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A Modified Analytic Function Space Feynman Integral of Functionals on Function Space*

SeungJun CHANG[1] HyunSoo CHUNG[2]

Abstract In this paper, the authors introduce a class of functionals. This class forms a Banach algebra for the special cases. The main purpose of this paper is to investigate some properties of the modified analytic function space Feynman integral of functionals in the class. Those properties contain various results and formulas which were not obtained in previous papers. Also, the authors establish some relationships involving the first variation via the translation theorem on function space. In particular, the authors establish the Fubini theorem for the modified analytic function space Feynman integral which was not obtained in previous researches yet.

Keywords Generalized Brownian motion process, Modified analytic Feynman integral, First variation, Cameron-Storvick type theorem, Fubini theorem
2000 MR Subject Classification 60J65, 28C20

1 Introduction

The function space $C_{a,b}[0,T]$, induced by a generalized Brownian motion, was introduced by Yeh in [19] and studied extensively in [6, 8–10, 12]. Various theories for the generalized analytic function space Feynman integral (generalized analytic Feynman integral) on function space have studied in many papers (see [6, 8-11]). However, the Fubini theorem for the generalized analytic function space Feynman integral was not established because the generalized Brownian motion has the nonzero mean function $a(t)$. In [9], the authors introduced a new concept of modified analytic function space Feynman integral and explained some physical phenomenon via the modified analytic function space Feynman integral. Also, they established various relationships for the modified analytic function space Feynman integral. Furthermore, they have established a version of Fubini theorem for the modified analytic function space Feynman integral for the special cases only.

In this paper, we establish the existence of the modified analytic function space Feynman integral of functionals in a class. We then obtain various relationships with respect to the modified analytic function space Feynman integral via the translation theorem. The end of

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[1]Department of Mathematics, Dankook University, Cheonan 330-714, Korea.
E-mail: sejchang@dankook.ac.kr
[2]Corresponding author. Department of Mathematics, Dankook University, Cheonan 330-714, Korea.
E-mail: hschung@dankook.ac.kr
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this paper, we give the general Fubini theorem with respect to modified analytic function space Feynman integral instead of the special cases.

The generalized Brownian motion process used in this paper and used in [6, 9–12] is non-stationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field (see [18]). While the Wiener process used in [1–5, 7, 13–17] is stationary in time and is free of drift.

2 Definitions and Preliminaries

In this section, we recall some definitions and properties from [6, 9–12, 19–20].

Let $D = [0, T]$ and let $(\boldsymbol{\Omega}, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\boldsymbol{\Omega}, \mathcal{B}, P)$ and D is called a generalized Brownian motion process if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with density function

$$W_n(\vec{t}, \vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1}))\Big)^{-\frac{1}{2}} \\ \times \ \exp\Big\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \Big\},$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L_2[0, T]$ and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

As explained in [20, pp.18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$. By Theorem 14.2 in [20, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions x on $[0, T]$ with $x(0) = 0$ under the sup-norm). Hence $(C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)$ is the function space induced by Y where $\mathcal{W}(C_{a,b}[0, T])$ is the collection of all Wiener measurable subsets of $C_{a,b}[0, T]$.

Let $L^2_{a,b}[0, T]$ be the Hilbert space of functions on $[0, T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0, T]$ induced by $a(\cdot)$ and $b(\cdot)$, i.e.,

$$L^2_{a,b}[0, T] = \Big\{v : \int_0^T v^2(s) \mathrm{d}b(s) < \infty \text{ and } \int_0^T v^2(s) \mathrm{d}|a|(s) < \infty\Big\},$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)\mathrm{d}[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], \|\cdot\|_{a,b}) \subset (L^2_{0,b}[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

A subset A of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\rho A \in \mathcal{W}(C_{a,b}[0,T])$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) (see [15]).

For $v \in L^2_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$ we let $\langle v, x\rangle = \int_0^T v(t)\mathrm{d}x(t)$ denote the Paley-Wiener-Zygmund (PWZ for short) stochastic integral, for more detailed see [6, 9–12].

Throughout this paper we will assume that each functional $F : C_{a,b}[0,T] \to \mathbb{C}$ we consider is scale-invariant measurable and

$$\int_{C_{a,b}[0,T]} |F(\rho x)|\mathrm{d}\mu(x) < \infty$$

for each $\rho > 0$.

In [9], the authors have pointed out the importance of modified analytic function space Feynman integral. They explained that the concept of modified analytic function space Feynman integral can be used to investigate some behaviors of the anharmonic oscillator in quantum mechanics. These explains tell us that our research is a meaningful subject.

We recall the definition of the modified analytic function space Feynman integral (AFSFI for short) (see [9]).

Definition 2.1 *Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0$ and $\mathrm{Re}(\lambda) \geq 0\}$. Let $h \in C_{a,b}[0,T]$ be given. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be such that for each $\lambda > 0$, the function space integral*

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x + c_\lambda h)\mathrm{d}\mu(x)$$

exists for all $\lambda > 0$ where c_λ is a real number which depends on λ. If there exists a function $J^*(\lambda)$ analytic in $D \subset \mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the *modified analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in D$ we write*

$$J^*(\lambda) = \int_{C_{a,b}[0,T]}^{an_\lambda^{c_\lambda},h} F(x)\mathrm{d}\mu(x).$$

Let $q \neq 0$ be a real number and let F be a functional such that $\int_{C_{a,b}[0,T]}^{an_\lambda^{c_\lambda},h} F(x)\mathrm{d}\mu(x)$ exists for all $\lambda \in D$. If the following limit exists, we call it the modified AFSFI *of F with parameter q and we write*

$$\int_{C_{a,b}[0,T]}^{anf_q^{c_q},h} F(x)\mathrm{d}\mu(x) = \lim_{\lambda\to -\mathrm{i}q} \int_{C_{a,b}[0,T]}^{an_\lambda^{c_\lambda},h} F(x)\mathrm{d}\mu(x),$$

where λ approaches $-\mathrm{i}q$ through values in D.

Remark 2.1 If $h(t) \equiv 0$ on $[0,T]$ or $c_\lambda = 0$, our modified AFSFI equals the concept of the generalized AFSFI, namely

$$\int_{C_{a,b}[0,T]}^{anf_q^{c_q},h} F(x)\mathrm{d}\mu(x) = \int_{C_{a,b}[0,T]}^{anf_q} F(x)\mathrm{d}\mu(x),$$

where $\int_{C_{a,b}[0,T]}^{anf_q} F(x)\mathrm{d}\mu(x)$ denotes the generalized AFSFI. Furthermore, in the setting of classical Wiener space (in our research, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$), our modified analytic Feynman integral, the generalized analytic Feynman integral and the analytic Feynman integral coincide.

The following is a well-known integration formula which is used several times in this paper. For each $\alpha \in \mathbb{C}$ and for $v \in L_{a,b}^2[0,T]$,

$$\int_{C_{a,b}[0,T]} \exp\{\alpha\langle v,x\rangle\}\mathrm{d}\mu(x) = \exp\Big\{\frac{\alpha^2}{2}(v^2,b') + \alpha(v,a')\Big\}, \tag{2.1}$$

where $(v^2,b') = \int_0^T v^2(s)\mathrm{d}b(s)$ and $(v,a') = \int_0^T v(s)\mathrm{d}a(s)$.

For each complex number α with $\mathrm{Re}(\alpha^2) \leq 0$, let $\mathcal{S}_\alpha \equiv \mathcal{S}_\alpha(L_{a,b}^2[0,T])$ be the class of functionals of the form

$$F(x) = \int_{L_{a,b}^2[0,T]} \exp\{\alpha\langle v,x\rangle\}\mathrm{d}f(v) \tag{2.2}$$

for s-a.e. $x \in C_{a,b}[0,T]$ such that for all $\rho > 0$,

$$\int_{L_{a,b}^2[0,T]} \exp\Big\{\rho\mathrm{Re}(\alpha)\int_0^T |v(t)|\mathrm{d}|a|(t)\Big\}|\mathrm{d}f(v)| < \infty, \tag{2.3}$$

where f is in $M(L_{a,b}^2[0,T])$, the class of all complex valued countably additive Borel measures on $\mathcal{B}(L_{a,b}^2[0,T])$.

Remark 2.2 We have the following observations as follows.

(1) For each $\rho > 0$,

$$\Big|\int_{C_{a,b}[0,T]} F(\rho x)\mathrm{d}\mu(x)\Big| \le \int_{L^2_{a,b}[0,T]} \exp\Big\{\rho \mathrm{Re}(\alpha)\int_0^T |v(t)|\mathrm{d}|a|(t)\Big\}|\mathrm{d}f(v)| < \infty.$$

This means that F is defined for s-a.e. $x \in C_{a,b}[0,T]$.

(2) If $\alpha = \mathrm{i}p$ for some p is in $\mathbb{R}$, then $\mathrm{Re}(\alpha) = 0$ and so

$$\Big|\int_{C_{a,b}[0,T]} F(\rho x)\mathrm{d}\mu(x)\Big| \le \|f\| < \infty.$$

That is to say, the condition (2.3) always holds. Furthermore, using the techniques similar to those used in [5], we can show that for each $\alpha \in \mathbb{C}$ with $\alpha = \mathrm{i}p, p \in \mathbb{R}$, the class $\mathcal{S}_\alpha$ is a Banach algebra with the norm

$$\|F\| = \|f\| = \int_{L^2_{a,b}[0,T]} |\mathrm{d}f(v)|, \quad f \in M(L^2_{a,b}[0,T]).$$

One can show that the correspondence $f \to F$ is injective, carries convolution into pointwise multiplication.

(3) In the setting of classical Wiener space (in our research, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$), the condition (2.3) always holds. Hence the class $\mathcal{S}_\alpha$ forms the Banach algebra for all nonzero complex number α with $\mathrm{Re}(\alpha^2) \le 0$.

3 Modified AFSFIs of Functionals in $\mathcal{S}_\alpha$

In this section we establish the existence of the modified AFSFI of functionals in $\mathcal{S}_\alpha$.

To establish the existence of modified AFSFI, we have to describe a region as a remark.

Remark 3.1 (1) Let $\gamma_1 = \eta + \mathrm{i}\zeta$ and $\gamma_2 = c + \mathrm{i}d$ be nonzero complex numbers with $\eta \le 0$ and $c \ge 0$. First, we note that

$$\mathrm{Re}\Big(\frac{\gamma_1}{\gamma_2}\Big) = \frac{\eta c + \zeta d}{c^2 + d^2} \le 0$$

implies that $\eta c + \zeta d \le 0$. This tells us that there are many nonzero complex numbers γ_1 and γ_2 so that $\mathrm{Re}\big(\frac{\gamma_1}{\gamma_2}\big) \le 0$. For example, if we take $\gamma_1 = -1 + \mathrm{i}$ and $\gamma_2 = 1 + \mathrm{i}$, then $\mathrm{Re}\big(\frac{\gamma_1}{\gamma_2}\big) = 0$. Also, if we take $\gamma_1 = -3 + 2\mathrm{i}$ and $\gamma_2 = 4 + 3\mathrm{i}$, then $\mathrm{Re}\big(\frac{\gamma_1}{\gamma_2}\big) = -6 \le 0$.

(2) Let α be a complex number with $\mathrm{Re}(\alpha^2) \le 0$ and let λ be an element of $\mathbb{C}_+$. Throughout this paper, we will consider a subregion Γ_α of $\mathbb{C}_+$, where

$$\Gamma_\alpha = \Big\{\lambda \in \mathbb{C}_+ : \mathrm{Re}\Big(\frac{\alpha^2}{\lambda}\Big) \le 0\Big\}. \tag{3.1}$$

In view of (1), the region Γ_α has sufficiently many complex numbers λ.

(3) Now we explain the region Γ_α for each α with $\mathrm{Re}(\alpha^2) \le 0$. Let $\alpha^2 = \eta + \mathrm{i}\zeta$ and $\lambda = c + \mathrm{i}d$ be complex numbers with $a \le 0$ and $c > 0$. Then for each α with $\mathrm{Re}(\alpha^2) \le 0$, we can describe the region Γ_α as follows.

1) When $d = 0$ or $\zeta = 0$, $\Gamma_\alpha = \mathbb{C}_+$.

2) When $d \neq 0$ and $\zeta \neq 0$, for a given α, if $\zeta > 0$, then the region Γ_α is given by $\{\lambda : d \leq -\frac{\eta}{\zeta}c\}$ and if $\zeta < 0$, then the region Γ_α is given by $\{\lambda : d \geq -\frac{\eta}{\zeta}c\}$.

3) The region Γ_α always contains all positive real numbers.

In our first lemma, we give the existence of the modified analytic function space integral of a functional F in $\mathcal{S}_\alpha$.

Lemma 3.1 *Let F be an element of $\mathcal{S}_\alpha$ such that the associated measure f satisfies the condition*

$$\begin{aligned}\int_{L^2_{a,b}[0,T]} &\exp\Big\{\mathrm{Re}(\alpha\lambda^{-\frac{1}{2}})\int_0^T |v(t)|\mathrm{d}|a|(t)\\&+\mathrm{Re}(c_\lambda)\|z_h\|_\infty\int_0^T |v(t)|\mathrm{d}b(t)\Big\}|\mathrm{d}f(v)|<\infty. \end{aligned}\tag{3.2}$$

Then the modified analytic function space integral $\int_{C_{a,b}[0,T]}^{an_\lambda^{c_\lambda,h}} F(x)\mathrm{d}\mu(x)$ of F exists and is equal to

$$\int_{L^2_{a,b}[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}(v^2,b')+\alpha\lambda^{-\frac{1}{2}}(v,a')+c_\lambda(vz_h,b')\Big\}\mathrm{d}f(v). \tag{3.3}$$

Proof First, for $z_h \in L_\infty[0,T]$, let $h(t) = \int_0^t z_h(s)\mathrm{d}b(s)$. Then $vz_h \in L^2_{a,b}[0,T]$ and $\langle v,h\rangle = (vz_h,b')$ for each $v \in L^2_{a,b}[0,T]$. Next, we note that for all $\lambda > 0$, using formula (2.1) and the Fubini theorem, it follows that

$$\begin{aligned}J(\lambda) &\equiv \int_{C_{a,b}[0,T]}\int_{L_{a,b}[0,T]} \exp\{\lambda^{-\frac{1}{2}}\alpha\langle v,x\rangle+\alpha c_\lambda(vz_h,b')\}\mathrm{d}f(v)\mathrm{d}\mu(x)\\&=\int_{L^2_{a,b}[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}(v^2,b')+\alpha\lambda^{-\frac{1}{2}}(v,a')+c_\lambda(vz_h,b')\Big\}\mathrm{d}f(v).\end{aligned}$$

Also, for all $\lambda > 0$, from the condition (3.2) and the fact that the real part of $\frac{\alpha^2}{2\lambda}$ is still non positive,

$$\begin{aligned}|J(\lambda)| \leq \int_{L^2_{a,b}[0,T]} &\exp\Big\{\lambda^{-\frac{1}{2}}\mathrm{Re}(\alpha)\int_0^T |v(t)|\mathrm{d}|a|(t)\\&+\mathrm{Re}(c_\lambda)\|z_h\|_\infty\int_0^T |v(t)|\mathrm{d}b(t)\Big\}|\mathrm{d}f(v)|<\infty.\end{aligned}$$

Finally, let

$$J^*(\lambda) = \int_{L^2_{a,b}[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}(v^2,b')+\alpha\lambda^{-\frac{1}{2}}(v,a')+c_\lambda(vz_h,b')\Big\}\mathrm{d}f(v),$$

where $\lambda \in \Gamma_\alpha$. Then, the function $J^*(\lambda)$ is well-defined on the region Γ_α. In fact,

$$|J^*(\lambda)| \leq \int_{L^2_{a,b}[0,T]} \exp\Big\{\mathrm{Re}(\alpha\lambda^{-\frac{1}{2}})\int_0^T |v(t)|\mathrm{d}|a|(t)$$

$$+\operatorname{Re}(c_\lambda)\|z_h\|_\infty\int_0^T|v(t)|\mathrm{d}b(t)\Big\}|\mathrm{d}f(v)|<\infty$$

for all $\lambda\in\Gamma_\alpha$. Also, $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$. Last, we will show that $J^*(\lambda)$ is analytic on Γ_α. Let Λ be any simple closed contour in Γ_α. Then using the Fubini theorem and the Cauchy theorem, we have

$$\begin{aligned}\int_\Lambda J^*(\lambda)\mathrm{d}\lambda&=\int_\Lambda\int_{L^2_{a,b}[0,T]}\exp\Big\{\frac{\alpha^2}{2\lambda}(v^2,b')+\alpha\lambda^{-\frac12}(v,a')+c_\lambda(vz_h,b')\Big\}\mathrm{d}f(v)\mathrm{d}\lambda\\&=\int_{L^2_{a,b}[0,T]}\int_\Lambda\exp\Big\{\frac{\alpha^2}{2\lambda}(v^2,b')+\alpha\lambda^{-\frac12}(v,a')+c_\lambda(vz_h,b')\Big\}\mathrm{d}\lambda\mathrm{d}f(v)\\&=0\end{aligned}$$

because the function $\exp\big\{\frac{\alpha^2}{2\lambda}(v^2,b')+\alpha\lambda^{-\frac12}(v,a')\big\}$ is analytic as a function of λ on Γ_α for each $\alpha\in\mathbb{C}$ with $\operatorname{Re}(\alpha^2)\le0$. Hence using the Morera's theorem $J^*(\lambda)$ is analytic on Γ_α and so we complete the proof of Lemma 3.1.

The following theorem is the first main theorem in this paper. In our next theorem, we give the existence of modified AFSFI of functionals in $\mathcal{S}_\alpha$.

Theorem 3.1 *Let q_0 be a nonzero real number. Let F and f be as in Lemma* 3.1 *and let q be a real number such that*

$$\begin{cases}|q|\ge|q_0|\ \textit{and}\ \operatorname{sign}(q)=-\operatorname{sign}(\operatorname{Im}(\alpha^2)), & \textit{if}\ \operatorname{Im}(\alpha^2)\ne0,\\|q|\ge|q_0|, & \textit{if}\ \operatorname{Im}(\alpha^2)=0,\end{cases}\tag{3.4}$$

where sign denotes the signum function defined by the formula $\operatorname{sign}(s)=\begin{cases}1, & \textit{if}\ s>0,\\-1, & \textit{if}\ s<0.\end{cases}$

Assume that

$$\begin{aligned}&\int_{L^2_{a,b}[0,T]}\exp\Big\{-\frac{\operatorname{Im}(\alpha^2)}{2|q_0|}(v^2,b')+\frac{M_\alpha}{\sqrt{2q_0}}\int_0^T|v(t)|\mathrm{d}|a|(t)\\&+\operatorname{Re}(c_q)\|z_h\|_\infty\int_0^T|v(t)|\mathrm{d}|a|(t)\Big\}|\mathrm{d}f(v)|<\infty,\end{aligned}\tag{3.5}$$

where $M_\alpha=|\operatorname{Re}(\alpha)-\operatorname{Im}(\alpha)|$. Then the modified AFSFI $\int^{anf_q^{c_q,h}}_{C_{a,b}[0,T]}F(x)\mathrm{d}\mu(x)$ *of F exists and is equal to*

$$\int_{L^2_{a,b}[0,T]}\exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}(v^2,b')+\alpha\Big(\frac{\mathrm{i}}{q}\Big)^{\frac12}(v,a')+c_q(vz_h,b')\Big\}\mathrm{d}f(v).\tag{3.6}$$

Proof It suffices to show that

$$\lim_{\lambda\to-\mathrm{i}q}J^*(\lambda)=\int_{L^2_{a,b}[0,T]}\exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}(v^2,b')+\alpha\Big(\frac{\mathrm{i}}{q}\Big)^{\frac12}(v,a')+c_q(vz_h,b')\Big\}\mathrm{d}f(v).\tag{3.7}$$

To do this, we recall the region Γ_α as in Remark 3.1. Then for all nonzero real number q satisfies the condition (3.4), there exists a sequence $\{\lambda_l\}_{l=1}^\infty$ in Γ_α such that $\lambda_l\to-\mathrm{i}q$ as $l\to\infty$. Then by Remark 3.1 and Lemma 3.1, we see that

$$|J^*(\lambda_l)| \le \int_{L^2_{a,b}[0,T]} \exp\Big\{\mathrm{Re}(\alpha\lambda_l^{-\frac{1}{2}})\int_0^T |v(t)|\mathrm{d}|a|(t) \\ + \mathrm{Re}(c_{\lambda_l})\|z_h\|_\infty \int_0^T |v(t)|\mathrm{d}b(t)\Big\}|\mathrm{d}f(v)| < \infty$$

for all $l = 1, 2, \cdots$. Hence using the dominated convergence theorem, for all nonzero real number q satisfies the condition (3.4),

$$\lim_{\lambda_l\to -\mathrm{i}q} J^*(\lambda_l) = \lim_{\lambda_l\to -\mathrm{i}q}\int_{L^2_{a,b}[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda_l}(v^2,b') + \alpha\lambda_l^{-\frac{1}{2}}(v,a') + c_{\lambda_l}(vz_h,b')\Big\}\mathrm{d}f(v) \\ = \int_{L^2_{a,b}[0,T]} \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}(v^2,b') + \alpha\Big(\frac{\mathrm{i}}{q}\Big)^{\frac{1}{2}}(v,a') + c_q(vz_h,b')\Big\}\mathrm{d}f(v),$$

which establishes (3.7) as desired. Also, we have

$$\Big|\int^{anf_q^{c_q,h}}_{C_{a,b}[0,T]} F(x)\mathrm{d}\mu(x)\Big| \\ \le \int_{L^2_{a,b}[0,T]} \exp\Big\{-\frac{\mathrm{Im}(\alpha^2)}{2|q_0|}(v^2,b') + \frac{\mathrm{Re}(\alpha)-\mathrm{Im}(\alpha)}{\sqrt{2q_0}}\int_0^T |v(t)|\mathrm{d}|a|(t) \\ + \mathrm{Re}(c_q)\|z_h\|_\infty \int_0^T |v(t)|\mathrm{d}|a|(t)\Big\}|\mathrm{d}f(v)| < \infty.$$

Hence we complete the proof of Theorem 3.1.

Remark 3.2 (1) From the definition of the class $\mathcal{S}_\alpha$, Lemma 3.1 and Theorem 3.1, we gave a condition to establish the existence of the modified analytic function space Feynman integral and modified AFSFI of F in $\mathcal{S}_\alpha$ respectively. But, we can give only one condition which contains these conditions as follows; let $M_1 = \rho\mathrm{Re}(\alpha), M_2 = \mathrm{Re}(\alpha\lambda^{-\frac{1}{2}}), M_3 = \mathrm{Re}(c_\lambda)\|z_h\|_\infty, M_4 = -\frac{\mathrm{Im}(\alpha^2)}{2|q_0|}, M_5 = \frac{M_\alpha}{\sqrt{2q_0}}$ and $M_6 = \mathrm{Re}(c_q)\|z_h\|_\infty$. Then all conditions (2.3), (3.2) and (3.5) are dominated by the condition

$$\int_{L^2_{a,b}[0,T]} \exp\{M_{\alpha,\lambda,q_0}(\|v\|_{a,b} + \|v\|^1_{a,b})\}|\mathrm{d}f(v)| < \infty, \tag{3.8}$$

where $M_{\alpha,\lambda,q_0} = \max\{|M_1|, \cdots, |M_6|\}$ and $\|v\|^1_{a,b}$ denotes the L_1-norm with respect to the a and b. Hence we can assume that for each F in $\mathcal{S}_\alpha$, F always satisfies the condition (3.8) above.

(2) If $\alpha = \mathrm{i}p$ for some $p \in \mathbb{R}$, then $M_{\alpha,\lambda,q_0} = \max\{|M_2|, |M_3|, |M_5|, |M_6|\}$ and

$$\int_{L^2_{a,b}[0,T]} \exp\{M_{\alpha,\lambda,q_0}\|v\|^1_{a,b}\}|\mathrm{d}f(v)| < \infty.$$

In particular, if $h(t) = 0$ on $[0,T]$, then $M_{\alpha,\lambda,q_0} = |M_2|$ and

$$\int_{L^2_{a,b}[0,T]} \exp\{M_2\|v\|^1_{a,b}\}|\mathrm{d}f(v)| < \infty.$$

4 Some Properties for the Modified AFSFI

In this section we give some relationships with respect to the modified AFSFI via the translation theorem on function space.

The following result was established in [11, p. 379].

Lemma 4.1 (Translation Theorem) *Let $z \in L^2_{a,b}[0,T]$ be given and let $x_0(t) = \int_0^t z(s)\mathrm{d}b(s)$ for $t \in [0,T]$. Assume that for $F : C_{a,b}[0,T] \to \mathbb{C}$,*

$$\int_{C_{a,b}[0,T]} |F(\rho x)|\mathrm{d}\mu(x) < \infty$$

for all non-zero real numbers ρ. Then

$$\begin{aligned}&\int_{C_{a,b}[0,T]} F(x + x_0)\mathrm{d}\mu(x)\\ &= \exp\Big\{ -\frac{1}{2}(z^2, b') - (z, a')\Big\} \int_{C_{a,b}[0,T]} F(x)\exp\{\langle z, x\rangle\}\mathrm{d}\mu(x). \qquad (4.1)\end{aligned}$$

We next give the definition of the first variation of a functional F on $C_{a,b}[0,T]$.

Definition 4.1 *Let F be a functional defined on $C_{a,b}[0,T]$. Then the first variation of F is defined by the formula*

$$\delta F(x \mid u) = \frac{\partial}{\partial k}F(x + ku)\Big|_{k=0}, \quad x, u \in C_{a,b}[0,T], \qquad (4.2)$$

if it exists.

We state an interesting observation involving the first variation.

Remark 4.1 (1) To establish the existence of the first variation of F in $\mathcal{S}_\alpha$, we give a condition for f as follows. For $F \in \mathcal{S}_\alpha$, we will assume that the associated measure f in $M(L^2_{a,b}[0,T])$ of F always satisfies the following inequality

$$\int_{L^2_{a,b}[0,T]} \|v\|_{a,b}|\mathrm{d}f(v)| < \infty. \qquad (4.3)$$

(2) First we could consider the following integral

$$\int_{L^2_{a,b}[0,T]} \alpha\langle v, u\rangle \exp\{\alpha\langle v, x\rangle\}\mathrm{d}f(v). \qquad (4.4)$$

Since $\mathrm{Re}(\alpha^2) \le 0$ and by an assumption (4.3),

$$\int_{L^2_{a,b}[0,T]} \alpha\langle v, u\rangle\mathrm{d}f(v) < \infty \qquad (4.5)$$

and

$$\int_{L^2_{a,b}[0,T]} \exp\{\alpha\langle v, x\rangle\}\mathrm{d}f(v)$$

exists for s-a.e. $x \in C_{a,b}[0,T]$. However, the integral (4.4) might not exist because the product of L_1-functionals might not be in L_1. Hence we should give a condition for f as follows. If $\mathrm{Re}(\alpha^2) \le 0$ and (4.3) holds then the integral (4.4) always exists.

In our next theorem, we obtain the formula for the first variation of functionals from $\mathcal{S}_\alpha$ into $\mathcal{S}_\alpha$.

Theorem 4.1 *Let F and f be as in Theorem* 3.1 *and let $u(t)=\int_0^t z_u(s)\mathrm{d}b(s)$ for some $z_u\in L_\infty[0,T]$. Assume that*

$$\Big|\frac{\partial}{\partial k}\exp\{\alpha\langle v,x+ku\rangle\}\Big|\le L(x), \tag{4.6}$$

where $L(x)$ is integrable on $C_{a,b}[0,T]$. Then the first variation $\delta F(x\mid u)$ of F exists and is equal to

$$\delta F(x\mid u)=\int_{L^2_{a,b}[0,T]}\alpha\langle v,u\rangle\exp\{\alpha\langle v,x\rangle\}\mathrm{d}f(v) \tag{4.7}$$

for s-a.e. $x\in C_0[0,T]$. Furthermore, as a function of x, δF is an element of $\mathcal{S}_\alpha$. In fact,

$$\delta F(x\mid u)=\int_{L^2_{a,b}[0,T]}\exp\{\alpha\langle v,x\rangle\}\mathrm{d}\phi(v),$$

where ϕ is an element of $M(L^2_{a,b}[0,T])$.

Proof Using (4.2) it follows that for s-a.e. $x\in C_{a,b}[0,T]$,

$$\begin{aligned}\delta F(x\mid u)&=\frac{\partial}{\partial k}\Big(\int_{L^2_{a,b}[0,T]}\exp\{\alpha\langle v,x\rangle+\alpha k\langle v,u\rangle\}\mathrm{d}f(v)\Big)\Big|_{k=0}\\&=\int_{L^2_{a,b}[0,T]}\alpha\langle v,u\rangle\exp\{\alpha\langle v,x\rangle\}\mathrm{d}f(v)\\&=\int_{L^2_{a,b}[0,T]}\exp\{\alpha\langle v,x\rangle\}\mathrm{d}\phi(v),\end{aligned} \tag{4.8}$$

where $\phi(E)=\int_E\alpha\langle v,u\rangle\mathrm{d}f(v)$ for $E\in\mathcal{B}(L^2_{a,b}[0,T])$. The first equality in (4.8) follows from condition (4.6) and so by using Remark 4.1, the all expressions in (4.8) exists. Hence we completes the proof of Theorem 4.2.

The following theorem is the second main result in this paper. In Theorem 4.2, we establish the existence of the modified AFSFI of the first variation for a functional F in $\mathcal{S}_\alpha$.

Theorem 4.2 *Let F,f,q and u be as in Theorem* 4.1. *Then the modified generalized* AFSFI $\int_{C_{a,b}[0,T]}^{anf_q^{c_q,h}}\delta F(x\mid u)\mathrm{d}\mu(x)$ *of $\delta F(x\mid u)$ exists and is equal to*

$$\int_{L^2_{a,b}[0,T]}\alpha\langle v,u\rangle\exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}(v^2,b')+\alpha\Big(\frac{\mathrm{i}}{q}\Big)^{\frac12}(v,a')+c_q(vz_h,b')\Big\}\mathrm{d}f(v). \tag{4.9}$$

Proof From Theorem 3.1 by replacing F with δF, we can prove Theorem 4.2.

To establish some relationships via the translation theorem, we need some facts as follows. For F and G be functionals on $C_{a,b}[0,T]$, $\delta(FG)(x\mid u)=\delta F(x\mid u)G(x)+F(x)\delta G(x\mid u)$ if it exists. Let F,u,h be as in Theorem 4.2 and let $G(x)=\exp\{\lambda c_\lambda\langle z_h,x\rangle\}$. Then we have

$$\delta G(x\mid u)=\lambda c_\lambda(z_hz_u,b')\exp\{\lambda c_\lambda\langle z_h,x\rangle\},$$

and so

$$\delta(FG)(x \mid u) = \delta F(x \mid u)\exp\{\lambda c_\lambda \langle z_h, x\rangle\} + \lambda c_\lambda (z_h z_u, b')F(x)\exp\{\lambda c_\lambda\langle z_h, x\rangle\}.$$

Thus we can conclude that

$$\begin{aligned}\delta F(x \mid u)\exp\{\lambda c_\lambda\langle z_h, x\rangle\} &= \delta(F(\cdot)\exp\{\lambda c_\lambda\langle z_h, \cdot\rangle\})(x \mid u)\\ &\quad - \lambda c_\lambda (z_h z_u, b')F(x)\exp\{\lambda c_\lambda\langle z_h, x\rangle\}. \qquad (4.10)\end{aligned}$$

The first relationship tells us that the modified AFSFI of the first variation of F in $\mathcal{S}_\alpha$ can be expressed by the modified AFSFIs without usage concept the first variation. It is called the modified Cameron-Storvick type theorem for the modified AFSFI.

Theorem 4.3 Relation 1 *Let F, q, h and u be as in Theorem 4.2. Then*

$$\begin{aligned}\int_{C_{a,b}[0,T]}^{anf_q^{c_q,h}} \delta F(x \mid u)\mathrm{d}\mu(x) &= \mathrm{i}qc_q(z_u z_h, b')\int_{C_{a,b}[0,T]}^{anf_q^{c_q,h}} F(x)\mathrm{d}\mu(x) - \mathrm{i}q\int_{C_{a,b}[0,T]}^{anf_q^{c_q,h}} \langle z_u, x\rangle F(x)\mathrm{d}\mu(x)\\ &\quad - (-\mathrm{i}q)^{\frac{1}{2}}(z_u, a')\int_{C_{a,b}[0,T]}^{anf_q^{c_q,h}} F(x)\mathrm{d}\mu(x). \qquad (4.11)\end{aligned}$$

Proof First, let $F_h(x) = F(x + c_\lambda h)$ and $G_\lambda(x) = F_h(\lambda^{-\frac{1}{2}}x)$. Using (4.2), for each $\lambda > 0$,

$$\begin{aligned}\int_{C_{a,b}[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + c_\lambda h \mid u)\mathrm{d}\mu(x) &= \frac{\partial}{\partial k}\int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x + c_\lambda h + ku)\mathrm{d}\mu(x)\Big|_{k=0}\\ &= \frac{\partial}{\partial k}\int_{C_{a,b}[0,T]} G_\lambda(x + x_0)\mathrm{d}\mu(x)\Big|_{k=0},\end{aligned}$$

where $x_0(t) = \int_0^t \lambda^{\frac{1}{2}} k z_u \mathrm{d}b(s)$. Applying the translation theorem in Lemma 4.1 to the functional G_λ, we have

$$\begin{aligned}&\int_{C_{a,b}[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + c_\lambda h \mid u)\mathrm{d}\mu(x)\\ &= \frac{\partial}{\partial k}\Big[\exp\Big\{-\frac{\lambda k^2}{2}(v^2, b') - \lambda^{\frac{1}{2}}k(v, a')\Big\}\int_{C_{a,b}[0,T]} G_\lambda(x)\exp\{\lambda^{\frac{1}{2}}k\langle z_u, x\rangle\}\mathrm{d}\mu(x)\Big]\Big|_{k=0}\\ &= -\lambda^{\frac{1}{2}}(z_u, a')\int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x + c_\lambda h)\mathrm{d}\mu(x)\\ &\quad + \int_{C_{a,b}[0,T]} \lambda^{\frac{1}{2}}\langle z_u, x\rangle F(\lambda^{-\frac{1}{2}}x + c_\lambda h)\mathrm{d}\mu(x)\\ &= \lambda\int_{C_{a,b}[0,T]} \langle z_u, \lambda^{-\frac{1}{2}}x + c_\lambda h\rangle F(\lambda^{-\frac{1}{2}}x + c_\lambda h)\mathrm{d}\mu(x)\\ &\quad - \lambda c_\lambda(z_u z_h, b')\int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x + c_\lambda h)\mathrm{d}\mu(x)\\ &\quad - \lambda^{\frac{1}{2}}(z_u, a')\int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x + c_\lambda h)\mathrm{d}\mu(x).\end{aligned}$$

It can be analytically λ in $\mathbb{C}_+$. As $\lambda \to -\mathrm{i}q$, we can obtain (4.11).

Throughout the next relationship, we establish that the modified generalized AFSFI of the first variation of F in $\mathcal{S}_\alpha$ is the generalized AFSFIs. That is to say, (4.12) tells that there is a connection between the modified AFSFI and generalized AFSFI.

Theorem 4.4 Relation 2 *Let F, h, q and u be as in Theorem 4.3. Then*

$$\begin{aligned}\int_{C_{a,b}[0,T]}^{anf_q^{c_q,h}} \delta F(x \mid u)\mathrm{d}\mu(x) &= \exp\Big\{\frac{\mathrm{i}qc_q^2}{2}(z_h, b') - (-\mathrm{i}q)^{\frac{1}{2}}c_q(z_h, a')\Big\} \\ &\quad\cdot \int_{C_{a,b}[0,T]}^{anf_q} \delta(F(\cdot)\exp\{-\mathrm{i}qc_q\langle z_h, \cdot\rangle\})(x \mid u)\mathrm{d}\mu(x) \\ &\quad+ \mathrm{i}qc_q(z_h z_u, b')\exp\Big\{\frac{\mathrm{i}qc_q^2}{2}(z_h, b') - (-\mathrm{i}q)^{\frac{1}{2}}c_q(z_h, a')\Big\} \\ &\quad\cdot \int_{C_{a,b}[0,T]}^{anf_q} F(x)\exp\{-\mathrm{i}qc_q\langle z_h, x\rangle\}\mathrm{d}\mu(x). \qquad (4.12)\end{aligned}$$

Proof Let $H_\lambda(x) = \delta F(\lambda^{-\frac{1}{2}}x \mid u)$ and let $x_0(t) = \int_0^t \lambda c_\lambda z_h(s)\mathrm{d}b(s)$. Then using (4.1)–(4.2), we have

$$\begin{aligned}&\int_{C_{a,b}[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + c_\lambda h \mid u)\mathrm{d}\mu(x) \\ &= \int_{C_{a,b}[0,T]} \delta F(\lambda^{-\frac{1}{2}}(x + \lambda^{\frac{1}{2}}c_\lambda h) \mid u)\mathrm{d}\mu(x) \\ &= \int_{C_{a,b}[0,T]} H_\lambda(x + x_0)\mathrm{d}\mu(x) \\ &= \exp\Big\{-\frac{\lambda c_\lambda^2}{2}(z_h^2, b') - \lambda^{\frac{1}{2}}c_\lambda(z_h, a')\Big\}\int_{C_{a,b}[0,T]} \delta F(\lambda^{-\frac{1}{2}}x \mid u)\exp\{\lambda^{\frac{1}{2}}c_\lambda\langle z_h, x\rangle\}\mathrm{d}\mu(x).\end{aligned}$$

Using (4.10), we have

$$\begin{aligned}&\int_{C_{a,b}[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + c_\lambda h \mid u)\mathrm{d}\mu(x) \\ &= \exp\Big\{-\frac{\lambda c_\lambda^2}{2}(z_h, b') - \lambda^{\frac{1}{2}}c_\lambda(z_h, a')\Big\} \\ &\cdot \int_{C_{a,b}[0,T]} \delta(F(\cdot)\exp\{\lambda c_\lambda\langle z_h, \cdot\rangle\})(\lambda^{-\frac{1}{2}}x \mid u)\mathrm{d}\mu(x) \\ &- \lambda c_\lambda(z_h z_u, b')\exp\Big\{-\frac{\lambda c_\lambda^2}{2}(z_h, b') - \lambda^{\frac{1}{2}}c_\lambda(z_h, a')\Big\} \\ &\cdot \int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x)\exp\{\lambda c_\lambda\langle z_h, \lambda^{-\frac{1}{2}}x\rangle\}\mathrm{d}\mu(x).\end{aligned}$$

It can be analytically λ in $\mathbb{C}_+$. As $\lambda \to -\mathrm{i}q$, we can obtain (4.12).

Remark 4.2 Applying Theorems 4.3–4.4, the right-side of (4.11)–(4.12), we can obtain a formula for the modified AFSFI and the generalized AFSFI. Also, we can apply the Cameron-Storvick type theorem used in [9, 12] to obtain another formula in first term in the right-side of (4.12).

5 More Properties for the Modified AFSFI

In this section we give some more relationships with respect to the modified AFSFI. In particular, we give the Fubini theorem with respect to the modified AFSFI for the general cases. Before do this, we will consider the following notations and formulas.

(1) We define a function to simply express many results and formulas in this paper. For $n \geq 2$, define a function $H_n : \widetilde{\mathbb{C}}_+^n \to \widetilde{\mathbb{C}}_+$ by

$$H_n(z_1, \cdots, z_n) = \sum_{j=1}^{n} z_j^{-\frac{1}{2}} - \Big(\sum_{j=1}^{n} z_j^{-1}\Big)^{\frac{1}{2}},$$

where $\sum\limits_{j=1}^{n} z_j^{-\frac{1}{2}} \neq 0$ and $\sum\limits_{j=1}^{n} z_j^{-1} \neq 0$. Note that H_n is a symmetric function for all $n = 2, 3, \cdots$.

(2) Let F be a $\mathbb{C}$-valued functional on $C_{a,b}[0,T]$ such that

$$\int_{C^2_{a,b}[0,T]} |F(\gamma x + \beta y)| \mathrm{d}(\mu \times \mu)(x, y) < \infty$$

for all nonzero real numbers γ and β. Then

$$\begin{aligned}
&\int_{C^2_{a,b}[0,T]} F(\gamma x + \beta y) \mathrm{d}(\mu \times \mu)(x, y) \\
=& \int_{C_{a,b}[0,T]} F(\sqrt{\gamma^2 + \beta^2} z + (\gamma + \beta - \sqrt{\gamma^2 + \beta^2}) a) \mathrm{d}\mu(z) \\
=& \int_{C_{a,b}[0,T]} F(\sqrt{\gamma^2 + \beta^2} z + H_2(\gamma^{-2}, \beta^{-2}) a) \mathrm{d}\mu(z). \qquad (5.1)
\end{aligned}$$

In [9], the author have established a Fubini theorem for the modified AFSFI as the special cases with respect to a instead of h only as follows:

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c_{q_2}}, a} \Big(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c_{q_1}}, a} F(x + y) \mathrm{d}\mu(x)\Big) \mathrm{d}\mu(y) \\
\doteq& \int_{C_{a,b}[0,T]}^{anf_{q_3}^{c_{q_3}}, a} F(z) \mathrm{d}\mu(z),
\end{aligned}$$

where $\doteq$ means that if either side exists, both sides exist and equality holds, $q_3 = \frac{q_1 q_2}{q_1 + q_2}$ and $c_{q_3} = H_2(-\mathrm{i}q_1, -\mathrm{i}q_2) + c_{q_1} + c_{q_2}$. However, by using an interesting formal used in the proof of Theorem 5.1, we can solve this problem. In our next theorem, we give the Fubini theorem for the modified AFSFI in general cases.

Theorem 5.1 *Let q_0 be a nonzero real number. Let q_1 and q_2 be real numbers whose satisfy the condition* (3.4) *with $q_1 + q_2 \neq 0$. Let F be an element of $\mathcal{S}_\alpha$ such that the associated measure f satisfies the condition* (3.8)*. Then*

$$\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c_{q_1}}, h_1} \Big(\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c_{q_2}}, h_2} F(x + y) \mathrm{d}\mu(x)\Big) \mathrm{d}\mu(y)$$

$$= \int_{C_{a,b}[0,T]}^{anf_{q_3}^{c_{q_3},h_3}} F(z)\mathrm{d}\mu(z)$$
$$= \int_{C_{a,b}[0,T]}^{anf_{q_2}^{c_{q_2},h_2}} \Big(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c_{q_1},h_1}} F(x+y)\mathrm{d}\mu(y)\Big)\mathrm{d}\mu(x), \tag{5.2}$$

where $q_3 = \frac{q_1 q_2}{q_1+q_2}$, $c_{q_3} = \sqrt{H_2^2(-\mathrm{i}q_1, -\mathrm{i}q_2) + c_{q_1}^2 + c_{q_2}^2} \neq 0$ *and*

$$h_3 = \frac{H_2(-\mathrm{i}q_1, -\mathrm{i}q_2)}{c_{q_3}} a + \frac{c_{q_1}}{c_{q_3}} h_1 + \frac{c_{q_2}}{c_{q_3}} h_2.$$

Further, they are given by the formula

$$\int_{L_{a,b}^2[0,T]} \exp\Big\{\Big(\frac{\mathrm{i}\alpha^2}{2q_1} + \frac{\mathrm{i}\alpha^2}{2q_2}\Big)(v^2, b') + \alpha\Big(\Big(\frac{\mathrm{i}}{q_1}\Big)^{\frac{1}{2}} + \Big(\frac{\mathrm{i}}{q_2}\Big)^{\frac{1}{2}}\Big)(v, a')$$
$$+ (c_{q_1} + c_{q_2})(vz_h, b')\Big\}\mathrm{d}f(v). \tag{5.3}$$

Proof First, we note that for each positive real numbers γ_1, γ_2 and γ_3, and $h_1, h_2, h_3 \in C_{a,b}[0,T]$, we have for $t \in [0,T]$,

$$\gamma_1 h_1(t) + \gamma_2 h_2(t) + \gamma_3 h_3(t)$$
$$= \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}\Big(\frac{\gamma_1 h_1(t)}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}} + \frac{\gamma_2 h_2(t)}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}} + \frac{\gamma_3 h_3(t)}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\Big)$$
$$\equiv \gamma h_0(t) \tag{5.4}$$

for some $\gamma \in \mathbb{R}$ and $h_0 \in C_{a,b}[0,T]$. Next, using equation (5.4) and (5.1) for $\lambda_1, \lambda_2 > 0$, we have

$$\int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} F(\lambda_1^{-\frac{1}{2}} x + \lambda_2^{-\frac{1}{2}} y + c_{\lambda_1} h_1 + c_{\lambda_2} h_2)\mathrm{d}\mu(x)\mu(y)$$
$$= \int_{C_{a,b}[0,T]} F(\sqrt{\lambda_1^{-1} + \lambda_2^{-1}} z + H_2(\lambda_1, \lambda_2)a + c_{\lambda_1} h_1 + c_{\lambda_2} h_2)\mathrm{d}\mu(z)$$
$$= \int_{C_{a,b}[0,T]} F(\sqrt{\lambda_1^{-1} + \lambda_2^{-1}} z + \gamma_3 h)\mathrm{d}\mu(z),$$

where

$$\gamma_3 = \sqrt{H_2^2(\lambda_1, \lambda_2) + c_{\lambda_1}^2 + c_{\lambda_2}^2}$$

and

$$h = \frac{H_2(\lambda_1, \lambda_2)}{\gamma_3} a + \frac{c_{\lambda_1}}{\gamma_3} h_1 + \frac{c_{\lambda_2}}{\gamma_3} h_2.$$

It can be analytically in λ_1 and λ_2 in $\mathbb{C}_+$ and as $\lambda_1 \to -\mathrm{i}q_1$ and $\lambda_2 \to -\mathrm{i}q_2$, we can establish (5.2). Finally, applying Theorem 3.1 repeatedly, we can obtain (5.3) as desired.

Combing Theorems 4.3–4.4 and 5.1, we have the following formulas.

(1) The first formula below is the modified Cameron-Storvick type theorem for the double modified AFSFIs.

$$\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c_{q_1},h_1}} \Big(\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c_{q_2},h_2}} \delta F(x+y \mid u)\mathrm{d}\mu(x)\Big)\mathrm{d}\mu(y)$$

$$
\begin{aligned}
&= \mathrm{i}\frac{q_1q_2}{q_1+q_2}c_{q_3}(z_uz_h,b')\int_{C_{a,b}[0,T]}^{anf_{q_3}^{c_{q_3},h_3}} F(z)\mathrm{d}\mu(z)\\
&\quad - \mathrm{i}\frac{q_1q_2}{q_1+q_2}\int_{C_{a,b}[0,T]}^{anf_{q_3}^{c_{q_3},h_3}} \langle z_u,z\rangle F(z)\mathrm{d}\mu(z) - \Big(-\mathrm{i}\frac{q_1q_2}{q_1+q_2}\Big)^{\frac{1}{2}}(z_u,a')\int_{C_{a,b}[0,T]}^{anf_{q_3}^{c_{q_3},h_3}} F(z)\mathrm{d}\mu(z).
\end{aligned}
$$

(2) The second formula below is the relationship between the double modified AFSFIs and generalized AFSFI.

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c_{q_1},h_1}} \Big(\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c_{q_2},h_2}} \delta F(x+y\mid u)\mathrm{d}\mu(x)\Big)\mathrm{d}\mu(y)\\
&= \exp\Big\{\frac{\mathrm{i}q_1q_2c_{q_3}^2}{2(q_1+q_2)}(z_h,b') - \Big(-\mathrm{i}\frac{q_1q_2}{q_1+q_2}\Big)^{\frac{1}{2}}c_{q_3}(z_h,a')\Big\}\\
&\quad\cdot\int_{C_{a,b}[0,T]}^{anf_{q_3}} \delta(F(\cdot)\exp\{-\mathrm{i}q_3c_{q_3}\langle z_h,\cdot\rangle\})(x\mid u)\mathrm{d}\mu(x)\\
&\quad + \mathrm{i}\frac{q_1q_2}{q_1+q_2}c_{q_3}(z_hz_u,b')\exp\Big\{\frac{\mathrm{i}q_1q_2c_{q_3}^2}{2(q_1+q_2)}(z_h,b') - \Big(-\mathrm{i}\frac{q_1q_2}{q_1+q_2}\Big)^{\frac{1}{2}}c_{q_3}(z_h,a')\Big\}\\
&\quad\cdot\int_{C_{a,b}[0,T]}^{anf_{q}} F(x)\exp\{-\mathrm{i}q_3c_{q_3}\langle z_h,x\rangle\}\mathrm{d}\mu(x).
\end{aligned}
$$

Remark 5.1 Using the mathematical induction, we also can establish all formulas and results for the n-dimensional version in this paper.

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