Chin. Ann. Math. Ser. B
41(1), 2020, 27–36
DOI: 10.1007/s11401-019-0183-z

Chinese Annals of Mathematics, Series B

Properties of Complex Oscillation of Solutions of a Class of Higher Order Linear Differential Equations*

Jianren LONG[1] Yezhou LI[2]

Abstract Let $A(z)$ be an entire function with $\mu(A) < \frac{1}{2}$ such that the equation $f^{(k)} + A(z)f = 0$, where $k \geq 2$, has a solution f with $\lambda(f) < \mu(A)$, and suppose that $A_1 = A + h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \mu(A)$. Then $g^{(k)} + A_1(z)g = 0$ does not have a solution g with $\lambda(g) < \infty$.

Keywords Complex differential equations, Entire function, Order of growth, Exponent of convergence of the zeros
2000 MR Subject Classification 34M10, 30D35

1 Introduction and Main Results

We shall assume that the reader is familiar with the standard notations and fundamental results of Nevanlinna theory of meromorphic functions (cf. [11, 13, 21]), such as $T(r, f)$, $N(r, f)$, $m(r, f)$ and $S(r, f) = o(T(r, f))$ outside a set of finite measure. In addition, for a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, the order of growth $\rho(f)$, lower order of growth $\mu(f)$ and exponent of convergence of the zeros $\lambda(f)$ are defined as follows, respectively,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r},$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r},$$

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N\Big(r, \dfrac{1}{f}\Big)}{\log r},$$

where $N\big(r, \frac{1}{f}\big)$ is the counting function of zeros of $f(z)$, defined by

$$N\Big(r, \frac{1}{f}\Big) = \int_0^r \frac{n\Big(t, \dfrac{1}{f}\Big) - n\Big(0, \dfrac{1}{f}\Big)}{t} \mathrm{d}t + n\Big(0, \frac{1}{f}\Big) \log r,$$

where $n\big(r, \frac{1}{f}\big)$ denotes the number of zeros of $f(z)$ in $|z| \leq r$, counting multiplicities.

Manuscript received July 7, 2017. Revised October 11, 2018.
[1]School of Computer Science and School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China; School of Mathematical Science, Guizhou Normal University, Guiyang 550001, China. E-mail: longjianren2004@163.com
[2]Corresponding author. School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China. E-mail: yezhouli@bupt.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11571049, 11501142, 11861023) and the Foundation of Science and Technology project of Guizhou Province of China (No. [2018]5769-05).

Remark 1.1 By [21], the exponent of convergence of zeros of $f(z)$ is also defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ n\Big(r, \dfrac{1}{f}\Big)}{\log r}.$$

Let A be an entire function. Suppose that $k \geq 2$ is an integer. Suppose that f_j $(j = 1, 2, \cdots, k)$ are solutions of the following complex differential equation:

$$f^{(k)} + A(z)f = 0. \tag{1.1}$$

In [12], Hille proved that any solutions of (1.1) are entire functions. In recent years, a lot of results have been done in the connection between the order of growth ρ of coefficient A and the exponent of convergence of the zeros λ of f_j $(j = 1, 2, \cdots, k)$, such as [1, 2, 4, 15–16, 20]. In particular it was shown in [2, 20] that, if $k = 2$ and A is transcendental entire function with $\rho(A) \leq \frac{1}{2}$, then (1.1) cannot have two linearly independent solutions f_1 and f_2, each with $\lambda(f_j)$ finite. A comparable result was proved for higher order equations in [16]. More results regarding complex oscillation of solution of linear differential equation can be found, for example, in [3, 6–10, 14, 18–19]. On the other hand, it is possible to have one solution f of (1.1) with no zeros, even for coefficients of very small growth. To see this, set $f = \mathrm{e}^B$ where B is an entire function. Then f solves (1.1) with $k = 2$ and $-A = \frac{f''}{f} = B'' + (B')^2$, as well as similar equations of higher order obtained by computing $\frac{f^{(k)}}{f}$ in terms of B. In [1], the author proved that small perturbations of such equations lead to the exponent of convergence of zeros of solution being at least the order of growth of the coefficient A.

Theorem 1.1 *Suppose that $A(z)$ is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (1.1) has a solution f with $\lambda(f) < \rho(A)$, and suppose that*

$$A_1 = A + h, \tag{1.2}$$

where $h(\not\equiv 0)$ is an entire function with $\rho(h) < \rho(A)$. Then

$$g^{(k)} + A_1(z)g = 0 \tag{1.3}$$

does not have a solution g with $\lambda(g) < \rho(A)$.

For a transcendental entire coefficient in (1.1), it seems interesting what conditions on $A(z)$ will guarantee that every solution $g \not\equiv 0$ of (1.3) has the infinite exponent of convergence of zeros? In this paper, using the similar idea in [1], we can obtain the following result, which shows that small perturbations of such equations lead to solutions whose zeros must have the infinite exponent of convergence.

Theorem 1.2 *Let $A(z)$ be a transcendental entire function of finite order with $\mu(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (1.1) has a solution f with $\lambda(f) < \mu(A)$, and suppose that A_1 satisfies (1.2) and $h \not\equiv 0$ is an entire function with $\rho(h) < \mu(A)$. Then the exponent of convergence of zeros of any nontrivial solutions of (1.3) is infinite.*

Remark 1.2 If $\mu(A) = \rho(A)$ in Theorem 1.2, then we will see a more general result than Theorem 1.1. At the same time, Theorem 1.2 contains also the case $\mu(A) < \rho(A)$.

By the proof of Theorem 1.2, we can easily see the following result.

Corollary 1.1 *Let $A(z)$ be a transcendental entire function of finite order with $\mu(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (1.1) has a solution f with finitely many zeros, and suppose that A_1 satisfies (1.2) and $h(\not\equiv 0)$ is an entire function with $\rho(h) < \mu(A)$. Then (1.3) does not have a nontrivial solution with finitely many zeros.*

The paper is organized as follows. In Section 2, we state and prove some lemmas. In Section 3, we prove Theorem 1.2.

2 Lemmas

In order to prove our theorem, we need the following definition and lemma.

Definition 2.1 (cf. [13]) *Let $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$ be the open disc in the complex plane. We say that countable union $\bigcup\limits_{n=1}^{\infty} B(z_n, r_n)$ is an R-set if $z_n \to \infty$ and $\Sigma r_n < \infty$.*

Lemma 2.1 (cf. [13]) *Suppose that $f(z)$ is a meromorphic function of finite order. Then there exists a positive integer N such that*

$$\left|\frac{f'(z)}{f(z)}\right| = O(|z|^N) \tag{2.1}$$

holds for large z outside of an R-set.

Before stating the following lemma, for $E \subset [0, \infty)$, we define the Lebesgue measure of E by $\mathrm{mes}(E)$, the logarithmic measure of $E \subset [1, \infty)$ by $m_l(E) = \int_E \frac{\mathrm{d}t}{t}$, and the upper and lower logarithmic density of $E \subset [1, \infty)$, respectively, by

$$\overline{\mathrm{logdens}}\, E = \limsup_{r\to\infty} \frac{m_l(E \cap [1, r])}{\log r}$$

and

$$\underline{\mathrm{logdens}}\, E = \liminf_{r\to\infty} \frac{m_l(E \cap [1, r])}{\log r}.$$

The proof of our theorem highly depends on the following lemma.

Lemma 2.2 (cf. [5]) *Let $f(z)$ be an entire function with $0 \leq \mu(f) < \frac{1}{2}$. Suppose that $m(r)$ is defined as*

$$m(r) = \inf_{|z|=r} \log |f(z)|. \tag{2.2}$$

If $\mu(f) \leq \sigma < \min\left(\rho(f), \frac{1}{2}\right)$, then the set $\{r : m(r) > r^{\sigma}\}$ has positive upper logarithmic density.

Moreover, we are going to use the following lemma, which gives an asymptotic representation for the logarithmic derivative of solutions of (1.1) with few zeros. It is a special case of a result from [16].

Lemma 2.3 (cf. [17]) *Let $A(z)$ be a transcendental entire function of finite order, and let E_1 be a subset of $[1, \infty)$ of infinite logarithmic measure and with the following property. For each $r \in E_1$, there exists an arc*

$$a_r = \{r\mathrm{e}^{it} : 0 \leq \alpha_r \leq t \leq \beta_r \leq 2\pi\}$$

of the circle $S(0,r)=\{z:|z|=r\}$ such that

$$\lim_{r\to\infty, r\in E_1}\frac{\min\{\log|A(z)|:z\in a_r\}}{\log r}=+\infty. \tag{2.3}$$

Let $k\geq 2$ and let f be a solution of (1.1) with $\lambda(f)<\infty$. Then there exists a subset $E_2\subset[1,\infty)$ of finite measure, such that for large $r\in E_0=E_1\backslash E_2$, we have

$$\frac{f'(z)}{f(z)}=c_rA(z)^{\frac{1}{k}}-\frac{k-1}{2k}\frac{A'(z)}{A(z)}+O(r^{-2}) \tag{2.4}$$

holds for all $z\in a_r$, where the constant c_r satisfies $c_r^k=-1$ and may depend on r, for a given $r\in E_0$ but not depend on z, and the branch of $A(z)^{\frac{1}{k}}$ is analytic on a_r (included in the case where a_r is the whole circle $S(0,r)$).

We note that E_2 has finite measure and so finite logarithmic measure, so E_0 has infinite logarithmic measure.

We will employ the following well-known representation for higher order logarithmic derivatives (cf. [11]).

Lemma 2.4 *Let $f(z)$ be an analytic function, and let $F=\frac{f'}{f}$. Then for $k\in\mathbb{N}$ we have*

$$\frac{f^{(k)}}{f}=F^k+\frac{k(k-1)}{2}F^{k-2}F'+P_{k-2}(F), \tag{2.5}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k\leq 2$ and has degree of $(k-2)$ when $k>2$.

Remark 2.1 By using Lemma 2.4, we shall see that it is possible to have a solution f of (1.1) with no zeros. Let $F=\frac{f'}{f}$, $f=\mathrm{e}^B$ where B is an entire function, obviously $F=B'$. It is clear to see that $f=\mathrm{e}^B$, which has no zeros, solves (1.1) with $-A=\frac{f^{(k)}}{f}$.

Lemma 2.5 *Let $A(z)$ be a transcendental entire function with $\mu(A)=\mu<\frac{1}{2}$ in the complex plane $\mathbb{C}$. Suppose that f is an entire function with $\lambda(f)<\mu$. Then there exists a set $E_3\subset[1,\infty)$ with $\overline{\mathrm{logdens}}\,E_3>0$, such that for any σ $\left(\mu\leq\sigma<\min\left(\frac{1}{2},\rho(A)\right)\right)$, we have*

$$\inf_{|z|=r\in E_3}\log|A(z)|>r^{\sigma}$$

and

$$\lim_{r\to\infty, r\in E_3}\frac{n\Big(r,\dfrac{1}{f}\Big)\log r}{T(r,A)}=0.$$

Proof By using Lemma 2.2, for any $\mu\leq\sigma<\min\left(\frac{1}{2},\rho(A)\right)$, we see that there exists a set $E_0\subset[1,\infty)$ with $\overline{\mathrm{logdens}}\,E_0>0$, where

$$E_0=\Big\{r>1:\inf_{|z|=r}\log|A(z)|>r^{\sigma}\Big\}. \tag{2.6}$$

Since $\lambda(f)<\sigma$, for any given $0<\varepsilon<\frac{\sigma-\lambda(f)}{2}$, there exists an $r_0>1$ such that

$$n\Big(r,\frac{1}{f}\Big)<r^{\lambda(f)+\varepsilon} \tag{2.7}$$

holds for all $r > r_0$. Set $E_3 = E_0 \cap [r_0, \infty)$. We claim that $\overline{\text{logdens}}\, E_3 > 0$. Note that

$$[r_0, \infty) = ([r_0, \infty) \cap E_0) \cup ([r_0, \infty) - E_0).$$

Thus

$$\begin{aligned}\overline{\text{logdens}}\, E_3 &\geq \underline{\text{logdens}}\, E_3 \\ &= \underline{\text{logdens}}[r_0, \infty) - \underline{\text{logdens}}([r_0, \infty) - E_0) \\ &\geq \underline{\text{logdens}}[r_0, \infty) - (1 - \overline{\text{logdens}}\, E_0) \\ &= \overline{\text{logdens}}\, E_0 > 0.\end{aligned}$$

By using (2.6)–(2.7) and $T(r, A) \leq \log^+ M(r, A) \leq 3T(2r, A)$, we have that

$$\frac{n\left(r, \dfrac{1}{f}\right) \log r}{T(r, A)} \leq \frac{r^{\lambda(f)+\varepsilon} \log r}{\left(\dfrac{r}{2}\right)^{\sigma}}$$

holds for any $r \in E_3$. So

$$\lim_{r\to\infty, r\in E_3} \frac{n\left(r, \dfrac{1}{f}\right) \log r}{T(r, A)} = 0.$$

The proof of Lemma 2.5 is completed.

3 Proof of Theorem 1.2

Suppose that (1.1) has a solution f with $\lambda(f) < \mu(A)$, (1.3) has a solution g with $\lambda(g) < \infty$, and note that $\rho(A) = \rho(A_1)$. Using (1.2) and $\rho(h) < \mu(A)$, let

$$f = P\mathrm{e}^U \tag{3.1}$$

and

$$g = Q\mathrm{e}^V, \tag{3.2}$$

where P, Q, U and V are entire functions which satisfy $\rho(P) = \lambda(f) < \infty$ and $\rho(Q) = \lambda(g) < \infty$, and $\max\{\rho(U), \rho(V)\} \leq \rho(A)$ (cf. [15]). Let

$$F = \frac{f'}{f} = \frac{P'}{P} + U', \quad G = \frac{g'}{g} = \frac{Q'}{Q} + V'. \tag{3.3}$$

Applying Lemma 2.4, we obtain

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2} F^{k-2} F' + P_{k-2}(F) \tag{3.4}$$

and

$$\frac{g^{(k)}}{g} = G^k + \frac{k(k-1)}{2} G^{k-2} G' + P_{k-2}(G), \tag{3.5}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree of $k-2$ when $k > 2$.

Choose σ such that

$$\max\{\lambda(f), \rho(h)\} < \mu(A) \leq \sigma < \min\Big(\rho(A), \frac{1}{2}\Big). \tag{3.6}$$

It follows from Lemma 2.5 that there exits a set $E_1 \subset [1, \infty)$ with $\overline{\text{logdens}}\, E_1 > 0$ such that

$$\inf_{|z|=r} \log |A(z)| > r^{\sigma} \tag{3.7}$$

holds for all $r \in E_1$. By Lemma 2.1, there exist $E_2 \subset [1, \infty)$ of finite measure and $M_1 \in \mathbb{N}$, such that

$$\Big|\frac{A'(z)}{A(z)}\Big| + \Big|\frac{P'(z)}{P(z)}\Big| + \Big|\frac{Q'(z)}{Q(z)}\Big| \leq r^{M_1}, \quad |z| = r \geq 1,\ r \notin E_2. \tag{3.8}$$

For large $|z| = r \in E_1$, by using (1.2) and (3.6)–(3.7), we also have

$$\log |A_1(z)| > \frac{r^{\sigma}}{2}. \tag{3.9}$$

The next step is to estimate $\frac{f'(z)}{f(z)}$ and $\frac{g'(z)}{g(z)}$ in terms of $A(z)$. We apply Lemma 2.3 to (1.1) and (1.3). Choose a_r to be the whole circle $|z| = r \in E_1$. This is possible since (3.7) and (3.9) imply that Lemma 2.3 holds. Hence for large $r \in E_0 = E_1 \backslash E_3$, where E_3 has finite measure and $E_3 \supset E_2$, by Lemma 2.3, the following is true:

$$\frac{f'(z)}{f(z)} = cA(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}), \quad |z| = r,\ c^k = -1, \tag{3.10}$$

and

$$\frac{g'(z)}{g(z)} = dA_1(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A_1'(z)}{A_1(z)} + O(r^{-2}), \quad |z| = r,\ d^k = -1. \tag{3.11}$$

Here c, d may depend on r, but not on z.

Next, we apply the binomial theorem to expand $A_1(z)^{\frac{1}{k}}$ and $\frac{A_1'(z)}{A_1(z)}$ in terms of $A(z)^{\frac{1}{k}}$ and $\frac{A'(z)}{A(z)}$. Using (3.6)–(3.7), we have

$$\Big|\frac{h(z)}{A(z)}\Big| \leq \frac{\mathrm{e}^{r^{\rho(h)+o(1)}}}{\mathrm{e}^{r^{\sigma}}} = o(1), \quad |z| = r \to \infty,\ r \in E_0,$$
$$\Big|\frac{h'(z)}{A(z)}\Big| \leq \frac{\mathrm{e}^{r^{\rho(h)+o(1)}}}{\mathrm{e}^{r^{\sigma}}} = o(1), \quad |z| = r \to \infty,\ r \in E_0.$$

By using the above inequalities, we get for $|z| = r \in E_0$, on suppressing the variable z for brevity,

$$A_1^{\frac{1}{k}} = (A+h)^{\frac{1}{k}} = A^{\frac{1}{k}}\Big(1 + \frac{h}{A}\Big)^{\frac{1}{k}} = A^{\frac{1}{k}}\Big(1 + O\Big(\frac{|h|}{|A|}\Big)\Big) \tag{3.12}$$

and

$$\frac{A_1'}{A_1} = \frac{A' + h'}{A + h} = \frac{A' + h'}{A\Big(1 + \dfrac{h}{A}\Big)} = \frac{A'}{A}\Big(1 + O\Big(\frac{|h|}{|A|}\Big)\Big) + o\Big(\frac{|h|}{|A|}\Big). \tag{3.13}$$

Using (3.11)–(3.13), we deduce that, for $|z| = r \in E_0$,

$$\frac{g'(z)}{g(z)} = dA(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}), \quad d^k = -1. \tag{3.14}$$

We recall Lemma 2.3 that c and d may depend on r, for given $r \in E_0$, but not depend on z. The following claim is key point to prove the Theorem 1.2.

Suppose that c and d are as in (3.10) and (3.14) respectively. Then we claim that $c = d$ for all large $r \in E_0$. In fact, we may write $d = c\omega$ where $\omega^k = 1$. Using (3.14), we obtain

$$\frac{g'(z)}{g(z)} = c\omega A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}), \quad \omega^k = 1. \tag{3.15}$$

Multiplying (3.10) by ω and subtracting (3.15), we get

$$\omega\Big(\frac{f'(z)}{f(z)} + \frac{k-1}{2k}\frac{A'(z)}{A(z)}\Big) = \frac{g'(z)}{g(z)} + \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}).$$

Integrating around $|z| = r_n$, where $r_n \to \infty$ as $n \to \infty$ and $r_n \in E_0$, we then find that

$$\omega\Big(n\Big(r_n, \frac{1}{f}\Big) + \frac{k-1}{2k}n\Big(r_n, \frac{1}{A}\Big)\Big) + o(1) = n\Big(r_n, \frac{1}{g}\Big) + \frac{k-1}{2k}n\Big(r_n, \frac{1}{A}\Big). \tag{3.16}$$

But the right-hand side of (3.16) must be positive since $n\big(r_n, \frac{1}{g}\big) \geq 0$ and $n\big(r_n, \frac{1}{A}\big) > 0$. This is because $N\big(r_n, \frac{1}{A}\big) = 0$, if $n\big(r_n, \frac{1}{A}\big) = 0$. Since $\inf\limits_{|z|=r_n} \log|A(z)|$ is very large for $r_n \to \infty, r_n \in E_0$, we get

$$m\Big(r_n, \frac{1}{A}\Big) = 0.$$

Hence

$$T\Big(r_n, \frac{1}{A}\Big) = 0.$$

Using the first fundamental theorem of Nevanlinna theory, we obtain

$$T(r_n, A) = O(1).$$

This contradicts the fact that A is transcendental and proves the claim that $n\big(r_n, \frac{1}{A}\big) > 0$. For the same reason, $n\big(r_n, \frac{1}{f}\big) + n\big(r_n, \frac{1}{A}\big)$ is a positive integer.

Since $n\big(r_n, \frac{1}{A}\big) \geq 1,\ n\big(r_n, \frac{1}{f}\big) + \frac{k-1}{2k}n\big(r_n, \frac{1}{A}\big) \geq \frac{k-1}{2k}$. Thus,

$$\Big|\mathrm{Im}\Big[\omega\Big(n\Big(r_n, \frac{1}{f}\Big) + \frac{k-1}{2k}n\Big(r_n, \frac{1}{A}\Big)\Big)\Big]\Big| \geq \frac{k-1}{2k}|\mathrm{Im}\,\omega|.$$

By (3.16), we get

$$\mathrm{Im}\Big[\omega\Big(n\Big(r_n, \frac{1}{f}\Big) + \frac{k-1}{2k}n\Big(r_n, \frac{1}{A}\Big)\Big)\Big] + \mathrm{Im}\,o(1) = 0.$$

Set $\delta = \inf\{|\operatorname{Im}\omega| : \omega^k = 1, \operatorname{Im}\omega \neq 0\}$. Obviously, $\delta > 0$. Thus for sufficiently large $r_n \in E_0$ and $|\operatorname{Im} o(1)| < \delta\frac{k-1}{2k}$, we get $\operatorname{Im}\omega = 0$. It follows from this and $\omega^k = 1$ that $\omega = 1$ or $\omega = -1$. From (3.16) it is impossible that $\omega = -1$. Therefore, $\omega = 1$ and $c = d$.

To complete the proof of theorem, by using (3.10), (3.14) and $c = d$, for $r \to \infty$ and $r \in E_0$, we get

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + o(1), \quad |z| = r.$$

And hence

$$n\Big(r, \frac{1}{f}\Big) = n\Big(r, \frac{1}{g}\Big) \tag{3.17}$$

holds for lager $r \in E_0$.

Using (3.3), we get

$$\frac{P'(z)}{P(z)} + U' = \frac{Q'(z)}{Q(z)} + V' + o(1).$$

Using (3.8), we see that

$$|U'(z) - V'(z)| \leq 2r^{M_1}$$

holds for $|z| = r$ and large $r \in E_0$. Since U and V are entire, we deduce that $Q_0(= U' - V')$ is a polynomial. Thus (3.3) becomes

$$F = G + M, \quad M = \frac{P'}{P} - \frac{Q'}{Q} + Q_0. \tag{3.18}$$

Using (1.1) and (3.4), we get

$$F^k + \frac{k(k-1)}{2} F^{k-2} F' + P_{k-2}(F) = -A, \tag{3.19}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree of at most $k-2$ when $k > 2$. Combining (1.2)–(1.3) and (3.5), we obtain

$$G^k + \frac{k(k-1)}{2} G^{k-2} G' + P_{k-2}(G) = -A - h. \tag{3.20}$$

Using (3.18)–(3.19), we get

$$(G + M)^k + \frac{k(k-1)}{2}(G + M)^{k-2}(G' + M') + P_{k-2}(G + M) = -A.$$

Expanding $(G + M)^k$ and $(G + M)^{k-2}$ by binomial theorem in above equality, we get

$$G^k + kMG^{k-1} + \frac{k(k-1)}{2} G^{k-2} G' + R_{k-2}(G, M) = -A, \tag{3.21}$$

where $R_{k-2}(G, M)$ is a polynomial in M, G and their derivatives, and has total degree of at most $k-2$ in G and its derivatives.

Combining (3.20) and (3.21), by the binomial theorem, we then get

$$h = kMG^{k-1} + S_{k-2}(G, M), \tag{3.22}$$

where $S_{k-2}(G, M)$ is a polynomial in G, M and their derivatives, of total degree of at most $k-2$ in G and its derivatives.

Now we claim that $M \not\equiv 0$. To prove the claim, we assume that $M \equiv 0$. Using (3.18), we get $F \equiv G$. Using (3.19)–(3.20), we have $h \equiv 0$. This contradicts the hypothesis $h \not\equiv 0$ and the claim follows.

Dividing (3.22) by MG^{k-2}, we get

$$kG + \frac{S_{k-2}(G, M)}{MG^{k-2}} = \frac{h}{MG^{k-2}}. \tag{3.23}$$

Suppose that $|G| > 1$. Now $\frac{S_{k-2}(G,M)}{MG^{k-2}}$ is a sum of terms

$$\frac{1}{MG^{k-2}} M^{j_0}(M')^{j_1} \cdots (M^{(k)})^{j_k} G^{q_0}(G')^{q_1} \cdots (G^{(k)})^{q_k},$$

where $q_0 + q_1 + \cdots + q_k \le k-2$ and hence such a term has modulus

$$\begin{aligned}
&|M|^{j_0+j_1+\cdots+j_k-1}\left|\frac{M'}{M}\right|^{j_1} \cdots \left|\frac{M^{(k)}}{M}\right|^{j_k} |G|^{q_0+q_1+\cdots+q_k-k+2}\left|\frac{G'}{G}\right|^{q_1} \cdots \left|\frac{G^{(k)}}{G}\right|^{q_k} \\
&\le |M|^{j_0+j_1+\cdots+j_k-1}\left|\frac{M'}{M}\right|^{j_1} \cdots \left|\frac{M^{(k)}}{M}\right|^{j_k}\left|\frac{G'}{G}\right|^{q_1} \cdots \left|\frac{G^{(k)}}{G}\right|^{q_k}.
\end{aligned} \tag{3.24}$$

Using (3.23)–(3.24), we get

$$\begin{aligned}
m(r, G) &\le c_0 m(r, M) + m\Big(r, \frac{1}{M}\Big) + m(r, h) + S(r, G) + S(r, M) \\
&\le c_1 T(r, M) + T(r, h) + S(r, G),
\end{aligned} \tag{3.25}$$

where c_j denote positive constants. Using (3.3), (3.7)–(3.8), (3.14), (3.17)–(3.18), (3.25) and Lemma 2.5, we deduce that

$$\begin{aligned}
m(r, A) &\le c_2(m(r, G) + \log r) \\
&\le c_3 T(r, M) + o(T(r, A)) \\
&\le c_3 N(r, M) + o(T(r, A)) \\
&\le c_3\Big(N\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{g}\Big)\Big) + o(T(r, A)) \\
&\le c_3\Big(n\Big(r, \frac{1}{f}\Big) + n\Big(r, \frac{1}{g}\Big)\Big)\log r + o(T(r, A)) \\
&\le 2c_3 n\Big(r, \frac{1}{f}\Big)\log r + o(T(r, A)) \\
&= o(T(r, A)) \\
&= o(m(r, A))
\end{aligned}$$

holds for large $r \in E_0$, where c_j denote positive constants. This is a contradiction. The proof of Theorem 1.2 is completed.

References

[1] Alotaibi, A., On complex oscillation theory, *Results Math.*, **47**, 2005, 165–175.

[2] Bank, S. and Laine, I., On the oscillation theory of $f'' + Af = 0$ where A is entire, *Trans. Amer. Math. Soc.*, **273**(1), 1982, 351–363.

[3] Bank, S., Laine, I. and Langley, J. K., On the frequency of zeros of solutions of second order linear differential equations, *Results Math.*, **10**, 1986, 8–24.

[4] Bank, S. and Langley, J. K., Oscillation theory for higher order linear differential equations with entire coefficients, *Complex Var. Theory Appl.*, **16**(2–3), 1991, 163–175.

[5] Barry, P., Some theorems related to the $\cos \pi\rho$ theorem, *Proc. London Math. Soc.*, **21**(3), 1970, 334–360.

[6] Chen, Z. X. and Gao, S. A., On complex oscillation property of solutions for higher-order periodic differential equations, *J. Inequal. Appl.*, **2007**, 2007, 13 pages, Article ID: 58189.

[7] Chen, Z. X. and Yang, C. C., Some further results on the zeros and growths of entire solutions of second order linear differential equations, *Kodai Math. J.*, **22**, 1999, 273–285.

[8] Chen, Z. X. and Yang, C. C., Quantitative estimation on the zeros and growths of entire solutions of linear differential equations, *Complex Var. Theory Appl.*, **42**(2), 2000, 119–133.

[9] Chiang, Y. M., Oscillation results on $y'' + Ay = 0$ in the complex domain with transcendental entire coefficients which have extremal deficiencies, *Proc. Edinburgh Math. Soc.,* (2), **38**(1), 1995, 13–34.

[10] Chiang, Y. M., Laine, I. and Wang, S. P., Oscillation results for some linear differential equations, *Math. Scand*, **77**(2), 1995, 209–224.

[11] Hayman, W., Meromorphic Functions, Clarendon Press, Oxford, 1964.

[12] Hille, E., Ordinary Differential Equations in the Complex Domain, Dover Publications Inc., Mineola, NY, 1997.

[13] Laine, I., Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter & Co., Berlin, New York, 1993.

[14] Laine, I. and Wu, P. C., On the osillation of certain second order linear differential equations, *Rev. Roumaine Math. Pures Appl.*, **44**(4), 1999, 609–615.

[15] Langley, J. K., Some oscillation theorems for higher order linear differential equations with entire coefficients of small growth, *Results Math.*, **20**, 1991, 517–529.

[16] Langley, J. K., On entire solutions of linear differential equations with one dominant coefficients, *Analysis*, **15**(2), 1995, 187–204.

[17] Langley, J. K., Postgraduate notes on complex analysis, http://www.maths.nott-ingham.ac.uk/personal/jkl/pg1.pdf.

[18] Latreuch, Z. and Belaïdi, B., Growth and oscillation of differential polynomials generated by complex differential equations, *Electro. J. Differential Equations*, **16**, 2013, 14 pages.

[19] Li, Y. Z. and Wang, J., Oscillation of solutions of linear differential equations, *Acta. Math. Sinica*, **24**(1), 2008, 167–176.

[20] Rossi, J., Second order differential equations with transcendental coefficients, *Proc. Amer. Math. Soc.*, **97**, 1986, 61–66.

[21] Yang, L., Value Distribution Theory, Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993.