

Geodesics in the Engel Group with a Sub-Lorentzian Metric — the Space-Like Case*

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Abstract Let E be the Engel group and D be a bracket generating left invariant distribution with a Lorentzian metric, which is a nondegenerate metric of index 1. In this paper, the author constructs a parametrization of a quasi-pendulum equation by Jacobi functions, and then gets the space-like Hamiltonian geodesics in the Engel group with a sub-Lorentzian metric.

Keywords Sub-Lorentzian metric, Engel Group, Geodesics
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1 Introduction

Sub-Riemannian geometry is an important branch of geometrical analysis. It has been applied in many fields, such as quantum physics, C-R geometry, control theory, sub-elliptic equations, and so on (see [1, 16–18]). Because of the degeneration of the metric, sub-Riemannian geometry enjoys many differences from the Riemannian one. For example, the Hausdorff dimension is larger than the manifold topological dimension, there exist strictly abnormal minimizers in sub-Riemannian manifolds (see [11, 15]). If we change the positively definite metric to an indefinite degenerate metric with index one, it leads to another geometry, sub-Lorentzian geometry. It is a special semi-sub-Riemannian geometry (see [14]). For more details about sub-Lorentzian geometry, the reader is referred to [4, 8, 12]. Since there are three kinds of horizontal curves (time-like, space-like, null) in sub-Lorentzian manifolds, researches on geodesics and reachable sets in sub-Lorentzian manifolds are more complicated than in sub-Riemannian manifolds. In [7], Chang, Markina, and Vasiliev have systematically studied the geodesics in an anti-de Sitter space with a sub-Lorentzian metric and a sub-Riemannian metric respectively. In [9], Grochowski computed reachable sets starting from a point in the Heisenberg sub-Lorentzian manifold on \mathbb{R}^3 . In [10], Grochowski studied the normal form for Martinet sub-Lorentzian structure of Hamiltonian type, and calculated reachable sets, future null conjugate and cut loci. It was shown in [13] that the Heisenberg group \mathbb{H} with a Lorentzian metric on \mathbb{R}^3 possesses the uniqueness of Hamiltonian geodesics of time-like or space-like type.

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The Engel group was first named by Cartan [6] in 1901. It is the simplest sub-Lorentzian manifold with nontrivial abnormal extremal trajectories, and is a Goursat manifold. In [2], Ardentov and Sachkov computed extremal trajectories on the sub-Riemannian Engel group. In [5], we computed nonspace-like geodesics on the Engel group with sub-Lorentzian metric, but did not calculate space-like geodesics. Because if we use the same method in the space-like case, the computation becomes too complicated. In this paper, we introduce a new method, construct a parameter transformation by Jacobi elliptic functions, and get the space-like Hamiltonian geodesics.

The paper is organized in the following way. In Section 2, we introduce some preliminaries as well as definitions of sub-Lorentzian manifolds, the Engel group. In Section 3, we give the Hamiltonian system, and compute the space-like abnormal geodesics. In Section 4, we introduce a new elliptic coordinate to solve the quasi pendulum equation, and obtain a complete description of the space-like Hamiltonian geodesics in the Engel group.

2 Preliminaries

Let M be a smooth n -dimensional manifold, D be a smooth distribution on M and g be a smooth varying Lorentzian metric on D . The triple (M, D, g) is called a sub-Lorentzian manifold.

We introduce a time orientation first. A vector $v \in D_p$ is said to be time-like if $g(v, v) < 0$; space-like if $g(v, v) > 0$ or $v = 0$; null (light-like) if $g(v, v) = 0$ and $v \neq 0$; and non-space-like if $g(v, v) \leq 0$. An absolutely continuous curve $\gamma(t)$ is said to be horizontal if its derivative $\gamma'(t)$ exists almost everywhere and lies in $D(\gamma(t))$. A horizontal curve is said to be time-like if its tangent vector is time-like a.e.. Similarly, space-like, null and non-space-like curves can be defined. By a time orientation of (M, D, g) , we mean a continuous time-like vector field on M . If X is a time orientation on (M, D, g) , then a non-space-like vector $v \in D_p$ is said to be future directed if $g(v, X(p)) < 0$, and past directed if $g(v, X(p)) > 0$. Throughout this paper, “f.d.” stands for “future directed”, “t.” for “time-like”, and “nspc.” for “non-space-like”. The sub-Lorentzian length of a horizontal curve l is defined by

$$l(\gamma(t)) = \int_0^1 \|\gamma'(t)\| dt = \int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|} dt.$$

A distribution $D \subset TM$ is called bracket generating if any local frame $\{X_i\}_{1 \leq i \leq r}$ for D , together with all of its iterated Lie brackets $[X_i, X_j], [X_i, [X_j, X_k]], \dots$ span the tangent bundle TM . Bracket generating distributions are sometimes also called completely nonholonomic or distributions that satisfying Hörmander’s condition.

Theorem 2.1 (Chow Theorem) *Fix a point $q \in M$. If the distribution $D \subset TM$ is bracket generating, then the set of points that can be connected to q by a horizontal curve is the component of M containing q .*

By the Chow theorem, we know that if D is bracket generating and M is connected, then any two points of M can be joined by a horizontal curve.

Now we introduce the Hamiltonian function with respect to the sub-Lorentzian structure

$$H(\lambda, u) = \frac{\nu}{2} \left(-u_1^2 + \sum_{i=2}^k u_i^2 \right) + \left\langle \lambda, \sum_{i=1}^k u_i X_i \right\rangle, \quad (2.1)$$

where $\lambda = (q, \xi) \in T_q^*M$, $u \in R^k$, $\nu = 0$ or -1 , $(\nu, \xi) \neq 0$. $\vec{H} \in T(T^*M)$ is the Hamiltonian vector field induced by H , λ_s is the flow of the vector field \vec{H} , i.e.,

$$\dot{\lambda}_s = \vec{H}(\lambda_s),$$

where u satisfies the maximum condition $\frac{\partial H}{\partial u} = 0$. If $\nu = 0$, the integral curve λ is called an abnormal lift, and the projection γ is an abnormal extremal. If $\nu = -1$, λ is called a normal lift, the projection γ is a normal geodesics (or a normal extremal). Immediately from the definition, we have the following lemma (cf. [5]).

Lemma 2.1 (cf. [5]) *The causal character (time-like, space-like, null) of normal sub-Lorentzian geodesics does not depend on time.*

At last, we describe the Engel group E . We consider the Engel group E with coordinates $q = (x_1, x_2, y, z) \in \mathbb{R}^4$. The group law is denoted by \odot and defined as

$$(x_1, x_2, y, z) \odot (x'_1, x'_2, y', z') \\ = \left(x_1 + x'_1, x_2 + x'_2, y + y' + \frac{x_1 x'_2 - x'_1 x_2}{2}, z + z' + \frac{x_2 x'_2}{2}(x_2 + x'_2) + x_1 y' + \frac{x_1 x'_2}{2}(x_1 + x'_1) \right).$$

A vector field X is said to be left-invariant if it satisfies $dL_q X(e) = X(q)$, where L_q denotes the left translation $p \rightarrow L_q(p) = q \odot p$ and e is the identity of E . This definition implies that any left-invariant vector field on E is a linear combination of the following vector fields:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial y}, & X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial y} + \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial z}, \\ X_3 &= \frac{\partial}{\partial y} + x_1 \frac{\partial}{\partial z}, & X_4 &= \frac{\partial}{\partial z}. \end{aligned} \quad (2.2)$$

The distribution $D = \text{span}\{X_1, X_2\}$ of E satisfies the bracket generating condition, since $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$. The Engel group is a nilpotent Lie group, since $[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0$. We define a smooth Lorentzian metric on D by

$$g(X_1, X_1) = -1, \quad g(X_2, X_2) = 1, \quad g(X_1, X_2) = 0. \quad (2.3)$$

X_1 is the time orientation.

The horizontal curve on the Engel group has the property (cf. [5]).

Lemma 2.2 (cf. [5]) *For a horizontal curve on the Engel group, the property of time-likeness (space-likeness, light-likeness) and future-directness (past-directness) is preserved under left translations.*

3 Abnormal Extremal Trajectories

In this section we write down the Hamiltonian system and compute the abnormal extremals. By Lemma 2.2 horizontal space-like curves are left invariant, so we can assume that the initial point is the origin, i.e., $x_1(0) = x_2(0) = y(0) = z(0) = 0$, and space-like initial velocity is $-u_1^2(0) + u_2^2(0) = 1$. Let us introduce the vector of costate variables $\eta = (\nu, \xi_1, \xi_2, \xi_3, \xi_4)$ and define the Hamiltonian function

$$H(\xi, q(t), u) = \nu \frac{-u_1^2 + u_2^2}{2} + \xi_1 u_1 + \xi_2 u_2 + \xi_3 \frac{x_1 u_2 - x_2 u_1}{2} + \xi_4 \frac{x_1^2 + x_2^2}{2} u_2. \quad (3.1)$$

From Pontryagin's maximum principle for this Hamiltonian function we obtain a Hamiltonian system for the costate variables

$$\dot{\xi}_1 = -H_{x_1} = -\frac{\xi_3 u_2}{2} - \xi_4 x_1 u_2, \quad \dot{\xi} = -H_{x_2} = \frac{\xi_3 u_1}{2} - \xi_4 x_2 u_2, \quad \dot{\xi}_3 = \dot{\xi}_4 = 0, \quad (3.2)$$

and the maximum condition

$$\max_{u \in \mathbb{R}^2} H(\xi(t), q(t), u(t)) = H(\xi(t), \hat{q}(t), \hat{u}(t)), \quad \nu \leq 0, \quad (3.3)$$

where $\hat{u}(t), \hat{q}(t)$ are the optimal processes, and the condition $\eta(t) \neq 0$ is the non-triviality of the costate variables.

Let $\nu = 0$. From the maximum condition (3.3) we obtain

$$H_{u_1} = \xi_1 - \frac{\xi_3 x_2}{2} = 0, \quad (3.4)$$

$$H_{u_2} = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4 (x_1^2 + x_2^2)}{2} = 0. \quad (3.5)$$

Differentiating equations (3.4)–(3.5), we obtain

$$0 = \dot{\xi}_1 - \frac{\xi_3 \dot{x}_2}{2} = \dot{\xi}_1 - \frac{\xi_3 u_2}{2} = -u_2(\xi_3 + \xi_4 x_1), \quad (3.6)$$

$$0 = \dot{\xi}_2 + \frac{\xi_3 \dot{x}_1}{2} + \xi_4(x_1 \dot{x}_1 + x_2 \dot{x}_2) = u_1(\xi_3 + \xi_4 x_1). \quad (3.7)$$

For the space-like case, $-u_1^2 + u_2^2 = 1$, so $\xi_3 + \xi_4 x_1 = 0$. If $\xi_4 = 0$, then $\xi_3 = 0$, and therefore $\xi = 0$. It is a contradiction with the non-triviality of the costate variables, hence $\xi_4 \neq 0$. In this case, we can get that $x_1 = \frac{-\xi_3}{\xi_4}$ is a constant, and $u_1 = 0$, $u_2 = \pm 1$, so the space-like abnormal extremal trajectories are the following expression:

$$\gamma(s) = \left(0, \pm s, 0, \pm \frac{s^3}{6} \right). \quad (3.8)$$

4 Normal Geodesics

Now we consider the normal case $\nu = -1$. It follows from the maximum condition (3.3) that $H_{u_1} = H_{u_2} = 0$. Hence

$$u_1 = -\left(\xi_1 - \frac{x_2 \xi_3}{2} \right), \quad u_2 = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4 (x_1^2 + x_2^2)}{2}. \quad (4.1)$$

Let $h_i = (\xi, X_i)$, $i = 1, 2, 3, 4$ be the Hamiltonians which are linear on fibers. It is easy to get

$$h_1 = \xi_1 - \frac{x_2 \xi_3}{2}, \quad h_2 = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4 (x_1^2 + x_2^2)}{2}, \quad h_3 = \xi_3 + x_1 \xi_4, \quad h_4 = \xi_4, \quad (4.2)$$

so $u_1 = -\zeta_1$ and $u_2 = \zeta_2$.

According to Pontryagin's maximum principle, we get the following Hamilton system:

$$\dot{x}_1 = -h_1, \quad (4.3)$$

$$\dot{x}_2 = h_2, \quad (4.4)$$

$$\dot{y} = \frac{1}{2}(x_2 h_1 + x_1 h_2), \quad (4.5)$$

$$\dot{z} = \frac{1}{2}h_2(x_1^2 + x_2^2), \quad (4.6)$$

$$\dot{h}_1 = -h_2 h_3, \quad (4.7)$$

$$\dot{h}_2 = -h_1 h_3, \quad (4.8)$$

$$\dot{h}_3 = -h_1 h_4, \quad (4.9)$$

$$\dot{h}_4 = 0. \quad (4.10)$$

For the space-like case, the curves are considered on the level surface $H = -h_1^2 + h_2 = 1$, whence we go over to another coordinate system (θ, c, α) :

$$h_1 = \sin h\theta, \quad h_3 = c,$$

$$h_2 = \cos h\theta, \quad h_4 = \alpha.$$

So in the normal case, the Hamiltonian system becomes

$$\dot{\theta} = -c, \quad (4.11)$$

$$\dot{c} = -\alpha \sin h\theta, \quad (4.12)$$

$$\dot{\alpha} = 0, \quad (4.13)$$

$$\dot{x}_1 = -\sin h\theta, \quad (4.14)$$

$$\dot{x}_2 = \cos h\theta, \quad (4.15)$$

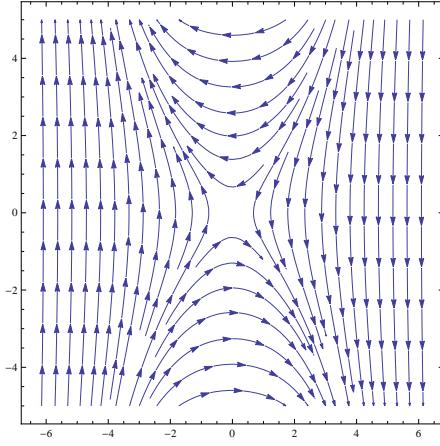
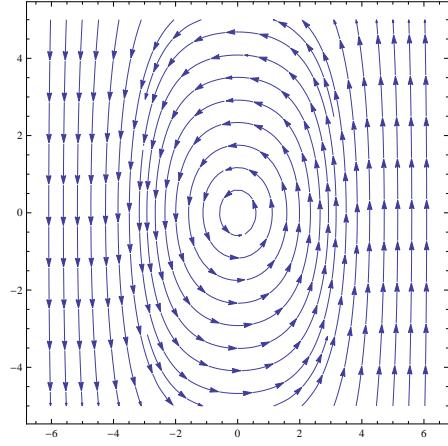
$$\dot{y} = \frac{1}{2}(x_2 \sin h\theta + x_1 \cos h\theta), \quad (4.16)$$

$$\dot{z} = \frac{1}{2} \cos h\theta(x_1^2 + x_2^2). \quad (4.17)$$

The vertical part is reduced to an equation which is quite similar to a pendulum one:

$$\ddot{\theta} = \alpha \sin h\theta, \quad \dot{\alpha} = 0. \quad (4.18)$$

For this reason, we call the formula quasi-pendulum equation. Figures 1–2 are phase graphs for $\alpha = 1$ and $\alpha = -1$.

Figure 1 $\ddot{\theta} = \sin h\theta$.Figure 2 $\ddot{\theta} = -\sin h\theta$.

Let us introduce the energy integral

$$E = \frac{1}{2}c^2 - \alpha \cos h\theta, \quad \dot{E} = 0.$$

The family of all normal geodesics is parameterized by the points of the set

$$C = \{(h_1, h_2, h_3, h_4) \in \mathbb{R}^4 \mid h_2^2 - h_1^2 = 1\} = \{(\theta, c, \alpha) \in \mathbb{R}^3\}.$$

We divide the set C into 10 parts as follows:

$$C = C_0^0 \cup C^0 \cup C_1^- \cup C_2^- \cup C_1^+ \cup C_2^+ \cup C_3^+ \cup C_4^+ \cup C_5^+ \cup C_6^+, \quad \lambda = (\theta, c, \alpha),$$

$$C_0^0 = \{\lambda \in C \mid \alpha = 0, c = 0\},$$

$$C^0 = \{\lambda \in C \mid \alpha = 0, c \neq 0\},$$

$$C_1^- = \{\lambda \in C \mid \alpha < 0, E \in (-\alpha, +\infty)\},$$

$$C_2^- = \{\lambda \in C \mid \alpha < 0, E = -\alpha\},$$

$$C_1^+ = \{\lambda \in C \mid \alpha > 0, E \in (-\infty, -\alpha)\},$$

$$C_2^+ = \{\lambda \in C \mid \alpha > 0, E \in (-\alpha, \alpha)\},$$

$$C_3^+ = \{\lambda \in C \mid \alpha > 0, E \in (\alpha, +\infty)\},$$

$$C_4^+ = \{\lambda \in C \mid \alpha > 0, E = \alpha\},$$

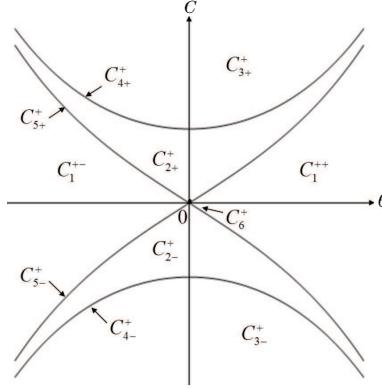
$$C_5^+ = \{\lambda \in C \mid \alpha > 0, E = -\alpha, c \neq 0\},$$

$$C_6^+ = \{\lambda \in C \mid \alpha > 0, E = -\alpha, c = 0\}.$$

The sets C_i^+ , $i = 1, 2, 3, 4, 5$ are further subdivided into subsets that depending on the sign of θ , c ,

$$C_{i+}^+ = C_i^+ \cap \{c > 0\}, \quad C_{i-}^+ = C_i^+ \cap \{c < 0\},$$

$$C_i^{++} = C_i^+ \cap \{\theta > 0\}, \quad C_i^{+-} = C_i^+ \cap \{\theta < 0\}.$$

Figure 3 Partition C for $\alpha > 0$.

We give the partitioning of a section of the cylinder $\{\lambda \in C, \alpha = \text{con } st > 0\}$ in Figure 3.

For subsets $C_0^0, C_0^-, C_2^-, C_6^+$, we compute directly. For $C_1^-, C_1^+, C_2^+, C_3^+, C_4^+$ and C_5^+ , we introduce new coordinates (φ, k) in the phase space of the quasi-pendulum. We call (φ, k) the elliptic coordinates. In this coordinates, the vector field $(-c, -\alpha \sin h\theta)$ is straightened out, and the quasi-pendulum equation takes a very simple form. That is to say, when we transform (θ, c) to (φ, k) , the quasi-pendulum equation

$$\begin{cases} \dot{\theta} = -c, \\ \dot{c} = -\alpha \sin h\theta \end{cases} \quad (4.19)$$

becomes

$$\begin{cases} \dot{\varphi} = 1, \\ \dot{k} = 0. \end{cases} \quad (4.20)$$

So the solution of the quasi-pendulum equation in the elliptic coordinates is

$$k = \text{con } st, \quad \varphi = \varphi_0 + t,$$

where φ_0 is the initial value, and expressions of space-like extremals can be obtained.

If $\lambda \in C_0^0$, we have $\theta \equiv \text{con } st$, and so is $\sin h\theta$. We assume $\sin h\theta = s_0$,

$$x_1(t) = -s_0 t, \quad (4.21)$$

$$x_2(t) = \sqrt{s_0^2 + 1} t, \quad (4.22)$$

$$y(t) \equiv 0, \quad (4.23)$$

$$z(t) = \frac{(2s_0^2 + 1)\sqrt{s_0^2 + 1}}{6} t^3. \quad (4.24)$$

If $\lambda \in C^0$, we have $c \equiv \text{con } st$, $\theta = \theta_0 - ct$,

$$x_1(t) = \frac{\cos h\theta - \cos h\theta_0}{c}, \quad (4.25)$$

$$x_2(t) = \frac{\sin h\theta_0 - \sin h\theta}{c}, \quad (4.26)$$

$$y(t) = \frac{ct - \sin h(ct)}{2c^2}, \quad (4.27)$$

$$\begin{aligned} z(t) = & \frac{1}{c^3} \left[-\frac{1}{3} \sin h^3\theta - \frac{1}{6} \sin h^3\theta_0 - \cos h^2\theta_0 \sin h\theta + \cos h^2\theta_0 \sin h\theta_0 \right. \\ & \left. + \frac{1}{2} \sin h\theta_0 \sin h^2\theta + \frac{1}{4} \cos h\theta_0 \sin h2\theta - \frac{1}{4} \cos h\theta_0 \sin h2\theta_0 - \frac{c}{2} (\cos h\theta_0)t \right]. \end{aligned} \quad (4.28)$$

If $\lambda \in C_1^-$, and the energy integral $E \in (-\alpha, +\infty)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{E + \alpha}{E - \alpha}} \in (0, 1), \quad (4.29)$$

$$\sin h\frac{\theta}{2} = -ksd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right), \quad \cos h\frac{\theta}{2} = nd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right), \quad (4.30)$$

$$c = 2\frac{k\sqrt{-\alpha}}{k'}cd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right), \quad \varphi \in \left[0, \frac{4k'K}{\sqrt{-\alpha}}\right], \quad (4.31)$$

where sd, nd, cd are elliptic functions, k is the modulus, k' is the complementary modulus. It is easy to check that in the elliptic coordinates, the vertical part of Hamiltonian system takes the form:

$$\dot{\varphi} = 1, \quad \dot{k} = 0, \quad \dot{\alpha} = 0, \quad (4.32)$$

and the solutions are

$$\varphi(t) = \varphi_0 + t, \quad k = \text{con } st, \quad \alpha = \text{con } st. \quad (4.33)$$

Now we compute expressions of space-like geodesics. By transformation (4.29)–(4.31), we get

$$\sin h\theta = 2 \sin h\frac{\theta}{2} \cos h\frac{\theta}{2} = -2ksd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right)nd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right), \quad (4.34)$$

$$\cos h\theta = 2 \sin h^2\frac{\theta}{2} + 1 = 2k^2sd^2\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right) + 1. \quad (4.35)$$

Substituting them into the horizontal part of the Hamiltonian system, we have

$$\begin{aligned} \dot{x}_1 &= 2ksd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right)nd\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right), \\ \dot{x}_2 &= 2k^2sd^2\left(\frac{\sqrt{-\alpha}\varphi}{k'}\right) + 1. \end{aligned}$$

By the following integral formulas:

$$\int sd u \ nd u du = -\frac{cd u}{k'^2}, \quad (4.36)$$

$$\int sd^2 u du = \frac{1}{k^2 k'^2} (E(u) - k'^2 u - k^2 sn u cd u), \quad (4.37)$$

we obtain

$$x_1 = -2 \frac{k}{\sqrt{-\alpha} k'} [cd(\chi_t) - cd(\chi_0)], \quad (4.38)$$

$$x_2 = \frac{2}{\sqrt{-\alpha} k'} [E(\chi_t) - E(\chi_0) - k^2 (sn(\chi_t)cd(\chi_t) - sn(\chi_0)cd(\chi_0))] - t, \quad (4.39)$$

where $\chi_t = \frac{\sqrt{-\alpha}}{k'} \varphi$, $\chi_0 = \frac{\sqrt{-\alpha}}{k'} \varphi_0$.

Since

$$\dot{y} = \frac{1}{2} (-x_2 \dot{x}_1 + x_1 \dot{x}_2), \quad (4.40)$$

$$\dot{z} = \frac{1}{2} (x_1^2 + x_2^2) \dot{x}_2, \quad (4.41)$$

we get

$$\begin{aligned} y &= \int x_1 \dot{x}_2 dt - \frac{x_1 x_2}{2} \\ &= -2 \frac{k}{\sqrt{-\alpha} k'} \int (cd\chi_t - cd\chi_0) (2k^2 sd^2 \chi_t + 1) dt - \frac{x_1 x_2}{2}. \end{aligned} \quad (4.42)$$

By (4.37) and formulas

$$\int cd u \, sd^2 u du = -\frac{1}{2k^3} \ln \left(\frac{1 + ksn u}{dn u} \right), \quad (4.43)$$

$$\int cd u du = \frac{1}{k} \ln \left(\frac{1 + ksn u}{dn u} \right), \quad (4.44)$$

we get

$$\begin{aligned} y &= -4 \frac{k^4}{\alpha} (sn\chi_t nd\chi_t - sn\chi_0 nd\chi_0) - 4 \operatorname{sgn} c \frac{k}{\alpha k'^2} [E(\chi_t) - E(\chi_0) - \sqrt{-\alpha} k' t \\ &\quad - k^2 (sn\chi_t cd\chi_t - sn\chi_0 cd\chi_0)] + 2 \operatorname{sgn} c \, cd\chi_0 \frac{k}{\sqrt{-\alpha} k'} t - \frac{x_1 x_2}{2}. \end{aligned} \quad (4.45)$$

Since

$$\dot{z} = \frac{1}{2} (x_1^2 + x_2^2) \dot{x}_2, \quad (4.46)$$

we get

$$\begin{aligned} z &= \int \frac{x_1^2 \dot{x}_2}{2} dt + \frac{x_2^3}{6} \\ &= -\frac{4k^2}{\alpha k'^2} \int (cd^2 \chi_t - 2cd\chi_0 cd\chi_t + cd\chi_0) (2k^2 sd^2 \chi_t + 1) dt + \frac{x_2^3}{6}. \end{aligned} \quad (4.47)$$

By (4.37), (4.43)–(4.44) and formulas

$$\begin{aligned} \int cd^2 u \, sd^2 u du &= \frac{1}{k'^2} \left[\frac{2k^2 - 2}{3k^4} u + \frac{2 - k^2}{3k^4} E(u) \right. \\ &\quad \left. + (2k^2 + 2k^4 - 1) sn u \, cd u - k^2 k'^2 sd u \, cd u \right], \end{aligned} \quad (4.48)$$

$$\int cd^2 u du = \frac{1}{k} \ln \left(\frac{1 + k s n u}{dn u} \right), \quad (4.49)$$

we get

$$\begin{aligned} z &= \frac{4k^2}{(-\alpha)^{\frac{3}{2}} k'} \left\{ \left(\frac{1}{3k^2} - cd^2 \chi_0 \right) \frac{\sqrt{-\alpha}}{k'} t + \left(\frac{2k^2 - 1}{3k^2 k'^2} + \frac{2}{k'^2} cd^2 \chi_0 \right) (E(\chi_t) - E(\chi_0)) \right. \\ &\quad \left. + \left[\frac{2k^2}{k'^2} (2k^2 + 2k^4 - 1) + 1 - \frac{2k^2}{k'^2} cd \chi_0 \right] (sn \chi_t cd \chi_t - sn \chi_0 cd \chi_0) + 2k^4 (sn \chi_t cd \chi_t nd \chi_t \right. \\ &\quad \left. - sn \chi_0 cd \chi_0 nd \chi_0) + 4k^3 cd \chi_0 (sn \chi_t nd^2 \chi_t - sn \chi_0 nd^2 \chi_0) \right\} + \frac{x_2^3}{6}. \end{aligned} \quad (4.50)$$

If $\lambda \in C_2^-$, we have $\theta = c \equiv 0$. Then we have

$$x_1(t) = 0, \quad x_2(t) = t, \quad y(t) = 0, \quad z(t) = \frac{1}{6} t^3. \quad (4.51)$$

If $\lambda \in C_1^+$, the energy integral $E \in (-\alpha, +\infty)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{E + \alpha}{E - \alpha}} \in (0, 1), \quad (4.52)$$

$$\sin h \frac{\theta}{2} = \frac{1}{k} \operatorname{sgn} \theta ds \left(\frac{\sqrt{\alpha}}{k} \varphi \right), \quad \cos h \frac{\theta}{2} = \frac{1}{k} ns \left(\frac{\sqrt{\alpha}}{k} \varphi \right), \quad (4.53)$$

$$c = \operatorname{sgn} \theta \frac{2}{k} \sqrt{\alpha} cs \left(\frac{\sqrt{\alpha}}{k} \varphi \right), \quad \varphi \in \left(0, \frac{2kK}{\sqrt{\alpha}} \right). \quad (4.54)$$

So we obtain

$$\dot{x}_1 = -\sin h \theta = -\frac{2}{k^2} \operatorname{sgn} \theta ds \left(\frac{\sqrt{\alpha}}{k} \varphi \right) ns \left(\frac{\sqrt{\alpha}}{k} \varphi \right), \quad (4.55)$$

$$\dot{x}_2 = \cos h \theta = \frac{2}{k^2} ds^2 \left(\frac{\sqrt{\alpha}}{k} \varphi \right) + 1. \quad (4.56)$$

Combining with the formulas

$$\int cs u du = \ln \left(\frac{1 - dn u}{sn u} \right), \quad (4.57)$$

$$\int cs^2 u du = -dn u cs u - E(u), \quad (4.58)$$

$$\int cs u ds^2 u du = -\frac{k^2}{2} ds^2 u du - \frac{1}{2} dn u ns^2 u, \quad (4.59)$$

$$\int cs^2 u ds^2 u du = \frac{2 - k'^2}{3} (E(u) + dn u cs u) - \frac{1}{3} ns^2 u dn u cs u - \frac{k'^2}{3} u, \quad (4.60)$$

$$\int ds^2 u du = k'^2 u - dn u cs u - E(u), \quad (4.61)$$

we get

$$x_1 = \operatorname{sgn} \theta \frac{2}{\sqrt{\alpha} k} (cs \phi_t - cs \phi_0), \quad (4.62)$$

$$x_2 = \frac{2}{\sqrt{\alpha} k} [dn \phi_0 cs \phi_0 - dn \phi_t cs \phi_t + E(\phi_0) - E(\phi_t)] + \frac{2k'^2}{k^2} t, \quad (4.63)$$

and further more,

$$\begin{aligned} y &= \int x_1 \dot{x}_2 dt - \frac{x_1 x_2}{2} \\ &= \operatorname{sgn} \theta \frac{2}{\sqrt{\alpha} k} \int (cs\phi_t - cs\phi_0) \left(\frac{2}{k^2} ds^2 \phi_t + 1 \right) dt - \frac{x_1 x_2}{2} \\ &= -\operatorname{sgn} \theta \frac{2}{\alpha k^2} (dn\phi_t ns^2 \phi_t - dn\phi_0 ns^2 \phi_0) - \operatorname{sgn} \theta \frac{4}{\alpha k^2} cs\phi_0 \left[\frac{\sqrt{\alpha} k'^2}{k} t \right. \\ &\quad \left. + dn\phi_0 cs\phi_0 - dn\phi_t cs\phi_t + E(\phi_0) - E(\phi_t) \right] - \operatorname{sgn} \theta \frac{\sqrt{\alpha}}{k} cs\phi_0 t - \frac{x_1 x_2}{2}, \end{aligned} \quad (4.64)$$

$$\begin{aligned} z &= \int \frac{x_1^2 \dot{x}_2}{2} dt + \frac{x_2^3}{6} \\ &= \frac{4}{k^2 \alpha} \int (cs\phi_t - cs\phi_0)^2 \left(\frac{2}{k^2} ds^2 \phi_t + 1 \right) dt + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha^{\frac{3}{2}} k} \left\{ \left(\frac{2 - k^2 - 6cs^2 \phi_0}{3k^2} \right) [E(\phi_t) - E(\phi_0) + dn\phi_t cs\phi_t - dn\phi_0 cs\phi_0] \right. \\ &\quad \left. - \frac{2}{3k^2} (ns^2 \phi_t dn\phi_t cs\phi_t - ns^2 \phi_0 dn\phi_0 cs\phi_0) - \operatorname{sgn} c \frac{\sqrt{\alpha}}{k} \left(-\frac{2k'^2}{3k^2} + \frac{2 - k^2}{k^2} cs^2 \phi_0 \right) t \right. \\ &\quad \left. + \frac{2}{k^2} cs\phi_0 (dn\phi_t ns^2 \phi_t - dn\phi_0 ns^2 \phi_0) \right\} + \frac{x_2^3}{6}, \end{aligned} \quad (4.65)$$

where $\phi_t = \frac{\sqrt{\alpha}}{k} \varphi$, $\phi_0 = \frac{\sqrt{\alpha}}{k} \varphi_0$.

If $\lambda \in C_2^+$, the energy integral $E \in (-\alpha, \alpha)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{\alpha - E}{2\alpha}} \in (0, 1), \quad (4.66)$$

$$\sin h \frac{\theta}{2} = -\operatorname{sgn} c k' sc(\sqrt{\alpha} \varphi), \quad \cos h \frac{\theta}{2} = dc(\sqrt{\alpha} \varphi), \quad (4.67)$$

$$c = 2\operatorname{sgn} c k' \sqrt{\alpha} n c(\sqrt{\alpha} \varphi), \quad \varphi \in \left(-\frac{K}{\sqrt{\alpha}}, \frac{K}{\sqrt{\alpha}} \right). \quad (4.68)$$

So we obtain

$$\dot{x}_1 = -\sin h \theta = -2\operatorname{sgn} c k' sc(\sqrt{\alpha} \varphi) dc(\sqrt{\alpha} \varphi), \quad (4.69)$$

$$\dot{x}_2 = \cos h \theta = 2k'^2 sc^2(\sqrt{\alpha} \varphi) + 1. \quad (4.70)$$

Combining with the formulas

$$\int nc u du = \frac{1}{k'} \ln \left(\frac{k' sn u + dn u}{cn u} \right), \quad (4.71)$$

$$\int nc^2 u du = \frac{1}{k'^2} [k'^2 u - E(u) + dn u sc u], \quad (4.72)$$

$$\int nc u sc^2 u du = \frac{1}{2k'^2} sc u dc u - \frac{1}{2k'^3} \ln \left(\frac{k' sn u + dn u}{cn u} \right), \quad (4.73)$$

$$\int nc^2 u sc^2 u du = \frac{1}{3k'^4} [(1 + k^2) E(u) - k'^2 u + dn u sc u (k'^2 nc^2 u - 1 - k^2)], \quad (4.74)$$

$$\int sc^2 u du = \frac{1}{k'^2} (dn u sc u - E(u)), \quad (4.75)$$

we get

$$x_1 = 2\operatorname{sgn} c \frac{k'}{\sqrt{\alpha}} (nc\psi_t - nc\psi_0), \quad (4.76)$$

$$x_2 = \frac{2}{\sqrt{\alpha}} [dn\psi_t sc\psi_t - dn\psi_0 sc\psi_0 - E(\psi_t) + E(\psi_0)] + t, \quad (4.77)$$

$$\begin{aligned} y &= \int x_1 \dot{x}_2 dt - \frac{x_1 x_2}{2} \\ &= 2\operatorname{sgn} c \frac{k'}{\sqrt{\alpha}} \int (nc\psi_t - nc\psi_0)(2k'^2 sc^2 \psi_t + 1) dt - \frac{x_1 x_2}{2} \\ &= 2\operatorname{sgn} c \frac{k'}{\alpha} \{ sc\psi_t dc\psi_t - sc\psi_0 dc\psi_0 - 2nc\psi_0 [dn\psi_t sc\psi_t - dn\psi_0 sc\psi_0 - E(\psi_t) \\ &\quad + E(\psi_0)] - \sqrt{\alpha} nc\psi_0 t \} - \frac{x_1 x_2}{2}, \end{aligned} \quad (4.78)$$

$$\begin{aligned} z &= \int \frac{x_1^2 \dot{x}_2}{2} dt + \frac{x_2^3}{6} \\ &= \frac{4k'^2}{\alpha} \int (nc\psi_t - nc\psi_0)^2 (2k'^2 sc^2 \psi_t + 1) dt + \frac{x_2^3}{6} \\ &= \frac{4k'^2}{\alpha^{\frac{3}{2}}} \left[\left(\frac{2(1+k^2)}{3k'^2} - \frac{1}{k'^2} - 2nc^2 \psi_0 \right) (E(\psi_t) - E(\psi_0) - dn\psi_t sc\psi_t + dn\psi_0 sc\psi_0) \right. \\ &\quad \left. + \left(\frac{1}{3} + nc^2 \psi_0 \right) \sqrt{\alpha} t - 2nc\psi_0 (sc\psi_t dc\psi_t - sc\psi_0 dc\psi_0) \right] + \frac{x_2^3}{6}, \end{aligned} \quad (4.79)$$

where

$$\psi_t = \sqrt{\alpha} \varphi, \quad \psi_0 = \sqrt{\alpha} \varphi_0.$$

If $\lambda \in C_3^+$, the energy integral $E \in (\alpha, +\infty)$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = \sqrt{\frac{E-\alpha}{E+\alpha}} \in (0, 1), \quad (4.80)$$

$$\sin h \frac{\theta}{2} = -\operatorname{sgn} c \operatorname{sc} \left(\frac{\sqrt{\alpha}}{k'} \varphi \right), \quad \cos h \frac{\theta}{2} = nc \left(\frac{\sqrt{\alpha}}{k'} \varphi \right), \quad (4.81)$$

$$c = \operatorname{sgn} c \frac{2}{k'} \sqrt{\alpha} \operatorname{dc} \left(\frac{\sqrt{\alpha}}{k'} \varphi \right), \quad \varphi = \left(-\frac{k' K}{\sqrt{\alpha}}, \frac{k' K}{\sqrt{\alpha}} \right). \quad (4.82)$$

So we obtain

$$\dot{x}_1 = -\sin h \theta = 2\operatorname{sgn} c \operatorname{sc} \left(\frac{\sqrt{\alpha}}{k'} \varphi \right) nc \left(\frac{\sqrt{\alpha}}{k'} \varphi \right), \quad (4.83)$$

$$\dot{x}_2 = \cos h \theta = 2sc^2 \left(\frac{\sqrt{\alpha}}{k'} \varphi \right) + 1. \quad (4.84)$$

Using (4.75) and the following formulas:

$$\int dc u du = \ln \left(\frac{1+sn u}{cn u} \right), \quad (4.85)$$

$$\int dc^2 u du = u - E(u) + dn u sc u, \quad (4.86)$$

$$\int dc u sc^2 u du = -\frac{1}{2} \ln \left(\frac{1+sn u}{cn u} \right) + \frac{1}{2} sc u nc u, \quad (4.87)$$

$$\begin{aligned} \int dc^2 u \ sc^2 u du &= \frac{1 - 2k^2}{3k'^2} E(u) - \frac{1}{3}u + \frac{1}{3}dc u \ sc u \ nc u \\ &\quad + \frac{1}{3k'^2} dc u \ sn u - \frac{2}{3}dc u \ sn u, \end{aligned} \quad (4.88)$$

we get

$$x_1 = 2\operatorname{sgn} c \frac{1}{\sqrt{\alpha} k'} (dc\delta_t - dc\delta_0), \quad (4.89)$$

$$x_2 = 2 \frac{1}{\sqrt{\alpha} k'} [dn\delta_t sc\delta_t - dn\delta_0 sc\delta_0 - E(\delta_t) + E(\delta_0)] + t, \quad (4.90)$$

$$\begin{aligned} y &= \int x_1 \dot{x}_2 dt - \frac{x_1 x_2}{2} \\ &= 2\operatorname{sgn} c \frac{1}{k' \sqrt{\alpha}} \int (dc\delta_t - dc\delta_0)(2sc^2\delta_t + 1) dt - \frac{x_1 x_2}{2} \\ &= 2\operatorname{sgn} c \frac{1}{\alpha} \left\{ sn\delta_t nc^2\delta_t - sn\delta_0 nc^2\delta_0 - 2 \frac{dc\delta_0}{k'^2} \left[dn\delta_t sc\delta_t - dn\delta_0 sc\delta_0 \right. \right. \\ &\quad \left. \left. - E(\delta_t) + E(\delta_0) - \frac{\sqrt{\alpha}}{k'} dc\delta_0 t \right] \right\} - \frac{x_1 x_2}{2}, \end{aligned} \quad (4.91)$$

$$\begin{aligned} z &= \int \frac{x_1^2 \dot{x}_2}{2} dt + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha k'^2} \int (dc\delta_t - dc\delta_0)^2 (2sc^2\delta_t + 1) dt + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha^{3/2} k'} \left[\left(\frac{-2 + k'^2}{3k'^2} + \frac{2dc^2\delta_0}{k'^2} \right) (E(\delta_t) - E(\delta_0)) + \left(\frac{1}{3} + dc^2\delta_0 \right) \frac{\sqrt{\alpha}}{k'} t \right. \\ &\quad \left. + \left(\frac{2 - k'^2}{3k'^2} + \frac{2dc^2\delta_0}{k'^2} \right) (dn\delta_t sc\delta_t - dn\delta_0 sc\delta_0) + \frac{2}{3} dc\delta_t sc\delta_t nc\delta_t \right. \\ &\quad \left. - \frac{2}{3} dc\delta_0 sc\delta_0 nc\delta_0 \right] + \frac{x_2^3}{6}, \end{aligned} \quad (4.92)$$

where

$$\delta_t = \frac{\sqrt{\alpha}}{k'} \varphi, \quad \delta_0 = \frac{\sqrt{\alpha}}{k'} \varphi_0.$$

If $\lambda \in C_4^+$, the energy integral $E = \alpha$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = 0, \quad (4.93)$$

$$\sin h \frac{\theta}{2} = -\operatorname{sgn} c \tan(\sqrt{\alpha} \varphi), \quad \cos h \frac{\theta}{2} = \sec(\sqrt{\alpha} \varphi), \quad (4.94)$$

$$c = 2\operatorname{sgn} c \sqrt{\alpha} \sec(\sqrt{\alpha} \varphi), \quad \varphi \in \left(-\frac{\pi}{2\sqrt{\alpha}}, \frac{\pi}{2\sqrt{\alpha}} \right). \quad (4.95)$$

So we obtain

$$\dot{x}_1 = -\sin \theta = 2\operatorname{sgn} c \tan(\sqrt{\alpha} \varphi) \sec(\sqrt{\alpha} \varphi), \quad (4.96)$$

$$\dot{x}_2 = \cos \theta = 2 \tan^2(\sqrt{\alpha} \varphi) + 1, \quad (4.97)$$

and then

$$x_1 = 2\operatorname{sgn} c \frac{1}{\sqrt{\alpha}} (\sec \psi_t - \sec \psi_0), \quad (4.98)$$

$$x_2 = \frac{2}{\sqrt{\alpha}}(\tan \psi_t - \tan \psi_0) - t, \quad (4.99)$$

$$\begin{aligned} y &= \int x_1 \dot{x}_2 dt - \frac{x_1 x_2}{2} \\ &= 2 \operatorname{sgn} c \frac{1}{\sqrt{\alpha}} \int (\sec \psi_t - \sec \psi_0)(2 \tan^2 \psi_t + 1) dt - \frac{x_1 x_2}{2} \\ &= 2 \operatorname{sgn} c \frac{1}{\alpha} [\tan \psi_t \sec \psi_t - \tan \psi_0 \sec \psi_0 - 2 \sec \psi_0 (\tan \psi_t - \psi_0) + \sec \psi_0 \sqrt{\alpha} t] \\ &\quad - \frac{x_1 x_2}{2}, \end{aligned} \quad (4.100)$$

$$\begin{aligned} z &= \int \frac{x_1^2 \dot{x}_2}{2} dt + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha} \int (\sec \psi_t - \sec \psi_0)^2 (2 \tan^2 \psi_t + 1) dt + \frac{x_2^3}{6} \\ &= \frac{4}{\alpha^{\frac{3}{2}}} \left\{ \frac{2}{3} (\tan^3 \psi_t - \tan^3 \psi_0) + (1 + 2 \sec^2 \psi_0) (\tan \psi_t - \tan \psi_0) - \sec^2 \psi_0 \sqrt{\alpha} t \right. \\ &\quad \left. - 2 \sec \psi_0 (\tan \psi_t \sec \psi_t - \tan \psi_0 \sec \psi_0) \right\} + \frac{x_2^3}{6}, \end{aligned} \quad (4.101)$$

where

$$\psi_t = \sqrt{\alpha} \varphi, \quad \psi_0 = \sqrt{\alpha} \varphi_0.$$

If $\lambda \in C_5^+$, the energy integral $E = -\alpha$, we introduce the elliptic coordinates (φ, k, α) as follows:

$$k = 1, \quad (4.102)$$

$$\sin h \frac{\theta}{2} = \operatorname{sgn} \theta \csc h(\sqrt{\alpha} \varphi), \quad \cos h \frac{\theta}{2} = \cot h(\sqrt{\alpha} \varphi), \quad (4.103)$$

$$c = 2 \operatorname{sgn} \theta \sqrt{\alpha} \operatorname{csch}(\sqrt{\alpha} \varphi), \quad \varphi \in (0, +\infty). \quad (4.104)$$

So we obtain

$$\dot{x}_1 = -\sin \theta = -2 \operatorname{sgn} \theta \csc h(\sqrt{\alpha} \varphi) \cot h(\sqrt{\alpha} \varphi), \quad (4.105)$$

$$\dot{x}_2 = \cos \theta = 2 \csc h^2(\sqrt{\alpha} \varphi) + 1, \quad (4.106)$$

and then

$$x_1 = 2 \operatorname{sgn} \theta \frac{1}{\sqrt{\alpha}} (\csc h \psi_t - \csc h \psi_0), \quad (4.107)$$

$$x_2 = -\frac{2}{\sqrt{\alpha}} (\cot h \psi_t - \cot h \psi_0) + t, \quad (4.108)$$

$$\begin{aligned} y &= \int x_1 \dot{x}_2 dt - \frac{x_1 x_2}{2} \\ &= 2 \operatorname{sgn} \theta \frac{1}{\sqrt{\alpha}} \int (\csc h \psi_t - \csc h \psi_0)(2 \csc h^2 \psi_t + 1) dt - \frac{x_1 x_2}{2} \\ &= \operatorname{sgn} \theta \frac{2}{\alpha} [-\cot h \psi_t \csc h \psi_t + \cot h \psi_0 \csc h \psi_0 + 2 \csc h \psi_0 (\cot h \psi_t - \cot h \psi_0) \\ &\quad + \sqrt{\alpha} \csc \psi_0 t] - \frac{x_1 x_2}{2}, \end{aligned} \quad (4.109)$$

$$\begin{aligned}
z &= \int \frac{x_1^2 \dot{x}_2}{2} dt + \frac{x_2^3}{6} \\
&= \frac{4}{\alpha} \int (\csc h \psi_t - \csc h \psi_0)^2 (2 \csc h^2 \psi_t + 1) dt + \frac{x_2^3}{6} \\
&= 4\alpha^{-\frac{3}{2}} \left\{ (1 - 2 \csc h^2 \psi_0) (\cot h \psi_t - \cot h \psi_0) - \frac{2}{3} (\cot h^3 \psi_t - \cot h^3 \psi_0) \right. \\
&\quad \left. - \operatorname{sgn} c \sqrt{\alpha} \csc h^2 \psi_0 t + 2 \csc h \psi_0 (\cot h \psi_t \csc h \psi_t - \cot h \psi_0 \csc h \psi_0) \right\} + \frac{x_2^3}{6}, \quad (4.110)
\end{aligned}$$

where

$$\psi_t = \sqrt{\alpha} \varphi, \quad \psi_0 = \sqrt{\alpha} \varphi_0.$$

If $\lambda \in C_6^+$, we have

$$\theta = c \equiv 0.$$

The expression is the same as the case of $\lambda \in C_2^-$.

Remark 4.1 System (4.11)–(4.17) has the symmetry

$$(\theta, c, \alpha, x_1, x_2, y, z, t) \mapsto \left(\theta, \frac{c}{\sqrt{|\alpha|}}, \pm 1, \sqrt{|\alpha|} x_1, \sqrt{|\alpha|} x_2, |\alpha| y, |\alpha|^{\frac{3}{2}} z, \sqrt{|\alpha|} t \right), \quad (4.111)$$

which transforms the variable φ and k as follows:

$$(\varphi, k, \alpha) \mapsto (\sqrt{|\alpha|} \varphi, k, \pm 1).$$

So we can also get space-like geodesics of the general case $\alpha \neq 0$ from the formulas for the special case $\alpha = \pm 1$.

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