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Piston Problems of Two-Dimensional Chaplygin Gas*

Shuxing CHEN[1] Aifang QU[2]

Abstract In this paper, the authors study the piston problem for the unsteady two-dimensional Euler system for a Chaplygin gas. The angle of the piston is allowed to vary in a wide range. The piston can be pushed forward into the static gas, or pulled back from the gas. The global existence of solution to the piston problem with any initial speed is established, and the structures of the global solutions are clearly described. The authors find that for the proceeding piston problem the front shock can be detached, attached or even adhere to the surface of the piston depending on the parameters of the flow and the piston; while for the receding problem the front rarefaction wave is always detached and the concentration will never occur.

Keywords Multi-Dimensional piston problem, Proceeding, Receding, Mach number
2000 MR Subject Classification 35L65, 35L67, 76L05, 76N10

1 Introduction

In this paper we study the two-dimensional piston problem for the Chaplygin gas. The piston problem is a basic prototype problem in the study of mathematical theory of compressible fluid dynamics (see [12, 23]). In one-dimensional case the problem is described as follows. Initially, the static gas with uniform pressure p_0 and density ρ_0 is assumed to be in an infinitely long tube enclosed by a piston at one end and open at the other end, then any motion of the piston will cause the corresponding motion of the gas in the tube. In particular, a shock wave will appear ahead of the piston if the piston is pushed forward into the gas, while a rarefaction wave will result in if the piston is pulled backward from the gas. The tube is often called a shock tube. Determining the state of the gas and the propagation of the nonlinear waves in the tube is called a piston problem. In [12–13, 23] the authors took the one-dimensional piston problem as a model to analyze the occurrence and the motion of the basic nonlinear waves in the compressible fluid, so that the importance of the piston problem is well known to relative researchers. One can refer to [4–5, 14–15] and the references therein for more related results.

If the shock tube is wide and the profile of the piston takes the shape of a wedge, then the motion of the gas is no longer one-dimensional. Moreover, the motion of the gas near

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[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: sxchen@fudan.edu.cn
[2]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.
E-mail: aifangqu@163.com afqu@shnu.edu.cn
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the tip of the piston can be locally described as a motion caused by the pushing or pulling of an infinite long wedge in the whole plane filled with gas. As in the one-dimensional case the piston problem in two-dimensional case will offer us much more opportunities to analyze the occurrence, propagation and interaction of nonlinear waves in two-dimensional space, which has rich structures and phenomena (see [1]). A good example is the work given by Volker Elling and Taiping Liu in [16], where the study of the two-dimensional piston problem is naturally linked with the problem on supersonic flow past a wedge. It is proved there when a sharp wedge suddenly hits the static gas with a constant speed, the global existence of the flow with an attached shock outside the wedge can be well determined, provided that the speed of the piston is supersonic and some restrictions on the flow parameters (see [16, (1.1)]) are satisfied. Another related work can be found in [8, 11], where the piston is an expanded disk in two-dimensional space and the expanding of the disk causes an expanding shock moving into the static gas.

It is anticipated to establish a general result for the two-dimensional piston problems without any restriction on the vertex angle of the piston and its speed moving into the static gas as in the one-dimensional case. In this paper we will give such an analysis for the piston moving in the Chaplygin gas, which amounts to the polytropic gas with $\gamma = -1$. One can refer to [2–3, 22] for more physical background of this kind of gas. Due to the linearly degenerate property of the Chaplygin gas the wave structure caused by a given motion is often simpler than that for the general polytropic gas with $\gamma \geq 1$. Hence this model allows us to obtain the global wave structure of the piston problem with initial data in a wide range. The result in this paper shows that the attached shock is present in the similar condition as that in [16] (see [6–7, 17, 21, 24–25]) for attached shock in steady case) and, furthermore, detached shock is also obtained for some kind of initial data as the vertex angle is suitably large. We believe that the result is helpful to study the similar problems for the general polytropic gas.

Let us describe the problem in more details. We first consider the proceeding piston problem. Assume that the gas is static initially, and a piston:

$$\{(x, y) \in \mathbb{R}^2 : x \geq |y| \cot \theta_0\} \tag{1.1}$$

with $\theta_0 \in (0, \frac{\pi}{2})$ moves from right into the gas on the left with uniform velocity $(-u_0, 0)$, where $u_0 > 0$ (see Figure 1). As we will see that different θ_0 may result in the solution with different structure for the same initial state. We would like to study the existence and the structure of the solution for a large class of initial data in this paper.

The two-dimensional inviscid adiabatic Euler equations take the form

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p = 0 \end{cases} \tag{1.2}$$

for $(t, \mathbf{x}) \in [0, \infty) \times R^2$. Here, $\rho, \mathbf{u} = (u, v), p$ are the density, the velocity and the pressure of the fluid, respectively. For the isentropic Chaplygin gas, the state equation is

$$p(\rho) = a^2 \Big(\frac{1}{\rho_*} - \frac{1}{\rho} \Big), \tag{1.3}$$

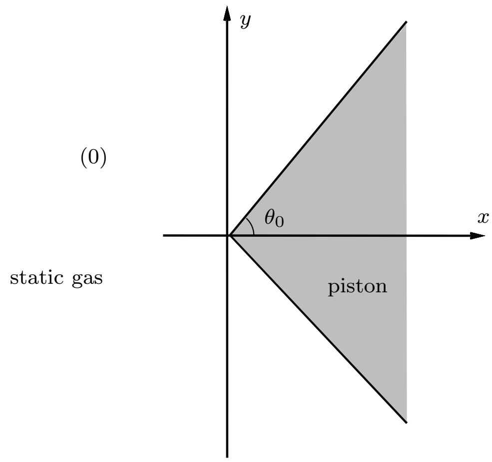

Figure 1 Two-dimensional piston problem.

where a and ρ_* are two positive constants. The sound speed of the gas is $c = \frac{a}{\rho}$.

The domain under consideration is

$$\Omega = \{(x, y, t) : x \le |y| \cot\theta_0 - u_0 t\}.$$

Its boundary is $\partial\Omega = W_u \cup W_l$ with

$$W_u = \{(x, y, t) : y \ge 0,\ y = (x + u_0 t)\tan\theta_0\}, \tag{1.4}$$

$$W_l = \{(x, y, t) : y < 0,\ y = -(x + u_0 t)\tan\theta_0\}. \tag{1.5}$$

We assume that the flow satisfies the following slip boundary condition

$$(u, v) \cdot \nu|_{\partial\Omega} = 0, \tag{1.6}$$

where ν is the outward unit normal of the boundary: $\nu|_{W_u} = (\sin\theta_0,\ -\cos\theta_0)$, $\nu|_{W_l} = (\sin\theta_0,\ \cos\theta_0)$.

Note that the above equations, the initial data as well as the boundary are invariant under the scaling

$$(x, y, t) \mapsto (\alpha x, \alpha y, \alpha t) \quad \text{for} \quad \alpha \ne 0.$$

Thus, we can seek self-similar solution with the form

$$\rho(x, y, t) = \rho(\xi, \eta), \quad (u, v)(x, y, t) = (u, v)(\xi, \eta) \quad \text{for} \quad (\xi, \eta) = \Big(\frac{x}{t}, \frac{y}{t}\Big).$$

By introducing such a transformation the system (1.2) is reduced to

$$\begin{cases} \operatorname{div}(\rho\mathbf{v}) + 2\rho = 0, \\ \operatorname{div}(\rho\mathbf{v} \otimes \mathbf{v}) + 3\rho\mathbf{v} + \nabla p(\rho) = 0, \end{cases} \tag{1.7}$$

with $\text{div} = \text{div}_{(\xi,\eta)}$, $\nabla = \nabla_{(\xi,\eta)}$, $\mathbf{v}(\xi,\eta) := \mathbf{u}(\xi,\eta) - (\xi,\eta)$ being the pseudovelocity. Correspondingly, the initial state of the gas becomes the far field condition:

$$(\rho, \mathbf{v})(\xi,\eta) = (\rho_0, -\xi, -\eta) \quad \text{as } \xi^2+\eta^2 \text{ is sufficiently large.} \tag{1.8}$$

In the (ξ,η) coordinates, the domain Ω becomes

$$\widetilde{\Omega} = \{(\xi,\eta) \in \mathbb{R}^2 : \xi \le |\eta| \cot\theta_0 - u_0\}.$$

Its boundaries are respectively

$$\widetilde{W_u} = \{(\xi,\eta) : \eta \ge 0, \eta = (\xi+u_0)\tan\theta_0\} \quad \text{and} \quad \widetilde{W_l} = \{(\xi,\eta) : \eta \le 0, \eta = -(\xi+u_0)\tan\theta_0\}.$$

Under the transformation $\widetilde{\xi} = \xi + u_0$, $\widetilde{\eta} = \eta$, the equations (1.7) are formally invariant. Correspondingly, the state of the gas in the far field should satisfy

$$(\widetilde{\rho},\ \widetilde{\mathbf{v}})(\widetilde{\xi},\widetilde{\eta}) = (\rho_0,\ u_0 - \widetilde{\xi},\ \ -\widetilde{\eta}).$$

The domain $\widetilde{\Omega}$ becomes

$$\{(\widetilde{\xi},\widetilde{\eta}) :\ \widetilde{\xi} \le |\widetilde{\eta}| \cot\theta_0\}$$

with boundaries

$$\widetilde{W_u} = \{(\widetilde{\xi},\ \widetilde{\eta}) :\ \widetilde{\eta} \ge 0, \widetilde{\eta} = \widetilde{\xi}\tan\theta_0\} \quad \text{and} \quad \widetilde{W_l} = \{(\widetilde{\xi},\widetilde{\eta}) : \widetilde{\eta} \le 0,\ \widetilde{\eta} = -\widetilde{\xi}\tan\theta_0\}.$$

Hereafter, we will go on in these new coordinates and drop "$\sim$" for simplification without confusion. This transformation makes the piston be fixed and the gas move to the piston with initial velocity $(u_0, 0)$. Thus we can equivalently consider the dynamical problem caused by uniformly moving gas hitting a fixed wedge with initial data $(\rho_0, u_0, 0)$.

Since the problem is symmetric with respect to the η-axis, it is sufficient to consider the problem in the upper half-plane $\eta > 0$ outside the wedge

$$\Lambda := \{\xi \le \eta\cot\theta_0, \eta \ge 0\}. \tag{1.9}$$

Then the piston problem in the (x, y, t)-coordinates can be reduced to the following boundary value problem in the self-similar coordinates (ξ,η).

Problem 1.1 Seek a solution (ρ, u, v) of system (1.7) in the self-similar domain Λ with the slip boundary condition on $\partial\Lambda$:

$$(u, v) \cdot \nu|_{\partial\Lambda} = 0, \tag{1.10}$$

and the far field condition at infinity:

$$(u, v)(\xi,\eta) \to (u_0, 0) \quad \text{as } \xi^2+\eta^2 \to +\infty. \tag{1.11}$$

In this paper, we have the following main result for the two-dimensional piston problem.

Theorem 1.1 *In the piston Problem* 1.1*, if a piston satisfying* (1.1) *moves from right into the static Chaplygin gas on the left with uniform velocity* $(-u_0, 0)$ *and* $u_0 > 0$*, then there exists a piecewise smooth flow field satisfying* (1.10) *and* (1.11) *for* $\theta_0 \in (0, \frac{\pi}{2})$*. In particular, if the Mach number* $M_0 = \frac{u_0}{c_0} < 1$*, then ahead of the piston there is a bow shock away from its tip; if* $\frac{1}{\sin\theta_0} > M_0 \geq 1$*, then there is a shock attached to the tip of the piston; and if* $M_0 \geq \frac{1}{\sin\theta_0}$*, then a part of gas will concentrates on the surface of the piston* (*called mass concentration*)*.*

One can also discuss the receding case, i.e., the piston recedes away from the gas ($u_0 < 0$). For this case we have the second conclusion.

Theorem 1.2 *If the piston problem satisfying* (1.1) *recedes away from the gas with uniform velocity* $(u_0, 0)$*, i.e.,* $u_0 < 0$*, then there exists a piecewise smooth flow field satisfying* (1.10) *and* (1.11) *for* $\theta_0 \in (0, \frac{\pi}{2})$*. There is always a rarefaction wave in front of the piston. Particularly, if* $\frac{c_0}{|u_0|} < \cos\theta_0 - \sin\theta_0$*, then there will appear an additional shock issuing from the tip of the piston and stopping at the sonic circle.*

The study of Problem 1.1 will be divided into three steps. First we determine the waves far away from the tip of the wedge. Due to the finite speed of wave propagation, the tip of the wedge has no influence on the flow field far away from it. Therefore, when we consider the flow field far away from the tip, the piston can be assumed as a half plane, i.e., $\theta_0 = \frac{\pi}{2}$. And it is equivalent to determine the positions of the wave front and the state of the gas behind the wave front for a one-dimensional problem. Next, we investigate the interaction of the incoming waves, and determine the position of all new resulting waves, as well as the states in the corresponding regions bounded by these new waves up to the sonic circles. Finally, we give the existence of the solution in the domain bounded by the sonic circle and the surface of the wedge by using the theory of elliptic equation.

For our discussion later, let us briefly recall some basic properties on the Chaplygin gas. These results will be extensively employed in this paper, and their proofs can be found in [9, 20, 22]. The basic facts for the self-similar solutions of Euler system for Chaplygin gas are:

(1) The state equation of the Chaplygin gas is

$$p(\rho) = a^2\Big(\frac{1}{\rho_*} - \frac{1}{\rho}\Big) \tag{1.12}$$

with ρ_* a given constant. Hence the sonic speed c is inversely proportional to the density ρ.

(2) All characteristics are linearly degenerate. The slope of shock is equal to the slope of corresponding characteristics. Any particle of the fluid moves across the shock with sonic speed as relative velocity.

(3) Any rarefaction wave degenerates to a characteristic, so that its width is zero and thus can be regarded as a shock with negative strength. Both the rarefaction waves and the shocks are called pressure waves.

(4) On the plane of the self-similar coordinate variables $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$, all pressure waves are tangential to the sonic circle of the state on both sides of the pressure wave.

(5) If two states U_l and U_r are connected by a slip line, then both (u_l, v_l) and (u_r, v_r) should be located on the line and $\rho_l = \rho_r$.

The above five properties allow us to be able to construct the self-similar solution of two-dimensional Euler system for Chaplygin gas in the hyperbolic region of the piston problem in this paper.

In the sequel, the letter c_i denotes the sonic speed corresponding to the state (ρ_i, u_i, v_i) without more explanation.

In the remaining part of this paper, our main effort is to prove Theorems 1.1 and 1.2. The paper is organized as follows. In Section 2 we study the case when the front of the moving body is a straight line with an inclined angle, and get the expression of the solution. In Sections 3 and 4 we consider the case when the moving body is a convex wedge. Particularly, in Section 3 we assume that the piston is pushed to the gas with the velocity $(u_0, 0)$ and $u_0 > 0$. By constructing the solution in the supersonic domain and proving the existence of solution in the subsonic domain, we complete the proof of Theorem 1.1. After that, in Section 4 we study the case that the piston is pulled away from the gas. We also establish the global existence of solution in this case and then complete the proof of Theorem 1.2. The proofs of these two theorems are the main contents of this paper. The conclusions are also summarized in two tables in Sections 3 and 4, respectively. In Section 5 we give a brief discussion on the case that the piston is a concave body, the description of which is given there. Finally, an appendix is given in Section 6 citing Grisvard's results on the regularity of the solutions of elliptic equations in a domain with corners.

2 Reduced Case: The Piston Reduces to a Half Plane

Consider the case that the piston is a moving half plane, making an inclination angle β with the x-axis. The initial data are

$$(\rho, u, v)(x, y, 0) = (\rho_0, u_0, 0), \quad x < y \cot \beta. \tag{2.1}$$

In this case, the problem is one-dimensional, it can be easily solved by using the method in [9–10]. Next we only list the corresponding results and omit the corresponding proof.

We should first determine the possible waves in front of the wall. The boundary condition (1.10) requires that $u \sin\beta - v\cos\beta = 0$ on $\partial\Lambda$. The fact point (5) in Section 1 indicates that the uniform flow connected to the wall by a slip line is impossible. In addition, all possible waves should locate on the left hand side of the wall requires $u_0 < c_0$. Thus only one wave denoted by L_{01} would be present in front of the wall. Directive calculation gives the location of the wave L_{01}:

$$\eta = \Big(\xi - u_0 + \frac{c_0}{\sin\beta}\Big) \tan\beta \tag{2.2}$$

as well as the state between the wave and the wall:

$$(\rho_1,\ u_1,\ v_1) = \Big(\frac{a}{c_0 - u_0 \sin\beta},\ u_0\cos^2\beta,\ u_0\cos\beta\sin\beta\Big). \tag{2.3}$$

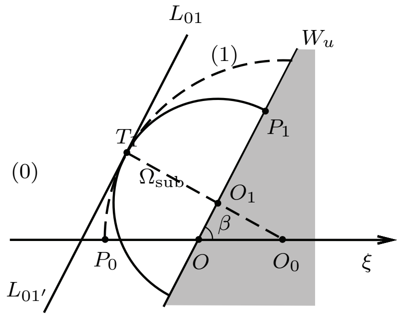

Figure 2 Normal case.

The shock L_{01} will terminate at a point T_1 of the sonic circle C_0, with

$$\begin{cases}\xi_{T_1} = u_0 - c_0 \sin\beta,\\ \eta_{T_1} = c_0 \cos\beta.\end{cases} \tag{2.4}$$

Another shock $L_{01'}$ coming from the opposite direction and having the same formula with L_{01} terminates at T_1, too. The state of the gas on the left hand side of $L_{01} \cup L_{01'}$ is U_0 and in the domain between the waves $L_{01} \cup L_{01'}$ and the wall is $(\frac{a}{c_0 - u_0 \sin\beta}, u_0 \cos^2\beta,\ u_0 \cos\beta \sin\beta)$. The subsonic domain is bounded by the sonic circle C_1 centered at the point

$$O_1(u_0 \cos^2\beta, u_0 \sin\beta \cos\beta)$$

with radius c_1 and the wall (see Figure 2).

Remark 2.1 (1) Note that the total mass on each segment between the shock wave and the wall parallel to ξ-axis is $\frac{a}{\sin\beta}$, which is independent of the initial data ρ_0 and u_0. It means that once the piston proceeds to the gas too quickly, say, $u_0 \geq \frac{c_0}{\sin\beta}$, then the shock and all gas between the shock and the wall adhere the wall and then form a concentration of mass like a Dirac measure. Such a phenomenon is called concentration (see [3, 22]).

(2) To avoid concentration, or in other words, c_1 makes sense if and only if

$$\frac{u_0}{c_0} < \frac{1}{\sin\beta} \quad \text{for } \beta \in \Big(0, \frac{\pi}{2}\Big). \tag{2.5}$$

If the initial Mach number is less than 1, then there exists a solution containing one shock in front of the piston, no matter what angle the piston makes with the ξ-axis. If the Mach number is greater than 1, the same conclusion also holds for suitably small β.

(3) For given static uniform gas, the faster the piston moves to the gas, the larger the strength of the shock wave is, as well as the density after the wave. In addition, if the piston moves to the gas with a fixed velocity, then the strength of the waves increases as the angle β, made by the velocity and the profile of the piston, increases. Furthermore, from (2.3) we know that as β increases from 0 to $\frac{\pi}{2}$, ρ_1 increases from ρ_0 to $\frac{a}{c_0 - u_0}$, while v_1 increases from 0 to $\frac{u_0}{2}$ if $\beta \in (0, \frac{\pi}{4})$, and decreases from $\frac{u_0}{2}$ to 0 if $\beta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

(4) The sonic circle C_0 intersects with the ξ-axis at the point P_0, which is the origin if $\frac{u_0}{c_0}=1$, and is on the right hand side of W_u if $\frac{u_0}{c_0}\in(1,\frac{1}{\sin\beta})$. It indicates that the origin may locate in the supersonic domain or the subsonic domain for different initial Mach number.

Remark 2.2 For the potential flow with state equation $p=\rho^\gamma,\gamma>1$, it is sufficient to consider the case $\beta=\frac{\pi}{2}$. The location of the shock wave is

$$\xi_0=-\frac{\rho_0 u_0}{\rho_1-\rho_0}. \tag{2.6}$$

From the Bernoulli's law we can get

$$\frac{u_0^2}{\rho_0^{\gamma-1}}=\frac{2\gamma}{\gamma-1}\frac{t-1}{t+1}(t^{\gamma-1}-1) \tag{2.7}$$

with $t=\frac{\rho_1}{\rho_0}\geq 1$.

Let $f(t)=\frac{t-1}{t+1}(t^{\gamma-1}-1)$, then for $t>1$,

$$f'(t)=\frac{2(t^{\gamma-1}-1)+(\gamma-1)(t+1)t^{\gamma-2}(t-1)}{(t+1)^2}>0. \tag{2.8}$$

Hence the inverse of $f(t)$ is well defined in $(1,+\infty)$. It means that for fixed ρ_0 any initial velocity u_0 corresponds to unique density ρ_1 behind a shock. In accordance, (2.6) gives the value of $|\xi_0|>0$. Therefore, the concentration phenomena could not occur for the polytropic gas.

3 Proceeding Piston Problem: $u_0>0$

In this case, the condition $\frac{u_0}{c_0}<\frac{1}{\sin\theta_0}$ is required in advance due to (1) and (2) of Remark 2.1.

By symmetry we only need to construct the solution on the upper half plane. As mentioned in Section 1, the waves far away from the origin are obtained by the normal case in Section 2 with $\beta=\theta_0$ directively. And there is one wave parallel to the upper wall W_u denoted still by L_{01} satisfying (2.2) (see Figure 3). The wave terminates at $T_1(u_0-c_0\sin\theta_0,c_0\cos\theta_0)$, the sonic point of both states U_0 and U_1. The state of the gas far away from the origin between W_u and L_{01} is

$$(\rho_1,u_1,v_1)=\Big(\frac{a}{c_0-u_0\sin\theta_0},u_0\cos^2\theta_0,u_0\cos\theta_0\sin\theta_0\Big). \tag{3.1}$$

It is possible that the tip of the wedge may locates in the supersonic region (outside the circle C_0). In this case, the uniform incoming flow does not satisfy the boundary condition, then new waves result in. According to this possibility, we consider the next three different cases specified by the initial data: (1) $0<\frac{u_0}{c_0}<1$; (2) $\frac{u_0}{c_0}=1$; (3) $\frac{u_0}{c_0}\in(1,\frac{1}{\sin\theta_0})$.

(1) If $0<\frac{u_0}{c_0}<1$, the left intersection point $P_0(u_0-c_0,0)$ of C_0 and the ξ-axis locates on the left hand side of W_u. The state of gas on the left hand side of the wave L_{01} can be kept unchanged up to its sonic circle C_0, i.e., $\widehat{P_0T_1}$ in Figure 3, due to the finite speed of wave

propagation. The coordinates of T_1 is given by (2.4). The sonic circles $C_0,\ C_1$, the ξ-axis and the wall W_u bound a domain denoted by $\Omega_{\rm sub}$ (see Figure 3). The point $P_1(u_0\cos^2\theta_0+(c_0-u_0\sin\theta_0)\cos\theta_0, u_0\cos\theta_0\sin\theta_0+(c_0-u_0\sin\theta_0)\sin\theta_0)$ is the intersection point of C_1 and W_u.

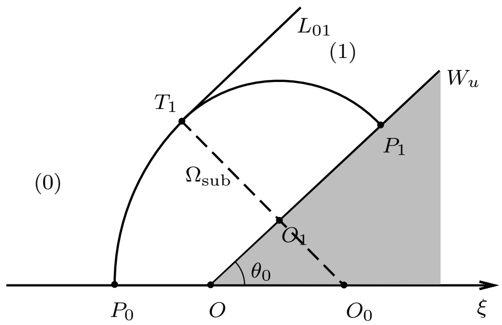

Figure 3 The piston proceeds to the gas with $M_0<1$.

Denote by Ω_0 the domain bounded by the part of the ξ-axis left of P_0, the arc $\widehat{P_0T_1}$ of the sonic circle C_0 and the wave L_{01}, then the state of the gas in Ω_0 is $(\rho_0,u_0,0)$. Denote by Ω_1 the domain bounded by the wave L_{01}, the arc $\widehat{P_1T_1}$ of the sonic circle C_1 and the wall W_u, then the state of the gas in Ω_1 is (ρ_1,u_1,v_1) given by (3.1).

So far, we have determined the supersonic domain composed of Ω_i and the state inside it for $i=0,1$ respectively. The subsonic domain $\Omega_{\rm sub}$ is bounded by the sonic circles $C_0,\ C_1$, the ξ-axis and the wall W_u. Hereafter, we denote the sonic part of the boundary of $\Omega_{\rm sub}$ by $\partial_1\Omega_{\rm sub}$ and the other part by $\partial_2\Omega_{\rm sub}$. We have here $\partial_1\Omega_{\rm sub}=\widehat{P_1T_1}\cup\widehat{T_1P_0}$ and $\partial_2\Omega_{\rm sub}=\overline{OP_1}\cup\overline{P_0O}$.

To get the global existence of solution to Problem 1.1, the state of the gas in $\Omega_{\rm sub}$ needs to be determined. We now focus on it. On the one hand, we know that the irrotational property could be kept forever for both steady and pseudosteady piecewise smooth flow of Chaplygin gas (see [22]). On the other hand, by the analysis in normal case, there is no slip line in front of the wall. Thus a velocity potential Φ can be introduced, such that the velocity $\mathbf{u}=\nabla\Phi$.

For isentropic irrotational flow, the Bernoulli's law reads in [12],

$$\Phi_t+\frac{1}{2}|\nabla_{(x,y)}\Phi|^2+i=\text{const}, \tag{3.2}$$

where i is the enthalpy satisfying $\mathrm{d}i=\frac{1}{\rho}\mathrm{d}p$. Then we have by (1.3)

$$\Phi_t+\frac{1}{2}|\nabla_x\Phi|-\frac{a^2}{2\rho^2}=\text{const}. \tag{3.3}$$

In addition, we have $\Phi(t,x,y)=t(\phi(\xi,\eta)+\frac{1}{2}(\xi^2+\eta^2))$ for some ϕ satisfying $\nabla_{(\xi,\eta)}\phi=\mathbf{v}(\xi,\eta)$, and (3.3) is reduced to

$$\frac{1}{2}|\nabla\phi|^2+\phi-\frac{a^2}{2\rho^2}=\text{const}. \tag{3.4}$$

Looking for self-similar solution, we have $|\mathbf{v}|^2 = \frac{a^2}{\rho^2}$ on the sonic part of the boundary of Ω_{sub}, which means that along this part the potential ϕ is constant, chosen as zero without loss of generality. Then (3.4) gives

$$\rho = \frac{a}{\sqrt{2\phi + |\nabla\phi|^2}}. \tag{3.5}$$

It together with the first equation of (1.2) gives

$$\operatorname{div}\frac{\nabla\phi}{\sqrt{2\phi + |\nabla\phi|^2}} + \frac{2}{\sqrt{2\phi + |\nabla\phi|^2}} = 0. \tag{3.6}$$

The boundary condition on $\partial_1\Omega_{\text{sub}}$ is $\phi = 0$, and on $\partial_2\Omega_{\text{sub}}$ is $\frac{\partial\phi}{\partial\nu} = 0$, where ν is the outward unit normal. Therefore, it holds the following boundary value problem in the subsonic domain.

$$\begin{cases} \operatorname{div}\dfrac{\nabla\phi}{\sqrt{2\phi + |\nabla\phi|^2}} + \dfrac{2}{\sqrt{2\phi + |\nabla\phi|^2}} = 0 & \text{in } \Omega_{\text{sub}}, \\ \phi = 0 & \text{on } \partial_1\Omega_{\text{sub}}, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial_2\Omega_{\text{sub}}. \end{cases} \tag{3.7}$$

Set $\psi = \sqrt{2\phi}$, then (3.6) becomes

$$\operatorname{div}\frac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}} + \frac{2}{\psi\sqrt{1 + |\nabla\psi|^2}} = 0, \tag{3.8}$$

and the boundary value problem (3.7) is reduced to

$$\begin{cases} \operatorname{div}\dfrac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}} + \dfrac{2}{\psi\sqrt{1 + |\nabla\psi|^2}} = 0 & \text{in } \Omega_{\text{sub}}, \\ \psi = 0 & \text{on } \partial_1\Omega_{\text{sub}}, \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial_2\Omega_{\text{sub}}. \end{cases} \tag{3.9}$$

For this problem we have the following result.

Lemma 3.1 *There exists a unique positive solution to* (3.9). *This solution belongs to* $C^{1,\alpha}(\overline{\Omega}_{\text{sub}} \setminus \partial_1\Omega_{\text{sub}}) \cap C(\overline{\Omega}_{\text{sub}})$ *with* $\alpha = \frac{\theta_0}{\pi - \theta_0}$, *and is* C^∞ *smooth both in* Ω_{sub} *and at the interior points of* $\partial_2\Omega_{\text{sub}}$ *except* O.

The basic idea of the proof of this lemma is referred to [22], but the appearance of the corner of the domain and the Neumann boundary condition requires some modification. For reader's convenience we write the detailed proof here.

By introducing parameters μ and with $0 \leq \mu \leq 2,\ \varepsilon > 0$, we consider the boundary value problem:

$$
\begin{cases}
\operatorname{div}\dfrac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}+\dfrac{\mu}{\psi\sqrt{1+|\nabla\psi|^2}}=0 & \text{in } \Omega_{\rm sub},\\
\psi=\varepsilon & \text{on } \partial_1\Omega_{\rm sub},\\
\dfrac{\partial\psi}{\partial\nu}=0 & \text{on } \partial_2\Omega_{\rm sub}.
\end{cases}
\tag{3.10}
$$

The problem (3.9) is the case $\mu=2$ and $\varepsilon=0$ for the problem (3.10). Denote by $\psi_{\mu,\varepsilon}$ the solution to (3.10), we will prove the existence of $\psi_{\mu,\varepsilon}$ for all $0\le\mu\le 2$ and $\varepsilon>0$, and then prove the existence of the limit of $\psi_{\mu,\varepsilon}$ as $\varepsilon\to 0$.

To simplify the notations we often denote $\Omega_{\rm sub}$ by Ω in the sequel.

For the case $\varepsilon>0$, $\psi_{0,\varepsilon}=\varepsilon$ is the solution of (3.10) with $\mu=0$. Let J_ε be the set of $\mu\in[0,2]$, such that the solution $\psi_{\mu,\varepsilon}$ exists and satisfies

$$
\varepsilon\le\psi_{\mu,\varepsilon}< C_\varepsilon,\quad |\nabla\psi_{\mu,\varepsilon}|< C_\varepsilon,
\tag{3.11}
$$

where C_ε is a positive constant depending on ε only. We are going to prove $J_\varepsilon\equiv[0,2]$. It can be derived by using the fact that J_ε is closed and open, because of $0\in J_\varepsilon$.

Before studying the property of the set J_ε, let us first derive the linearized problem of (3.10). By a direct computation we have the linearized problem for unknown function v as

$$
\begin{cases}
Lv+\Sigma c_i\dfrac{\partial v}{\partial x_i}+dv=f & \text{in } \Omega,\\
v=0 & \text{on } \partial_1\Omega_{\rm sub},\\
\dfrac{\partial v}{\partial\nu}=0 & \text{on } \partial_2\Omega_{\rm sub},
\end{cases}
\tag{3.12}
$$

where

$$
\begin{aligned}
&Lv=(1+\psi_{x_2}^2)v_{x_1x_1}-2\psi_{x_1}\psi_{x_2}v_{x_1x_2}+(1+\psi_{x_1}^2)v_{x_2x_2},\\
&c_i=-\frac{\mu}{\psi}\psi_{x_i},\\
&d=-\frac{\mu}{\psi^2}(1+|\nabla\psi|^2).
\end{aligned}
$$

Obviously, the operator L is uniformly elliptic under the assumption (3.11) and $d<0$. Notice that the boundary $\partial_1\Omega_{\rm sub}$ is of $C^{1,1}$ and piecewise C^2, while the boundary $\partial_2\Omega_{\rm sub}$ consists of $\overline{OP_0}$ and $\overline{OP_1}$ forming an angle $\pi-\theta$. Besides, $\partial_1\Omega_{\rm sub}$ perpendicularly intersects $\partial_2\Omega_{\rm sub}$ at P_0 and P_1, so that by using reflection the corners P_0 and P_1 can be eliminated. The point O is the corner of the domain Ω, which has to be considered more carefully.

Return to consider the property of the set J_ε. To prove the openness of J_ε we assume that $\mu_0\in J_\varepsilon$ and the corresponding solution to (3.10) is $\psi_{\mu_0,\varepsilon}$. Taking the linearized problem at $\psi_{\mu_0,\varepsilon}$ as in (3.12), it is shown in Appendix that such a linearized problem gives a one-to-one mapping from $f\in C^{-1+\alpha}(\Omega)$ to $v\in C^{1+\alpha}(\Omega)$, and the solution v satisfies the estimate

$$
\|v\|_{C^{1+\alpha}(\Omega)}\le C\|f\|_{C^{-1+\alpha}(\Omega)},
\tag{3.13}
$$

where $C^{-1+\alpha}(\Omega)$ is defined as a set of functions f, which can be decomposed as

$$f = \sum_{i=1,2} \frac{\partial f_i}{\partial x_i} + f_0, \quad f_i \in C^\alpha(\Omega) \quad \text{for} \;\; 0 \le i \le 2,$$

while the $C^{-1+\alpha}(\Omega)$ norm is defined by

$$\|f\|_{-1+\sigma,\Omega} = \inf \sum_{i=0}^{2} \|f_i\|_{\sigma,\Omega}.$$

The above argument means that the mapping defined by the linearized operator is invertible at $(\mu_0, \psi_{\mu_0,\varepsilon})$. Thus by the implicit function theorem there exists a neighborhood of μ_0, such that the mapping $\mu \mapsto \psi_{\mu,\varepsilon}$ is well defined there. This fact implies the openness of J_ε at μ_0.

To prove the closeness of J_ε we take a sequence $\{\mu_m\} \subset J_\varepsilon$ with $\mu_m \to \mu$. We notice that the operator in (3.10) (multiplied by a factor $(1+|\nabla\psi|^2)^{\frac{3}{2}}$) is uniformly elliptic and satisfies the maximum principle. Then the $C^{1+\alpha}$ estimate as shown in Appendix implies the bounds of $\psi_{\mu_m,\varepsilon}$ in $C^{1+\alpha}(\overline{\Omega})$, which is independent of m. Hence we can choose a subsequence $\{\mu_{m_k}\}$, such that the corresponding $\{\psi_{\mu_{m_k},\varepsilon}\}$ is convergent in $C^{1,\alpha'}$ with $\alpha' < \alpha$, and the limit $\psi_{\mu,\varepsilon}$ is in $C^{1,\alpha}$. It implies $\mu \in J_\varepsilon$.

Hence $J_\varepsilon \equiv [0,2]$, which gives the existence of $\psi_{2,\varepsilon}$.

Now let us study the limit of $\psi_{2,\varepsilon}$ as $\varepsilon \to 0$. Since ψ can neither reach its maximum inside Ω nor on $\partial_2\Omega_{\text{sub}}\backslash\{O\}$, then by the maximum principle we have $\psi_{2,\varepsilon} \ge \varepsilon$.

Take a family of functions $\phi^{r,m}(x) = \frac{1}{2}(r^2 - |x-m|^2)$. Obviously, $\phi^{r,m}(x)$ is a solution of (3.7) for any $m \in R^2$, $r > 0$. Correspondingly, the function $\psi^{r,m} = \sqrt{2\phi^{r,m}}$ satisfies (3.8), if $\phi^{r,m}(x) > 0$. Moreover, $\psi^{r,m}$ satisfies $\psi \ge \varepsilon$ on $\partial_1\Omega_{\text{sub}}$ if $\Omega \subset B_{\sqrt{r^2-\varepsilon^2}}(m)$, and satisfies $\psi \le \varepsilon$ on $\partial_2\Omega_{\text{sub}}$ if $B_{\sqrt{r^2-\varepsilon^2}}(m) \subset \Omega$.

Consider the normal derivatives $\frac{\partial\psi^{r,m}}{\partial\nu}$. On the line $\theta = \theta_0$, the outward unit normal direction is $(\sin\theta_0, -\cos\theta_0)$. Then

$$\frac{\partial\phi^{r,m}}{\partial\nu} = (m_1 - x_1)\sin\theta_0 - (m_2 - x_2)\cos\theta_0 = m_1\sin\theta_0 - m_2\cos\theta_0. \tag{3.14}$$

Obviously, $\frac{\partial\phi^{r,m}}{\partial\nu}\big|_{\theta=\theta_0} > 0$, if $m \in \Sigma' = \{\theta_0 + \pi < \theta < 2\pi\}$. Similarly, $\frac{\partial\phi^{r,m}}{\partial\nu}\big|_{\theta=\pi} > 0$. Therefore, $\frac{\partial\psi^{r,m}}{\partial\nu}$ satisfies the same inequalities on $\theta = \theta_0$ and $\theta = \pi$. It implies that $\psi^{r,m}$ is a supersolution to (3.9), as $m \in \Sigma'$ and $\Omega \subset B_{\sqrt{r^2-\varepsilon^2}}(m)$. Taking ψ^+_ε as the infimum of all these $\psi^{r,m}$, we have $\varepsilon \le \psi_{2,\varepsilon} \le \psi^+_\varepsilon$.

On the other hand, if m locates in $\Sigma = \{\theta_0 < \theta < \pi\}$ and $B_{\sqrt{r^2-\varepsilon^2}}(m) \subset \Omega$, then $\psi^{r,m}$ satisfies the equation (3.8) in Ω, $\psi \le \varepsilon$ on $\partial_1\Omega_{\text{sub}}$ and $\frac{\partial\psi}{\partial\nu} < 0$ on $\partial_2\Omega_{\text{sub}}$. Hence $\psi^{r,m}$ is a subsolution to (3.9), so that $\psi^\varepsilon \ge \psi^{r,m}$. Denoting by ψ^-_ε the supremum of all these $\psi^{r,m}$, we have $\psi_{2,\varepsilon} \ge \psi^-_\varepsilon \ge 0$.

In view of the fact that $\psi_{2,\varepsilon}$ monotonically decreases as $\varepsilon \to 0$, and is bounded below, then $\psi(x) = \lim\limits_{\varepsilon\to 0} \psi_{2,\varepsilon}(x)$ are well defined. To prove that $\phi(x) = \frac{1}{2}\psi(x)^2$ is actually the solution to (3.7), we have to establish the uniform boundedness of $\|\nabla\phi_{2,\varepsilon}\|_{L^\infty}$.

First, the normal vector of the boundaries $\theta = \theta_0$ and $\theta = \pi$ are different, and $\frac{\partial \phi_{2,\varepsilon}}{\partial \nu}\big|_{\partial_2\Omega_{\rm sub}} = 0$, then $\nabla\phi_{2,\varepsilon} = 0$ at 0.

To derive the uniform boundedness of $\nabla\phi_{2,\varepsilon}$ inside Ω, we define $z_\varepsilon = \frac{1}{2}|\nabla\phi_{2,\varepsilon}|^2$ and establish the boundedness of it. To simplify notations we will simply denote $\phi_{2,\varepsilon}$ and z_ε by ϕ and z in the sequel. Expanding (3.7) we have

$$L\phi + 2z + 4\phi = 0, \tag{3.15}$$

where $L = (2\phi + 2z)\Delta - \nabla\phi \otimes \nabla\phi : D^2$. Acting the operator ∇ to (3.15) and then multiplying by $\nabla\phi$, we have

$$\nabla\phi \cdot \nabla L\phi + 2\nabla\phi \cdot \nabla z + 8z = 0.$$

Direct calculation gives

$$\begin{aligned}\nabla L\phi = &(2\nabla\phi + 2\nabla z)\Delta\phi + (2\phi + 2z)\nabla\Delta\phi \\ &- (\nabla\phi_x^2\phi_{xx} + 2\nabla(\phi_x\phi_y)\phi_{xy} + \nabla\phi_y^2\phi_{yy}) - \nabla\phi \otimes \nabla\phi : D^2\nabla\phi.\end{aligned}$$

$$\nabla z \cdot \nabla Lz = 4z\Delta z + 2\Delta\phi\nabla\phi \cdot \nabla z + Lz - (2\phi + 2z)\mathrm{Tr}(D^2\phi)^2 - |\nabla z|^2.$$

Then z satisfies

$$Lz + (2\Delta\phi\nabla\phi - \nabla z + 2\nabla\phi) \cdot \nabla z + (4\Delta\phi - 2\mathrm{Tr}(D^2\phi)^2 + 8)z = 2\phi\mathrm{Tr}(D^2\phi)^2, \tag{3.16}$$

and the right hand side is obviously nonnegative. (3.16) shows that if the coefficient of z is negative, we can use the maximum principle to estimate z. However, we do not know the sign of $4\mathrm{Tr}(D^2\phi) - 2\mathrm{Tr}(D^2\phi)^2 + 8$. To make remedy we use (3.15) and replace the variable z by $z + \alpha\phi$. By adding (3.16) with (3.15) multiplied by α, we obtain

$$L(z + \alpha\phi) + (2\Delta\phi\nabla\phi + 2\nabla\phi) \cdot \nabla z + (4\Delta\phi - 2\mathrm{Tr}(D^2\phi)^2 + 8 + 2\alpha)z + 4\alpha\phi \geq 0. \tag{3.17}$$

That is

$$L(z + \alpha\phi) + (2\Delta\phi\nabla\phi + 2\nabla\phi) \cdot \nabla(z + \alpha\phi) + 2Mz + 4\alpha\phi \geq 0, \tag{3.18}$$

where $M = -\mathrm{Tr}(D^2\phi)^2 + 2(1 - \alpha)\Delta\phi - \alpha^2 - \alpha - 4$.

Note that by reflection any point on $\partial_2\Omega_{\rm sub}$ except the corner O can be treated as the inner point of the domain Ω, since the boundary condition on $\partial_2\Omega_{\rm sub}$ is homogeneous Neumann type condition, and the equation satisfied by ϕ is reflection symmetry with respect to the two straight sides of $\partial_2\Omega_{\rm sub}$ (see [18, Lemma 6.18]). Similarly, the points $\partial_1\Omega_{\rm sub} \cap \partial_2\Omega_{\rm sub}$ can be treated as the inner points of $\partial_1\Omega_{\rm sub}$. Now if $z + \alpha\phi$ arrive at local maximum at some point y_0 inside Ω or on $\partial_2\Omega_{\rm sub} \setminus \{O\}$, then $\nabla(z + \alpha\phi) = 0$ and $L(z + \alpha\phi) \leq 0$. Hence

$$Mz \geq -2\alpha\phi \tag{3.19}$$

there. Due to $\Delta\phi = \mathrm{Tr}(D^2\phi)$, we can obtain $M \le -\frac{1}{4}$ by choosing $\alpha = \frac{5}{2}$. Furthermore, $\nabla\phi = 0$ due to the boundary condition on $\partial_2\Omega_{\mathrm{sub}}$, then we have

$$\begin{aligned}\sup_{\Omega}\Big(z+\frac{5}{2}\phi\Big) &\le \max\Big[\sup_{\partial_1\Omega_{\mathrm{sub}}\cup O}\Big(z+\frac{5}{2}\phi\Big), \frac{45}{2}\sup_{\Omega}\phi_+\Big] \\ &\le \max\Big[\sup_{\partial_1\Omega_{\mathrm{sub}}}\Big(z+\frac{5}{2}\phi\Big), \frac{45}{2}\sup_{\Omega}\phi_+\Big]. \qquad (3.20)\end{aligned}$$

Therefore, $\nabla\phi$ (a simplified notation of $\nabla\phi_{2,\varepsilon}$) is bounded on the whole $\overline{\Omega}$.

Summing up, $\phi_{2,\varepsilon}$ and $\psi_{2,\varepsilon}$ are convergent as $\varepsilon \to 0_+$, and the limit of them are the solutions of (3.7) and (3.10), respectively. Furthermore, by using classical elliptic theory (see [18]) the solution is C^∞ inside of Ω. Moreover, at any point Q on $\partial_2\Omega_{\mathrm{sub}} \setminus O$ we can use reflection to reduce the case to that for an interior point of domain, because the boundary condition is simply the Neumann condition. Hence ψ is also C^∞ at the point Q.

If $\theta_0 \to \frac{\pi}{2}$, then we have

$$\xi_{T_1} = u_0 - c_0\sin\theta_0 \to u_0 - c_0, \quad \eta_{T_1} = c_0\cos\theta_0 \to 0. \qquad (3.21)$$

In other words, T_1 will coincide with P_0, and L_{01} will be vertical to the ξ-axis. The limit case is nothing but the solution of the normal symmetric case.

(2) If $\frac{u_0}{c_0} = 1$, then the intersection point P_0 of C_0 and the ξ-axis coincides with the origin (see Figure 4). The domain Ω_0 with state of gas $(\rho_0, u_0, 0)$ is bounded by the negative ξ-axis, $\widehat{OT_1}$ and the wave L_{01}. The domain Ω_1 with state of gas (ρ_1, u_1, v_1) is just the same as above. The difference from the case $\frac{u_0}{c_0} < 1$ is that P_0 here coincides with the origin O. Hence the boundary of the domain Ω_{sub} is $\partial_1\Omega_{\mathrm{sub}} = \widehat{P_1T_1} \cup \widehat{T_1O}$ and $\partial_2\Omega_{\mathrm{sub}} = \overline{P_1O}$. The angle formed by the arc $\widehat{OT_1}$ and the straight line $\overline{OP_1}$ is $\theta_1 = \frac{\pi}{2} - \theta_0$. Since the boundary conditions assigned there are of Dirichlet type on $\widehat{OT_1}$ and of Neumann type on $\overline{OP_1}$, then the local regularity of the solutions of Laplacian there should be $C^{1+\alpha}$ with $\alpha = \frac{\pi}{2\theta_1} = \frac{\theta_0}{\pi-\theta_0}$ (see [19]). Therefore, by using the same argument we know that Lemma 2.1 also holds in the recent case.

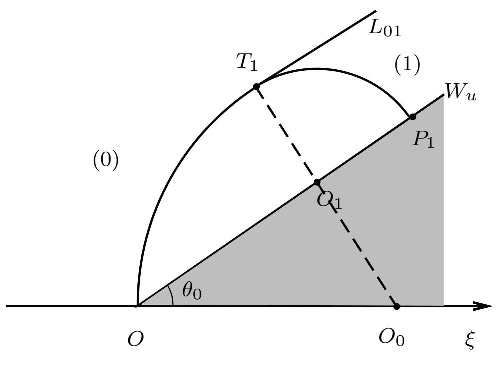

Figure 4 The piston proceeds to the gas with $M_0 = 1$.

If $\theta_0 \to \frac{\pi}{2}$, then $\xi_{T_1} = u_0 - c_0\sin\theta_0 \to u_0 - c_0 = 0$, $\eta_{T_1} = c_0\cos\theta_0 \to 0$. In this case, the gas will ultimately form a concentration on the surface of the wall. The state away from the wall is constant $(\rho_0, u_0, 0)$.

(3) If $\frac{u_0}{c_0} \in (1, \frac{1}{\sin\theta_0})$, then $c_0 - u_0 < 0$, the point P_0 locates on the right hand side of W_u. The origin is not enclosed by the sonic circle C_0. Note that the initial data and boundary condition do not satisfy the compatibility condition near the origin. Thus a new wave L_{02} will result in from the origin (see Figure 5). It will terminate at a point T_2 of C_0. Then we have $\overline{OT_2} \perp \overline{O_0T_2}$. Denote by U_2 the state near the origin and lies between L_{02} and W_u, and denote by O_2 the intersection point of W_u and $\overline{O_0T_2}$. Here we indicate that (ξ_{O_2}, η_{O_2}) is nothing but (u_2, v_2). Based on this analysis, we can determine the location of the wave L_{02}, the domain bounded by L_{02}, W_u and the circle C_2, as well as the state U_2 inside the domain as follows.

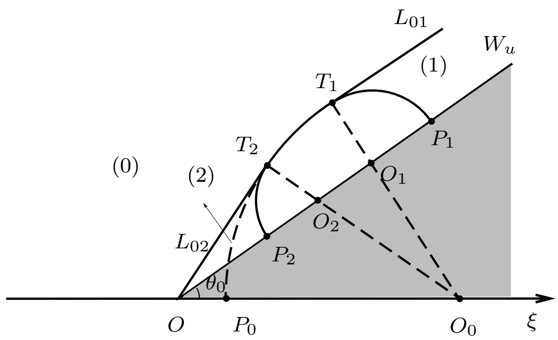

Figure 5 The piston proceeds to the gas with $M_0 \in \big(1, \frac{1}{\sin\theta_0}\big)$.

The angle made by $\overline{O_0T_2}$ and the ξ-axis is $\arccos\frac{c_0}{u_0}$. Then

$$|\overline{OO_2}| = \frac{u_0 \sin\left(\arccos\dfrac{c_0}{u_0}\right)}{\cos\left(\dfrac{\pi}{2} - \theta_0 - \arccos\dfrac{c_0}{u_0}\right)} = \frac{u_0\sqrt{u_0^2 - c_0^2}}{c_0\sin\theta_0 + \sqrt{u_0^2 - c_0^2}\cos\theta_0}.$$

The coordinates of O_2 are

$$\begin{cases} \xi_{O_2} = |\overline{OO_2}|\cos\theta_0 = \dfrac{u_0\sqrt{u_0^2 - c_0^2}\cos\theta_0}{c_0\sin\theta_0 + \sqrt{u_0^2 - c_0^2}\cos\theta_0}, \\ \eta_{O_2} = |\overline{OO_2}|\sin\theta_0 = \dfrac{u_0\sqrt{u_0^2 - c_0^2}\sin\theta_0}{c_0\sin\theta_0 + \sqrt{u_0^2 - c_0^2}\cos\theta_0}. \end{cases} \tag{3.22}$$

Thus, we have

$$(u_2, v_2) = \Big(\frac{u_0\sqrt{u_0^2 - c_0^2}\cos\theta_0}{c_0\sin\theta_0 + \sqrt{u_0^2 - c_0^2}\cos\theta_0}, \frac{u_0\sqrt{u_0^2 - c_0^2}\sin\theta_0}{c_0\sin\theta_0 + \sqrt{u_0^2 - c_0^2}\cos\theta_0}\Big). \tag{3.23}$$

In addition,

$$|\overline{OT_2}| = \sqrt{u_0^2 - c_0^2}.$$

The coordinates of the terminal point T_2 of wave L_{02} are

$$\begin{cases} \xi_{T_2} = |\overline{OT_2}| \sin\left(\arccos\dfrac{c_0}{u_0}\right) = \dfrac{u_0^2 - c_0^2}{u_0}, \\ \eta_{T_2} = |\overline{OT_2}| \cos\left(\arccos\dfrac{c_0}{u_0}\right) = \dfrac{c_0\sqrt{u_0^2 - c_0^2}}{u_0}. \end{cases} \tag{3.24}$$

Meanwhile, the sonic speed of the gas with state U_2 is

$$\begin{aligned} c_2 &= |\overline{OT_2}| \tan\angle T_2OO_2 \\ &= |\overline{OT_2}| \tan\left(\frac{\pi}{2} - \theta_0 - \arccos\frac{c_0}{u_0}\right) \\ &= \frac{\sqrt{u_0^2 - c_0^2}(c_0\cos\theta_0 - \sin\theta_0\sqrt{u_0^2 - c_0^2})}{c_0\sin\theta_0 + \cos\theta_0\sqrt{u_0^2 - c_0^2}}. \end{aligned} \tag{3.25}$$

The sonic circle of state U_2 is $C_2: \ (\xi - u_2)^2 + (\eta - v_2)^2 = c_2^2$. It intersects with W_u at a point P_2, with

$$\begin{cases} \xi_{P_2} = \xi_{O_2} - c_2\cos\theta_0, \\ \eta_{P_2} = \eta_{O_2} - c_2\sin\theta_0. \end{cases} \tag{3.26}$$

The boundary of the domain $\Omega_{\rm sub}$ in this case is

$$\partial_1\Omega_{\rm sub} = \widehat{P_2T_2} \cup \widehat{T_2T_1} \cup \widehat{T_1P_1}, \tag{3.27}$$

corresponding to the arcs of sonic circles C_2, C_0 and C_1 respectively, and

$$\partial_2\Omega_{\rm sub} = \overline{P_1P_2}. \tag{3.28}$$

The domain Ω_0 with state $(\rho_0, u_0, 0)$ is bounded by the negative ξ-axis, the wave L_{02}, the arc $\widehat{T_2T_1}$ and the wave L_{01}. The domain Ω_1 with state (ρ_1, u_1, v_1) given by (3.1) is bounded by the wave L_{01}, $\widehat{T_1P_1}$ and W_u. The domain Ω_2 with state (ρ_2, u_2, v_2) is bounded by the wave L_{02}, the wall W_u and $\widehat{P_2T_2}$. Here $\rho_2 = \frac{1}{c_2}$ and (u_2, v_2) are given by (3.25) and (3.23), respectively. The points T_1, T_2 and P_2 are given by (2.4), (3.24) and (3.26), respectively.

The remaining work is to solve the corresponding boundary value problems in the subsonic domain $\Omega_{\rm sub}$. It can be reduced to

$$\begin{cases} {\rm div}\dfrac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}} + \dfrac{2}{\psi\sqrt{1+|\nabla\psi|^2}} = 0 & \text{in } \Omega_{\rm sub}, \\ \psi = 0 & \text{on } \widehat{P_2T_2} \cup \widehat{T_2T_1} \cup \widehat{T_1P_1}, \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \overline{P_1P_2}. \end{cases} \tag{3.29}$$

Since the corners of the elliptic domain are P_1 and P_2, which can be eliminated by reflection. Therefore, as did in the case (1) we can solve the problem (3.29), so that have the following conclusion.

Lemma 3.2 *There exists a unique positive solution to* (3.29)*. This solution belongs to* $C^{1,\alpha}(\overline{\Omega}_{\rm sub}\setminus\partial_1\Omega_{\rm sub})\cap C(\overline{\Omega}_{\rm sub})$ *for any* $\alpha<1$ *and is* C^∞ *smooth both in* $\Omega_{\rm sub}$ *and at the interior points of* $\partial_2\Omega_{\rm sub}$*.*

We also notice that there will appear concentration due to Remark 2.1 (1), if $\frac{u_0}{c_0}\geq\frac{1}{\sin\theta_0}$.

In summary, if the initial Mach number is less than 1, there is a shock present in front of the upper wall of the wedge. The shock is detached in which case the flow near the tip of the wedge is subsonic. As $\theta_0\to\frac{\pi}{2}$, it coincides with the normal symmetric case. If the initial Mach number is greater than or equal to 1, then the shock in front of the piston is attached. More precisely, the shock has a straight part near the tip as $\frac{u_0}{c_0}\in(1,\frac{1}{\sin\theta_0})$. Finally, the shock simply adheres to the surface of the piston, if $\frac{u_0}{c_0}\geq\frac{1}{\sin\theta_0}$. Hence Theorem 1.1 is established. The conclusion can also be illustrated by the following Table 1. The concentration here is in the same sense as that in [3].

Table 1 The wave picture for the proceeding piston problem.

Mach number	wave structure
$\dfrac{u_0}{c_0}<1$	detached shock (Figure 3)
$\dfrac{u_0}{c_0}=1$	attached shock curved at the tip (Figure 4)
$\dfrac{u_0}{c_0}\in\Big(1,\dfrac{1}{\sin\theta_0}\Big)$	attached shock with straight part at the tip (Figure 5)
$\dfrac{u_0}{c_0}\geq\dfrac{1}{\sin\theta_0}$	concentration

4 Receding Piston Problem: $u_0<0$

Consider the case that the piston is pulled back from the gas, i.e., $u_0<0$. We also only need to analyze the motion of the gas on the upper half plane. The analysis on the normal case in Section 2 for $u_0>0$ is also available for $u_0<0$. Since the condition (2.5) is satisfied for all $u_0<0$, then the only assumption in this part is

$$u_0<0,\quad \theta_0\in\Big(0,\frac{\pi}{2}\Big). \tag{4.1}$$

As before, we start with determining the waves far away from the origin. It also can be treated as the normal case in Section 2 with $\beta=\theta_0$. There is one wave denoted by L_{01} parallel to the upper wall W_u (see Figure 6). Its location is given by (2.2). It terminates at the point T_1 given by (2.4). Denote by U_1 the state of the gas far away from the origin between W_u

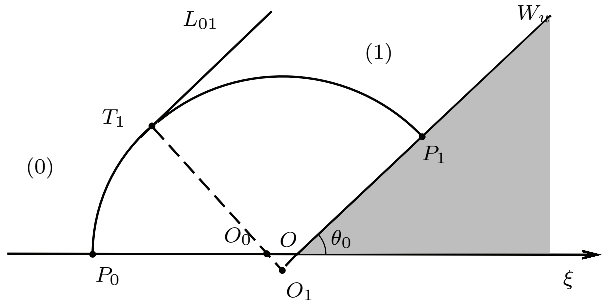

Figure 6 The piston is pulled back with $-\frac{c_0}{u_0} > \cos\theta_0 - \sin\theta_0$.

and L_{01}. Then the solution U_1 is given by (3.1). Correspondingly, the sonic speed there is $c_1 = c_0 - u_0 \sin\theta_0$.

Remark 4.1 Noting that $u_0 < 0$, we have $\rho_1 < \rho_0$, which means that the wave L_{01} is a rarefaction wave. The faster the piston recedes from the gas, the smaller the density ρ_1 is, and the stronger the rarefaction wave is. However, the density ρ_1 is always positive, no matter how large the value $|u_0|$ is. It means that vacuum ($\rho = 0$) will never appear here for the Chaplygin gas.

The center of the sonic circle C_1, denoted by $O_1(u_0\cos^2\theta_0, u_0\cos\theta_0\sin\theta_0)$, locates below the ξ-axis, the radius of C_1 is $|\overline{O_1P_1}| = c_1 = c_0 - u_0\sin\theta_0$. The comparison of the length of $\overline{O_1P_1}$ and $\overline{OO_1}$ determines the wave structure of the receding piston problem. Obviously, $|\overline{O_1P_1}| \le |\overline{OO_1}|$ gives $c_0 - u_0\sin\theta_0 \le -u_0\cos\theta_0$, i.e.,

$$-\frac{c_0}{u_0} \le \cos\theta_0 - \sin\theta_0. \tag{4.2}$$

Since $u_0 < 0$, (4.2) is impossible for $\theta_0 > \frac{\pi}{4}$. The above analysis tell us that the tip of the wedge may locates in the supersonic domain (outside C_1). In this case the uniform state U_1 does not satisfy the boundary condition, so that a new wave may take place due to the influence of the tip of the wedge. Hence we consider the following three different cases specified by the initial data: (1) $-\frac{c_0}{u_0} > \cos\theta_0 - \sin\theta_0$; (2) $-\frac{c_0}{u_0} = \cos\theta_0 - \sin\theta_0$; (3) $-\frac{c_0}{u_0} < \cos\theta_0 - \sin\theta_0$.

(1) If $-\frac{c_0}{u_0} > \cos\theta_0 - \sin\theta_0$, the sonic circle C_1 will intersect with W_u at a point P_1 (see Figure 6). Then

$$|\overline{OP_1}| = c_0 - u_0\sin\theta_0 + u_0\cos\theta_0 = c_0 + u_0(\cos\theta_0 - \sin\theta_0), \tag{4.3}$$

and the coordinates of the point P_1 are

$$\begin{cases} \xi_{P_1} = (c_0 + u_0(\cos\theta_0 - \sin\theta_0))\cos\theta_0, \\ \eta_{P_1} = (c_0 + u_0(\cos\theta_0 - \sin\theta_0))\sin\theta_0. \end{cases} \tag{4.4}$$

We can construct the solution in the same way as that for $u_0 > 0, \frac{u_0}{c_0} < 1$. Far away from the origin point, there is a wave denoted by L_{01} parallel to the wall. The equation of the wave is given by (2.2). The state of gas on the left hand side of L_{01} will be constant until its sonic circle C_0, i.e., $\widehat{P_0T_1}$ in Figure 6 because of the finite speed of wave propagation. Here, P_0 is $(u_0 - c_0, 0)$, T_1 is given by (2.4) and P_1 is given by (4.4). The state of gas between L_{01} and W_u is constant U_1 given by (3.1) until the sonic circle C_1, i.e., $\widehat{T_1P_1}$. The structure of the solution is depicted in Figure 6. In this case, the boundary of the domain $\Omega_{\rm sub}$ is $\partial_1\Omega_{\rm sub} = \widehat{P_0T_1} \cup \widehat{T_1P_1}$ and $\partial_2\Omega_{\rm sub} = \overline{P_0O} \cup \overline{OP_1}$. Lemma 2.1 is available to this case.

Obviously, for $\theta_0 < \frac{\pi}{4}$, the condition $-\frac{c_0}{u_0} > \cos\theta_0 - \sin\theta_0$ always holds. Besides, if $\theta_0 \to \frac{\pi}{2}$, T_1 will coincides with P_0 and L_{01} will be vertical to the ξ-axis, which is just the solution of the normal symmetric case.

(2) If $-\frac{c_0}{u_0} = \cos\theta_0 - \sin\theta_0$, then the sonic circle C_1 intersects with W_u at the tip O because of $\xi_{P_1} = 0$. The domain Ω_0 with state $(\rho_0, u_0, 0)$ is the same as above. It differs from case (1) in that the point P_1 coincides with O, and Ω_1 is bounded by L_{01}, $\widehat{T_1O}$ and W_u; the boundary of the subsonic domain $\Omega_{\rm sub}$ is $\partial_1\Omega_{\rm sub} = \widehat{P_0T_1} \cup \widehat{T_1O}$, and $\partial_2\Omega_{\rm sub} = \overline{P_0O}$. By Lemma 3.1 we get the existence of solution in the subsonic domain $\Omega_{\rm sub}$. The structure of the solution is depicted in Figure 7. (3) If $-\frac{c_0}{u_0} < \cos\theta_0 - \sin\theta_0$, then there is no intersection point of C_1

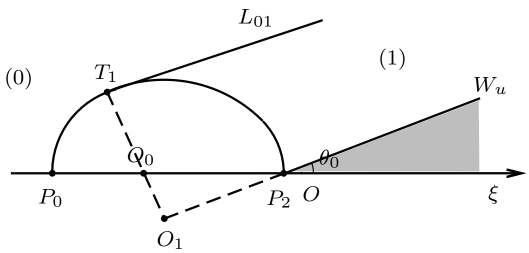

Figure 7 The piston is pulled back with $-\frac{c_0}{u_0} = \cos\theta_0 - \sin\theta_0$.

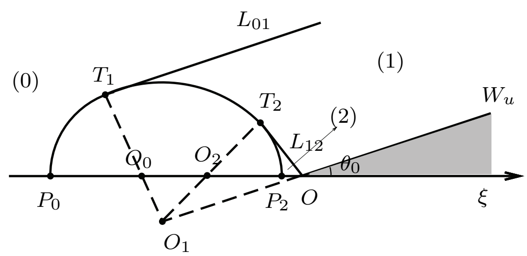

Figure 8 The piston is pulled back with $-\frac{c_0}{u_0} < \cos\theta_0 - \sin\theta_0$.

and W_u because the point P_1 given by (4.4) locates below the ξ-axis. In accordance, the tip of the wedge is not enclosed by the sonic circle C_1. The state U_1 inside Ω_1 is still given by (3.1). Since the state U_1 does not satisfy the boundary condition on the ξ-axis, then a new wave L_{12} arises from the origin (see Figure 8). The wave L_{12} is tangent to C_1 at a point T_2. We have

$$\begin{aligned}|\overline{OT_2}| &= \sqrt{u_0^2\cos^2\theta_0 - c_1^2}\\ &= \sqrt{u_0^2\cos^2\theta_0 - c_0^2 - u_0^2\sin^2\theta_0 + 2c_0u_0\sin\theta_0}\\ &= \sqrt{u_0^2\cos 2\theta_0 - c_0^2 + 2c_0u_0\sin\theta_0}.\end{aligned}\tag{4.5}$$

Denote by O_2 the intersection of $\overline{O_1T_2}$ with the ξ-axis. The angle made by $\overline{OO_2}$ and $\overline{OT_2}$ is

$$\angle O_2OT_2 = \arcsin\frac{c_1}{|u_0|\cos\theta_0} - \theta_0 = -\arcsin\frac{c_1}{u_0\cos\theta_0} - \theta_0,\tag{4.6}$$

and the coordinates of the point T_2 are

$$\begin{cases}\xi_{T_2} = -|\overline{OT_2}|\cos\angle O_2OT_2,\\ \eta_{T_2} = |\overline{OT_2}|\sin\angle O_2OT_2.\end{cases}\tag{4.7}$$

Also, we have the coordinates of the point O_2,

$$\begin{cases}\xi_{O_2} = -\dfrac{|\overline{OT_2}|}{\cos\angle O_2OT_2},\\ \eta_{O_2} = 0.\end{cases}\tag{4.8}$$

The domain Ω_0 with state U_0 is the same as above. The domain Ω_1 now is bounded by the wave L_{01}, the arc $\widehat{T_1T_2}$ of the sonic circle C_1, the wave L_{12} (i.e., $\overline{OT_2}$) and the wall W_u. Denote by Ω_2 the domain bounded by the wave L_{12}, the arc $\widehat{T_2P_2}$ and the ξ-axis. The state of gas in Ω_2 is (ρ_2, u_2, v_2) with

$$\begin{cases}u_2 = \xi_{O_2} = -\dfrac{\sqrt{u_0^2\cos 2\theta_0 - c_0^2 + 2c_0u_0\sin\theta_0}}{\cos\left(\arcsin\dfrac{c_1}{u_0\cos\theta_0} + \theta_0\right)},\\ v_2 = 0,\\ c_2 = |\overline{OT_2}|\tan\angle O_2OT_2\\ \quad = -\sqrt{u_0^2\cos 2\theta_0 - c_0^2 + 2c_0u_0\sin\theta_0}\tan\left(\arcsin\dfrac{c_1}{u_0\cos\theta_0} + \theta_0\right).\end{cases}\tag{4.9}$$

Here, the point T_1 is given by (2.4), T_2 is given by (4.7), the coordinates of P_2 is $(u_2 + c_2, 0)$. Since $c_2 < c_1$, then $\rho_2 > \rho_1$. Notice that near the wave L_{12} the particles of the fluid move from Ω_1 to Ω_2, i.e., from a domain with lower density to a domain with higher density. It shows that the wave L_{12} is a shock.

The boundary of the subsonic domain Ω_{sub} is $\partial_1\Omega_{\text{sub}} = \widehat{P_0T_1} \cup \widehat{T_1T_2} \cup \widehat{T_2P_2}$ and $\partial_2\Omega_{\text{sub}} = \overline{P_2P_0}$. This case corresponds to that the tip of the piston is sharp and the initial Mach number of the gas relative to the piston is large. By Lemma 3.2 we can obtain the existence of solution in the domain Ω_{sub} and then get the global existence of solution for the Problem 1.1 in the case $u_0 < 0$.

Table 2 The wave structure for the receding piston problem.

Initial data	wave structure
$-\dfrac{c_0}{u_0} > \cos\theta_0 - \sin\theta_0$	detachedrarefactionwave, thetiplocatesinthesubsonic domain(Figure6)
$-\dfrac{c_0}{u_0} = \cos\theta_0 - \sin\theta_0$	detachedrarefactionwave, thetiplocatesonthesonic circle(Figure7)
$-\dfrac{c_0}{u_0} < \cos\theta_0 - \sin\theta_0$	detachedmainrarefactionwavewithashockissuingfrom thetip(Figure8)

We can rewrite the condition $-\frac{c_0}{u_0} < \cos\theta_0 - \sin\theta_0$ as $-\frac{c_0}{u_0} < \sin(\frac{\pi}{2} - \theta_0) - \sin\theta_0$, the right hand side of which equals $2\sin(\frac{\pi}{4} - \theta_0)\cos\frac{\pi}{4} = \sqrt{2}\sin(\frac{\pi}{4} - \theta_0)$. Hence the condition in the case (3) can be replaced by $\theta_0 < \frac{\pi}{4} - \arcsin\frac{c_0}{\sqrt{2}|u_0|}$.

In summary, for the receding piston problem the concentration will never appear. The main pressure wave is a rarefaction wave, which is always detached. Meanwhile, when the vertex angle of the piston is small (less than $\frac{\pi}{4} - \arcsin\frac{c_0}{\sqrt{2}|u_0|}$), a shock will arise at the tip, and it terminates at the sonic circle. The conclusion can also be represented by the following Table 2.

5 Concave Piston Problem

In this final part, we briefly discuss another possibility of the piston problem, i.e., the head of the piston forms an superior angle. Locally, the piston can be replaced by a body having a superior angle with two infinitely long sides. It is a domain outside an ordinary wedge (see Figure 9). This problem is called a concave piston problem, while the problems discussed in Sections 3 and 4 are called convex piston problems correspondingly.

Consider the symmetric case, then we only need to study the motion of the gas on the upper half plane. In the upper half plane the boundary of the piston is $y = x\tan\theta_0$ with $\theta_0 \in (\frac{\pi}{2}, \pi)$. Under the self-similar scaling and a shift transformation as stated in Section 1, we could fix the piston and consider the problem in the domain

$$\Lambda = \{\xi \le \eta\cot\theta_0, \eta \ge 0\}, \quad \frac{\pi}{2} < \theta_0 < \pi. \tag{5.1}$$

For the concave piston problem, due to the property of the finite speed of wave propagation, the state of the gas far away from the origin can also be treated as a one-dimensional problem. There is a wave L_{01}, parallel to the surface W_u of the piston, coming from infinity. For the preceding case the wave is a shock, and for the receding case the wave is a rarefaction wave. The equation of L_{10} is given by (2.2) with $\beta = \theta_0$.

Different from the convex piston problems discussed in Sections 3 and 4, the wave L_{10} will be reflected by the ξ-axis (in the whole (ξ, η) plane, the reflection of L_{01} by the ξ-axis amounts to the interaction of L_{01} with its symmetric image $L_{01'}$). Meanwhile, the vertex O generally locates in the subsonic region of the gas, so that no wave will issue from it.

Furthermore, due to the different combination of the flow parameters (c_0, u_0) and the angle θ_0 of the piston, there will appear quite different wave structures. The related wave may stop

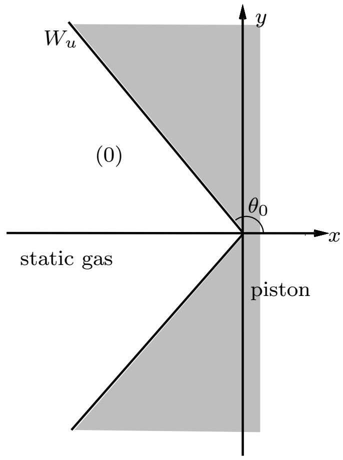

Figure 9 Concave piston with $\theta_0 \in \left(\frac{\pi}{2}, \pi\right)$.

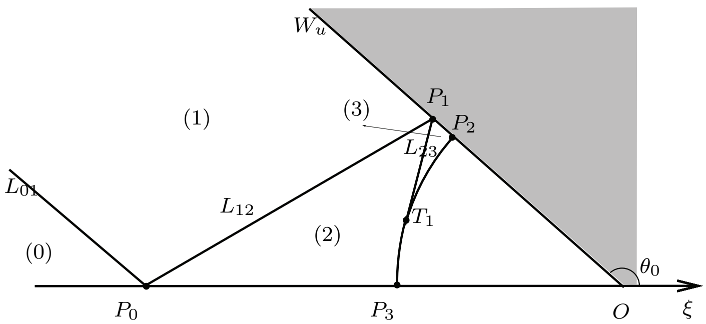

Figure 10 A typical case for concave piston problem.

at somewhere on the sonic circle, or it may be reflected by the surface of the piston again and then continue its propagation. Figure 10 shows a typical case of the wave structure for the parameters satisfying

$$\theta_0 \in \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right), \quad \frac{u_0}{c_0} \in \left(\frac{1+\cot\theta_0}{\sin\theta_0}, \frac{1-\cot\frac{\theta_0}{2}}{\sin\theta_0}\right). \tag{5.2}$$

The wave structure in this typical case is shown in the following. Here we only give the result and omit the related calculations.

The wave L_{01} reflects on the ξ-axis at $P_0(u_0 - \frac{c_0}{\sin\theta_0}, 0)$, and the reflected wave L_{12} is

$$\eta = \left(\xi - u_0 + \frac{c_0}{\sin\theta_0}\right)\tan\alpha_1 \tag{5.3}$$

with $\alpha_1 = 2\arctan(-\tan\theta_0 + \frac{u_0}{c_0}\sin\theta_0\tan\theta_0) - \pi + \theta_0$. The resulting wave L_{12} reflects again on W_u at $P_1(\xi_{P_1}, \eta_{P_1})$ with

$$\xi_{P_1} = \frac{\left(u_0 - \dfrac{c_0}{\sin\theta_0}\right)\tan\alpha_1}{\tan\alpha_1 - \tan\theta_0}, \quad \eta_{P_1} = \frac{\left(u_0 - \dfrac{c_0}{\sin\theta_0}\right)\tan\alpha_1\tan\theta_0}{\tan\alpha_i - \tan\theta_0},$$

and forms a new wave $L_{23}: \eta = (\xi - \xi_{P1})\tan(\alpha_1 - 2\alpha_2) + \eta_{P_1}$, where

$$\alpha_2 = \arctan\frac{c_2}{\ell_1}, \quad \ell_1 = c_2\cot\alpha_1 - \frac{\eta_{P_1}}{\sin\alpha_1}.$$

L_{23} terminates at a sonic point $T_1(\xi_{T_1}, \eta_{T_1})$ with

$$\xi_{T_1} = u_2 - c_2\cos\theta_*, \quad \eta_{T_1} = c_2\sin\theta_*,$$

where $\theta_* = 2\alpha_2 - \alpha_1 - \frac{\pi}{2}$. The domain Λ is divided into supersonic part and subsonic part by the sonic arcs $\widehat{P_2T_1} \cup \widehat{T_1P_3}$. The flow on the left hand side of these sonic arcs is supersonic, and the flow on the right hand side is subsonic. Then the global solution can be obtained like the discussion for the convex piston.

For the cases corresponding to other combination of parameters, the wave structure is different, while the analysis is similar. Hence we will not give all details and leave it to readers.

6 Appendix: Hölder Estimate for Elliptic Operators in Curvilinear Polygon

For the elliptic equations defined in a domain with corners, the regularity of their solutions at corners are generally worse than that near other points on the boundary. There are many studies on the regularity of these solutions near corners. Here we refer some results in [19] for our required applications.

As a typical case Grisvard in [19, p. 182] first studied the solutions of boundary value problems for Laplace equation in a polygon Ω surrounded by straight lines Γ_j $(j = 1, \cdots, N)$,

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_j \text{ with } j \in \mathsf{D}, \\ \dfrac{\partial u}{\partial \nu_{\mathbf{j}}} + \beta_j \dfrac{\partial u}{\partial \tau_{\mathbf{j}}} = 0 & \text{on } \Gamma_j \text{ with } j \in \mathsf{N}, \end{cases} \tag{6.1}$$

where $\nu_{\mathbf{j}}$ is the normal direction of Γ_j, $\tau_{\mathbf{j}}$ is the tangential direction of Γ_j, $\mu_{\mathbf{j}} = \nu_{\mathbf{j}} + \beta_j\tau_{\mathbf{j}}$, and we use the notation $\Gamma_j = \Gamma_{j-N}$.

Define

$$\phi_j = \begin{cases} \arctan\beta_j, & \text{if } j \in \mathsf{N}, \\ \dfrac{\pi}{2}, & \text{if } j \in \mathsf{D}. \end{cases}$$

Let ω_j be the angle formed by Γ_j and Γ_{j+1} and define

$$\lambda_{j,m} = \frac{\phi_j - \phi_{j+1} + m\pi}{\omega_j}.$$

Let r_j, θ_j $(j = 1, \cdots, N)$ be the local coordinates near $S_j = \Gamma_j \cap \Gamma_{j+1}$, and let η_j be the cut-off function in the neighborhood of S_j, and define

$$\mathsf{S}_{j,m}(r_j e^{i\theta_j}) = r_j^{-\lambda_{j,m}}\cos(\lambda_{j,m}\theta_j + \phi_{j+1})\eta_j(r_j e^{i\theta_j}),$$

if $\lambda_{j,m}$ is not integer; and

$$\mathsf{S}_{j,m}(r_j e^{i\theta_j}) = r_j^{-\lambda_{j,m}}[\log r_j\cos(\lambda_{j,m}\theta_j + \phi_{j+1}) + \theta_j\sin(\lambda_{j,m}\theta_j + \phi_{j+1})]\eta_j(r_j e^{i\theta_j}),$$

if $\lambda_{j,m}$ is an integer.

Then the following conclusions hold.

Theorem A.1 (see [19, Theorem 6.4.2.4]) *Assume that* $0<\sigma<1$ *and that* $\frac{1}{\pi}(\phi_{j+1}-\phi_j-(2+\sigma)\omega_i)$ *is not an integer for any* j. *Then there exists a constant* C *such that for any* $C^{2+\sigma}(\overline{\Omega})$ *solution of the problem* (6.1), *the following estimate holds*

$$\|u\|_{C^{2+\sigma}(\overline{\Omega})}\le C(\|\Delta u\|_{C^{\sigma}(\overline{\Omega})}+\|u\|_{C^{1+\sigma}(\overline{\Omega})}). \tag{6.2}$$

Theorem A.2 (see [19, Theorem 6.4.2.5]) *Assume that* D *is not empty and that at least two of the vectors* μ_{j} *are linearly independent. Assume that* $0<\sigma<1$ *and* $\frac{1}{\pi}(\phi_{j+1}-\phi_j-(2+\sigma)\omega_j)$ *is not an integer for any* j. *Then for each* $f\in C^{\sigma}(\overline{\Omega})$ *with* $0<\sigma<1$, *there exists a solution* u *of* (6.1) *and numbers* $c_{j,m}$ *such that*

$$u-\sum_{-(\sigma+2)<\lambda_{j,m}<0}c_{j,m}S_{j,m}\in C^{2,\sigma}(\overline{\Omega}) \tag{6.3}$$

and u *is the solution of* (6.1).

These two propositions can be extended to their $C^{1+\alpha}$ version. That is the following two results.

Theorem A′.1 *Assume that* $0<\sigma<1$ *and that* $\frac{1}{\pi}(\phi_{j+1}-\phi_j-(1+\sigma)\omega_i)$ *is not an integer for any* j. *Then there exists a constant* C *such that for any* $C^{1+\sigma}(\overline{\Omega})$ *solution of the problem* (6.1), *the following estimate holds*

$$\|u\|_{C^{1+\sigma}(\overline{\Omega})}\le C(\|\Delta u\|_{C^{-1+\sigma}(\overline{\Omega})}+\|u\|_{C^{\sigma}(\overline{\Omega})}), \tag{6.4}$$

where $\|\cdot\|_{C^{-1+\alpha}}$ *is defined in* §3.

Theorem A′.2 *Assume that* D *is not empty and that at least two of the vectors* μ_{j} *are linearly independent. Assume that* $0<\sigma<1$ *and* $\frac{1}{\pi}(\phi_{j+1}-\phi_j-(1+\sigma)\omega_j)$ *is not an integer for any* j. *Then for each* $f\in C^{\sigma}(\overline{\Omega})$, *there exists a solution* u *of* (6.1) *and numbers* $c_{j,m}$ *such that*

$$u-\sum_{-(\sigma+1)<\lambda_{j,m}<0}c_{j,m}S_{j,m}\in C^{1,\sigma}(\overline{\Omega}), \tag{6.5}$$

and u *is the solution of* (6.1).

The proof of Theorem A′.1 is similar to the proof of Theorem A.1 given in [19]. It consists of three main steps: Localizing the discussing to each corner, mapping each angular domain to a band by a coordinate transformation and applying the known results on the estimates for elliptic equation. The main difference of the proof for Theorem A′.1 to that for Theorem A.1 is that one should apply the weak solution theory for elliptic equations rather than classical Schauder theory. Therefore, next we only present the main point on the difference. Meanwhile, since the proof of Theorem A′.2 is quite similar to that for Theorem A.2, then we omit it here.

Let us keep notations given in the beginning of the Appendix. As defined in [19], $P^{m,\sigma}(\overline{\Omega})$ is the subspace of $C^{m+\sigma}(\overline{\Omega})$ for all $u\in C^{m+\sigma}$ satisfying

$$D^{\alpha}u(S_i)=0\quad\text{for } |\alpha|\le m,\ 1\le j\le N.$$

The norm of $P^{m,\sigma}(\overline{\Omega})$ is defined as

$$\|u\|_{P^{m,\sigma}(\overline{\Omega})}=\sum_{|\alpha|\le m}\inf\rho^{|\alpha|-m-\sigma}|D^\alpha u|+\sum_{|\alpha|=m}\inf\frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^\sigma},\tag{6.6}$$

where ρ is the distance of a given point to the corners. Then the estimate (6.4) can be derived from

$$\|u\|_{P^{1,\sigma}}\le C(\|f\|_{C^{-1+\sigma}}+\|u\|_{C^\sigma}).\tag{6.7}$$

In order to prove (6.7) we introduce a partition of unity $1=\sum\limits_{j=0}^{N}\eta_j$, where η_0 is supported in a domain away from boundary, while each η_j $(j\ne 0)$ equals 1 near the vertex S_j and equals 0 near S_ℓ $(\ell\ne j)$. Now if we can prove

$$\|\eta_j u\|_{P^{1,\sigma}}\le C\|\Delta(\eta_j u)\|_{C^{-1+\sigma}}\tag{6.8}$$

for each j, then the estimate (6.7) holds. In fact, by using (6.8) we have

$$\begin{aligned}\|u\|_{P^{1,\sigma}}&=\|\sum\eta_j u\|_{P^{1,\sigma}}\le\sum\|\eta_j u\|_{P^{1,\sigma}}\\&\le C\sum\|\Delta(\eta_j u)\|_{C^{-1+\sigma}}\le C\sum(\|\eta_j\Delta u\|_{C^{-1+\sigma}}+\|u\|_{C^\sigma})\\&\le C(\|f\|_{C^{-1+\sigma}}+\|u\|_{C^\sigma}).\end{aligned}$$

Replacing $\eta_j u$ by v we need to prove

$$\|v\|_{P^{1,\sigma}}\le C\|\Delta v\|_{C^{-1+\sigma}},\tag{6.9}$$

where $v\in P^{1,\sigma}$ is defined in a singular domain with vertex angle ω_j and compactly supported, v also satisfies the boundary conditions as shown in (6.1).

According to the definition of $C^{-1+\alpha}$ norm, if Δv can be rewritten as $\sum\limits_\ell\frac{\partial g_\ell}{\partial x_\ell}$, then (6.9) means

$$\|v\|_{P^{1,\sigma}}\le C\sum_\ell\|g_\ell\|_{C^\sigma(\Omega)}.\tag{6.10}$$

As did in [19] one can use coordinates transformation

$$x=\mathrm{e}^t\cos\theta,\quad y=\mathrm{e}^t\sin\theta\tag{6.11}$$

to transform the angular domain with vertex S_j and vertex angle ω_j to a band $B:\{-\infty<t<\infty,\ 0<\theta<\omega_j\}$. Accordingly, let $w=\mathrm{e}^{-(\sigma+1)t}v(\mathrm{e}^{t+\mathrm{i}\theta})$, it will satisfy

$$D_t^2w+D_\theta^2w+2(\sigma+1)D_tw+(\sigma+1)^2w=\mathrm{e}^{-(\sigma+1)t}(v_{tt}+v_{\theta\theta})\ (\triangleq k).\tag{6.12}$$

Then (6.10) is reduced to

$$\|w\|_{C^{1+\sigma}(B)}\le C\|k\|_{C^{-1+\sigma}(B)}.\tag{6.13}$$

Since B does not contain any singular point on its boundary yet, then (6.13) can be obtained by using the boundedness of Fourier multiplier (or the estimates of weak solutions of elliptic equation (see [18])). Returning to the original coordinates we obtain (6.9).

Finally, since $P^{1,\sigma}$ contains a subspace of $C^{1,\sigma}(\overline{\Omega})$ with finite codimension, then (6.9) implies (6.4) immediately.

References

[1] Ben-Dor, G., Shock Wave Reflection Phenomena, Springer-Verlag, New York, 1991.

[2] Bento, M., Bertolami, O. and Sen, A., Generalized Chaplygin gas, accelerated expansion, and dark-energy-matter unification, *Physical Review D*, **66**, 2002, 043507.

[3] Brenier, Y., Solutions with concentration to the Riemann problem for the one-dimensional Chaplygin gas equations, *Journal of Mathematical Fluid Mechanics*, **7**, 2005, 326–331.

[4] Chen, G., Multidimensional Conservation Laws: Overview, Problems, and Perspective, Springer US, Boston, MA, 2011, 23–72.

[5] Chen, G. and Feldman, M., Global solutions of shock reflection by large-angle wedges for potential flow, *Ann of Math*, **171**, 2010, 1067–1182.

[6] Chen, S., Existence of local solution to supersonic flow around a three dimensional wing, *Advances in Appl. Math.*, **13**, 1992, 273–304.

[7] Chen, S., Existence of stationary supersonic flow past a pointed body, *Archive Rat. Mech. Anal.*, **156**, 2001, 141–181.

[8] Chen, S., A singular multi-dimensional piston problem in compressible flow, *Journal of Differential Equations*, **189**, 2003, 292–317.

[9] Chen, S. and Qu, A., Two-dimensional Riemann problems for Chaplygin gas, *SIAM Journal on Mathematical Analysis*, **44**, 2012, 2146–2178.

[10] Chen, S. and Qu, A., Riemann boundary value problems and reflection of shock for the Chaplygin gas, *Science in China A: Mathematics*, **55**, 2012, 671–685.

[11] Chen, S., Wang, Z. and Zhang, Y., Global existence of shock front solution to axially symmetric piston problem in compressible flow, *Z. Angew. Math. Phys.*, **59**, 2008, 434–456,

[12] Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves, Applied Mathematical Sciences, **21**, Springer-Verlag, New York, 1976.

[13] Dafermos, C. M., Hyperbolic Conservation Laws in Continuum Physics, **325**, 3rd ed., Springer-Verlag, Berlin, Heidelberg, 2010.

[14] Ding. M., Kuang, J. and Zhang, Y., Global stability of rarefaction wave to the 1-D piston problem for the compressible full Euler equations, *J. Math. Anal. Appl.*, DOI: 10.1016/j.jmaa.2016.11.059.

[15] Ding, M. and Li, Y., An Overview of Piston Problems in Fluid Dynamics, Hyperbolic Conservation Laws and Related Analysis with Applications, Springer Proceedings in Mathematics & Statistics, **49**, Springer-Verlag, Berlin, Heidelberg, 2014.

[16] Elling, V. and Liu, T.-P., Supersonic flow onto a solid wedge, *Communications on Pure and Applied Mathematics*, **61**, 2008, 1347–1448.

[17] Fang. B. and Xiang, W., The uniqueness of transonic shocks in supersonic flow past a 2-D wedge, *J. Math. Anal. Appl.*, **437**, 2016, 194–213.

[18] Gilbarg, D. and Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 Edition, Grundlehren Der Mathematischen Wissenschaften, **224**, Springer-Verlag, New York, 2003.

[19] Grisvard, P., Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, **24**, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[20] Guo, L., Sheng, W. and Zhang, T., The two-dimensional Riemann problem for isentropic Chaplygin gas dynamic system, *Commun. Pure Appl. Anal.*, **9**, 2010, 431–458.

[21] Liu, L., Xu, G., and Yuan, H., Stability of spherically symmetric subsonic flows and transonic shocks under multidimensional perturbations, *Adv. Math.*, **291**, 2016, 696–757.

[22] Serre, D., Multidimensional shock interaction for a Chaplygin gas, *Arch. Ration. Mech. Anal.*, **191**, 2009, 539–577.

[23] Smoller, J., Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1994.

[24] Xin, Z. and Yin, H., Global multidimensional shock wave for the steady supersonic flow past a three-dimensional curved cone, *Anal. Appl.* (*Singap.*), **4**, 2006, 101–132.

[25] Zhang, Y., Steady supersonic flow past an almost straight wedge with large vertex angle, *Journal of Differential Equations*, **192**, 2003, 1–46.