ensures

$$\sup_{j\in\mathbb{Z}}2^{j(\frac{1}{2}+\epsilon)}\|[\dot\Delta_j\mathbb{P},\mu(\theta)]d\|_{L^1_t(L^6)}\leq C\|\theta\|_{L^\infty_t(B^{\frac{1}{2}+\epsilon}_{\infty,\infty})}\|\nabla u\|_{L^1_t(L^6)}$$
$$\leq C\overline{C}\|\theta\|_{L^\infty_t(B^{\frac{1}{2}+\epsilon}_{\infty,\infty})}. \tag{5.45}$$

Substituting (5.44) and (5.45) into (5.43) leads to (5.40), and we complete the proof of Lemma 5.2.

With Lemma 5.2, we can prove the a priori $L^1(\mathbb{R}^+;\dot B^1_{\infty,1}(\mathbb{R}^3))$ estimate for u.

Proposition 5.6 *Under the assumptions of Theorem* 1.1*, one has* (2.7) *for any* $t\in[0,T^*[$.

Proof By virtue of (2.2), we only need to prove the estimate of $\|u\|_{L^1(t_0,t;\dot B^1_{\infty,1})}$ for any $t<T^*$. As a matter of fact, for any integer N and $p\in]6,8]$ satisfying (5.39), we deduce from Lemma 3.1 and Lemma 5.2 that

$$\|u\|_{L^1(t_0,t;\dot B^1_{\infty,1})}\leq\|\nabla u\|_{L^1(t_0,t;L^p)}+\sum_{0<j\leq N}2^{\frac{3j}{p}}\|\dot\Delta_j\nabla u\|_{L^1(t_0,t;L^p)}+\sum_{j\geq N}2^{\frac{j}{2}}\|\dot\Delta_j\nabla u\|_{L^1(t_0,t;L^6)}$$
$$\leq C(2^{\frac{3N}{p}}\|\nabla u\|_{L^1(t_0,t;L^p)}+2^{-N\varepsilon}\|u\|_{\widetilde L^1(t_0,t;\dot B^{\frac{3}{2}+\epsilon}_{6,\infty})})$$
$$\leq C(\|u(t_0)\|_{H^1}(1+\|u(t_0)\|^{1-2\varepsilon}_{H^1}\|u_0\|^{2\varepsilon}_{\dot B^0_{3,1}})$$
$$+2^{\frac{3N}{p}}\|\nabla u\|_{L^1_t(L^p)}+\overline{C}2^{-N\varepsilon}\|\theta\|_{\widetilde L^\infty(t_0,t;\dot B^{\frac{1}{2}+\varepsilon}_{\infty,\infty})}). \tag{5.46}$$

However as

$$\|\theta\|_{\widetilde L^\infty(t_0,t;\dot B^{\frac{1}{2}+\varepsilon}_{\infty,\infty})}\leq\|\theta(t_0)\|_{\dot B^{\frac{1}{2}+\varepsilon}_{\infty,\infty}}\exp(C\|u\|_{L^1(t_0,t;\dot B^1_{\infty,1})}), \tag{5.47}$$

we get, by taking

$$N\sim\frac{C}{\varepsilon\ln 2}\|u\|_{L^1(t_0,t;\dot B^1_{\infty,1})}$$

in (5.46), that

$$\|u\|_{L^1(t_0,t;\dot B^1_{\infty,1})} \le C(\|u(t_0)\|_{H^1}(1+\|u(t_0)\|_{H^1}^{1-2\varepsilon}\|u_0\|_{\dot B^0_{3,1}}^{2\varepsilon}) + \overline{C}\|\theta_0\|_{B^{\frac12+\varepsilon}_{\infty,\infty}} + \|\nabla u\|_{L^1(t_0,t;L^p)}\exp(C\|u\|_{L^1_t(\dot B^1_{\infty,1})})). \tag{5.48}$$

Therefore, in view of (2.3) and (2.6), whenever $\|u_0\|_{\dot B^0_{3,1}}$ is so small that

$$C\overline{C}^{\frac{3}{p-3}}\|u_0\|_{\dot B^0_{3,1}}^{\frac{p-6}{p-3}}\exp(2C^2\|u(t_0)\|_{H^1}(1+\|u(t_0)\|_{H^1}^{1-2\varepsilon}\|u_0\|_{\dot B^0_{3,1}}^{2\epsilon}) + 2C^2\overline{C}\|\theta_0\|_{B^{\frac12+\varepsilon}_{\infty,\infty}}) \le \frac{C}{2}(\|u(t_0)\|_{H^1}(1+\|u(t_0)\|_{H^1}^{1-2\varepsilon}\|u_0\|_{\dot B^0_{3,1}}^{2\epsilon}) + \overline{C}\|\theta_0\|_{B^{\frac12+\varepsilon}_{\infty,\infty}}), \tag{5.49}$$

we infer from (5.48) that

$$\|u\|_{L^1(t_0,t;\dot B^1_{\infty,1})} \le 2C(\|u(t_0)\|_{H^1}(1+\|u(t_0)\|_{H^1}^{1-2\varepsilon}\|u_0\|_{\dot B^0_{3,1}}^{2\varepsilon}) + \overline{C}\|\theta_0\|_{B^{\frac12+\varepsilon}_{\infty,\infty}}),$$

which together with (2.2) leads to (2.7). This completes the proof of the proposition.

6 The Proof of Theorem 1.1

6.1 Existence part of Theorem 1.1

The proof to the existence part of Theorem 1.1 basically follows the following strategy. We begin by solving an appropriate approximate problem, and then we provide uniform estimates for such approximate solutions, and finally we prove the convergence of such approximate solution sequence to a solution of (1.4) through a standard compactness argument.

Lemma 6.1 *Let $(\theta, u, \nabla\Pi)$ be a smooth enough solution of (1.4) on $[0,T^*[$. Then under the assumption that*

$$\|\mu(\theta_0)-1\|_{L^\infty} \le \varepsilon_0 \tag{6.1}$$

for ε_0 sufficiently small, one has for any $t<T^*$,

$$\|u\|_{\widetilde L^\infty_t(\dot B^0_{3,1})} + \|\theta\|_{\widetilde L^\infty_t(\dot B^1_{3,1})} + \|u\|_{L^1_t(\dot B^2_{3,1})} + \|\nabla\Pi\|_{L^1_t(\dot B^0_{3,1})} \le C(\|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|_{\dot B^1_{3,1}} + \sqrt{t}\|u_0\|_{\dot B^0_{3,1}}\|\theta_0\|^2_{\dot B^{\frac12}_{\infty,\infty}})\exp(C\|\nabla u\|_{L^1_t(L^\infty)}). \tag{6.2}$$

Proof We first notice that u solves

$$\begin{cases}\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\Pi = 2\mathrm{div}((\mu(\theta)-1)d),\\ \mathrm{div}\, u = 0,\\ u|_{t=0} = u_0,\end{cases}$$

from which, we get, by using Proposition 4.2, that

$$\|u\|_{\widetilde L^\infty_t(\dot B^0_{3,1})} + \|u\|_{L^1_t(\dot B^2_{3,1})} + \|\nabla\Pi\|_{L^1_t(\dot B^0_{3,1})} \lesssim \|u_0\|_{\dot B^0_{3,1}} + \int_0^t \|\nabla u(t')\|_{L^\infty}\|u(t')\|_{\dot B^0_{3,1}}\mathrm{d}t' + \|(\mu(\theta)-1)d\|_{L^1_t(\dot B^1_{3,1})},$$

whereas it follows from product laws in Besov spaces that

$$\|(\mu(\theta)-1)d\|_{L^1_t(\dot{B}^1_{3,1})} \lesssim \|\mu(\theta)-1\|_{L^\infty_t(L^\infty)}\|u\|_{L^1_t(\dot{B}^2_{3,1})} + \int_0^t \|\nabla u(t')\|_{L^\infty}\|(\mu(\theta)-1)(t')\|_{\dot{B}^1_{3,1}}\mathrm{d}t',$$

which together with (6.1) ensures that

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})} + \|u\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\Pi\|_{L^1_t(\dot{B}^0_{3,1})}\\ &\leq C\Big(\|u_0\|_{\dot{B}^0_{3,1}} + \int_0^t \|\nabla u(t')\|_{L^\infty}(\|u(t')\|_{\dot{B}^0_{3,1}} + \|\theta(t')\|_{\dot{B}^1_{3,1}})\mathrm{d}t'\Big).\end{aligned} \tag{6.3}$$

On the other hand, we get, by first applying the operator $\dot{\Delta}_j$ to the θ equation of (1.4) and then taking the L^2 inner product of the resulting equation with $|\dot{\Delta}_j\theta|\dot{\Delta}_j\theta$, that

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j\theta(t)\|^3_{L^3} + (u\cdot\nabla\dot{\Delta}_j\theta \mid |\dot{\Delta}_j\theta|\dot{\Delta}_j\theta)_{L^2} + ([\dot{\Delta}_j; u]\nabla\theta \mid |\dot{\Delta}_j\theta|\dot{\Delta}_j\theta)_{L^2} = 0.$$

As $\operatorname{div} u = 0$, by applying (2) of (3.6) for $\alpha = \frac{1}{2}$, we write

$$\begin{aligned}\|\dot{\Delta}_j\theta\|_{L^\infty_t(L^3)} &\leq \|\dot{\Delta}_j\theta_0\|_{L^3} + \|[\dot{\Delta}_j; u]\nabla\theta\|_{L^1_t(L^3)}\\ &\leq Cd_j 2^{-j}\Big(\|\theta_0\|_{\dot{B}^1_{3,1}} + \int_0^t (\|\nabla u(t')\|_{L^\infty}\|\theta(t')\|_{\dot{B}^1_{3,1}} + \|u(t')\|_{\dot{B}^{\frac{3}{2}}_{3,1}}\|\theta(t')\|_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}})\mathrm{d}t'\Big),\end{aligned}$$

which together with

$$\|a\|_{\dot{B}^{\frac{3}{2}}_{3,1}} \lesssim \|a\|^{\frac{1}{2}}_{\dot{B}^2_{3,1}}\|\nabla a\|^{\frac{1}{2}}_{L^3}$$

and Definition 3.2 implies that

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})} \leq C\Big(&\|\theta_0\|_{\dot{B}^1_{3,1}} + \int_0^t (\|\nabla u(t')\|_{L^\infty}\|\theta(t')\|_{\dot{B}^1_{3,1}}\\ &+ \|\nabla u(t')\|_{L^3}\|\theta(t')\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}})\mathrm{d}t'\Big) + \frac{1}{2}\|u\|_{L^1_t(\dot{B}^2_{3,1})}.\end{aligned} \tag{6.4}$$

Summing up (6.3) and (6.4), we obtain

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})} + \|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})} + \|u\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\Pi\|_{L^1_t(\dot{B}^0_{3,1})}\\ &\leq C\Big(\|u_0\|_{\dot{B}^0_{3,1}} + \|\theta_0\|_{\dot{B}^1_{3,1}} + \int_0^t \|\nabla u(t')\|_{L^\infty}(\|u(t')\|_{\dot{B}^0_{3,1}} + \|\theta(t')\|_{\dot{B}^1_{3,1}})\mathrm{d}t'\\ &\quad + \int_0^t \|\nabla u(t')\|_{L^3}\|\theta(t')\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}\mathrm{d}t'\Big).\end{aligned} \tag{6.5}$$

Yet due to (2.1) for $p = 6$, we have

$$\|\nabla u\|_{L^2_t(L^3)} \leq C\|u_0\|_{\dot{B}^0_{3,1}}.$$

Then applying Gronwall's lemma to (6.5) and using the fact that

$$\|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{1}{2}}_{\infty,\infty})} \leq \|\theta_0\|_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}\exp(C\|\nabla u\|_{L^1_t(L^\infty)}),$$

we conclude the proof of (6.2).

Now let us recall the following lemma from [1].

Lemma 6.2 (see [1, Lemma 4.2]) *Let $s\in\mathbb{R}$ and $(p,r)\in[1,\infty[^2$. Let $G\in\dot B^s_{p,r}(\mathbb{R}^3)$. Then there exists $G^n\in H^\infty(\mathbb{R}^3)$, such that for all $\varepsilon>0$, there exists some $n_0\in\mathbb{N}$ such that*

$$\|G^n-G\|_{\dot B^s_{p,r}}\le\varepsilon\quad\text{for all}\ \ n\ge n_0.$$

If moreover $G=(G_1,G_2,G_3)$ and $\operatorname{div}G=0$, we can choose $G^n=(G_1^n,G_2^n,G_3^n)$ so that $\operatorname{div}G^n=0$.

Proof (The Existence Part of Theorem 1.1) By virtue of Lemma 6.2, we can find θ_0^n, $u_0^n\in H^\infty(\mathbb{R}^3)$ for $n\in\mathbb{N}$ so that

$$\begin{aligned}&\|\mu(\theta_0^n)-1\|_{L^\infty}\lesssim\|\mu(\theta_0)-1\|_{L^\infty}+\|\theta_0^n-\theta_0\|_{\dot B^1_{3,1}}\lesssim\varepsilon_0,\\&\|\theta_0^n\|_{\dot B^{(\frac12)_+}_{\infty,\infty}}\lesssim\|\theta_0\|_{\dot B^{(\frac12)_+}_{\infty,\infty}},\quad\|u_0^n\|_{\dot B^0_{3,1}}\le2\varepsilon\quad\text{and}\quad\operatorname{div}u_0^n=0.\end{aligned}\tag{6.6}$$

Then according to [1, Theorem 1.1], we deduce that the System (1.4) with the initial data (θ_0^n,u_0^n) admits a unique local in time solution $(\theta^n,u^n,\nabla\Pi^n)$ on $[0,T_n^*[$ verifying

$$\begin{aligned}&\theta^n\in\mathcal{C}([0,T_n^*[;\,H^{s+1}(\mathbb{R}^3)),\quad u^n\in\mathcal{C}([0,T_n^*[;\,H^s(\mathbb{R}^3))\cap\widetilde L^1_{T^n_*}(H^{s+2}),\\&\text{and}\quad\nabla\Pi^n\in L^1([0,T_n^*[;\,H^s(\mathbb{R}^3))\quad\text{for any }s>\frac12.\end{aligned}$$

Moreover, whenever $\varepsilon_0,\varepsilon$ are small enough in (6.6), we deduce from Proposition 5.6 that there exists a positive constant C_0, which depends on $\|\theta_0\|_{B^1_{3,1}\cap B^{(\frac12)_+}_{\infty,\infty}}$ and $\|u_0\|_{\dot H^{-2\delta}\cap\dot B^0_{3,1}}$ so that

$$\|u^n\|_{L^1([0,t];\dot B^1_{\infty,1})}\le C_0\quad\text{for any }t<T_n^*,\tag{6.7}$$

from which and (5.47), Lemma 6.1, we infer that $(\theta^n,u^n,\nabla\Pi^n)$ is uniformly bounded in $(\widetilde L^\infty_t(B^1_{3,1})\cap\widetilde L^\infty_t(B^{(\frac12)_+}_{\infty,\infty}))\times(\widetilde L^\infty_t(\dot B^0_{3,1})\cap L^1_t(\dot B^2_{3,1}))\times L^1_t(\dot B^0_{3,1})$ for any fixed $t<T_n^*$. This implies that $T_n^*=\infty$. To prove that there is a subsequence of $\{(\theta^n,u^n,\nabla\Pi^n)\}_{n\in\mathbb{N}}$, which converges to a solution $(\theta,u,\nabla\Pi)$ of (1.4), which satisfies (1.7), we only need to use a standard compactness argument of Lions-Aubin's lemma. Since this argument is rather classical, we shall not present the details here. One may check similar argument from page 582 to page 583 of [1] for details.

6.2 Uniqueness part of Theorem 1.1

Let $(\theta_i,u_i,\nabla\Pi_i)$, for $i=1,2$ be two solutions of (1.4), which satisfy (1.7). We denote

$$(\delta\theta,\delta u,\nabla\delta\Pi)\stackrel{\text{def}}{=}(\theta_2-\theta_1,u_2-u_1,\nabla\Pi_2-\nabla\Pi_1).$$

Then due to (1.4), the system for $(\delta\theta,\delta u,\delta\nabla\Pi)$ reads

$$\begin{cases}\partial_t\delta\theta+u_2\cdot\nabla\delta\theta=-\delta u\cdot\nabla\theta_1,\\\partial_t\delta u+(u_2\cdot\nabla)\delta u-\Delta\ u+\nabla\delta\Pi=\delta F,\\\operatorname{div}\delta u=0,\\(\delta\theta,\delta u)|_{t=0}=(0,0),\end{cases}\tag{6.8}$$

where δF is determined by

$$\delta F\stackrel{\text{def}}{=}-(\delta u\cdot\nabla)u_1+2\operatorname{div}((\mu(\theta_2)-\mu(\theta_1))d(u_1))+2\operatorname{div}((\mu(\theta_2)-1)d(\delta u)).\tag{6.9}$$

We first deduce from the transport equation of (6.8) that

$$\|\delta\theta\|_{L^\infty_t(L^3)} \le \|\delta u\|_{L^1_t(L^\infty)}\|\nabla\theta_1\|_{L^\infty_t(L^3)} \le C\|\theta_1\|_{L^\infty_t(\dot B^1_{3,1})}\|\delta u\|_{L^1_t(\dot B^1_{3,1})}. \tag{6.10}$$

Yet notice that

$$\|\delta u\|_{L^1_t(\dot B^1_{3,1})} \le C\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}\log\Big(\mathrm{e}+\frac{\|\delta u\|_{\widetilde L^1_t(\dot B^0_{3,\infty})}+\|\delta u\|_{\widetilde L^1_t(\dot B^2_{3,\infty})}}{\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}}\Big)$$

and

$$\|\delta u\|_{\widetilde L^1_t(\dot B^0_{3,\infty})} \le t\sum_{i=1}^2\|u_i\|_{L^\infty_t(\dot B^0_{3,1})} \quad\text{and}\quad \|\delta u\|_{\widetilde L^1_t(\dot B^2_{3,\infty})} \le \sum_{i=1}^2\|u_i\|_{L^1_t(\dot B^2_{3,1})},$$

we infer

$$\|\delta\theta\|_{L^\infty_t(L^3)} \le C\|\theta_1\|_{L^\infty_t(\dot B^1_{3,1})}\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}\log\Big(\mathrm{e}+\frac{\alpha(t)}{\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}}\Big) \tag{6.11}$$

with

$$\alpha(t) \stackrel{\text{def}}{=} t\sum_{i=1}^2\|u_i\|_{L^\infty_t(\dot B^0_{3,1})}+\sum_{i=1}^2\|u_i\|_{L^1_t(\dot B^2_{3,1})}.$$

On the other hand, it follows from (6.8) and Proposition 4.2 that

$$\|\delta u\|_{\widetilde L^\infty_t(\dot B^{-1}_{3,\infty})}+\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}+\|\nabla\delta\Pi\|_{\widetilde L^1_t(\dot B^{-1}_{3,\infty})} \le C\|\delta F\|_{\widetilde L^1_t(\dot B^{-1}_{3,\infty})}\exp(C\|u_2\|_{L^1_t(\dot B^2_{3,1})}). \tag{6.12}$$

While we deduce from (6.9) and standard product laws in Besov spaces (see [7]) that

$$\begin{aligned}\|\delta F\|_{\widetilde L^1_t(\dot B^{-1}_{3,\infty})} &\lesssim \|u_1\|_{L^1_t(\dot B^2_{3,1})}\|\delta u\|_{\widetilde L^\infty_t(\dot B^{-1}_{3,\infty})}+\int_0^t\|\nabla u_1(t')\|_{L^\infty}\|\delta\theta(t')\|_{L^3}\mathrm{d}t'\\ &\quad+\|\mu(\theta_0)-1\|_{L^\infty}\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}+\|\mu(\theta_2)-1\|_{\widetilde L^2_t(\dot B^1_{3,1})}\|\nabla\delta u\|_{\widetilde L^2_t(\dot B^{-1}_{3,\infty})}.\end{aligned}$$

Observing that

$$\|\mu(\theta_2)-1\|_{\widetilde L^2_t(\dot B^1_{3,1})}\|\nabla\delta u\|_{\widetilde L^2_t(\dot B^{-1}_{3,\infty})} \le C\sqrt{t}\|\mu(\theta_2)-1\|_{\widetilde L^\infty_t(\dot B^1_{3,1})}\|\delta u\|^{\frac12}_{\widetilde L^\infty_t(\dot B^{-1}_{3,\infty})}\|\delta u\|^{\frac12}_{\widetilde L^1_t(\dot B^1_{3,\infty})}$$

and $u_1\in L^1_t(\dot B^2_{3,1})$, we can find some positive function $\zeta(t)$ with $\lim\limits_{t\to0}\zeta(t)=0$ so that

$$\begin{aligned}\|\delta F\|_{\widetilde L^1_t(\dot B^{-1}_{3,\infty})} &\le C\int_0^t\|\nabla u_1(t')\|_{L^\infty}\|\delta\theta(t')\|_{L^3}\mathrm{d}t'\\ &\quad+C(\zeta(t)+\|\mu(\theta_0)-1\|_{L^\infty})(\|\delta u\|_{\widetilde L^\infty_t(\dot B^{-1}_{3,\infty})}+\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}).\end{aligned}$$

Resuming the above inequality into (6.12), we obtain for $t\le\mathfrak{t}_1$,

$$\|\delta u\|_{\widetilde L^\infty_t(\dot B^{-1}_{3,\infty})}+\|\delta u\|_{\widetilde L^1_t(\dot B^1_{3,\infty})}+\|\nabla\delta\Pi\|_{\widetilde L^1_t(\dot B^{-1}_{3,\infty})} \le C\int_0^t\|\nabla u_1(t')\|_{L^\infty}\|\delta\theta(t')\|_{L^3}\mathrm{d}t'.$$

As for $\alpha\ge0$ and $x\in(0,1]$, there holds

$$\ln(\mathrm{e}+\alpha x^{-1}) \le (1-\ln x)\ln(\mathrm{e}+\alpha),$$

therefore, by virtue of (6.11), we eventually find for $t \le \mathfrak{t}_1$,

$$\begin{aligned} W(t) &\stackrel{\text{def}}{=} \|\delta u\|_{\widetilde{L}^\infty_t(\dot B^{-1}_{3,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(\dot B^{1}_{3,\infty})} + \|\nabla \delta \Pi\|_{\widetilde{L}^1_t(\dot B^{-1}_{3,\infty})} \\ &\le C\int_0^t \|\nabla u_1(\tau)\|_{L^\infty} \|\delta u\|_{\widetilde{L}^1_\tau(\dot B^{1}_{3,\infty})}(1-\ln \|\delta u\|_{\widetilde{L}^1_\tau(\dot B^{1}_{3,\infty})})\mathrm{d}\tau. \end{aligned} \tag{6.13}$$

Since $x \mapsto x(1-\log x)$ is increasing on $]0,1]$, we obtain for $t \le \mathfrak{t}_1$,

$$W(t) \le C\int_0^t \|\nabla u_1(\tau)\|_{L^\infty} W(\tau)(1-\ln W(\tau))\mathrm{d}\tau. \tag{6.14}$$

Due to

$$\int_0^1 \frac{\mathrm{d}r}{r(1-\ln r)} = +\infty,$$

applying Osgood lemma to (6.14) leads to

$$W(t) = 0 \quad \text{for } t \le \mathfrak{t}_1.$$

This proves the uniqueness part of Theorem 1.1 for $t \le \mathfrak{t}_1$. The global in time uniqueness of solutions to (1.4), which satisfies (1.7), can be obtained by a boot-strap argument. This completes the proof of Theorem 1.1.

7 The a Priori Estimates Related to the System (1.9)

In this section, we shall present the a priori estimates, which we need to prove Theorem 1.2, for the System (1.9). As in Section 5, we shall first present the short time a priori estimates.

7.1 The short time estimates for smooth enough solutions of (1.9)

Proposition 7.1 *Let (θ, u) be a smooth enough solution of* (1.9) *on* $[0, T^*[$. *We assume that $\|\theta_0\|_{L^1}$ and $\|u_0\|_{\dot B^0_{3,1}}$ are so small that*

$$\|\theta_0\|_{L^\infty} \le C\|\theta_0\|_{L^1}^{\frac{\alpha}{3+\alpha}} \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}^{\frac{3}{3+\alpha}} \le \eta_1, \tag{7.1}$$

and that

$$\begin{aligned} &\|\theta_0\|_{L^1} + \|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|_{L^1}^{\frac16}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac56} + \|\theta_0\|_{L^1}^{\frac13}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac23} + \|\theta_0\|_{L^1}^{\frac{\alpha}{6+\alpha}}(\|\theta_0\|_{\dot B^1_{3,1}} \\ &+ \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}})^{\frac{6}{6+\alpha}} \le \eta_2 \end{aligned} \tag{7.2}$$

for some sufficiently small constants η_1, η_2. Then one has (2.8).

Proof We first rewrite the velocity equation of (1.9) as

$$u(t) = \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}(-\mathrm{div}(u\otimes u) + \mathrm{div}(2(\mu(\theta)-1)d) + \theta e_3)(t')\mathrm{d}t', \tag{7.3}$$

applying $\dot\Delta_j$ to the above equation and using Lemmas 3.1 and 3.2, we get

$$\begin{aligned} \|\dot\Delta_j u(t)\|_{L^\infty} \le \mathrm{e}^{-ct2^{2j}}\|\dot\Delta_j u_0\|_{L^\infty} + C\int_0^t \mathrm{e}^{-c(t-t')2^{2j}}(2^j(\|\dot\Delta_j(u\otimes u)\|_{L^\infty} \\ + \|\dot\Delta_j((\mu(\theta)-1)d)\|_{L^\infty}) + \|\dot\Delta_j\theta\|_{L^\infty})(t')\mathrm{d}t', \end{aligned} \tag{7.4}$$

from which, for $p(t) \stackrel{\text{def}}{=} 6(1+\|\nabla u\|_{L^1_t(L^\infty)})$, we get, by a similar derivation of (5.12), that

$$\begin{aligned}\|u\|_{L^1_t(\dot B^1_{\infty,1})} \le \|u_0\|_{\dot B^{-1}_{\infty,1}} &+ C(t\|\theta\|_{L^\infty_t(\dot B^0_{3,1})} + \|u\|_{L^\infty_t(L^{3,\infty})}\|u\|_{L^1_t(\dot B^1_{\infty,1})}\\ &+ \|\theta_0\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{\infty,1})} + \|\theta\|_{L^\infty_t(\dot B^0_{\infty,1})}\|\nabla u\|_{L^1_t(L^\infty)}\\ &+ \|\theta\|_{L^\infty_t(\dot B^{\frac{3}{p(t)}}_{p(t),1})}\|\nabla u\|_{L^1_t(L^\infty)}(1+\|\nabla u\|_{L^1_t(L^\infty)})). \end{aligned}\tag{7.5}$$

Yet it follows from [19] that

$$\|\theta\|_{L^\infty_t(\dot B^0_{p,1})} \le C\|\theta_0\|_{\dot B^0_{p,1}}(1+\|\nabla u\|_{L^1_t(L^\infty)}) \quad \text{for } p=3,\,\infty.$$

And it is easy to observe from the proof of [2, Lemma 3.1] that this lemma still holds for smooth enough solutions of (4.5), so that we have

$$\|\theta\|_{L^\infty_t(\dot B^{\frac{3}{p(t)}}_{p(t),1})} \le C\|\theta_0\|_{\dot B^{\frac{3}{p(t)}}_{p(t),1}}(1+\|\nabla u\|_{L^\infty}).$$

Whereas according to Definition 3.1, for any fixed integer N, one has

$$\begin{aligned}\|\theta_0\|_{\dot B^{\frac{3}{p(t)}}_{p(t),1}} &\le \sum_{j\le N} 2^{3j}\|\dot\Delta_j\theta_0\|_{L^1} + \sum_{j\ge N}(2^j\|\dot\Delta_j\theta_0\|_{L^3})^{\frac{3}{p(t)}}(2^{j\alpha}\|\dot\Delta_j\theta_0\|_{L^\infty})^{1-\frac{3}{p(t)}}2^{-j\alpha(1-\frac{3}{p(t)})}\\ &\le C(2^{3N}\|\theta_0\|_{L^1} + 2^{-N\alpha(1-\frac{3}{p(t)})}(\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}))\\ &\le C(2^{3N}\|\theta_0\|_{L^1} + 2^{-\frac{N\alpha}{2}}(\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}})),\end{aligned}$$

where we used the fact that $1-\frac{3}{p(t)} \ge \frac12$ in the last step. Taking N in the above inequality so that

$$2^{N(3+\frac{\alpha}{2})} \sim \frac{\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}}{\|\theta_0\|_{L^1}},$$

we obtain

$$\|\theta_0\|_{\dot B^{\frac{3}{p(t)}}_{p(t),1}} \le C\|\theta_0\|_{L^1}^{\frac{\alpha}{6+\alpha}}(\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}})^{\frac{6}{6+\alpha}}.$$

Along the same line, we have

$$\|\theta_0\|_{\dot B^0_{3,1}} \le C\|\theta_0\|_{L^1}^{\frac13}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac23} \quad \text{and} \quad \|\theta_0\|_{L^\infty} \le \|\theta_0\|_{\dot B^0_{\infty,1}} \le C\|\theta_0\|_{L^1}^{\frac{\alpha}{3+\alpha}}\|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}^{\frac{3}{3+\alpha}}. \tag{7.6}$$

Hence by virtue of (7.1), we deduce from (7.5) that

$$\begin{aligned}\|u\|_{L^1_t(\dot B^1_{\infty,1})} \le C(&\|u_0\|_{\dot B^0_{3,1}} + t\|\theta_0\|_{L^1}^{\frac13}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac23}(1+\|u\|_{L^1_t(\dot B^1_{\infty,1})})\\ &+ \|u\|_{L^\infty_t(L^{3,\infty})}\|u\|_{L^1_t(\dot B^1_{\infty,1})} + \|\theta_0\|_{L^1}^{\frac{\alpha}{3+\alpha}}\|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}^{\frac{3}{3+\alpha}}\|u\|^2_{L^1_t(\dot B^1_{\infty,1})}\\ &+ \|\theta_0\|_{L^1}^{\frac{\alpha}{6+\alpha}}(\|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}})^{\frac{6}{6+\alpha}}\|u\|_{L^1_t(\dot B^1_{\infty,1})}(1+\|u\|_{L^1_t(\dot B^1_{\infty,1})})^2).\end{aligned}\tag{7.7}$$

To estimate $\|u\|_{L^\infty_1(L^{3,\infty})}$, we need the following lemma, which we admit for the time being.

Lemma 7.1 *Under the assumptions of Proposition* 7.1, *one has*

$$\|u\|_{L^\infty_1(L^{3,\infty})} \le C(\|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|_{L^1} + \|\theta_0\|_{L^1}^{\frac16}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac56}) \le C\eta_2. \tag{7.8}$$

Thanks to (7.2) and (7.8), we infer from (7.7) that

$$\|u\|_{L^1_1(\dot B^1_{\infty,1})}\leq C(\|u_0\|_{\dot B^0_{3,1}}+\|\theta_0\|_{L^1}^{\frac13}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac23}+\eta_2(\|u\|^2_{L^1_1(\dot B^1_{\infty,1})}+\|u\|^3_{L^1_1(\dot B^1_{\infty,1})})),$$

which leads to (2.8), and this completes the proof of Proposition 7.1.

Proposition 7.1 is proved provided that we present the proof of Lemma 7.1, which we give as follows.

Proof of Lemma 7.1 In fact, it follows from (7.3) and [26, Lemma 23] that

$$\begin{aligned}\|u\|_{L^\infty_t(L^{3,\infty})}\lesssim \|u_0\|_{L^{3,\infty}}+\|u\|^2_{L^\infty_t(L^{3,\infty})}+\Big\|\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}\mathrm{div}((\mu(\theta)-1)d)(t')\mathrm{d}t'\Big\|_{L^\infty_t(L^{3,\infty})}\\ +\Big\|\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}(\theta e_3)(t')\mathrm{d}t'\Big\|_{L^\infty_t(L^{3,\infty})}.\end{aligned}\tag{7.9}$$

Applying Lemma 4.2 for $p=6$ yields

$$\begin{aligned}\Big\|\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}\mathrm{div}((\mu(\theta)-1)d)(t')\mathrm{d}t'\Big\|_{L^\infty_t(L^3)}&\leq C\|(\mu(\theta)-1)d\|_{L^{2,1}_t(L^3)}\\ &\leq C\|\mu(\theta)-1\|_{L^\infty_t(L^\infty)}\|\nabla u\|_{L^{2,1}_t(L^3)}.\end{aligned}$$

Whereas due to (7.3), we get, by a similar proof of (5.8) and (5.9), that

$$\|u\|_{L^{4,2}_t(L^6)}\lesssim \|S(t)u_0\|_{L^{4,2}_t(L^6)}+\|u\|^2_{L^4(L^6)}+\|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L^2_t(L^3)}+\||D|^{-1}\theta\|_{L^2_t(L^3)}$$

and

$$\|\nabla u\|_{L^{2,1}_t(L^3)}\lesssim \|\nabla S(t)u_0\|_{L^{2,1}_t(L^3)}+\|u\|^2_{L^{4,2}_t(L^6)}+\|1-\mu(\theta_0)\|_{L^\infty}\|\nabla u\|_{L^{2,1}_t(L^3)}+\||D|^{-1}\theta\|_{L^{2,1}_t(L^3)},$$

so that thanks to (7.1) and (7.6), one has

$$\|u\|_{L^{4,2}_t(L^6)}+\|\nabla u\|_{L^{2,1}_t(L^3)}\leq C(\|u_0\|_{\dot B^0_{3,1}}+\|u\|^2_{L^{4,2}_t(L^6)}+\||D|^{-1}\theta\|_{L^{2,1}_t(L^3)}).$$

Since for $p\geq q$, there holds (see [18])

$$\|1_{[0,t]}\|_{L^{p,q}}=\Big(\int_0^t s^{\frac qp-1}\mathrm{d}s\Big)^{\frac pq}=\Big(\frac pq\Big)^{\frac pq}t,$$

we have

$$\begin{aligned}\||D|^{-1}\theta\|_{L^{2,1}_t(L^3)}\lesssim\|1\|_{L^{2,1}_t}\||D|^{-1}\theta\|_{L^\infty_t(L^3)}&\lesssim \|1_{[0,t]}\|_{L^{2,1}}\|\theta\|_{L^\infty_t(L^1\cap L^3)}\\ &\lesssim \sqrt{t}\|\theta_0\|_{L^1\cap L^3}.\end{aligned}$$

As a result, it comes out

$$\|u\|_{L^{4,2}_t(L^6)}+\|\nabla u\|_{L^{2,1}_t(L^3)}\leq C(\|u_0\|_{\dot B^0_{3,1}}+\|u\|^2_{L^{4,2}_t(L^6)}+\sqrt{t}\|\theta_0\|_{L^1\cap L^3}).$$

In particular, if

$$\|u_0\|_{\dot B^0_{3,1}}+\|\theta_0\|_{L^1\cap L^3}\ll 1,$$

we infer

$$\|u\|_{L_1^{4,2}(L^6)}+\|\nabla u\|_{L_1^{2,1}(L^3)}\lesssim\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|_{L^1\cap L^3}.$$

Finally it is easy to observe that

$$\begin{aligned}\Big\|\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}(\theta e_3)\mathrm{d}t'\Big\|_{L_t^\infty(L^{3,\infty})}&\lesssim\Big\|\int_0^t \mathrm{e}^{(t-t')\Delta}|D||D|^{-1}\mathbb{P}(\theta e_3)\mathrm{d}t'\Big\|_{L_t^\infty(L^{3,\infty})}\\&\lesssim\||D|^{-1}\theta\|_{L_t^\infty(L^{\frac32,\infty})}\lesssim\|\theta\|_{L_t^\infty(L^1)}\lesssim\|\theta_0\|_{L^1}.\end{aligned}\tag{7.10}$$

Inserting the above two inequalities into (7.9), we arrive at

$$\begin{aligned}\|u\|_{L_1^\infty(L^{3,\infty})}\le C(&\|u_0\|_{L^{3,\infty}}+\|u\|^2_{L_1^\infty(L^{3,\infty})}\\&+\|\theta_0\|_{L^\infty}(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|_{L^1\cap L^3})+\|\theta_0\|_{L^1}).\end{aligned}\tag{7.11}$$

Note from (7.2) that

$$\begin{aligned}&\|u_0\|_{L^{3,\infty}}+\|\theta_0\|_{L^\infty}(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|_{L^1\cap L^3})+\|\theta_0\|_{L^1}\\&\le C(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|_{L^1}+\|\theta_0\|_{L^1}^{\frac16}\|\theta_0\|^{\frac56}_{\dot{B}^1_{3,1}})\le C\eta_2,\end{aligned}$$

from which and (7.11), we conclude the proof of (7.8).

Proposition 7.2 *Let (θ,u) be a smooth enough solution of (1.9) on $[0,T^*[$. Then under the assumptions of Proposition 7.1 and*

$$(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|^{\frac13}_{L^1}\|\theta_0\|^{\frac23}_{\dot{B}^1_{3,1}})(\|\theta_0\|_{\dot{B}^1_{3,1}}+\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}})+\|\theta_0\|_{L^1}\|\theta_0\|^2_{\dot{B}^\alpha_{\infty,\infty}}\le\eta_3\tag{7.12}$$

for some sufficiently small η_3, we have

$$\begin{aligned}&\|u\|_{\widetilde{L}_1^\infty(\dot{B}^0_{3,1})}+\|u\|_{L_1^1(\dot{B}^2_{3,1})}\\&\le C(\|\theta_0\|_{L^1}\|\theta_0\|^2_{\dot{B}^\alpha_{\infty,\infty}}+(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|^{\frac13}_{L^1}\|\theta_0\|^{\frac23}_{\dot{B}^1_{3,1}})(1+\|\theta_0\|_{\dot{B}^1_{3,1}})).\end{aligned}\tag{7.13}$$

Proof Let us denote $U(1)\overset{\text{def}}{=}\|u\|_{\widetilde{L}_1^\infty(\dot{B}^0_{3,1})}+\|u\|_{L_1^1(\dot{B}^2_{3,1})}$. Then we get, by a similar derivation of (6.3), that

$$U(1)\le C(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta\|_{L_1^1(\dot{B}^0_{3,1})}+(\|u\|_{\widetilde{L}_1^\infty(\dot{B}^0_{3,1})}+\|\theta\|_{\widetilde{L}_1^\infty(\dot{B}^1_{3,1})})\|\nabla u\|_{L_1^1(L^\infty)}),$$

from which, (7.2) and (2.8), we infer

$$U(1)\le C(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|^{\frac13}_{L^1}\|\theta\|^{\frac23}_{L_1^\infty(\dot{B}^1_{3,1})}+\|\theta\|_{\widetilde{L}_1^\infty(\dot{B}^1_{3,1})}(\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|^{\frac13}_{L^1}\|\theta_0\|^{\frac23}_{\dot{B}^1_{3,1}})).$$

However it follows from part (4) of Lemma 4.3 and Proposition 7.1 that

$$\|\theta\|_{\widetilde{L}_1^\infty(\dot{B}^1_{3,1})}\le C(\|\theta_0\|_{\dot{B}^1_{3,1}}+\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}\|u\|_{L_1^1(\dot{B}^{2-\alpha}_{3,1})}).$$

So that we obtain

$$\begin{aligned}U(1)\le C((&\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|^{\frac13}_{L^1}\|\theta_0\|^{\frac23}_{\dot{B}^1_{3,1}})(1+\|\theta_0\|_{\dot{B}^1_{3,1}}+\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}\|u\|_{L_1^1(\dot{B}^{2-\alpha}_{3,1})})\\&+\|\theta_0\|^{\frac13}_{L^1}\|\theta_0\|^{\frac23}_{\dot{B}^\alpha_{\infty,\infty}}\|u\|^{\frac23}_{L_1^1(\dot{B}^{2-\alpha}_{3,1})}).\end{aligned}\tag{7.14}$$

Yet by using interpolation and Young's inequalities, one has for any $\sigma > 0$, there exists some positive constant $C_\sigma > 0$ such that

$$\|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}^{\frac{2}{3}}\|u\|_{L^1_1(\dot B^{2-\alpha}_{3,1})}^{\frac{2}{3}} \leq \sigma U(1) + C_\sigma\|\theta_0\|_{L^1}\|\theta_0\|^2_{\dot B^\alpha_{\infty,\infty}},$$

and

$$\begin{aligned}&(\|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac{2}{3}})\|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}\|u\|_{L^1_1(\dot B^{2-\alpha}_{3,1})}\\ &\leq \sigma\|u\|_{L^1_t(\dot B^2_{3,1})} + C_\sigma\|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}^{\frac{2}{\alpha}}(\|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac{2}{3}})^{\frac{2}{\alpha}}U(1).\end{aligned}$$

Resuming the above two inequalities into (7.14), we conclude the proof of (7.13) if there holds (7.12) and the assumptions of Proposition 7.1.

Corollary 7.1 *Let (θ, u) be a smooth enough solution of (1.9) on $[0, T^*[$. Let η_2 and η_3 be given by (7.2) and (7.12) respectively. Then under the assumptions of Proposition 7.2, we can find some $t_1 \in]\frac{1}{2}, 1[$ so that there holds (2.9).*

Proof We first deduce from (7.13) that

$$\|u\|_{L^2_1(\dot B^1_{3,1})} \leq C\|u\|_{\widetilde L^\infty_1(\dot B^0_{3,1})}^{\frac{1}{2}}\|u\|_{L^1_1(\dot B^2_{3,1})}^{\frac{1}{2}} \leq C(\eta_2 + \eta_3), \tag{7.15}$$

which ensures that there exists some $\bar t \in]0, \frac{1}{2}[$ so that

$$\|u(\bar t)\|_{\dot B^0_{3,1}\cap\dot B^1_{3,1}} \leq C(\eta_2 + \eta_3). \tag{7.16}$$

Next we claim that

$$\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})} \leq C(\eta_2 + \eta_3)(1 + \|\theta_0\|_{\dot B^1_{3,1}} + \|\theta_0\|_{\dot B^\alpha_{\infty,\infty}}). \tag{7.17}$$

Indeed we first infer from (7.4) that

$$\begin{aligned}\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})} \leq{}& \|u(\bar t)\|_{\dot B^1_{3,1}} + C(\|\theta\|_{\widetilde L^2(\bar t,1;\dot B^0_{3,1})} + \|u\otimes u\|_{\widetilde L^\infty(\bar t,1;\dot B^0_{3,1})}\\ &+ \|T_{(\mu(\theta)-1)}d\|_{\widetilde L^\infty(\bar t,1;\dot B^0_{3,1})} + \|T_d(\mu(\theta)-1)\|_{\widetilde L^\infty(\bar t,1;\dot B^0_{3,1})}\\ &+ \|R(\mu(\theta)-1,d)\|_{\widetilde L^2(\bar t,1;\dot B^1_{3,1})}).\end{aligned}$$

Note that by using (1) of (3.6) and (7.8), we have

$$\|u\otimes u\|_{\widetilde L^\infty(\bar t,1;\dot B^0_{3,1})} \leq C\|u\|_{L^\infty_1(L^{3,\infty})}\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})} \leq C\eta_2\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})}$$

and

$$\begin{aligned}\|T_{(\mu(\theta)-1)}d\|_{\widetilde L^\infty(\bar t,1;\dot B^0_{3,1})} &\leq C\|\mu(\theta)-1\|_{L^\infty_1(L^\infty)}\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})}\\ &\leq C\|\theta_0\|_{L^\infty}\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})} \leq C\eta_1\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})}.\end{aligned}$$

Hence by virtue of (7.1), we obtain

$$\begin{aligned}\|u\|_{\widetilde L^\infty(\bar t,1;\dot B^1_{3,1})} \leq{}& C(\|u(\bar t)\|_{\dot B^1_{3,1}} + \|\theta\|_{\widetilde L^2(\bar t,1;\dot B^0_{3,1})}\\ &+ \|T_d(\mu(\theta)-1)\|_{\widetilde L^\infty(\bar t,1;\dot B^0_{3,1})} + \|R(\mu(\theta)-1,d)\|_{\widetilde L^2(\bar t,1;\dot B^1_{3,1})}),\end{aligned}$$

from which, and the para-product estimates (see e.g. [7]), we infer

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty(\bar{t},1;\dot{B}^1_{3,1})} &\le C(\|u(\bar{t})\|_{\dot{B}^1_{3,1}} + \|\theta\|_{\widetilde{L}^\infty(\bar{t},1;\dot{B}^0_{3,1})} + \|\theta\|_{\widetilde{L}^\infty_1(\dot{B}^0_{\infty,1})}\|u\|_{\widetilde{L}^\infty(\bar{t},1;\dot{B}^1_{3,1})} \\ &\quad + \|\theta\|_{\widetilde{L}^\infty_1(\dot{B}^1_{3,1})}\|u\|_{\widetilde{L}^2(\bar{t},1;\dot{B}^1_{3,1})}). \end{aligned} \tag{7.18}$$

However, it follows from (2.8) that

$$\|\theta\|_{\widetilde{L}^\infty_1(\dot{B}^0_{\infty,1})} \le C\|\theta_0\|_{\dot{B}^0_{\infty,1}}(1+\|\nabla u\|_{L^1_1(L^\infty)}) \le C\|\theta_0\|_{L^1}^{\frac{\alpha}{3+\alpha}}\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}^{\frac{3}{3+\alpha}} \le C\eta_1,$$

$$\|\theta\|_{\widetilde{L}^\infty_1(\dot{B}^0_{3,1})} \le C\|\theta_0\|_{\dot{B}^0_{3,1}}(1+\|\nabla u\|_{L^1_1(L^\infty)}) \le C\|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{2}{3}} \le C\eta_2.$$

While part (4) of Lemma 4.3 and (7.13) ensures that

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty_1(\dot{B}^1_{3,1})} + \|\theta\|_{L^1_1(\dot{B}^{1+s}_{3,1})} &\le C(\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}\|u\|_{L^1_1(\dot{B}^{2-\alpha}_{3,1})}) \\ &\le C(\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}). \end{aligned} \tag{7.19}$$

Resuming the above estimates into (7.18) and using (7.13) leads to (7.17).

On the other hand, it follows from (7.19) that there exists some $t_1 \in]\frac{1}{2}, 1[$ so that there holds the second inequality of (2.9). And (7.17) ensures the first inequality of (2.9). This completes the proof of Corollary 7.1.

7.2 The $L^1(\mathbb{R}^+; \dot{B}^1_{\infty,1})$ estimate for the velocity field

As a convention in the remaining of this subsection, we always denote t_1 to be the positive time determined by Corollary 7.1.

Proposition 7.3 *Let $3-\sqrt{6} < s \le 1$, and (θ, u) be a smooth enough solution of the System (1.9) on $[0, T^*[$. Then under the assumptions of Proposition 7.2 and*

$$\|u(t_1)\|_{\dot{B}^0_{3,1}\cap\dot{B}^1_{3,1}} + \|\theta_0\|_{L^1} + \|\theta_0\|_{L^1}^{\frac{1}{2}(1-\frac{3}{p_s})}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{1}{2}(1+\frac{3}{p_s})} + \|\theta_0\|_{L^1}^{1-\frac{3}{p_s}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{3}{p_s}} \le \eta_4 \tag{7.20}$$

for some small enough constant η_4, and where

$$p_s \stackrel{\text{def}}{=} \frac{3(5-s)}{3-s} \in \Big]\frac{9}{2}, 6\Big], \tag{7.21}$$

we have

$$\begin{aligned}\|u\|_{L^1_t(\dot{B}^1_{\infty,1})} &\le C(\|u_0\|_{\dot{B}^0_{3,1}} + \|u(t_1)\|_{\dot{B}^0_{3,1}}(1+\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}}) \\ &\quad + \sqrt{t}(\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}})^2 + \langle t\rangle\|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{2}{3}}) \quad \textit{for } s=1, \end{aligned} \tag{7.22}$$

and for $s \in \big]3-\sqrt{6}, 1[$,

$$\begin{aligned}\|u\|_{L^1_t(\dot{B}^1_{\infty,1})} &\le C(\|u_0\|_{\dot{B}^0_{3,1}} + \langle t\rangle\|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{2}{3}} + \|u(t_1)\|_{\dot{B}^0_{3,1}} \\ &\quad + (\|\theta(t_1)\|_{B^{\frac{3}{2}}_{3,\infty}} + \|\theta_0\|_{\dot{B}^1_{3,1}}(\|u(t_1)\|_{\dot{B}^1_{3,1}} + t^{\frac{2(p_s-3)}{p_s}}\|\theta_0\|_{L^1}^{\frac{2(6-p_s)}{3p_s}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{5p_s-12}{3p_s}})) \\ &\quad \times (\|u(t_1)\|_{\dot{B}^0_{3,1}\cap\dot{B}^1_{3,1}} + \eta_4(t^{\frac{6-p_s}{p_s}} + t^{\frac{3}{p_s}}) + t^{\frac{9}{2p_s}}\|\theta_0\|_{L^1}^{\frac{2(p_s-3)}{3p_s}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{p_s+6}{3p_s}})). \end{aligned} \tag{7.23}$$

We remark that for $s \in]3-\sqrt{6},1]$, there holds

$$\|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}} + \|\theta(t_1)\|_{B^{\frac{3}{2}}_{3,\infty}} \lesssim \|\theta(t_1)\|_{L^1} + \|\theta(t_1)\|_{\dot{B}^{1+s}_{3,1}},$$

which is bounded according to (2.9).

Proof of Proposition 7.3 We admit Lemmas 2.1 and 2.2 for the time being and continue the proof of Proposition 7.3.

As a matter of fact, due to (2.8), we only need to prove (7.22) and (7.23) for $\|u\|_{L^1(t_1,t;\dot{B}^1_{\infty,1})}$. Indeed by virtue of (2.10) and Lemma 3.2, we deduce that

$$\begin{aligned}\|u\|_{L^1(t_1,t;\dot{B}^{1+\frac{3}{p_s}}_{p_s,1})} \leq C(&\|u(t_1)\|_{\dot{B}^{-1+\frac{3}{p_s}}_{p_s,1}} + \|u\otimes u\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})} \\ &+ \|(\mu(\theta)-1)\nabla u\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})} + \|\theta\|_{L^1_t(\dot{B}^{-1+\frac{3}{p_s}}_{p_s,1})}).\end{aligned} \tag{7.24}$$

Note that for $s \in \big]3-\sqrt{6},1]$, p_s given by (7.21) belongs to $]3,6]$. Hence by part (1) of Lemma 4.3, we have

$$\begin{aligned}\|\theta\|_{L^1_t(\dot{B}^{-1+\frac{3}{p_s}}_{p_s,1})} &\leq t\Big(\sum_{j\leq N} 2^{2j}\|\dot{\Delta}_j\theta\|_{L^\infty_t(L^1)} + \sum_{j>N} 2^{j(-1+\frac{3}{p_s})}\|\dot{\Delta}_j\theta\|_{L^\infty_t(L^{p_s})}\Big) \\ &\leq Ct(2^{2N}\|\theta_0\|_{L^1} + 2^{-N(1-\frac{3}{p_s})}\|\theta_0\|_{L^{p_s}}).\end{aligned}$$

Choosing N in the above inequality so that

$$2^{3N(1-\frac{1}{p_s})} \sim \frac{\|\theta_0\|_{L^{p_s}}}{\|\theta_0\|_{L^1}},$$

we obtain

$$\|\theta\|_{L^1_t(\dot{B}^{-1+\frac{3}{p_s}}_{p_s,1})} \leq Ct\|\theta_0\|_{L^1}^{\frac{p_s-3}{3(p_s-1)}}\|\theta_0\|_{L^{p_s}}^{\frac{2p_s}{3(p_s-1)}} \leq Ct\|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{2}{3}}.$$

Whereas applying (1) of (3.6) and para-product estimates (see [7] for instance) leads to

$$\|u\otimes u\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})} \leq C\|u\|_{L^\infty(t_1,t;L^{3,\infty})}\|u\|_{L^1(t_1,t;\dot{B}^{1+\frac{3}{p_s}}_{p_s,1})},$$

and

$$\begin{aligned}&\|(\mu(\theta)-1)\nabla u\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})} \\ &\leq C(\|\mu(\theta)-1\|_{L^\infty_t(L^\infty)}\|u\|_{L^1(t_1,t;\dot{B}^{1+\frac{3}{p_s}}_{p_s,1})} + \|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})}) \\ &\leq C(\|\theta_0\|_{L^\infty}\|u\|_{L^1(t_1,t;\dot{B}^{1+\frac{3}{p_s}}_{p_s,1})} + \|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})}),\end{aligned}$$

where in the last step, we used (1.5) so that

$$\|\mu(\theta)-1\|_{L^\infty_t(L^\infty)} \leq C\|\theta\|_{L^\infty_t(L^\infty)} \leq C\|\theta_0\|_{L^\infty} \leq C\|\theta_0\|_{L^1}^{\frac{\alpha}{3+\alpha}}\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}^{\frac{3}{3+\alpha}}. \tag{7.25}$$

Resuming the above estimates into (7.24) and using (7.1) and (2.11) gives rise to

$$\|u\|_{L^1(t_1,t;\dot{B}^{1+\frac{3}{p_s}}_{p_s,1})} \leq C(\|u(t_1)\|_{\dot{B}^0_{3,1}} + t\|\theta_0\|_{L^1}^{\frac{1}{3}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{2}{3}} + \|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})}). \tag{7.26}$$

It remain to handle the last term in (7.26). In the case when $s = 1$, p_1 given by (7.21) equals 6, and then we get, by using para-product estimate, that

$$\|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{1}{2}}_{6,1})} \le C\|(\mu(\theta)-1)\|_{\widetilde{L}^4(t_1,t;\dot{B}^1_{6,\infty})}\|\nabla u\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^0_{6,1})}.$$

Yet it follows from part (3) of Lemma 4.3 and (7.31) below that

$$\begin{aligned}\|\theta\|_{\widetilde{L}^4(t_1,t;\dot{B}^1_{6,\infty})} &\le \|\theta(t_1)\|_{\dot{B}^{\frac{3}{4}}_{6,\infty}} + C\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^4_t(L^6)}\\ &\le C(\|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}} + \|\theta_0\|_{\dot{B}^1_{3,1}}\eta_4).\end{aligned} \tag{7.27}$$

Whereas similar to (7.24), we infer from (2.10) that

$$\begin{aligned}\|u\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^1_{6,1})} \le C(&\|u(t_1)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} + \|u\|_{L^\infty(t_1,t;L^{3,\infty})}\|u\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^1_{6,1})}\\ &+ \|\theta_0\|_{L^\infty}\|u\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^1_{6,1})} + \|T_{\nabla u}(\mu(\theta)-1)\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^0_{6,1})}).\end{aligned}$$

Note that para-product estimates (see [7]) along with (7.27) and (7.31) below ensures that

$$\begin{aligned}\|T_{\nabla u}(\mu(\theta)-1)\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^0_{6,1})} &\le C\|(\mu(\theta)-1)\|_{\widetilde{L}^4(t_1,t;\dot{B}^1_{6,\infty})}\|u\|_{\widetilde{L}^2(t_1,t;\dot{B}^{\frac{1}{2}}_{6,1})}\\ &\le C\sqrt{t}\|\theta\|_{\widetilde{L}^4(t_1,t;\dot{B}^1_{6,\infty})}(\|u\|_{L^4(t_1,t;L^6)} + \|\nabla u\|_{L^4(t_1,t;L^6)})\\ &\le C\sqrt{t}(\|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}} + \|\theta_0\|_{\dot{B}^1_{3,1}}).\end{aligned}$$

Hence by virtue of (7.1) and (7.31), we infer

$$\|u\|_{\widetilde{L}^{\frac{4}{3}}(t_1,t;\dot{B}^1_{6,1})} \le C(\|u(t_1)\|_{\dot{B}^0_{3,1}} + \sqrt{t}(\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}})).$$

We thus obtain

$$\begin{aligned}\|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{1}{2}}_{6,1})} \le C(&\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}})\\ &\times(\|u(t_1)\|_{\dot{B}^0_{3,1}} + \sqrt{t}(\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta(t_1)\|_{\dot{B}^{\frac{5}{4}}_{3,\infty}})).\end{aligned}$$

Resuming the above estimate into (7.26) leads to (7.22).

When $3-\sqrt{6} < s < 1$, which corresponds to p_s given by (7.21) satisfying $\frac{3}{p_s} < s < 1$ and $3 < p_s < 6$, we get, by using para-product estimate (see [7]), that

$$\|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})} \le C\|(\mu(\theta)-1)\|_{\widetilde{L}^{\frac{p_s}{p_s-3}}(t_1,t;\dot{B}^s_{\frac{3p_s}{6-p_s},\infty})}\|u\|_{\widetilde{L}^{\frac{p_s}{3}}(t_1,t;\dot{B}^{1-s+\frac{3}{p_s}}_{\frac{3p_s}{p_s-3},1})}. \tag{7.28}$$

Yet it follows from part (3) of Lemma 4.3 that

$$\begin{aligned}\|\theta\|_{\widetilde{L}^{\frac{p_s}{p_s-3}}(t_1,t;\dot{B}^s_{\frac{3p_s}{6-p_s},\infty})} &\le C(\|\theta(t_1)\|_{B^{\frac{3s}{p_s}}_{\frac{3p_s}{6-p_s},\infty}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^{\frac{p_s}{p_s-3}}(t_1,t;L^{\frac{3p_s}{6-p_s}})})\\ &\le C(\|\theta(t_1)\|_{B^{\frac{3}{2}}_{3,\infty}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^{\frac{p_s}{p_s-3}}(t_1,t;L^{\frac{3p_s}{6-p_s}})}).\end{aligned} \tag{7.29}$$

And according to Definition 3.1 and Lemma 3.3, for any integer L, we write

$$\begin{aligned}\|a\|_{\dot{B}^{1-s+\frac{3}{p_s}}_{\frac{3p_s}{p_s-3},1}} &= \sum_{j\le L} 2^{j(1-s+\frac{6}{p_s})}\|\dot{\Delta}_j a\|_{L^3} + \sum_{j>L} 2^{-j(s-\frac{3}{p_s})}\|\dot{\Delta}_j\nabla a\|_{L^{\frac{3p_s}{p_s-3}}}\\ &\le C(2^{L(1-s+\frac{6}{p_s})}\|a\|_{L^{3,\infty}} + 2^{-L(s-\frac{3}{p_s})}\|\nabla a\|_{L^{\frac{3p_s}{p_s-3}}}).\end{aligned}$$

Taking L so that

$$2^{L(1+\frac{3}{p_s})} \sim \frac{\|\nabla a\|_{L^{\frac{3p_s}{p_s-3}}}}{\|a\|_{L^{3,\infty}}}$$

in the above inequality, we obtain

$$\|a\|_{\dot{B}^{1-s+\frac{3}{p_s}}_{\frac{3p_s}{p_s-3},1}} \le C\|a\|_{L^{3,\infty}}^{\frac{sp_s-3}{p_s+3}}\|\nabla a\|_{L^{\frac{3p_s}{p_s-3}}}^{\frac{p_s(1-s)+6}{p_s+3}},$$

from which, we infer

$$\|u\|_{\widetilde{L}^{\frac{p_s}{3}}(t_1,t;\dot{B}^{1-s+\frac{3}{p_s}}_{\frac{3p_s}{p_s-3},1})} \le C(t^{\frac{3}{p_s}}\|u\|_{L^\infty(t_1,t;L^{3,\infty})} + \|\nabla u\|_{L^{\frac{p_s}{3}}(t_1,t;L^{\frac{3p_s}{p_s-3}})}). \tag{7.30}$$

Resuming the estimates (7.29), (7.30) into (7.28) and using Lemmas 2.1 and 2.2, we arrive at

$$\begin{aligned}&\|T_{\nabla u}(\mu(\theta)-1)\|_{L^1(t_1,t;\dot{B}^{\frac{3}{p_s}}_{p_s,1})}\\ &\le C(\|\theta(t_1)\|_{B^{\frac{3}{2}}_{3,\infty}} + \|\theta_0\|_{L^\infty}(\|u(t_1)\|_{\dot{B}^1_{3,1}} + t^{\frac{2(p_s-3)}{p_s}}\|\theta_0\|_{L^1}^{\frac{2(6-p_s)}{3p_s}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{5p_s-12}{3p_s}}))\\ &\quad\times(\eta_4(t^{\frac{6-p_s}{p_s}} + t^{\frac{3}{p_s}}) + \|u(t_1)\|_{\dot{B}^0_{3,1}\cap\dot{B}^1_{3,1}} + t^{\frac{9}{2p_s}}\|\theta_0\|_{L^1}^{\frac{2(p_s-3)}{3p_s}}\|\theta_0\|_{\dot{B}^1_{3,1}}^{\frac{p_s+6}{3p_s}}).\end{aligned}$$

Substituting the above estimate into (7.26) and using (2.8) gives rise to (7.23). This completes the proof of Proposition 7.3.

We now turn to the proof of Lemmas 2.1 and 2.2.

Proof of Lemma 2.1 Note that for $s\in]0,1]$, p_s given by (7.21) belongs to $]3,6]$. Then we get, by a similar derivation of (5.4) and (5.5) that

$$\begin{aligned}\|u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})} &\le C(\|u(t_1)\|_{\dot{B}^{\frac{3}{p_s}-1}_{p_s,\frac{2p_s}{p_s-3}}} + \|u\|^2_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})}\\ &\quad + \|\mu(\theta)-1\|_{L_t^{\frac{2p_s}{p_s-3}}(L^{p_s})}\|\nabla u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})} + \|\theta\|_{L_t^{\frac{2p_s}{p_s-3}}(L^{p_s})}),\end{aligned}$$

and

$$\begin{aligned}\|\nabla u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})} &\le C(\|u(t_1)\|_{\dot{B}^{\frac{3}{p_s}}_{p_s,1}} + \|u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})}\|\nabla u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})}\\ &\quad + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})} + \|\theta\|_{L_t^{\frac{p_s}{p_s-3}}(L^{\frac{p_s}{2}})}).\end{aligned}$$

By virtue of Part (1) and Part (2) of Lemma 4.3, we have

$$\begin{aligned}&\|\theta\|_{L_t^{\frac{2p_s}{p_s-3}}(L^{p_s})} \le \|\theta\|_{L_t^2(L^{\frac{6}{3-s}})}^{1-\frac{3}{p_s}}\|\theta\|_{L_t^\infty(L^\infty)}^{\frac{3}{p_s}} \le C\|\theta\|_{L_t^2(\dot{H}^{\frac{s}{2}})}^{1-\frac{3}{p_s}}\|\theta_0\|_{L^\infty}^{\frac{3}{p_s}} \le C\|\theta_0\|_{L^2}^{1-\frac{3}{p_s}}\|\theta_0\|_{L^\infty}^{\frac{3}{p_s}},\\ &\|\theta\|_{L_t^{\frac{p_s}{p_s-3},1}(L^{\frac{p_s}{2}})} \le \|\theta\|_{L_t^2(L^{\frac{6}{3-s}})}^{2(1-\frac{3}{p_s})}\|\theta\|_{L_t^\infty(L^\infty)}^{\frac{6}{p_s}-1} \le C\|\theta_0\|_{L^2}^{2(1-\frac{3}{p_s})}\|\theta_0\|_{L^\infty}^{\frac{6}{p_s}-1},\end{aligned}$$

and

$$\|\mu(\theta)-1\|_{L_t^{\frac{2p_s}{p_s-3}}(L^{p_s})} \le C\|\theta\|_{L_t^{\frac{2p_s}{p_s-3}}(L^{p_s})} \le C\|\theta_0\|_{L^1}^{\frac12(1-\frac{3}{p_s})}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac12(1+\frac{3}{p_s})}.$$

Therefore, whenever there hold (7.1) and (7.20), we have

$$\begin{aligned}
&\|u\|_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})} + \|\nabla u\|_{L^{\frac{2p_s}{p_s-3},1}(t_1,t;L^{p_s})}\\
&\le C(\|u(t_1)\|_{\dot B^0_{3,1}\cap \dot B^1_{3,1}}\\
&\quad + \|\theta_0\|_{L^1}^{\frac12(1-\frac{3}{p_s})}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac12(1+\frac{3}{p_s})} + \|\theta_0\|_{L^1}^{1-\frac{3}{p_s}}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac{3}{p_s}}) \le c\eta_4.
\end{aligned} \tag{7.31}$$

On the other hand, since p_s given by (5.39) is greater than $\frac92$, we get, by applying Lemma 4.2, that

$$\begin{aligned}
\Big\|\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}\mathrm{div}((\mu(\theta)-1)d)\mathrm{d}t'\Big\|_{L^\infty(t_1,t;L^3)} &\le C\|(\mu(\theta)-1)d\|_{L^{\frac{p_s}{p_s-3},1}(t_1,t;L^{\frac{p_s}{2}})}\\
&\le C\|\theta\|_{L_t^{\frac{2p_s}{p_s-3}}(L^{p_s})}\|\nabla u\|_{L^{\frac{2p_s}{p_s-3},1}(t_1,t;L^{p_s})}.
\end{aligned}$$

Then by virtute of (7.9), (7.10) and (7.31), we conclude

$$\|u\|_{L^\infty(t_1,t;L^{3,\infty})} \le C(\|u(t_1)\|_{\dot B^0_{3,1}} + \|u\|^2_{L^\infty(t_1,t;L^{3,\infty})} + \|\theta_0\|_{L^1}^{\frac12(1-\frac{3}{p_s})}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac12(1+\frac{3}{p_s})}\eta_4 + \|\theta_0\|_{L^1}).$$

We thus infer from (7.20) that

$$\|u\|_{L^\infty(t_1,t;L^{3,\infty})} \le C(\|u(t_1)\|_{\dot B^0_{3,1}} + \|\theta_0\|_{L^1} + \|\theta_0\|_{L^1}^{\frac12(1-\frac{3}{p_s})}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac12(1+\frac{3}{p_s})}),$$

which leads to (2.11).

Proof of Lemma 2.2 Thanks to (7.3), we get, by applying Proposition 4.1 and Lemma 4.2, that

$$\begin{aligned}
\|\nabla u\|_{L^{\frac{p_s}{p_s-3}}(t_1,t;L^{\frac{3p_s}{6-p_s}})} &\le C(\|u(t_1)\|_{\dot B^{\frac{6}{p_s}-1}_{\frac{3p_s}{6-p_s},\frac{p_s}{p_s-3}}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^{\frac{p_s}{p_s-3}}(t_1,t;L^{\frac{3p_s}{6-p_s}})}\\
&\quad + \|u\|^2_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{\frac{6p_s}{6-p_s}})} + \|\theta\|_{L_t^{\frac{p_s}{2(p_s-3)}}(L^{\frac{3p_s}{2(6-p_s)}})}).
\end{aligned} \tag{7.32}$$

Note that for $q\in]3,\infty[$, we have

$$\|f\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^{3,\infty}(\mathbb{R}^3)}^{\frac12}\|\nabla f\|_{L^{\frac q2}(\mathbb{R}^3)}^{\frac12}.$$

Applying the above inequality for $q=\frac{6p_s}{6-p_s}$ gives

$$\|u\|^2_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{\frac{6p_s}{6-p_s}})} \le C\|u\|_{L^\infty(t_1,t;L^{3,\infty})}\|\nabla u\|_{L^{\frac{p_s}{p_s-3}}(t_1,t;L^{\frac{3p_s}{6-p_s}})}.$$

Resuming the above estimate into (7.32) and using (2.11) and the fact that

$$\|\theta\|_{L_t^{\frac{p_s}{2(p_s-3)}}(L^{\frac{3p_s}{2(6-p_s)}})} \le t^{\frac{2(p_s-3)}{p_s}}\|\theta\|_{L_t^\infty(L^{\frac{3p_s}{2(6-p_s)}})} \le t^{\frac{2(p_s-3)}{p_s}}\|\theta_0\|_{L^{\frac{3p_s}{2(6-p_s)}}},$$

we obtain the first inequality of (2.12).

Similar to the proof of (7.32), one has

$$
\begin{aligned}
\|\nabla u\|_{L^{\frac{p_s}{3}}(t_1,t;L^{\frac{3p_s}{p_s-3}})} \leq C\Big(&\|u(t_1)\|_{\dot B^{1-\frac{6}{p_s}}_{\frac{3p_s}{p_s-3},\frac{p_s}{3}}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^{\frac{p_s}{3}}(t_1,t;L^{\frac{3p_s}{p_s-3}})} + \|u\|^2_{L^{\frac{2p_s}{3}}(t_1,t;L^{\frac{6p_s}{p_s-3}})}\\
&+ \Big\|\nabla\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}(\theta e_3)(t')\mathrm{d}t'\Big\|_{L_t^{\frac{p_s}{3}}(L^{\frac{3p_s}{p_s-3}})}\Big). \qquad (7.33)
\end{aligned}
$$

By using interpolation inequality that

$$
\|a\|_{L^{\frac{6p_s}{p_s-3}}} \leq C\|a\|_{L^{p_s}}^{\frac{3p_s-9}{2p_s}}\|\nabla a\|_{L^{p_s}}^{\frac{9-p_s}{2p_s}}
$$

and (7.31), we obtain

$$
\begin{aligned}
\|u\|^2_{L^{\frac{2p_s}{3}}(t_1,t;L^{\frac{6p_s}{p_s-3}})} &\leq Ct^{\frac{6-p_s}{p_s}}\|u\|^2_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{\frac{6p_s}{p_s-3}})}\\
&\leq Ct^{\frac{6-p_s}{p_s}}\|u\|^{\frac{3p_s-9}{p_s}}_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})}\|\nabla u\|^{\frac{9-p_s}{p_s}}_{L^{\frac{2p_s}{p_s-3}}(t_1,t;L^{p_s})} \leq C\eta_4 t^{\frac{6-p_s}{p_s}}.
\end{aligned}
$$

While applying Lemma 4.2 gives rise to

$$
\begin{aligned}
\Big\|\nabla\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}(\theta e_3)(t')\mathrm{d}t'\Big\|_{L_t^{\frac{p_s}{3}}(L^{\frac{3p_s}{p_s-3}})} &\leq Ct^{\frac{3}{2p_s}}\Big\|\nabla\int_0^t \mathrm{e}^{(t-t')\Delta}\mathbb{P}(\theta e_3)(t')\mathrm{d}t'\Big\|_{L_t^{\frac{2p_s}{3}}(L^{\frac{3p_s}{p_s-3}})}\\
&\leq Ct^{\frac{3}{2p_s}}\|\theta\|_{L_t^{\frac{p_s}{3}}(L^{\frac{3p_s}{2(p_s-3)}})} \leq Ct^{\frac{9}{2p_s}}\|\theta\|_{L_t^\infty(L^{\frac{3p_s}{2(p_s-3)}})}\\
&\leq Ct^{\frac{9}{2p_s}}\|\theta_0\|_{L^1}^{\frac{2(p_s-3)}{3p_s}}\|\theta_0\|_{\dot B^1_{3,1}}^{\frac{p_s+6}{3p_s}}.
\end{aligned}
$$

Resuming the above two estimates into (7.33) and using the assumption (7.1), we get the second inequality of (2.12). This completes the proof of Lemma 2.2.

8 Proof of Theorem 1.2

8.1 Existence part of Theorem 1.2

As in Subsection 6.1, we first present the following a priori estimate.

Lemma 8.1 *Let (θ, u) be a sufficiently smooth solution of* (1.9) *on $[0, T^*[$. We assume that there holds* (7.1). *Then for any $\alpha \in]0,1[$, one has*

$$
\begin{aligned}
&\|u\|_{\widetilde L_t^\infty(\dot B^0_{3,1})} + \|u\|_{L_t^1(\dot B^2_{3,1})} + \|\nabla\Pi\|_{\widetilde L_t^1(\dot B^0_{3,1})}\\
&\leq C(\|u_0\|_{\dot B^0_{3,1}} + t^3\|\theta_0\|_{L^1} + \|\theta_0\|_{\dot B^1_{3,1}})\exp(C\|\nabla u\|_{L_t^1(L^\infty)})\\
&\quad\times\exp\Big(Ct\|\theta_0\|^{\frac{2}{\alpha}}_{\dot B^\alpha_{\infty,\infty}}\exp\Big(\frac{2C}{\alpha}\|\nabla u\|_{L_t^1(L^\infty)}\Big)\Big). \qquad (8.1)
\end{aligned}
$$

Proof We first write the u equation of (1.9) as

$$
\partial_t u + \mathrm{div}(u\otimes u) - \Delta u + \nabla\Pi = \theta e_3 + \mathrm{div}(2(\mu(\theta)-1)d),
$$

from which and Proposition 4.2, we deduce

$$
\begin{aligned}
&\|u\|_{\widetilde{L}_t^\infty(\dot{B}^0_{3,1})} + \|u\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\Pi\|_{\widetilde{L}^1_t(\dot{B}^0_{3,1})}\\
&\lesssim \|u_0\|_{\dot{B}^0_{3,1}} + \|\theta\|_{L^1_t(\dot{B}^0_{3,1})} + \|(\mu(\theta)-1)d\|_{L^1_t(\dot{B}^1_{3,1})} + \int_0^t \|\nabla u(t')\|_{L^\infty}\|u(t')\|_{\dot{B}^0_{3,1}}\,\mathrm{d}t'.
\end{aligned}
$$

Yet by using Bony's decomposition, we have

$$
\begin{aligned}
\|(\mu(\theta)-1)d\|_{L^1_t(\dot{B}^1_{3,1})} &\le \|T_d(\mu(\theta)-1)\|_{L^1_t(\dot{B}^1_{3,1})} + \|T'_{(\mu(\theta)-1)}d\|_{L^1_t(\dot{B}^1_{3,1})}\\
&\le C(\|\nabla u\|_{L^1_t(L^\infty)}\|\mu(\theta)-1\|_{L^\infty_t(\dot{B}^1_{3,1})} + \|\mu(\theta)-1\|_{L^\infty_t(L^\infty)}\|u\|_{L^1_t(\dot{B}^2_{3,1})}),
\end{aligned}
$$

and it is easy to observe that

$$
\|\theta(t)\|_{\dot{B}^0_{3,1}} \le C\|\theta(t)\|_{L^1}^{\frac{1}{3}}\|\theta(t)\|_{\dot{B}^1_{3,1}}^{\frac{2}{3}}.
$$

Hence by virtue of (7.1) and (7.25), we obtain

$$
\begin{aligned}
&\|u\|_{\widetilde{L}_t^\infty(\dot{B}^0_{3,1})} + \|u\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\Pi\|_{\widetilde{L}^1_t(\dot{B}^0_{3,1})}\\
&\lesssim \|u_0\|_{\dot{B}^0_{3,1}} + t^3\|\theta_0\|_{L^1} + \|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})}(1+\|\nabla u\|_{L^1_t(L^\infty)}) + \int_0^t \|\nabla u(t')\|_{L^\infty}\|u(t')\|_{\dot{B}^0_{3,1}}\,\mathrm{d}t',
\end{aligned}
$$

which together with part (4) of Lemma 4.3 implies that

$$
\begin{aligned}
\|u\|_{\widetilde{L}_t^\infty(\dot{B}^0_{3,1})} + \|u\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\Pi\|_{\widetilde{L}^1_t(\dot{B}^0_{3,1})} &\lesssim \|u_0\|_{\dot{B}^0_{3,1}} + \int_0^t \|\nabla u(t')\|_{L^\infty}\|u(t')\|_{\dot{B}^0_{3,1}}\,\mathrm{d}t'\\
&\quad + t^3\|\theta_0\|_{L^1} + (\|\theta_0\|_{\dot{B}^1_{3,1}} + \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}\|u\|_{L^1_t(\dot{B}^{2-\alpha}_{3,1})})\\
&\quad \times \exp(C\|\nabla u\|_{L^1_t(L^\infty)}). \qquad (8.2)
\end{aligned}
$$

However, it follows from interpolation and Young's inequalities that for any $\sigma > 0$, there exists some $C_\sigma > 0$ so that

$$
\begin{aligned}
\|u\|_{L^1_t(\dot{B}^{2-\alpha}_{3,1})}\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}\exp(C\|\nabla u\|_{L^1_t(L^\infty)}) &\le \sigma\|u\|_{L^1_t(\dot{B}^2_{3,1})} + C_\sigma\|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}^{\frac{2}{\alpha}}\\
&\quad \times \exp\Big(\frac{2C}{\alpha}\|\nabla u\|_{L^1_t(L^\infty)}\Big)\int_0^t \|u(t')\|_{\dot{B}^0_{3,1}}\,\mathrm{d}t'.
\end{aligned}
$$

We thus infer from (8.2) that

$$
\begin{aligned}
&\|u\|_{\widetilde{L}_t^\infty(\dot{B}^0_{3,1})} + \|u\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\Pi\|_{\widetilde{L}^1_t(\dot{B}^0_{3,1})}\\
&\lesssim \|u_0\|_{\dot{B}^0_{3,1}} + t^3\|\theta_0\|_{L^1} + \int_0^t \|\nabla u(t')\|_{L^\infty}\|u(t')\|_{\dot{B}^0_{3,1}}\,\mathrm{d}t'\\
&\quad + \|\theta_0\|_{\dot{B}^1_{3,1}}\exp(C\|\nabla u\|_{L^1_t(L^\infty)}) + \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}}^{\frac{2}{\alpha}}\exp\Big(\frac{2C}{\alpha}\|\nabla u\|_{L^1_t(L^\infty)}\Big)\int_0^t \|u(t')\|_{\dot{B}^0_{3,1}}\,\mathrm{d}t'.
\end{aligned}
$$

Applying Gronwall's lemma to the above inequality leads to (8.1).

We now turn to the existence part of Theorem 1.2.

The existence part of Theorem 1.2 We basically follow the same line as that in Subsection 6.1. Firstly by virtue of Lemma 6.2, we can find $\theta_0^n,\ u_0^n \in H^\infty(\mathbb{R}^3)$ for $n \in \mathbb{N}$ so that

$$\|\theta_0^n\|_{L^1} \le 2\eta_0, \quad \|\theta_0^n\|_{\dot{B}^\alpha_{\infty,\infty}\cap\dot{B}^1_{3,1}} \lesssim \|\theta_0\|_{\dot{B}^\alpha_{\infty,\infty}\cap\dot{B}^1_{3,1}}, \quad \|u_0^n\|_{\dot{B}^0_{3,1}} \le 2\eta \quad \text{and} \quad \operatorname{div} u_0^n = 0. \tag{8.3}$$

Then according to [1, Theorem 1.1], we deduce that the System (1.9) with the initial data (θ_0^n, u_0^n) admits a unique local in time solution $(\theta^n, u^n, \nabla\Pi^n)$ on $[0, T_n^*[$ verifying

$$\theta^n \in \mathcal{C}([0,T_n^*[;\ H^{s+1}(\mathbb{R}^3)), \quad u^n \in \mathcal{C}([0,T_n^*[;\ H^s(\mathbb{R}^3)) \cap \widetilde{L}^1_{T_n^*}(H^{s+2})$$

and

$$\nabla\Pi^n \in L^1(]0,T_n^*[;\ H^s(\mathbb{R}^3)) \quad \text{for any } s > \frac{1}{2}.$$

Moreover, whenever η_0, η are small enough in (1.10), we deduce from Proposition 7.3 that there exists a positive constant C_1, which depends on $\|\theta_0\|_{B^1_{3,1}\cap B^\alpha_{\infty,\infty}}$ so that

$$\|u^n\|_{L^1([0,t];\dot{B}^1_{\infty,1})} \le C_1 \quad \text{for any } t < T_n^*, \tag{8.4}$$

from which and Lemma 8.1, we infer that $(\theta^n, u^n, \nabla\Pi^n)$ is uniformly bounded in $\widetilde{L}^\infty_t(\dot{B}^1_{3,1}) \times (\widetilde{L}^\infty_t(\dot{B}^0_{3,1}) \cap L^1_t(\dot{B}^2_{3,1})) \times L^1_t(\dot{B}^0_{3,1})$ for any fixed $t < T_n^*$. This implies that

$$T_n^* = \infty.$$

To prove that there is a subsequence of $\{(\theta^n, u^n, \nabla\Pi^n)\}_{n\in\mathbb{N}}$, which converges to a solution $(\theta, u, \nabla\Pi)$ of (1.9), which satisfies (1.11), we need to use a standard compactness argument of Lions-Aubin's lemma, which we shall not present the details here. One may check similar argument from page 582 to page 583 of [1] for details.

8.2 Uniqueness part of Theorem 1.2

Let $(\theta_i, u_i, \nabla\Pi_i)$, for $i = 1, 2$ be two solutions of (1.9) which satisfy (1.11). We denote

$$(\delta\theta, \delta u, \nabla\delta\Pi) \stackrel{\text{def}}{=} (\theta_2 - \theta_1, u_2 - u_1, \nabla\Pi_2 - \nabla\Pi_1).$$

Then thanks to (1.9), the system for $(\delta\theta, \delta u, \delta\nabla\Pi)$ reads

$$\begin{cases} \partial_t\delta\theta + u_2\cdot\nabla\delta\theta + |D|^s\delta\theta = -\delta u\cdot\nabla\theta_1, \\ \partial_t\delta u + (u_2\cdot\nabla)\delta u - \Delta\delta u + \nabla\delta\Pi = \delta G, \\ \operatorname{div}\delta u = 0, \\ (\delta\theta, \delta u)|_{t=0} = (0,0), \end{cases} \tag{8.5}$$

where δG is determined by

$$\delta G = \delta F + \delta\theta e_3$$

for δF given by (6.9).

Then similar to (6.10), we first deduce from the transport diffusion equation of (8.5) and part (1) of Lemma 4.3 that

$$\|\delta\theta(t)\|_{L^\infty_t(L^3)} \le C\|\theta_1\|_{L^\infty_t(\dot{B}^1_{3,1})}\|\delta u\|_{L^1_t(\dot{B}^1_{3,1})}$$

and

$$\begin{aligned}\|\delta\theta\|_{L^\infty_t(\dot{B}^{-1}_{3,\infty})} &\le C\|\delta\theta\|_{L^\infty_t(L^{\frac{3}{2}})} \le C\|\nabla\theta_1\|_{L^\infty_t(L^3)}\|\delta u\|_{L^1_t(L^3)}\\ &\le C\sqrt{t}\|\theta_1\|_{L^\infty_t(\dot{B}^1_{3,1})}\|\delta u\|^{\frac12}_{L^\infty(\dot{B}^{-1}_{3,\infty})}\|\delta u\|^{\frac12}_{\widetilde{L}^1(\dot{B}^1_{3,\infty})}.\end{aligned} \tag{8.6}$$

While a similar derivation of (6.13) yields for some small enough positive time $\mathfrak{t}_2$ and for $t \le \mathfrak{t}_2$,

$$\begin{aligned}W(t) &\stackrel{\text{def}}{=} \|\delta u\|_{\widetilde{L}^\infty_t(\dot{B}^{-1}_{3,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(\dot{B}^1_{3,\infty})} + \|\nabla\delta\Pi\|_{\widetilde{L}^1_t(\dot{B}^{-1}_{3,\infty})}\\ &\le C\Big(\int_0^t \|\nabla u_1(\tau)\|_{L^\infty}\|\delta u\|_{\widetilde{L}^1_\tau(\dot{B}^1_{3,\infty})}(1-\ln\|\delta u\|_{\widetilde{L}^1_\tau(\dot{B}^1_{3,\infty})})\mathrm{d}\tau + \|\delta\theta\|_{L^1_t(\dot{B}^{-1}_{3,\infty})}\Big).\end{aligned} \tag{8.7}$$

Resuming the Estimate (8.6) into (8.7) ensures that for some small enough positive time $\mathfrak{t}_2$ and $t \le \mathfrak{t}_2$,

$$W(t) \le C\int_0^t \|\nabla u_1(\tau)\|_{L^\infty} W(\tau)(1-\ln W(\tau))\mathrm{d}\tau. \tag{8.8}$$

With (8.8), we can follow the same line as that in Subsection 6.2 to complete the uniqueness part of Theorem 1.2.

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