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Generalized Weighted Morrey Estimates for Marcinkiewicz Integrals with Rough Kernel Associated with Schrödinger Operator and Their Commutators

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Abstract Let $L=-\Delta+V(x)$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential $V(x)$ belonging to the reverse Hölder class. The aim of this paper is to give generalized weighted Morrey estimates for the boundedness of Marcinkiewicz integrals with rough kernel associated with Schrödinger operator and their commutators. Moreover, the boundedness of the commutator operators formed by BMO functions and Marcinkiewicz integrals with rough kernel associated with Schrödinger operators is discussed on the generalized weighted Morrey spaces. As its special cases, the corresponding results of Marcinkiewicz integrals with rough kernel associated with Schrödinger operator and their commutators have been deduced, respectively. Also, Marcinkiewicz integral operators, rough Hardy-Littlewood (H-L for short) maximal operators, Bochner-Riesz means and parametric Marcinkiewicz integral operators which satisfy the conditions of our main results can be considered as some examples.

Keywords Marcinkiewicz operator, Rough kernel Schrödinger operator generalized weighted Morrey space, Commutator, BMO
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1 Introduction

In this paper we consider the differential Schrödinger operator

$$L=-\Delta+V(x)\quad \text{on } \mathbb{R}^n,\quad n\geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class RH_q for some exponent $q\geq \frac{n}{2}$; that is, a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to RH_q $(q>1)$ if there exists a constant C such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V(x)^q \mathrm{d}x\Big)^{\frac{1}{q}} \leq \frac{C}{|B|}\int_B V(x)\mathrm{d}x \tag{1.1}$$

holds for every ball $B\subset\mathbb{R}^n$ (see [15, 17]). Obviously, $RH_{q_2}\subset RH_{q_1}$, if $q_1<q_2$.

We introduce the definition of the reverse Hölder index of V as $q_0=\sup\{q: V\in RH_q\}$. It is worth pointing out that the RH_q class is that, if $V\in RH_q$ for some $q>1$, then there exists

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$\varepsilon > 0$, which depends only on n and the constant C in (1.1), such that $V \in RH_{q+\varepsilon}$. Therefore, under the assumption $V \in RH_{\frac{n}{2}}$, we may conclude $q_0 > \frac{n}{2}$. Throughout this paper, we always assume that $0 \neq V \in RH_n$.

The Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}}.$$

Stein [18] first introduced the operator μ_Ω and proved that μ_Ω is of type (p,p) $(1 < p \le 2)$ and of weak type $(1,1)$ in the case of $\Omega \in Lip_\gamma(S^{n-1})$ $(0 < \gamma \le 1)$.

Similar to the Marcinkiewicz integral operator μ_Ω, one defines the Marcinkiewicz integral operator with rough kernel $\mu^L_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu^L_{j,\Omega} f(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\le t} |\Omega(x-y)| K^L_j(x,y) f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial_{x_j}} L^{-\frac{1}{2}}$, $j = 1, \cdots, n$. In particular, when $V = 0$, $K^\Delta_j(x,y) = \widetilde{K^\Delta_j}(x,y)|x-y| = \frac{\frac{x_j - y_j}{|x-y|}}{|x-y|^{n-1}}$ and $\widetilde{K^\Delta_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial_{x_j}} \Delta^{-\frac{1}{2}}$, $j = 1, \cdots, n$. In this paper, we write $K^\Delta_j(x,y) = K_j(x,y)$ and $\mu_{j,\Omega} = \mu^\Delta_{j,\Omega}$ and so $\mu^\Delta_{j,\Omega}$ is defined by

$$\mu_{j,\Omega} f(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\le t} |\Omega(x-y)| K_j(x,y) f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}}.$$

Obviously, $\mu_{j,\Omega}$ are classical Marcinkiewicz functions with rough kernel.

Now we give the definition of the commutator generalized by μ_Ω and b by

$$\mu_{\Omega,b}(f)(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}}.$$

On the other hand, for $b \in L^{\mathrm{loc}}_1(\mathbb{R}^n)$, denote by B the multiplication operator defined by $\mathrm{B}f(x) = b(x)f(x)$ for any measurable function f. If $\mu^L_{j,\Omega}$ is a linear operator on some measurable function space, then the commutator formed by B and $\mu^L_{j,\Omega}$ is defined by

$$\mu^L_{j,\Omega,b} f(x) = [b, \mu^L_{j,\Omega}] f(x) := (\mathrm{B}\mu^L_{j,\Omega} - \mu^L_{j,\Omega}\mathrm{B}) f(x) = b(x)\, \mu^L_{j,\Omega} f(x) - \mu^L_{j,\Omega}(bf)(x).$$

The commutators we are interested in here are of the form

$$\mu^L_{j,\Omega,b} f(x) = [b, \mu^L_{j,\Omega}] f(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\le t} |\Omega(x-y)| K^L_j(x,y) [b(x) - b(y)] f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}}.$$

It is worth noting that for a constant C, if $\mu^L_{j,\Omega}$ is linear we have

$$[b + C, \mu^L_{j,\Omega}] f = (b + C)\mu^L_{j,\Omega} f - \mu^L_{j,\Omega}((b + C)f)$$

$$
\begin{aligned}
&= b\mu_{j,\Omega}^{L} f + C\mu_{j,\Omega}^{L} f - \mu_{j,\Omega}^{L}(bf) - C\mu_{j,\Omega}^{L} f \\
&= [b, \mu_{j,\Omega}^{L}] f.
\end{aligned}
$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b \in \mathrm{BMO}$ (bounded mean oscillation space) has had the most historical significance.

The classical Morrey space was introduced by Morrey in [14], since then a large number of investigations have been given to them by mathematicians. Recently, some authors established the boundedness of some Marcinkiewicz integrals associated with Schrödinger operator on the Morrey type spaces from a various point of view provided that the nonnegative potential V belonging to the reverse Hölder class (see [1, 9, 17]). Motivated by these results, our aim in this paper is to establish the boundedness for the Marcinkiewicz integrals with rough kernel and their commutators associated with Schrödinger operator on generalized weighted Morrey spaces provided that the nonnegative potential V belonging to the reverse Hölder class.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The rough Hardy-Littlewood (H-L for short) maximal operator M_Ω and its commutator are defined by

$$
M_\Omega f(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |\Omega(x-y)||f(y)| \mathrm{d}y,
$$

$$
M_{\Omega,b}(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int\limits_{B(x,t)} |b(x)-b(y)||\Omega(x-y)||f(y)| \mathrm{d}y,
$$

respectively.

The structure of this paper is as the following. The first section is devoted to the introduction. In Section 2, the definitions of basic spaces such as weighted Lebesgue, weighted Morrey and generalized weighted Morrey spaces and the relationship between these spaces have been considered. The Section 3 and Section 4 are devoted to the proofs of main results. In last section, we have applied Theorem 3.2 and Theorem 4.3 (our main results) to several particular operators such as Marcinkiewicz integral operators, rough H-L maximal operators, Bochner-Riesz means and parametric Marcinkiewicz integral operators.

At last, we make some conventions on notation. Throughout this paper, C denotes a positive constant that is independent of the main parameters, but whose value may vary from line to line. The expression $F \lesssim G$ means that there exists a positive constant C such that $F \le CG$. If $F \lesssim G$ and $G \lesssim F$, we write $F \approx G$ and say that F and G are equivalent. We will also denote the conjugate exponent of $p > 1$ by $p' = \frac{p}{p-1}$ and $q > 1$ by $q' = \frac{q}{q-1}$. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For any set E, χ_E denotes its characteristic function, if E is also measurable and w is a weight, $w(E) := \int\limits_E w(x)\mathrm{d}x$. Also, throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

We have $CB(x,r)=B(x,Cr)$ for $C>0$. Finally, we use the notation

$$f_{B(x,r)}=\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}f(y)\mathrm{d}y.$$

2 Definitions and Preliminaries

In this section, we recall the definitions of basic spaces such as weighted Lebesgue, weighted Morrey and generalized weighted Morrey spaces and the relationship between these spaces which has been considered. We also present some basic facts about weight functions that we use in the following sections.

A locally integrable and positive function defined on $\mathbb{R}^n$ is called a weight. We first recall the definition of weighted Lebesgue spaces.

Definition 2.1 (Weighted Lebesgue Space) *Let $1\le p\le\infty$ and given a weight $w(x)\in A_p(\mathbb{R}^n)$, we shall define weighted Lebesgue spaces as*

$$L_p(w)\equiv L_p(\mathbb{R}^n,w)=\Big\{f:\|f\|_{L_{p,w}}=\Big(\int\limits_{\mathbb{R}^n}|f(x)|^p w(x)\mathrm{d}x\Big)^{\frac{1}{p}}<\infty\Big\},\quad 1\le p<\infty,$$

$$L_{\infty,w}\equiv L_\infty(\mathbb{R}^n,w)=\Big\{f:\|f\|_{L_{\infty,w}}=\operatorname*{esssup}_{x\in\mathbb{R}^n}|f(x)|w(x)<\infty\Big\}.$$

Here and after, A_p denotes the Muckenhoupt classes (see [8]). We now define the fundamental classes of weights known as the Muckenhoupt classes.

Definition 2.2 *Provided that A_p denotes the Muckenhoupt classes (see [8]), for $1<p<\infty$, a locally integrable function $w:\mathbb{R}^n\to(0,\infty)$ is said to be an $A_p(\mathbb{R}^n)$ weight if*

$$\begin{aligned}[w]_{A_p}&:=\sup_B[w]_{A_p(B)}\\&=\sup_B\Big(\frac{1}{|B|}\int\limits_B w(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int\limits_B w(x)^{-\frac{p'}{p}}\mathrm{d}x\Big)^{\frac{p}{p'}}<\infty,\end{aligned}\tag{2.1}$$

where the supremum is taken with respect to all the balls B and $p'=\frac{p}{p-1}$. The condition (2.1) is called the A_p-condition, and the weights which satisfy it are called A_p-weights. The property of the A_p-weights implies that generally speaking, we should check whether a weight w satisfies an A_p-condition or not. The expression $[w]_{A_p}$ is also called characteristic constant of w. Similarly, we shall give the definitions of the Muckenhoupt classes A_p with $p=1,\infty$. A locally integrable function $w:\mathbb{R}^n\to(0,\infty)$ is said to be an $A_1(\mathbb{R}^n)$ weight if

$$\begin{aligned}[w]_{A_1}&:=\sup_B[w]_{A_1(B)}\\&=\sup_B\Big(\frac{1}{|B|}\int\limits_B w(y)\mathrm{d}y\Big)\operatorname*{esssup}_{x\in B}w(x)^{-1}<\infty\end{aligned}\tag{2.2}$$

holds for all balls B. A weight belonging to the set

$$A_\infty=\bigcup_{1\le p<\infty}A_p$$

is said to be a Muckenhoupt A_∞ weight. It is also known that the monotone property $A_p \subset A_q \subset A_\infty$ holds for every constants $1 \le p < q < \infty$.

By (2.1), we have

$$(w^{-\frac{p'}{p}}(B))^{\frac{1}{p'}} = \|w^{-\frac{1}{p}}\|_{L_{p'}(B)} \le C|B|w(B)^{-\frac{1}{p}} \tag{2.3}$$

for $1 < p < \infty$. Suppose that $w \in A_p(\mathbb{R}^n)$, by the definition of $A_p(\mathbb{R}^n)$, we know that $w^{1-p'} \in A_{p'}(\mathbb{R}^n)$. Note that

$$\Big(\operatorname*{essinf}_{x\in E} f(x)\Big)^{-1} = \operatorname*{esssup}_{x\in E} \frac{1}{f(x)} \tag{2.4}$$

is true for any real-valued nonnegative function f and is measurable on E (see [20, page 143]) and (2.2); we get

$$\begin{aligned}\|w^{-1}\|_{L_\infty(B)} &= \operatorname*{esssup}_{x\in B} \frac{1}{w(x)} \\ &= \frac{1}{\operatorname*{essinf}_{x\in B} w(x)} \le C|B|w(B)^{-1}.\end{aligned} \tag{2.5}$$

Then, Komori and Shirai [12] introduced a version of the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of the weighted Lebesgue space $L_p(w)$, and investigated the boundedness of classical operators in harmonic analysis (see [12] for details).

Definition 2.3 (Weighted Morrey Space) *Let $1 \le p < \infty$, $0 < \kappa < 1$ and w be a weight function. Then the weighted Morrey space $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^n, w)$ is defined by*

$$\begin{aligned}&L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^n, w) \\ &\quad = \Big\{f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n) : \|f\|_{L_{p,\kappa}(w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{L_{p,w}(B(x,r))} < \infty\Big\}.\end{aligned}$$

Furthermore, the weak weighted Morrey space $WL_{p,\kappa}(w) \equiv WL_{p,\kappa}(\mathbb{R}^n, w)$ is defined by

$$\begin{aligned}&WL_{p,\kappa}(w) \equiv WL_{p,\kappa}(\mathbb{R}^n, w) \\ &\quad = \Big\{f \in WL^{\mathrm{loc}}_{p,w}(\mathbb{R}^n) : \|f\|_{WL_{p,\kappa}(w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{WL_{p,w}(B(x,r))} < \infty\Big\}.\end{aligned}$$

Remark 2.1 Alternatively, we could define the weighted Morrey spaces with cubes instead of balls. Hence we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Remark 2.2 (1) When $w \equiv 1$ and $\kappa = \frac{\lambda}{n}$ with $0 \le \lambda \le n$, then the weighted Morrey space is reduced to the ordinary Morrey space.

(2) If $\kappa = 0$, then the weighted Morrey space is reduced to the weighted Lebesgue space.

On the other hand, the generalized weighted Morrey spaces $M_{p,\varphi}(w)$, which is a natural extension of the weighted Morrey space $L_{p,\kappa}(w)$ were been introduced by Guliyev [7] as follows.

Definition 2.4 (Generalized Weighted Morrey Space) *Let $1 \le p < \infty$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. Then, the generalized weighted Morrey space $M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$ is defined by*

$$\begin{aligned} M_{p,\varphi}(w) &\equiv M_{p,\varphi}(\mathbb{R}^n, w) \\ &= \Big\{ f \in L^{\rm loc}_{p,w}(\mathbb{R}^n) : \|f\|_{M_{p,\varphi}(w)} \\ &= \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} < \infty \Big\}. \end{aligned}$$

Furthermore, the weak generalized weighted Morrey space $WM_{p,\varphi}(w) \equiv WM_{p,\varphi}(\mathbb{R}^n, w)$ is defined by

$$\begin{aligned} WM_{p,\varphi}(w) &\equiv WM_{p,\varphi}(\mathbb{R}^n, w) \\ &= \Big\{ f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n) : \|f\|_{WM_{p,\varphi}(w)} \\ &= \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty \Big\}. \end{aligned}$$

Remark 2.3 (1) When $w \equiv 1$, then the generalized weighted Morrey space is reduced to the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$, then the generalized weighted Morrey space is reduced to the weighted Morrey space.

(3) When $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 \le \lambda \le n$, then the generalized weighted Morrey space is reduced to the ordinary Morrey and weak Morrey space, respectively.

(4) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then the generalized weighted Morrey space is reduced to the weighted Lebesgue space.

3 Marcinkiewicz Integrals with Rough Kernel Associated with Schrödinger Operator $\mu^L_{j,\Omega}$ on the Generalized Weighted Morrey Spaces $M_{p,\varphi}(w)$

In this section we prove boundedness of the operators $\mu^L_{j,\Omega}$, $j = 1, \cdots, n$ on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ by using the following Lemma 3.1 and (2.4).

We first prove the following Theorem.

Theorem 3.1 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$ and $V \in RH_n$. Then, for every $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$ the inequality*

$$\|\mu^L_{j,\Omega}(f)\|_{L_{p,w}} \lesssim \|f\|_{L_{p,w}} \tag{3.1}$$

holds.

Proof The statement of Theorem 3.1 follows by the following inequality

$$\mu^L_{j,\Omega} f(x) \le \mu_{j,\Omega} f(x) + C M_\Omega f(x) \quad \text{a.e. } x \in \mathbb{R}^n,$$

and the boundedness of operators $\mu_{j,\Omega}$ and M_Ω on $L_p(w)$ (see [4, 6]) in the same manner as in the proof of Theorem 5 in [1].

Before we give the proof of Theorem 3.2, we need the following lemma.

Lemma 3.1 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$ and $V \in RH_n$.*

If $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$, then the inequality

$$\|\mu_{j,\Omega}^L(f)\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t} \tag{3.2}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_{p,w}^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $p = 1 < q \le \infty$ the inequality

$$\|\mu_{j,\Omega}^L(f)\|_{WL_{1,w}(B(x_0,r))} \lesssim w(B(x_0,r)) \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))} w(B(x_0,t))^{-1} \frac{\mathrm{d}t}{t} \tag{3.3}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_{1,w}^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof For any $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{(2B)^C}(y), \quad r > 0, \tag{3.4}$$

and have

$$\|\mu_{j,\Omega}^L(f)\|_{L_{p,w}(B)} \le \|\mu_{j,\Omega}^L(f_1)\|_{L_{p,w}(B)} + \|\mu_{j,\Omega}^L(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_p(w)$, $\mu_{j,\Omega}^L(f_1) \in L_p(w)$ and from the boundedness of $\mu_{j,\Omega}^L$ on $L_p(w)$ (see Theorem 3.1) it follows that:

$$\|\mu_{j,\Omega}^L(f_1)\|_{L_{p,w}(B)} \le \|\mu_{j,\Omega}^L(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|f_1\|_{L_{p,w}(\mathbb{R}^n)} = C\|f\|_{L_{p,w}(2B)}, \tag{3.5}$$

where constant $C > 0$ is independent of f.

By the Hölder's inequality,

$$|B(x_0,r)| \lesssim w(B(x_0,r))^{\frac{1}{p}} \|w^{-\frac{1}{p}}\|_{L_{p'}(B(x_0,r))}. \tag{3.6}$$

Then, for $q' < p < \infty$, it is clear that $w \in A_{\frac{p}{q'}}$ implies $w \in A_p$, by (3.6) and (2.3) we have

$$\begin{aligned}
\|f\|_{L_{p,w}(2B)} &\approx |B(x_0,r)| \|f\|_{L_{p,w}(B(x_0,2r))} \int_{2r}^{\infty} \frac{\mathrm{d}t}{t^{n+1}} \lesssim |B(x_0,r)| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\
&\lesssim w(B(x_0,r))^{\frac{1}{p}} \|w^{-\frac{1}{p}}\|_{L_{p'}(B(x_0,r))} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\
&\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-\frac{1}{p}}\|_{L_{p'}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\
&\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{3.7}$$

By (3.7), we get

$$\|\mu^L_{j,\Omega}(f_1)\|_{L_{p,w}(B)} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \tag{3.8}$$

To estimate $\|\mu^L_{j,\Omega}(f_2)\|_{L_{p,w}(B(x_0,r))}$, we first need to prove $\mu^L_{j,\Omega}$ satisfies the following inequality:

$$\sup_{x\in B} |\mu^L_{j,\Omega}(f_2(x))| \lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-\frac{n}{q'}} \Big(\int\limits_{B(x_0,2^{j+1}r)} |f(y)|^{q'} dy \Big)^{\frac{1}{q'}}. \tag{3.9}$$

Indeed, let $\Delta_i = B(x_0, 2^{j+1}r) \diagdown B(x_0, 2^j r)$ and $x \in B(x_0,r)$. By the Hölder's inequality, we have

$$\begin{aligned} \sup_{x\in B} |\mu^L_{j,\Omega}(f_2(x))| &\le \sup_{x\in B} \Big| \int\limits_{(2B)^C} \frac{|f(y)||\Omega(x-y)'|}{|x_0-y|^n} dy \Big| \\ &\le \sup_{x\in B} \sum_{j=1}^{\infty} \Big(\int\limits_{\Delta_i} |\Omega(x-y)'|^q dy \Big)^{\frac{1}{q}} \Big(\int\limits_{\Delta_i} \frac{|f(y)|^{q'}}{|x-y|^{nq'}} dy \Big)^{\frac{1}{q'}}. \end{aligned}$$

When $x \in B(x_0,r)$ and $y \in \Delta_i$, then by a direct calculation, we can see that $2^{j-1}r \le |y-x| < 2^{j+1}r$. Hence,

$$\Big(\int\limits_{\Delta_i} |\Omega(x-y)'|^q dy \Big)^{\frac{1}{q}} \lesssim \|\Omega\|_{L_q(S^{n-1})} |B(x_0, 2^{j+1}r)|^{\frac{1}{q}}. \tag{3.10}$$

We also note that if $x \in B = B(x_0,r)$, $y \in (2B)^C = B^C(x_0, 2r)$, then $|y-x| \approx |y-x_0|$. Consequently,

$$\Big(\int\limits_{\Delta_i} \frac{|f(y)|^{q'}}{|x-y|^{nq'}} dy \Big)^{\frac{1}{q'}} \le \frac{1}{|B(x_0,2^{j+1}r)|} \Big(\int\limits_{B(x_0,2^{j+1}r)} |f(y)|^{q'} dy \Big)^{\frac{1}{q'}}. \tag{3.11}$$

Combining (3.10) and (3.11), we get (3.9).

Let $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$. Since $\mu^L_{j,\Omega}$ satisfies (3.9), it follows from the Hölder's inequality that

$$\begin{aligned} \sup_{x\in B} |\mu^L_{j,\Omega}(f_2(x))| &\lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-\frac{n}{q'}} \|f\|_{L_{p,w}(B(x_0,2^{j+1}r))} \|w^{-\frac{1}{p}}\|_{L_{q'(\frac{p}{q'})'}(B(x_0,2^{j+1}r))} \\ &\lesssim \sum_{j=1}^{\infty} \int\limits_{2^{j+1}r}^{2^{j+2}r} (2^{j+1}r)^{-(1+\frac{n}{q'})} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-\frac{1}{p}}\|_{L_{q'(\frac{p}{q'})'}(B(x_0,t))} dt \\ &\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-\frac{1}{p}}\|_{L_{q'(\frac{p}{q'})'}(B(x_0,t))} \frac{dt}{t^{1+\frac{n}{q'}}}. \end{aligned} \tag{3.12}$$

Note that $w \in A_{\frac{p}{q'}}$, by (2.3) we get

$$\|w^{-\frac{1}{p}}\|_{L_{q'(\frac{p}{q'})'}(B(x_0,t))} \lesssim t^{\frac{n}{q'}} w(B(x_0,t))^{-\frac{1}{p}}. \tag{3.13}$$

Then by (3.12)–(3.13),

$$\sup_{x\in B}|\mu^L_{j,\Omega}(f_2(x))| \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}. \tag{3.14}$$

When $q'=p$, then $w\in A_1$. Then for any $p>1$, by (3.9) and (2.5) we have

$$\begin{aligned}
\sup_{x\in B}|\mu^L_{j,\Omega}(f_2(x))| &\lesssim \sum_{j=1}^{\infty}(2^{j+1}r)^{-\frac{n}{p}}\Big(\int_{B(x_0,2^{j+1}r)}|f(y)|^p\mathrm{d}y\Big)^{\frac{1}{p}} \\
&\lesssim \sum_{j=1}^{\infty}(2^{j+1}r)^{-\frac{n}{p}}\Big(\int_{B(x_0,2^{j+1}r)}|f(y)|^p w(x)\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\operatorname*{essinf}_{x\in B(x_0,2^{j+1}r)} w(x)\Big)^{-\frac{1}{p}} \\
&\lesssim \sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}r}\|f\|_{L_{p,w}(B(x_0,2^{j+1}r))} w(B(x_0,2^{j+1}r))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t} \\
&\lesssim \sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}r}\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t} \\
&\lesssim \int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t}.
\end{aligned} \tag{3.15}$$

Hence, for all $p\in[1,\infty)$ by (3.14)–(3.15) we get

$$\|\mu^L_{j,\Omega}(f_2)\|_{L_{p,w}(B)} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}. \tag{3.16}$$

Combining (3.8) and (3.16) we complete the proof of (3.2).

On the other hand, let $p=1<q\le\infty$. From the weak $(1,1)$ boundedness of T_Ω and (3.7) it follows that

$$\begin{aligned}
\|\mu^L_{j,\Omega}(f_1)\|_{WL_{1,w}(B)} &\le \|\mu^L_{j,\Omega}(f_1)\|_{WL_{1,w}(\mathbb{R}^n)} \lesssim \|f_1\|_{L_{1,w}(\mathbb{R}^n)} \\
&= \|f\|_{L_{1,w}(2B)} \lesssim w(B(x_0,r)) \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))} w(B(x_0,t))^{-1} \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{3.17}$$

Then from (3.16)–(3.17) we get (3.3), which completes the proof.

In the following theorem (our main result), we get the boundedness of the operators $\mu^L_{j,\Omega}$, $j=1,\cdots,n$ on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$.

Theorem 3.2 *Let $\Omega\in L_q(S^{n-1})$, $1<q\le\infty$, $\Omega(\mu x)=\Omega(x)$ for any $\mu>0, x\in\mathbb{R}^n\setminus\{0\}$ and $V\in RH_n$. Let also, for $q'<p$ and $w\in A_{\frac{p}{q'}}$, the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{essinf}_{t<\tau<\infty}\varphi_1(x,\tau)w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}}\frac{\mathrm{d}t}{t} \le C\,\varphi_2(x,r), \tag{3.18}$$

where C does not depend on x and r.

Then the operators $\mu_{j,\Omega}^L$, $j=1,\cdots,n$ are bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p>1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$. Moreover, we have for $p>1$,

$$\|\mu_{j,\Omega}^L(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)},$$

and for $p=1$,

$$\|\mu_{j,\Omega}^L(f)\|_{WM_{1,\varphi_2}(w)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

Proof Since $f\in M_{p,\varphi_1}(w)$, then from (2.4) and the fact that the norm $\|f\|_{L_{p,w}(B(x_0,t))}$ is a non-decreasing function with respect to t, we get

$$\begin{aligned}\frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\operatorname*{essinf}\limits_{0<t<\tau<\infty}\varphi_1(x_0,\tau)w(B(x_0,\tau))^{\frac{1}{p}}} &\leq \operatorname*{esssup}_{0<t<\tau<\infty}\frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\varphi_1(x_0,\tau)w(B(x_0,\tau))^{\frac{1}{p}}}\\ &\leq \sup_{\tau>0,x_0\in\mathbb{R}^n}\frac{\|f\|_{L_{p,w}(B(x_0,\tau))}}{\varphi_1(x_0,\tau)w(B(x_0,\tau))^{\frac{1}{p}}} \lesssim \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned}\tag{3.19}$$

For $q'<p<\infty$, since (φ_1,φ_2) satisfies (3.18), we have

$$\begin{aligned}&\int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t}\\ &\approx \int_r^\infty \frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)w(B(x_0,\tau))^{\frac{1}{p}}}\frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)w(B(x_0,\tau))^{\frac{1}{p}}}{w(B(x_0,t))^{\frac{1}{p}}}\frac{\mathrm{d}t}{t}\\ &\lesssim \|f\|_{M_{p,\varphi_1}(w)}\int_r^\infty \frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)w(B(x_0,\tau))^{\frac{1}{p}}}{w(B(x_0,t))^{\frac{1}{p}}}\frac{\mathrm{d}t}{t}\\ &\lesssim \|f\|_{M_{p,\varphi_1}(w)}\varphi_2(x_0,r). \end{aligned}\tag{3.20}$$

Then by (3.2) and (3.20), we get

$$\begin{aligned}\|\mu_{j,\Omega}^L(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x_0\in\mathbb{R}^n,r>0}\varphi_2(x_0,r)^{-1}w(B(x_0,r))^{-\frac{1}{p}}\|\mu_{j,\Omega}^L(f)\|_{L_{p,w}(B(x_0,r))}\\ &\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\varphi_2(x_0,r)^{-1}\int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t}\\ &\lesssim \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned}$$

For the case of $1=p<q$, we can also use the same method, so we omit the details. This completes the proof of Theorem 3.2.

When $\Omega\equiv 1$, from Theorem 3.2 we get the following corollary.

Corollary 3.1 *Let $1\leq p<\infty$, $w\in A_p$, $V\in RH_n$ and the pair (φ_1,φ_2) satisfies condition (3.18). Let also the operators μ_j^L, $j=1,\cdots,n$ are bounded on $L_p(w)$ for $p>1$ and bounded from $L_1(w)$ to $WL_1(w)$. Then the operators μ_j^L, $j=1,\cdots,n$ are bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p>1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

Remark 3.1 Note that, in the case of $w \equiv 1$, Theorem 3.2 has been proved in [1].

Let $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$, $w \in A_\infty$ and $V \in RH_n$. Then for $q' \le p$, $p \ne 1$ and $w \in A_{\frac{p}{q'}}$, the pair (φ_1, φ_2) satisfies condition (3.18). Hence, from Theorem 3.2 we get the following new result.

Corollary 3.2 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$, $0 < \kappa < 1$ and $V \in RH_n$. Let also the operators $\mu^L_{j,\Omega}$, $j = 1, \cdots, n$ are bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. For $q' \le p$, $p \ne 1$ and $w \in A_{\frac{p}{q'}}$, the pair (φ_1, φ_2) satisfies condition* (3.18). *Then the operators $\mu^L_{j,\Omega}$, $j = 1, \cdots, n$ are bounded on the weighted Morrey spaces $L_{p,\kappa}(w)$ for $p > 1$ and bounded from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.*

4 Commutators of Marcinkiewicz Integrals with Rough Kernel Associated with Schrödinger Operator $\mu^L_{j,\Omega}$ on the Generalized Weighted Morrey Spaces $M_{p,\varphi}(w)$

In this section we prove the boundedness of the operators $\mu^L_{j,\Omega,b}$, $j = 1, \cdots, n$ with $b \in \mathrm{BMO}(\mathbb{R}^n)$ on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ by using the following Lemma 4.4 and (2.4).

Spaces of Bounded Mean Oscillation (BMO for short) have been, and continue to be, of great interest and a subject of intense research in harmonic analysis. One of the most fascinating aspects of BMO spaces is their self-improvement properties, which go back to the work of John and Nirenberg in [11]. Functions of BMO were also introduced by John and Nirenberg [11], in connection with differential equations. The definition on $\mathbb{R}^n$ reads as follows.

Definition 4.1 (see [11]) *The space* $\mathrm{BMO}(\mathbb{R}^n)$ *of functions of bounded mean oscillation consists of locally summable functions with finite semi-norm*

$$\|b\|_* \equiv \|b\|_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}| \mathrm{d}y < \infty, \tag{4.1}$$

where $b_{B(x,r)}$ is the mean value of the function b on the ball $B(x,r)$ and $\|b\|_*$ is called the BMO*-norm of b, and it becomes a norm on after dividing out the constant functions. Bounded functions are in* BMO *and a* BMO*-function is locally in $L_p(\mathbb{R})$ for every $p < \infty$. Typical examples of* BMO*-functions are of the form* $\log|P|$ *with a polynomial on $\mathbb{R}^n$. Furthermore,* BMO *is a bit like the space L_∞, but L_∞ is a subspace of* BMO*. Indeed,*

$$\begin{aligned} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}| \mathrm{d}y &\le \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y)| \mathrm{d}y + \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b_{B(x,r)}| \mathrm{d}y \\ &= \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y)| \mathrm{d}y + |b_{B(x,r)}| \\ &\le 2\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y)| \mathrm{d}y \le 2\|b\|_{L_\infty}. \end{aligned}$$

As a result, since $\|b\|_* \le 2\|b\|_{L_\infty}$, $L_\infty(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ is valid.

Remark 4.1 (see [9]) The fact that precisely the mean value $b_{B(x,r)}$ figures in (4.1) is inessential and one gets an equivalent seminorm if $b_{B(x,r)}$ is replaced by an arbitrary constant c :

$$\|b\|_* \approx \sup_{r>0} \inf_{c\in\mathbb{C}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y)-c| \mathrm{d}y.$$

In 1961, John and Nirenberg [11] established the following deep property of functions from BMO.

Theorem 4.1 (see [11]) *If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $B(x,r)$ is a ball, then*

$$|\{x \in B(x,r) \,:\, |b(x)-b_{B(x,r)}| > \xi\}| \le |B(x,r)| \exp\Big(-\frac{\xi}{C\|b\|_*}\Big), \quad \xi > 0,$$

where C depends only on the dimension n.

By Theorem 4.1, it is easy to get the following.

Lemma 4.1 *Let $w \in A_\infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then for any $p \ge 1$ we have*

$$\Big(\frac{1}{w(B)} \int\limits_B |b(y)-b_B|^p w(y) \mathrm{d}y\Big)^{\frac{1}{p}} \lesssim \|b\|_*.$$

Lemma 4.2 (see [13]) *Let b be a function in $\mathrm{BMO}(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{|B(x,r_1)|} \int\limits_{B(x,r_1)} |b(y)-b_{B(x,r_2)}|^p \mathrm{d}y\Big)^{\frac{1}{p}} \lesssim \Big(1+\Big|\ln\frac{r_1}{r_2}\Big|\Big)\|b\|_*.$$

By Lemmas 4.1–4.2, it is easily to prove the following result.

Lemma 4.3 (see [8]) *Let $w \in A_\infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w(B(x,r_1))} \int\limits_{B(x,r_1)} |b(y)-b_{B(x,r_2)}|^p w(y) \mathrm{d}y\Big)^{\frac{1}{p}} \lesssim \Big(1+\Big|\ln\frac{r_1}{r_2}\Big|\Big)\|b\|_*.$$

We first prove the following Theorem.

Theorem 4.2 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$, $V \in RH_n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, for every $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$, there is a constant C independent of f such that*

$$\|\mu^L_{j,\Omega,b}(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}. \tag{4.2}$$

Proof The statement of Theorem 4.2 follows by the following inequality

$$\mu^L_{j,\Omega,b} f(x) \le \mu_{j,\Omega,b} f(x) + C M_{\Omega,b} f(x) \quad \text{a.e. } x \in \mathbb{R}^n, \tag{4.3}$$

and the boundedness of operators $M_{\Omega,b}$ and $\mu_{j,\Omega,b}$ on $L_p(w)$ (see [2, 5]) in the same manner as in the proof of Theorem 3 in [9].

We present the following lemma, which is the heart of the proof of Theorem 4.3.

Lemma 4.4 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$, $V \in RH_n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, for $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$ the inequality*

$$\begin{aligned}
&\|\mu^L_{j,\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))} \\
&\lesssim \|b\|_* \, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}
\end{aligned} \tag{4.4}$$

holds for any ball $B(x_0,r)$ and for all $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. As in the proof of Lemma 3.1, we represent function f in form (3.4) and have

$$\|\mu^L_{j,\Omega,b}(f)\|_{L_{p,w}(B)} \le \|\mu^L_{j,\Omega,b}(f_1)\|_{L_{p,w}(B)} + \|\mu^L_{j,\Omega,b}(f_2)\|_{L_{p,w}(B)}.$$

For $q' < p$ and $w \in A_{\frac{p}{q'}}$, from the boundedness of $\mu^L_{j,\Omega,b}$ on $L_p(w)$ (see Theorem 4.2) it follows that

$$\begin{aligned}
\|\mu^L_{j,\Omega,b}(f_1)\|_{L_{p,w}(B)} &\le \|\mu^L_{j,\Omega,b}(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \\
&\lesssim \|b\|_* \|f_1\|_{L_{p,w}(\mathbb{R}^n)} = \|b\|_* \|f\|_{L_{p,w}(2B)}.
\end{aligned}$$

As in the proof of (3.7), we get

$$\|\mu^L_{j,\Omega,b}(f_1)\|_{L_{p,w}(B)} \lesssim \|b\|_* \, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}.$$

We now turn to deal with the term $\|\mu^L_{j,\Omega,b}(f_2)\|_{L_{p,w}(B)}$. For any given $x \in B(x_0,r)$, we have

$$\begin{aligned}
|\mu^L_{j,\Omega,b}(f_2(x))| &\lesssim |b(x) - b_{B(x_0,r)}||\mu^L_{j,\Omega}(f_2(x))| + |\mu^L_{j,\Omega}((b - b_{B(x_0,r)})f_2)(x)| \\
&= J_1 + J_2.
\end{aligned}$$

Since $\mu^L_{j,\Omega}$ satisfies (3.9), by (3.14)–(3.15) we get

$$J_1 \lesssim |b(y) - b_{B(x_0,r)}| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}.$$

Applying Lemma 4.3, we get

$$\|J_1\|_{L_{p,w}(B(x_0,r))} \lesssim \|b\|_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}.$$

Now, let us estimate J_2. When $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, it follows from (3.9) that

$$J_2 \lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-\frac{n}{q'}} \Big(\int_{B(x_0,2^{j+1}r)} |(b(y) - b_{B(x_0,r)})f(y)|^{q'} \mathrm{d}y \Big)^{\frac{1}{q'}}.$$

Set $\nu = \frac{p}{q'}$. From $w \in A_\nu$, we know $w^{1-\nu'} \in A_{\nu'}$. Since $q' < p$, it follows from Hölder's inequality that

$$\begin{aligned}&\Big(\int\limits_{B(x_0,2^{j+1}r)} |(b(y)-b_{B(x_0,r)})|^{q'}|f(y)|^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\\ &\lesssim \|f\|_{L_{p,w}(B(x_0,2^{j+1}r))}\|b(\cdot)-b_{B(x_0,r)}\|_{L_{q'\nu'}(w^{1-\nu'},B(x_0,2^{j+1}r))}.\end{aligned}$$

Then

$$\begin{aligned}J_2 &\lesssim \sum_{j=1}^{\infty}(2^{j+1}r)^{-\frac{n}{q'}}\|f\|_{L_{p,w}(B(x_0,2^{j+1}r))}\|b(\cdot)-b_{B(x_0,r)}\|_{L_{q'\nu'}(w^{1-\nu'},B(x_0,2^{j+1}r))}\\ &\lesssim \sum_{j=1}^{\infty}\Big(1+\ln\frac{2^{j+1}r}{r}\Big)(2^{j+1}r)^{-\frac{n}{q'}}\|f\|_{L_{p,w}(B(x_0,2^{j+1}r))}\|b(\cdot)-b_{B(x_0,r)}\|_{L_{q'\nu'}(w^{1-\nu'},B(x_0,2^{j+1}r))}\\ &\lesssim \sum_{j=1}^{\infty}\int\limits_{2^{j+1}r}^{2^{j+2}r}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\|b(\cdot)-b_{B(x_0,r)}\|_{L_{q'\nu'}(w^{1-\nu'},B(x_0,t))}\frac{\mathrm{d}t}{t^{\frac{n}{q'}+1}}\\ &\lesssim \int\limits_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\|b(\cdot)-b_{B(x_0,r)}\|_{L_{q'\nu'}(w^{1-\nu'},B(x_0,t))}\frac{\mathrm{d}t}{t^{\frac{n}{q'}+1}}.\end{aligned}$$

Since $w^{-\frac{\nu'}{q'}} = w^{1-\nu'} \in A_{\nu'}$, by (3.12), we know

$$(w^{1-\nu'}(B(x_0,t)))^{\frac{1}{q'\nu'}} \lesssim t^{\frac{n}{\nu}}w(B(x_0,t))^{-\frac{1}{p}}. \tag{4.5}$$

Using Lemma 4.3 and by the fact that $w \in A_\nu$ and (4.5), we thus obtain

$$\begin{aligned}&\Big(\int\limits_{B(x_0,2^{j+1}r)} |(b(y)-b_{B(x_0,r)})|^{q'\nu'}w^{1-\nu'}(y)\mathrm{d}y\Big)^{\frac{1}{q'\nu'}}\\ &\lesssim \|b\|_*\Big(1+\ln\frac{t}{r}\Big)(w^{1-\nu'}(B(x_0,t)))^{\frac{1}{q\nu'}}\\ &\lesssim \|b\|_*\Big(1+\ln\frac{t}{r}\Big)t^{\frac{n}{q'}}w(B(x_0,t))^{-\frac{1}{p}}.\end{aligned} \tag{4.6}$$

Then by (4.6),

$$J_2 \lesssim \|b\|_*\int\limits_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t}.$$

Hence,

$$\|J_2\|_{L_{p,w}(B(x_0,r))} \lesssim \|b\|_*\,w(B(x_0,r))^{\frac{1}{p}}\int\limits_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t}.$$

Summing up $\|J_1\|_{L_{p,w}(B(x_0,r))}$ and $\|J_2\|_{L_{p,w}(B(x_0,r))}$ for all $p \in (1,\infty)$, we get

$$\|\mu^L_{j,\Omega,b}(f_2)\|_{L_{p,w}(B(x_0,r))}$$

$$\lesssim \|b\|_* \, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}.$$

Finally, we have

$$\|\mu^L_{j,\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))}$$
$$\lesssim \|b\|_* \, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}.$$

This completes the proof of Lemma 4.4.

Now we can give the following theorem (our main result).

Theorem 4.3 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$, $V \in RH_n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Let also, for $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{\mathrm{d}t}{t} \le C\,\varphi_2(x,r), \tag{4.7}$$

where C does not depend on x and r.

Then, the operators $\mu^L_{j,\Omega,b}$, $j = 1, \cdots, n$ are bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|\mu^L_{j,\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof Similar to the proof of Theorem 3.2, by (3.19), we have

$$\int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t}$$
$$\approx \int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau) w(B(x_0,\tau))^{\frac{1}{p}}} \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau) w(B(x_0,\tau))^{\frac{1}{p}}}{w(B(x_0,t))^{\frac{1}{p}}} \frac{\mathrm{d}t}{t}$$
$$\lesssim \|f\|_{M_{p,\varphi_1}(w)} \int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau) w(B(x_0,\tau))^{\frac{1}{p}}}{w(B(x_0,t))^{\frac{1}{p}}} \frac{\mathrm{d}t}{t}.$$

For $q' < p < \infty$, since (φ_1, φ_2) satisfies (4.7), we know

$$\int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{\mathrm{d}t}{t} \lesssim \|f\|_{M_{p,\varphi_1}(w)} \varphi_2(x_0,r). \tag{4.8}$$

Then by (4.4) and (4.8), we get

$$\|\mu^L_{j,\Omega,b}(f)\|_{M_{p,\varphi_2}(w)}$$
$$= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} w(B(x_0,r))^{-\frac{1}{p}} \|\mu^L_{j,\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))}$$

$$\lesssim \|b\|_* \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int\limits_r^\infty \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{\mathrm{d}t}{t}$$
$$\lesssim \|b\|_*\|f\|_{M_{p,\varphi_1}(w)}.$$

This completes the proof of Theorem 4.3.

When $\Omega\equiv 1$, from Theorem 4.3 we get the following corollary.

Corollary 4.1 *Let $1<p<\infty$, $w\in A_p$, $V\in RH_n$, $b\in \mathrm{BMO}(\mathbb{R}^n)$ and the pair (φ_1,φ_2) satisfies condition (4.7). Let also the operators $\mu^L_{j,b}$, $j=1,\cdots,n$ are bounded on $L_p(w)$. Then the operators $\mu^L_{j,b}$, $j=1,\cdots,n$ are bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

Remark 4.2 Note that, in the case of $w\equiv 1$, Theorem 4.3 has been proved in [9].

Let $\varphi_1(x,r)=\varphi_2(x,r)\equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, $0<\kappa<1$, $w\in A_\infty$ and $V\in RH_n$. Then for $q'<p$ and $w\in A_{\frac{p}{q'}}$, the pair (φ_1,φ_2) satisfies condition (4.7). Hence, from Theorem 4.3 we get the following new result.

Corollary 4.2 *Let $\Omega\in L_q(S^{n-1})$, $1<q\le\infty$, $\Omega(\mu x)=\Omega(x)$ for any $\mu>0, x\in\mathbb{R}^n\setminus\{0\}$, $V\in RH_n$, $0<\kappa<1$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$. Let also the operators $\mu^L_{j,\Omega,b}$, $j=1,\cdots,n$ are bounded on $L_p(w)$ for $p>1$. For $q'<p$ and $w\in A_{\frac{p}{q'}}$, the pair (φ_1,φ_2) satisfies condition (4.7). Then the operators $\mu^L_{j,\Omega,b}$, $j=1,\cdots,n$ are bounded on the weighted Morrey spaces $L_{p,\kappa}(w)$ for $p>1$.*

5 Some Applications

In this section, we shall apply Theorem 3.2 and Theorem 4.3 to several particular operators such as Marcinkiewicz integral operators, rough H-L maximal operators, Bochner-Riesz means and parametric Marcinkiewicz integral operators.

5.1 Marcinkiewicz integral operators

Let $\Omega\in L_q(S^{n-1})$, $1<q\le\infty$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$. Then by [4–5], for every $q'<p$ and $w\in A_{\frac{p}{q'}}$, there is a constant C independent of f such that

$$\|\mu_\Omega(f)\|_{L_{p,w}}\le C\|f\|_{L_{p,w}}$$

and

$$\|\mu_{\Omega,b}(f)\|_{L_{p,w}}\le C\|b\|_*\|f\|_{L_{p,w}}.$$

Theorem 5.1 *Let $\Omega\in L_q(S^{n-1})$, $1<q\le\infty$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$. Let also $q'<p<\infty$ and $w\in A_{\frac{p}{q'}}$.*

If the pair (φ_1,φ_2) satisfies condition (3.18), then we have for $p>1$,

$$\|\mu_\Omega(f)\|_{M_{p,\varphi_2}(w)}\lesssim\|f\|_{M_{p,\varphi_1}(w)},$$

and for $p=1$,

$$\|\mu_\Omega(f)\|_{WM_{1,\varphi_2}(w)}\lesssim\|f\|_{M_{1,\varphi_1}(w)}.$$

If the pair (φ_1, φ_2) *satisfies condition* (4.7), *then we have for* $p > 1$,

$$\|\mu_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof In the proof of Theorem 5.1, we will check whether μ_Ω only satisfies (3.9). We know that if $x \in B = B(x_0, r)$, $y \in (2B)^C = B^C(x_0, 2r)$ and $\Delta_i = B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)$ $(j \geq 1)$, then

$$t \geq |x-y| \geq |y-x_0| - |x-x_0| \geq 2^{j-1}r.$$

Then, by Minkowski's inequality we get

$$\begin{aligned}
\mu_\Omega(f\chi_{(2B)^C})(x) &= \Big(\int_0^\infty \Big| \int_{(2B)^C \cap \{y:|x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} \\
&= \Big(\int_0^\infty \sum_{j=1}^\infty \Big| \int_{(\Delta_i)^C \cap \{y:|x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty \Big(\int_{\Delta_i} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \mathrm{d}y \Big) \Big(\int_{2^{j-1}r}^\infty \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty (2^{j+1}r)^{-1} \int_{\Delta_i} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \mathrm{d}y. \qquad (5.1)
\end{aligned}$$

When $\Omega \in L_\infty(S^{n-1})$, then we have

$$\sup_{x \in B} |\mu_\Omega(f\chi_{(2B)^C})(x)| \lesssim \sum_{j=1}^\infty (2^{j+1}r)^{-n} \int_{B(x_0, 2^{j+1}r)} |f(y)| \mathrm{d}y. \qquad (5.2)$$

When $\Omega \in L_q(S^{n-1})$, $1 < q < \infty$, then by Hölder's inequality,

$$\int_{\Delta_i} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \mathrm{d}y \lesssim \Big(\int_{\Delta_i} |\Omega(x-y)'|^q \mathrm{d}y \Big)^{\frac{1}{q}} \Big(\int_{\Delta_i} \frac{|f(y)|^{q'}}{|x-y|^{(n-1)q'}} \mathrm{d}y \Big)^{\frac{1}{q'}}. \qquad (5.3)$$

Hence, it follows from (3.10), (5.1), (5.3) that

$$\sup_{x \in B} |\mu_\Omega(f\chi_{(2B)^C})(x)| \lesssim \sum_{j=1}^\infty (2^{j+1}r)^{-\frac{n}{q'}} \Big(\int_{B(x_0, 2^{j+1}r)} |f(y)|^{q'} \mathrm{d}y \Big)^{\frac{1}{q'}}. \qquad (5.4)$$

Combining (5.2) with (5.4) and since the rest of the proof is the same as the proof of Theorem 3.2 and Theorem 4.3, the proof of Theorem 5.1 is completed.

5.2 Rough H-L maximal operators

Duoandikoetxea [6] and Alvarez et al. [2] proved the following results, respectively.

Let $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, for every $q' \leq p < \infty$ and $w \in A_{\frac{p}{q'}}$, there is a constant C independent of f such that

$$\|M_\Omega(f)\|_{L_{p,w}} \leq C\|f\|_{L_{p,w}}$$

and

$$\|M_{\Omega,b}(f)\|_{L_{p,w}} \leq C\|b\|_*\|f\|_{L_{p,w}}.$$

Theorem 5.2 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let also $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$.*

If the pair (φ_1, φ_2) satisfies condition (3.18), then we have for $p > 1$,

$$\|M_\Omega(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)},$$

and for $p = 1$,

$$\|M_\Omega(f)\|_{WM_{1,\varphi_2}(w)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

If the pair (φ_1, φ_2) satisfies condition (4.7), then we have for $p > 1$,

$$\|M_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_*\|f\|_{M_{p,\varphi_1}(w)}.$$

Proof Similar to the proof of Theorem 5.1, we will check whether M_Ω only satisfies (3.9). Let $x \in B = B(x_0, r)$, $y \in (2B)^C = B^C(x_0, 2r)$ and $\Delta_i = B(x_0, 2^{j+1}r) \diagdown B(x_0, 2^j r)$ $(j \geq 1)$. Note that, if $\Delta_i \cap \{y : |x-y| \leq t\} \neq \emptyset$, then

$$t > |x-y| \geq |y-x_0| - |x-x_0| \geq 2^{j+1}r - r \geq C2^{j+1}r.$$

Thus,

$$t^{-n} \leq C(2^{j+1}r)^{-n}.$$

Hence, for any $t > 0$,

$$\begin{aligned} & t^{-n} \int\limits_{(2B)^C \cap \{y:|x-y|<t\}} |\Omega(x-y)||f(y)|\mathrm{d}y \\ & \lesssim \sum_{j=1}^{\infty} t^{-n} \int\limits_{\Delta_i \cap \{y:|x-y|<t\}} |\Omega(x-y)||f(y)|\mathrm{d}y \\ & \lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-n} \int\limits_{\Delta_i} |\Omega(x-y)||f(y)|\mathrm{d}y. \end{aligned}$$

By Hölder's inequality, the above expression is majorized by

$$\lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-n} \Big(\int\limits_{\Delta_i} |\Omega(x-y)|^q \mathrm{d}y \Big)^{\frac{1}{q}} \Big(\int\limits_{B(x_0, 2^{j+1}r)} |f(y)|^{q'} \mathrm{d}y \Big)^{\frac{1}{q'}}.$$

Applying (3.10), we get

$$\begin{aligned} & t^{-n} \int\limits_{(2B)^C \cap \{y:|x-y|<t\}} |\Omega(x-y)||f(y)|\mathrm{d}y \\ & \lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-\frac{n}{q'}} \Big(\int\limits_{B(x_0, 2^{j+1}r)} |f(y)|^{q'} \mathrm{d}y \Big)^{\frac{1}{q'}} \end{aligned}$$

for any $t > 0$. This means that

$$M_\Omega(f\chi_{(2B)^C})(x) \lesssim \sum_{j=1}^{\infty}(2^{j+1}r)^{-\frac{n}{q'}}\Big(\int\limits_{B(x_0,2^{j+1}r)}|f(y)|^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}$$

holds for any $x_0 \in \mathbb{R}^n$ and $r > 0$ and since the rest of the proof is the same as the proof of Theorem 3.2 and Theorem 4.3, the proof of Theorem 5.2 is completed.

5.3 Bochner-Riesz means

Bochner-Riesz means were first introduced by Bochner [3] in connection with summation of multiple Fourier series and played an important role in harmonic analysis. The Bochner-Riesz means of order $\delta > 0$ in $\mathbb{R}^n (n \geq 2)$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$(B_R^\delta f)^\Lambda(\xi) = \Big(1 - \frac{|\xi|^2}{R^2}\Big)_+ \widehat{f}(\xi), \quad 0 < R < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f and $A_+ = \max(A, 0)$. We recall that the Bochner-Riesz means can be expressed as convolution operators (see [19])

$$B_R^\delta f(x) = (f * \phi_{\frac{1}{R}})(x),$$

where $\phi_{\frac{1}{R}}(x) = R^n f(Rx)$, and for all $\delta \geq \frac{n-1}{2}$ the kernel ϕ can be represented as (see [19])

$$\phi(x) \lesssim (1 + |x|)^{-n-(\delta-\frac{n-1}{2})}. \tag{5.5}$$

The associated maximal Bochner-Riesz operator is defined by

$$B_*^\delta(f)(x) = \sup_{R>0} |B_R^\delta f(x)|.$$

When $\delta > \frac{n-1}{2}$, it is well-known that (see [19])

$$B_*^\delta(f)(x) \lesssim M(f)(x).$$

Then, by the boundedness of maximal function $M(f)$ on $L_{p,w}$, we know that if $w \in A_p$ $(1 < p < \infty)$, then for all $\delta \geq \frac{n-1}{2}$,

$$\|B_*^\delta(f)\|_{L_{p,w}} \lesssim \|f\|_{L_{p,w}}$$

holds.

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < R < \infty$. Consider the commutator $[b, B_R^\delta]$ defined by

$$[b, B_R^\delta](f)(x) = b(x)B_R^\delta f(x) - B_R^\delta(bf)(x) = \int\limits_{\mathbb{R}^n}[b(x) - b(y)]\phi_{\frac{1}{R}}(x - y)f(y)\mathrm{d}y.$$

The maximal operator $[b, B_*^\delta]$ associated with the commutator is defined by

$$[b, B_*^\delta](f)(x) = \sup_{R>0} |[b, B_R^\delta](f)(x)|.$$

Note that $B_R^\delta f(x) \le B_*^\delta(f)(x)$, then if $w \in A_p$ $(1<p<\infty)$, the following

$$\|B_R^\delta(f)\|_{L_{p,w}} \lesssim \|f\|_{L_{p,w}}$$

holds for all $\delta \ge \frac{n-1}{2}$. Thus, by the boundedness criterion for the commutators of linear operators, we see that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, B_R^\delta]$ is also bounded on $L_{p,w}$ for all $1<p<\infty$ and $w \in A_p$.

Theorem 5.3 *Let $\delta \ge \frac{n-1}{2}$ and $1<p<\infty$. Let also $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $w \in A_p$.*
If the pair (φ_1,φ_2) satisfies condition (3.18), then we have for $p>1$,

$$\|B_*^\delta(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)},$$

and for $p=1$,

$$\|B_*^\delta(f)\|_{WM_{1,\varphi_2}(w)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

If the pair (φ_1,φ_2) satisfies condition (4.7), then we have for $p>1$,

$$\|[b, B_R^\delta](f)\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof As in the proof of Theorem 5.1, we will check whether B_R^δ and B_*^δ only satisfy (3.9). Note that when $\delta \ge \frac{n-1}{2}$, then by (5.5), we get

$$|\phi(x)| \lesssim |x|^{-n}.$$

We also note that if $x \in B = B(x_0,r)$, $y \in (2B)^C = B^C(x_0,2r)$, then $|x-y| \approx |x-x_0|$. Thus,

$$\begin{aligned}
\sup_{x\in B} |B_R^\delta(f\chi_{(2B)^C})(x)| &\le \sup_{x\in B} |B_*^\delta(f\chi_{(2B)^C})(x)| \\
&= \sup_{x\in B}\sup_{R>0} |(f\chi_{(2B)^C}) * \phi_{\frac{1}{R}}(x)| \\
&\lesssim \sup_{x\in B}\sup_{R>0} \int\limits_{(2B)^C} \frac{R^n}{(R|x-y|)^n} |f(y)| \mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-n} \int\limits_{B(x_0,2^{j+1}r)} |f(y)| \mathrm{d}y.
\end{aligned}$$

This means that B_R^δ and B_*^δ satisfy (3.9) and since the rest of the proof is the same as the proof of Theorem 3.2 and Theorem 4.3, the proof of Theorem 5.3 is completed.

5.4 Parametric Marcinkiewicz integral operators

For $0<\rho<n$, in 1960, Hörmander [10] defined the parametric Marcinkiewicz integral operator of higher dimension as

$$\mu_\Omega^\rho(f)(x) = \Big(\int\limits_0^\infty |F_{\Omega,t}^\rho(x)|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \Big)^{\frac{1}{2}},$$

where

$$F^{\rho}_{\Omega,t}(x) = \int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\mathrm{d}y,$$

and proved that it is of type (p,p) for $1 < p < \infty$ and of weak type $(1,1)$ when $\Omega \in Lip_{\gamma}(S^{n-1})$ $(0 < \gamma \le 1)$. When $\rho = 1$, we simply denote it by μ_{Ω}. It is well known that the operator μ_{Ω} was defined by Stein in [18].

Let b be a locally integrable function, the commutator generated by parametric Marcinkiewicz integral operator μ^{ρ}_{Ω} and b is defined by

$$[b,\mu^{\rho}_{\Omega}](f)(x) = \Big(\int\limits_0^{\infty}\Big| \int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]f(y)\mathrm{d}y\Big|^2 \frac{\mathrm{d}t}{t^{2\rho+1}}\Big)^{\frac{1}{2}}, \quad 0 < \rho < n.$$

In [16], the weighted boundedness of parametric Marcinkiewicz integral and its commutator with rough kernels were considered.

Theorem 5.4 (see [16]) *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \rho < n$. Then, for every $q' \le p < \infty$ and $w \in A_{\frac{p}{q'}}$, there is a constant C independent of f such that*

$$\|\mu^{\rho}_{\Omega}(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}$$

and

$$\|[b,\mu^{\rho}_{\Omega}](f)\|_{L_{p,w}} \le C\|b\|_*\|f\|_{L_{p,w}}.$$

Theorem 5.5 *Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \rho < n$. Let also $q' < p < \infty$ and $w \in A_{\frac{p}{q'}}$.*

If the pair (φ_1, φ_2) satisfies condition (3.18), then we have for $p > 1$,

$$\|\mu^{\rho}_{\Omega}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)},$$

and for $p = 1$,

$$\|\mu^{\rho}_{\Omega}(f)\|_{WM_{1,\varphi_2}(w)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

If the pair (φ_1, φ_2) satisfies condition (4.7), then we have for $p > 1$,

$$\|[b,\mu^{\rho}_{\Omega}](f)\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_*\|f\|_{M_{p,\varphi_1}(w)}.$$

Proof The statement of Theorem 5.5 follows by Lemma 3.1 and Lemma 4.4 in the same manner as in the proof of Theorem 3.2 and Theorem 4.3.

References

[1] Akbulut, A. and Kuzu, O., Marcinkiewicz integrals with rough kernel associated with Schrödinger operator on vanishing generalized Morrey Spaces, *Azerb. J. Math.*, **4**, 2014, 40–54.

[2] Alvarez, J., Bagby, R. J., Kurtz, D. S. and Pérez, C., Weighted estimates for commutators of linear operators, *Studia Math.*, **104**, 1993, 195–209.

[3] Bochner, S., Summation of multiple Fourier series by spherical means, *Trans. Amer. Math. Soc.*, **40**, 1936, 175–207.

[4] Ding, Y., Fan, D. and Pan, Y., Weighted boundedness for a class of rough Marcinkiewicz integrals, *Indiana Univ. Math. J.*, **48**, 1999, 1037–1055.

[5] Ding, Y., Lu, S. Z. and Yabuta, K., On commutators of Marcinkiewicz integrals with rough kernel, *J. Math. Anal. Appl.*, **275**, 2002, 60–68.

[6] Duoandikoetxea, J., Weighted norm inequalities for homogeneous singular integrals, *Trans. Amer. Math. Soc.*, **336**, 1993, 869–880.

[7] Guliyev, V. S., Generalized weighted Morrey spaces and higher order commutators of sublinear operators, *Eurasian Math. J.*, **3**, 2012, 33–61.

[8] Gürbüz, F., Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, *Canad. Math. Bull.*, **60**, 2017, 131–145.

[9] Gürbüz, F., Vanishing generalized Morrey spaces and commutators of Marcinkiewicz integrals with rough kernel associated with Schrödinger operator, arXiv:1603.03469 [math.CA].

[10] Hörmander, L., Translation invariant operators, *Acta Math.*, **104**, 1960, 93–139.

[11] John, F. and Nirenberg, L., On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, **14**, 1961, 415–426.

[12] Komori, Y. and Shirai, S., Weighted Morrey spaces and a singular integral operator, *Math. Nachr.*, **282**, 2009, 219–231.

[13] Lin, Y. and Lu, S. Z., Strongly singular Calderón-Zygmund operators and their commutators, *Jordan J. Math. Stat. (JJMS)*, **1**, 2008, 31–49.

[14] Morrey, C. B., On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, **43**, 1938, 126–166.

[15] Shen, Z., L_p estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier (Grenoble)*, **45**, 1995, 513–546.

[16] Shi, X. and Jiang, Y., Weighted boundedness of parametric Marcinkiewicz integral and higher order commutator, *Anal. Theory Appl.*, **25**, 2009, 25–39.

[17] Shu, Y. and Wang, M., The fractional maximal operator and Marcinkiewicz integrals associated with Schrödinger operators on Morrey spaces with variable exponent, *Anal. Theory Appl.*, **31**, 2015, 77–89.

[18] Stein, E. M., On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz, *Trans. Amer. Math. Soc.,* **88**, 1958, 430–466.

[19] Stein, E. M. and Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton Mathematical Series, Princeton University Press, Princeton, N.J., 1971.

[20] Wheeden, R. L. and Zygmund, A., Measure and Integral: An Introduction to Real Analysis, **43**, Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1977.