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The Schwarzian Derivative of Harmonic Mappings in the Plane*

Liping NIE[1] Zongxin YANG[1]

Abstract In this paper, the authors introduce a definition of the Schwarzian derivative of any locally univalent harmonic mapping defined on a simply connected domain in the complex plane. Using the new definition, the authors prove that any harmonic mapping f which maps the unit disk onto a convex domain has Schwarzian norm $\|S_f\| \le 6$. Furthermore, any locally univalent harmonic mapping f which maps the unit disk onto an arbitrary regular n-gon has Schwarzian norm $\|S_f\| \le \frac{8}{3}$.

Keywords Schwarzian derivative, Schwarzian norm, Harmonic mapping
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1 Introduction

The Schwarzian derivative of an analytic function was first introduced by Schwarz, when he sought to generalize the Schwarz-Christoffel formula to conformal mappings of polygons bounded by circular arcs. More recently, Nehari [10], Chuaqui-Duren-Osgood [1–3] and Osgood-Stowe [11] have developed many important and interesting results in terms of the Schwarzian derivative and Schwarzian norm of analytic functions.

The classical Schwarzian derivative of a locally univalent analytic function f is defined by

$$\mathbb{S}_f = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2. \tag{1.1}$$

The key property is its invariance under postcomposition with Möbius transformations. If $T(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$ is a Möbius transformation, then $\mathbb{S}_{T\circ f} = \mathbb{S}_f$. This is special case of the transformation formula

$$\mathbb{S}_{g\circ f} = (\mathbb{S}_g \circ f)(f')^2 + \mathbb{S}_f, \tag{1.2}$$

since $\mathbb{S}_T = 0$ for Möbius transformation T. Also note that if T is Möbius transformation, then

$$\mathbb{S}_{f\circ T} = (\mathbb{S}_f \circ T)(T')^2.$$

Manuscript received November 8, 2016. Revised September 3, 2017.
[1]Institute of Mathematics and Informatics, Jiangxi Normal University, Nanchang 330022, China.
E-mail: NLP@163.com zxyang@jxnu.edu.cn
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Furthermore, let f be an analytic function in the unit disk $U = \{z \mid |z| < 1\}$, then the Schwarzian norm of f is defined by the formula

$$\|\mathbb{S}_f\| = \sup_{z\in U}\{(1-|z|^2)^2|\mathbb{S}_f(z)|\}. \tag{1.3}$$

A definition of the Schwarzian derivative for a more general class of complex-valued harmonic functions was presented by Chuaqui, Duren and Osgood in [1]. The formula was derived by passing the minimal surface associated locally with a given harmonic function and appealing to a definition given in [11] for the Schwarzian derivative of a conformal mapping between arbitrary Riemannian manifolds. Specially, recall that a complex-valued harmonic function in a simply connected domain has a canonical representation $f = h + \overline{g}$ (where h and g are analytic functions) that is unique to an additive constant. Any harmonic mapping $f = h + \overline{g}$ with $|h'|+|g'| \neq 0$ lifts to a mapping $\widehat{f}$ onto a minimal surface defined by conformal parameters if and only if the dilatation $\omega = \frac{g'}{h'}$ equals the square of an analytic function q. In other words, $\omega = q^2$ for some analytic function q. For such mapping f, the Schwarzian derivative presented in [1] is defined by the formula

$$Sf = 2\{(\log\lambda)_{zz} - ((\log\lambda)_z)^2\}, \tag{1.4}$$

where $\lambda = |h'| + |g'|$ and

$$(\log\lambda)_z = \frac{\partial(\log\lambda)}{\partial z} = \frac{1}{2}\Big(\frac{\partial(\log\lambda)}{\partial x} - \mathrm{i}\frac{\partial(\log\lambda)}{\partial y}\Big).$$

In terms of the canonical representation $f = h + \overline{g}$ and the dilatation $\omega = q^2$,

$$Sf = \mathbb{S}_h + \frac{2\overline{q}}{1+|q|^2}\Big(q'' - q'\frac{h''}{h'}\Big) - 4\Big(\frac{q'\overline{q}}{1+|q|^2}\Big)^2, \tag{1.5}$$

where $\mathbb{S}_h$ is the classical Schwarzian derivative (1.1) of the analytic function h.

Later, Hernández and Martín gave another definition of Schwarzian derivative for the sense-preserving harmonic mappings in [7] by changing the conformal metric $\lambda = |h'| + |g'|$ with $\lambda = \sqrt{J_f}$, where $J_f = |f_z|^2 - |f_{\overline{z}}|^2$ is the Jacobian determinant of f. Then, by the formula

$$S(f) = 2\{(\log\lambda)_{zz} - ((\log\lambda)_z)^2\},$$

and $\omega = \frac{g'}{h'}$, they obtained

$$S(f) = \mathbb{S}_h + \frac{\overline{\omega}}{1-|\omega|^2}\Big(\omega'\frac{h''}{h'} - \omega''\Big) - \frac{3}{2}\Big(\frac{\omega'\overline{\omega}}{1-|\omega|^2}\Big)^2. \tag{1.6}$$

Now, in this present paper, we will introduce another new definition of the Schwarzian derivative of harmonic mapping f by using the conformal metric $\lambda = \sqrt{|h'|^2 + |g'|^2}$. Then, similarly, by the formula

$$S_f = 2\{(\log\lambda)_{zz} - ((\log\lambda)_z)^2\}$$

and $\omega = \frac{g'}{h'}$, we get

$$S_f = \mathbb{S}_h + \frac{\overline{\omega}}{1+|\omega|^2}\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}\Big(\frac{\omega'\overline{\omega}}{1+|\omega|^2}\Big)^2, \tag{1.7}$$

which is the new Schwarzian derivative formula we will use throughout this paper.

In terms of these three kinds of definitions of Schwarzian derivative of harmonic mappings, we can easily find that they have one thing in common, that is, if f is an analytic function, then $\lambda = |f'|$, so that

$$\log\lambda = \frac{1}{2}(\log f' + \log \overline{f}'),$$

which derives $(\log\lambda)_z = \frac{1}{2}\frac{f''}{f'}$. Therefore, all these three kinds of definitions of Schwarzian derivative of harmonic mappings become the form

$$\begin{aligned} S_f &= 2\{(\log\lambda)_{zz} - ((\log\lambda)_z)^2\} \\ &= \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2, \end{aligned}$$

which is in agreement with the classical Schwarzian derivative of analytic functions, the formula (1.1). Also, it should be emphasized that, in our new definition of Schwarzian derivative of harmonic mappings, the conformal metric $\lambda = \sqrt{|h'|^2 + |g'|^2}$ has Gaussian curvature $K \le 0$, so that the underlying harmonic mapping might be lifted to a minimal surface.

Indeed, by the general formula for Gaussian curvature

$$K = -\lambda^{-2}\Delta(\log\lambda),$$

where $\lambda = \sqrt{|h'|^2 + |g'|^2}$ and Δ denotes the Laplacian. Recall that

$$\Delta = 4\frac{\partial^2}{\partial\overline{z}\partial z},$$

a straightforward calculation shows that

$$K = -\frac{4|\omega'|^2}{|h'|^2(1+|\omega|^2)^3} \le 0.$$

In the next section, we will discuss some properties of the new Schwarzian derivative of locally univalent harmonic mappings in a simply connected domain. In the last part, we mainly discuss some properties of the new Schwarzian norm and estimate the Schwarzian norm of the convex harmonic mapping f defined on the unit disk, and we obtain $\|S_f\| \le 6$. We also consider some special convex harmonic mappings defined on the unit disk, such as, the convex harmonic mapping f which maps the unit disk onto the region inside a regular n-gon inscribed in the unit circle, and we obtain that $\|S_f\| \le \frac{8}{3}$ for such convex harmonic mapping f. Furthermore, in terms of Theorem 3.5, we have also get that any convex mapping f, which maps the unit disk onto a regular n-gon has Schwarzian norm $\|S_f\| \le \frac{8}{3}$.

2 Some Properties of the New Schwarzian Derivative

In this section, we are going to discuss some properties of the new Schwarzian derivative (1.7) for locally univalent harmonic mappings in a simply connected domain. And , there is no loss of generality, we assume that the harmonic mappings in this section are sense-preserving.

2.1 The chain rule

Let f be a sense-preserving harmonic mapping in a simply connected domain $\Omega \subset \mathbb{C}$. It is well known that if φ is a locally univalent analytic function for which the composition $f \circ \varphi$ is defined, then the function $f \circ \varphi$ is again a sense-preserving harmonic mapping. A straightforward calculation shows that

$$S_{f\circ\varphi} = (S_f \circ \varphi)(\varphi')^2 + S_\varphi, \tag{2.1}$$

which is a direct generalization of the chain rule (1.2) for the Schwarzian derivative of analytic functions. And it's not difficult to verify that $S_{T\circ f} = S_f$ for any Möbius transformation T.

Furthermore, from this formula, it is not difficult to find that if A is an affine mapping of the form $A(z) = az + b\overline{z} + c$ with $a, b, c \in \mathbb{C}$, the composition $A \circ f$ is harmonic (and sense-preserving if $a \neq 0$ and $\left|\frac{b}{a}\right| < 1$). In this case, the Schwarzian derivative (1.7) of $A \circ f$ may be not equal to that of f. But we can give two sufficient conditions for $S_{A\circ f} = S_f$.

First, if $A(z) = az + b\overline{z} + c$ with $a, b, c \in \mathbb{C}$, and $f = h + \overline{g}$ with dilatation $\omega = \frac{g'}{h'}$, then

$$A \circ f = (ah + bg) + \overline{(\overline{a}g + \overline{b}h)},$$

and let $F = A \circ f = H + \overline{G}$, where $H = ah + bg$ and $G = \overline{a}g + \overline{b}h$. Thus,

$$H' = (a + b\omega)h', \quad G' = (\overline{b} + \overline{a}\omega)h'.$$

In terms of formula

$$S_F = 2(\log \lambda_F)_{zz} - ((\log \lambda_F)_z)^2,$$

where $\lambda_F = \sqrt{|H'|^2 + |G'|^2} = \sqrt{|h'|^2(|a + b\omega|^2 + |\overline{b} + \overline{a}\omega|^2)}$.

A straightforward calculation shows

$$\begin{aligned} S_F = \mathbb{S}_h &+ \frac{(|a|^2 + |b|^2)\omega''\overline{\omega}}{|a + b\omega|^2 + |\overline{b} + \overline{a}\omega|^2} - \frac{h''}{h'}\Big(\frac{2\overline{a}b + (|a|^2 + |b|^2)\omega'\overline{\omega}}{|a + b\omega|^2 + |\overline{b} + \overline{a}\omega|^2}\Big) \\ &- \frac{3}{2}\Big(\frac{2\overline{a}b + (|a|^2 + |b|^2)\omega'\overline{\omega}}{|a + b\omega|^2 + |\overline{b} + \overline{a}\omega|^2}\Big)^2. \end{aligned}$$

From this formula, it is not difficult to find $S_F = S_f$ if $b = 0$. Indeed, if $b = 0$, then $A(z) = az + c$ is a conformal affine mapping.

Another sufficient condition is that the conformal metrics $\lambda_f = \sqrt{|h'|^2 + |g'|^2}$ and $\lambda_F = \sqrt{|H'|^2 + |G'|^2}$ are homothetic. That is, $\lambda_F = c\lambda_f$ for some constant $c > 0$.

Indeed,

$$S_F = 2(\log \lambda_F)_{zz} - ((\log \lambda_F)_z)^2$$

and

$$S_f = 2(\log \lambda_f)_{zz} - ((\log \lambda_f)_z)^2,$$

then, if $S_F = S_f$, a short calculation shows

$$\Big(\log \frac{\lambda_F}{\lambda_f}\Big)_{zz} + (\log \lambda_F \lambda_f)_z \Big(\log \frac{\lambda_f}{\lambda_F}\Big)_z = 0.$$

It is not difficult to see that this equality holds if $\lambda_F = c\lambda_f$ for some constant $c > 0$.

The next lemma will be very useful in the proof of one upper bound of the Schwarzian norm of convex harmonic mappings defined on the unit disk, which is one of the main results of the next section.

Lemma 2.1 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the simply connected domain Ω with dilatation ω, where h and g are analytic in Ω. Then, for all $z_0 \in \Omega$,*

$$S_f(z_0) = S_{h+\overline{\omega(z_0)}g}(z_0). \tag{2.2}$$

The proof of this lemma will be given in the next section.

2.2 Analytic Schwarzian derivative

First, recall that if a sense-preserving harmonic mapping f in the simply connected domain Ω has constant dilatation, then $f = h + \alpha\overline{h} + \gamma$ for some complex-constant α and γ (with $|\alpha| < 1$) and some locally univalent function h that is analytic in Ω.

Next, we will use the new Schwarzian derivative to characterize harmonic mappings with analytic Schwarzian derivative.

Theorem 2.1 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the simply connected domain Ω, then, S_f is analytic if and only if f has constant dilatation.*

Proof If $f = h + \alpha\overline{h} + \gamma$ for some locally univalent analytic mapping h, some $|\alpha| < 1$ and some $\gamma \in \mathbb{C}$, then $S_f \equiv \mathbb{S}_h$ in Ω and the result follows.

Suppose that a sense-preserving harmonic mapping $f = h + \overline{g}$ with dilatation ω has analytic Schwarzian derivative S_f defined by

$$S_f = \mathbb{S}_h + \frac{\overline{\omega}}{1+|\omega|^2}\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}\Big(\frac{\omega'\overline{\omega}}{1+|\omega|^2}\Big)^2.$$

If $\omega \equiv a, a \in U$, then $f = h + \alpha\overline{h} + \gamma$ for some locally univalent analytic function h and we are done. In order to get a contradiction, we assume that ω is not constant in Ω. After multiplying the last equation by $(1 + |\omega|^2)^2$, we obtain

$$(\mathbb{S}_h - S_f)(1 + |\omega|^2)^2 + \overline{\omega}(1 + |\omega|^2)\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}(\omega'\overline{\omega})^2 = 0. \tag{2.3}$$

Rewriting (2.3) in terms of 1, $\overline{\omega}$ and $\overline{\omega}^2$, and denoting $u = \mathbb{S}_h - S_f$, we have

$$u + \overline{\omega}\Big(2u\omega + \omega'' - \omega'\frac{h''}{h'}\Big) + \overline{\omega}^2\Big(u\omega^2 + \omega\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}(\omega')^2\Big) = 0. \tag{2.4}$$

Differentiate with respect to $\overline{z}$ in (2.4) to get

$$\overline{\omega'}\Big(2u\omega + \omega'' - \omega'\frac{h''}{h'}\Big) + 2\overline{\omega\ \omega'}\Big(u\omega^2 + \omega\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}(\omega')^2\Big) = 0. \tag{2.5}$$

Now, since ω is not constant, there exists a disk $D(z_0, R)$ with center z_0 and radius $R > 0$ contained in Ω, where $\omega' \neq 0$. We can divide both sides of (2.5) by $\overline{\omega'}$, and then take derivative with respected to $\overline{z}$ again to obtain

$$\overline{\omega'}\Big(u\omega^2 + \omega\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}(\omega')^2\Big) = 0 \quad \text{in } D(z_0, R).$$

Keeping in mind that $\omega' \neq 0$ in $D(z_0, R)$, we see that

$$u\omega^2 + \omega\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}(\omega')^2 \equiv 0. \tag{2.6}$$

Hence, from (2.5) we have

$$2u\omega + \omega'' - \omega'\frac{h''}{h'} = 0, \tag{2.7}$$

which implies, by (2.4), that $u = \mathbb{S}_h - S_f \equiv 0$ in $D(z_0, R) \subset \Omega$. But in this case, using (2.7), we obtain

$$\omega'' - \omega'\frac{h''}{h'} = 0, \tag{2.8}$$

and substituting in (2.6), we get $\omega' \equiv 0$ in $D(z_0, R)$. This is a contradiction. Hence, $\omega \equiv \alpha \in U$ and $f = h + \alpha\overline{h} + \gamma$, as we wanted to prove.

Recall that a locally univalent analytic function f in the simply connected domain Ω is a Möbius transformation if and only if $\mathbb{S}_f \equiv 0$ in Ω. A locally univalent harmonic Möbius transformation is a harmonic mapping of the form $f = \alpha T + \beta\overline{T} + \gamma$, where T is a classical Möbius transformation were characterized in [1] in terms of the Schwarzian derivative (1.1). The following corollary also characterizes the harmonic Möbius transformations in terms of S_f.

Corollary 2.1 *A locally univalent mapping $f = h + \overline{g}$ equals a harmonic Möbius transformation if and only if $S_f \equiv 0$, where h and g are analytic functions.*

Proof Since $S_{\overline{f}} = S_f$, we can assume that f is sense-preserving. If f is a Möbius transformation, then $S_f = 0$. Conversely, if $S_f = 0$, by Theorem 2.1, $f = h + \alpha\overline{h} + \gamma$ for some constants $\alpha \in U$ and $\gamma \in \mathbb{C}$, and for some locally univalent analytic function h with $\mathbb{S}_h = 0$. Hence, h is a Möbius transformation.

2.3 Harmonic Schwarzian derivative

In this part, we will discuss the harmonic Schwarzian derivative.

The classical Schwarzian derivative (1.1) of a locally univalent analytic function φ is also analytic. In the case of harmonic mappings $f = h + \overline{g}$, we have an analogous result, we state this as a theorem.

Theorem 2.2 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the simply connected domain Ω. If S_f is harmonic, then S_f is analytic.*

Proof By a straightforward calculation,

$$\frac{\partial^2 S_f}{\partial z\partial\overline{z}} = \varphi_1'\frac{\overline{\omega'}}{(1+|\omega|^2)^2} + 9\frac{(\omega')^3(\overline{\omega})^2\overline{\omega'}}{(1+|\omega|^2)^4} - \varphi_2\frac{2\overline{\omega\omega'}}{(1+|\omega|^2)^3}, \tag{2.9}$$

where

$$\varphi_1 = \omega'' - \omega'\frac{h''}{h'}, \quad \varphi_2 = \varphi_1\omega' + 3\omega'\omega''.$$

Assume that $\Delta S_f \equiv 0$ in Ω. Hence, according to formula

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

the right-hand side of (2.9) is identically zero in Ω. After multiplying (2.9) by $(1+|\omega|^2)^4$, we get

$$0 \equiv \varphi_1' \overline{\omega'}(1+|\omega|^2)^2 + 9(\omega')^3(\overline{\omega})^2\overline{\omega'} - 2\varphi_2\overline{\omega}\overline{\omega'}(1+|\omega|^2). \tag{2.10}$$

If the dilatation ω of f is constant, then the Schwarzian derivative of f is analytic. If ω is not constant, there exists a disk $D(z_0, R) \subset \Omega$, where $\omega' \neq 0$. Dividing (2.10) by $\overline{\omega'}$, we obtain

$$0 \equiv \varphi_1'(1+|\omega|^2)^2 + 9(\omega')^3(\overline{\omega})^2 - 2\varphi_2\overline{\omega}(1+|\omega|^2),$$

which is equivalent to

$$0 \equiv \varphi_1' + (2\varphi_1'\omega - 2\varphi_2)\overline{\omega} + (\varphi_1'\omega^2 + 9(\omega')^3 - 2\varphi_2\omega)(\overline{\omega})^2. \tag{2.11}$$

Taking derivative with respect to $\overline{z}$ in (2.11) and dividing by $\overline{\omega'}$, we have

$$0 \equiv (2\varphi_1'\omega - 2\varphi_2) + 2\overline{\omega}(\varphi_1'\omega^2 + 9(\omega')^3 - 2\varphi_2\omega) \quad \text{in } D(z_0, R).$$

If $\varphi_1'\omega^2 + 9(\omega')^3 - 2\varphi_2\omega \neq 0$, then $\overline{\omega}$ must be constant (in this case ω would be both analytic and anti-analytic). This is a contradiction since we are assuming that $\omega' \neq 0$ in $D(z_0, R)$. Hence, $\varphi_1'\omega^2 + 9(\omega')^3 - 2\varphi_2\omega \equiv 0$ (in some disk $D \subset D(z_0, R)$). This fact implies that $\varphi_1'\omega - \varphi_2 = 0$ in D. Therefore, using (2.11), we obtain that $\varphi_2 = \varphi_1' = 0$ as well and we conclude that ω is constant in D. This proves the theorem.

3 The Schwarzian Norm of Harmonic Mappings Defined on the Unit Disk

Let f be a harmonic mapping in the unit disk U, using the new definition for Schwarzian derivative (1.7) of harmonic mappings, we define the Schwarzian norm of f by the formula

$$\|S_f\| = \sup_{z \in U}\{(1-|z|^2)^2|S_f(z)|\}. \tag{3.1}$$

3.1 Properties of Schwarzian norm

First, it should be observed that Schwarzian norm is Möbius invariant. In other words, $\|S_{f\circ T}\| = \|S_f\|$ if T is any Möbius self-mapping of the disk.

In this part, we mainly discuss the new Schwarzian norm (3.1) of convex harmonic mappings in the unit disk.

First, recall that Kraus [8] proved that if f is analytic and univalent in the unit disk U, then $\|\mathbb{S}_f\| \leq 6$, where the Schwarzian norm $\|\mathbb{S}_f\|$ is defined by the formula (1.3). Another related result due to Nehari [9] states that if f is convex that is, f is univalent in the unit disk, and the domain $f(U)$ is convex, then $\|\mathbb{S}_f\| \leq 2$. Both constants 2 and 6 are sharp.

Indeed, the analytic Koebe mapping

$$K(z) = \frac{z}{(1-z)^2}, \quad z \in U$$

is univalent in the unit disk. It maps U onto the full plane minus the part of the negative real axis from $-\frac{1}{4}$ to infinity and has Schwarzian derivative

$$\mathbb{S}_K(z) = -\frac{6}{(1-z^2)^2}.$$

Hence, $\|\mathbb{S}_K\| = 6$. Note that $|\mathbb{S}_f(x)|(1-x^2)^2 = 6$ for all real numbers $-1 < x < 1$.

Furthermore, the analytic and univalent strip mapping L, defined in the unit disk by $L(z) = \frac{1}{2}\log\frac{1+z}{1-z}$, is convex. It has Schwarzian derivative

$$\mathbb{S}_L(z) = \frac{2}{(1-z^2)^2}.$$

The same constant 2 also appears in the Nehari criterion for univalence. This criterion states that if the Schwarzian norm of a locally univalent analytic function f in the unit disk U is bounded by 2, then f is univalent in U.

In terms of the Schwarzian derivative defined by (1.5), namely, it was proved in [2] that there exists a constant C_1 such that

$$\|Sf\| = \sup_{z\in U}\{(1-|z|^2)^2|Sf(z)|\} \le C_1,$$

for any univalent sense-preserving harmonic mapping $f = h + \overline{g}$ in the unit disk with dilatation $\omega = \frac{g'}{h'} = q^2$ for some analytic function q (with $|q| < 1$) in U. The sharp value of C_1 is unknown.

There is an analogous result in our new Schwarzian norm (3.1), we state this as a theorem.

Theorem 3.1 *Let $f = h + \overline{g}$ be a univalent harmonic mapping in the unit disk, then there exists a constant C_2 such that*

$$\|S_f\| \le C_2.$$

The sharp value of C_2 is unknown.

Proof Since $S_{A\circ f} = S_f$ for any conformal mapping A, we may assume that $f \in S_H$, where S_H denotes the class of all sense-preserving harmonic mappings of the unit disk with $h(0) = g(0) = 1 - h'(0) = 0$. Notice that f is univalent and ϕ is an automorphism of the disk with $\phi(0) = z$, then $f\circ\phi$ is also univalent and $|S_{f\circ\phi}(0)| = (1-|z|^2)^2|S_f(z)|$. Hence, it is not difficult to verify that

$$\sup_{f\in S_H}\|S_f\| = \sup_{f\in S_H}|S_f(0)|.$$

Now, fix an arbitrary function $f \in S_H$. According to [13, Theorem 9], $f_R(z) = f(Rz)$ is a univalent convex harmonic mapping. Therefore, $|S_{f_R}(0)| \le 6$ by Theorem 3.4, which will be introduced in the next section. Using the chain rule, we see that

$$R^2|S_f(0)| = |S_{f_R}(0)| \le 6.$$

Therefore

$$\sup_{f\in S_H}|S_f(0)|\le\frac{6}{R^2}.$$

This proves the theorem.

If we just consider the locally univalent harmonic mappings in the unit disk, then we have the result below.

Theorem 3.2 *Let $f=h+\overline{g}$ be a locally univalent sense-preserving harmonic mapping in the unit disk U. Then, $\|S_f\|\le\infty$ if and only if $\|\mathbb{S}_h\|\le\infty$.*

Proof First suppose that $\|\mathbb{S}_h\|\le\infty$. By hypothesis, f has analytic dilatation $\omega=\frac{g'}{h'}$, and $|\omega(z)|<1$ in the unit disk. Hence

$$\frac{|\overline{\omega}(z)|}{1+|\omega(z)|^2}\le\frac{1}{2},\quad z\in U,$$

and so it follows from (1.7) that

$$|S_f(z)|\le|\mathbb{S}_h(z)|+\frac{1}{2}|\omega''(z)|+\frac{1}{2}|\omega'(z)|\Big|\frac{h''(z)}{h'(z)}\Big|+\frac{3}{8}|\omega'(z)|^2.$$

By the Schwarz-Pick lemma,

$$|\omega'(z)|\le\frac{|\omega'(z)|}{1-|\omega(z)|^2}\le\frac{1}{1-|z|^2}.$$

If a function φ is analytic in the unit disk and $|\varphi(z)|\le\frac{1}{1-|z|^2}$, then it follows from Cauchy's integral formula that $|\varphi'(z)|\le\frac{4}{(1-|z|^2)^2}$. We apply this to the function $\varphi=\omega'$ to see that

$$|\omega''(z)|\le\frac{4}{(1-|z|^2)^2}.$$

Finally, a result of Pommerenke [12, p. 133] asserts that

$$(1-|z|^2)\Big|\frac{h''(z)}{h'(z)}\Big|\le2+2\Big(1+\frac{1}{2}\|\mathbb{S}_h\|\Big)^{\frac{1}{2}}.\tag{3.2}$$

Putting the estimates together, we conclude that

$$\|S_f\|\le\|\mathbb{S}_h\|+\Big(1+\frac{1}{2}\|\mathbb{S}_h\|\Big)^{\frac{1}{2}}+\frac{27}{8}<\infty.$$

Conversely, suppose $\|S_f\|\le\infty$, the formula (1.7) and preceding estimates show that

$$|\mathbb{S}_h(z)|\le|S_f(z)|+\frac{19}{8(1-|z|^2)^2}+\frac{1}{2(1-|z|^2)}\Big|\frac{h''(z)}{h'(z)}\Big|.\tag{3.3}$$

In order to use Pommerenke's estimate of $\frac{h''}{h'}$, we apply (3.3) to the dilated function $f_r=h_r+\overline{g_r}$, where $0<r<1$ and $f_r(z)=f(rz)$. Note that $S_{f_r}(z)=r^2S_f(rz)$, so that

$$(1-|z|^2)^2|S_{f_r}(z)|\le(1-|rz|^2)^2|S_f(rz)|\le\|S_f\|.$$

Because $\|\mathbb{S}_{h_r}\|$ is finite for each $r<1$, we can apply (3.3) to f_r and invoke (3.2) to infer that

$$(1-|z|^2)^2|\mathbb{S}_{h_r}(z)| \leq \|S_f(z)\| + \Big(1+\frac{1}{2}\|\mathbb{S}_{h_r}\|\Big)^{\frac{1}{2}} + \frac{27}{8},$$

or

$$\|\mathbb{S}_{h_r}\| - \Big(1+\frac{1}{2}\|\mathbb{S}_{h_r}\|\Big)^{\frac{1}{2}} \leq \|S_f\| + \frac{27}{8}.$$

Now, let $r\to 1$ to conclude that $\|\mathbb{S}_h\| \leq \infty$.

Indeed, in terms of this proof and inserting the estimate in [3] that $\|\mathbb{S}_h\| < 19,204$, we obtain $\|S_f\| < 19,306$ for any univalent harmonic mapping f in the unit disk.

Finally, we observe that univalent harmonic mappings with range convex in one direction have finite Schwarzian norm $\|S_f\| \leq 38.375$. These mappings are obtained by a known process of shearing conformal mappings whose range is convex in one direction [6, Section 3.4].

Theorem 3.3 *Suppose a function φ is analytic and univalent in the unit disk, and its range is convex in the horizontal direction. Let $f = h+\overline{g}$ be the the harmonic shear of φ in the horizontal direction with dilatation $\omega = \frac{g'}{h'}$, where ω in analytic and $|\omega(z)|<1$ in the unit disk. Then, $\|S_f\| \leq 38.375$.*

Proof From the process of shear construction, we have $h-g=\varphi$ and $h' = \frac{\varphi'}{1-\omega}$. A straightforward calculation yields the formula

$$\begin{aligned} S_f = \mathbb{S}_\varphi + \frac{\overline{\omega}}{1+|\omega|^2}\Big(\omega'' - \omega'\frac{\varphi''}{\varphi'} - \frac{(\omega')^2}{1-\omega}\Big) - \frac{3}{2}\Big(\frac{\omega'\overline{\omega}}{1+|\omega|^2}\Big)^2 \\ - \frac{\omega'}{1-\omega}\frac{\varphi''}{\varphi'} + \frac{(\omega')^2+2\omega''(1-\omega)}{2(1-\omega)^2}. \end{aligned}$$

The preceding estimates for ω and its derivatives in the proof of Theorem 3.2 can now be applied to derive the inequality

$$\begin{aligned} |S_f(z)| \leq |\mathbb{S}_\varphi(z)| + \frac{19}{8(1-|z|^2)^2} + \frac{1}{2(1-|z|^2)}\Big|\frac{\varphi''(z)}{\varphi'(z)}\Big| + \frac{1}{2(1-|z|^2)}\Big|\frac{\omega'(z)}{1-\omega(z)}\Big| \\ + \Big|\frac{\omega'(z)}{1-\omega(z)}\Big|\Big|\frac{\varphi''(z)}{\varphi'(z)}\Big| + \frac{1}{2}\Big|\frac{\omega'(z)}{1-\omega(z)}\Big|^2 + \Big|\frac{\omega''(z)}{1-\omega(z)}\Big|. \end{aligned}$$

Since

$$\Big|\frac{\omega'(z)}{1-\omega(z)}\Big| \leq \frac{|\omega'(z)|}{1-|\omega(z)|} \leq \frac{2|\omega'(z)|}{1-|\omega(z)|^2} < \frac{2}{1-|z|^2},$$

and let $\psi = \frac{\omega'}{2(1-\omega)}$, by Cauchy's integral formula that $|\psi'(z)| \leq \frac{4}{(1-|z|^2)^2}$, we have

$$\Big|\frac{\omega''(z)}{1-\omega(z)} + \Big(\frac{\omega'(z)}{1-\omega(z)}\Big)^2\Big| \leq \frac{8}{(1-|z|^2)^2},$$

which implies that

$$\Big|\frac{\omega''(z)}{1-\omega(z)}\Big| \leq \frac{12}{(1-|z|^2)^2}.$$

Furthermore, since φ is analytic univalent function, a standard inequality (see [5, p. 32]) shows that

$$\Big|\frac{\varphi''(z)}{\varphi'(z)}\Big| \leq \frac{6}{1-|z|^2},$$

and Kraus' theorem gives $(1-|z|^2)^2|\mathbb{S}_\varphi(z)| \le 6$, combining these estimates, we find that

$$\|S_f\| \le 6 + \frac{19}{8} + 3 + 1 + 12 + 2 + 12 = 38.375.$$

This proves the theorem.

The harmonic shear of the Koebe function $K(z) = z(1-z)^{-2}$with dilatation $\omega(z) = z$ is $f = h + \overline{g}$, where

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \quad g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}$$

under the assumption that $h(0) = g(0) = 0$. A straightforward calculation yields that

$$h'(z) = \frac{1+z}{(1-z)^4},$$

so that

$$S_f(z) = \frac{-19 - 10z - 3z^2}{2(1-z^2)^2} - \frac{\overline{z}(5+3z)}{(1+|z|^2)(1-z^2)} - \frac{3}{2}\Big(\frac{\overline{z}}{1+|z|^2}\Big)^2,$$

from which an easy calculation gives $\|S_f\| = 25.5$. These results are unchanged if the Koebe function $K(z) = z(1-z)^{-2}$ is sheared with dilatation $\omega(z) = \mathrm{e}^{\mathrm{i}\theta} z$ for any $\theta \in [0, 2\pi)$. Therefore, since the Koebe function maximizes the Schwarzian norm for analytic univalent functions, it is reasonable to conjecture that $\|S_f\| \le 25.5$ for all univalent harmonic mappings in the unit disk.

3.2 Schwarzian norm of convex harmonic mappings

Recall that Hernández and Martín have proved in [7, Proposition 2] that if $f = h + \overline{g}$ is a convex harmonic mapping in the unit disk U, then $\|S(f)\| \le 6$, where the Schwarzian norm $\|S(f)\|$ is defined by the formula

$$\|S(f)\| = \sup_{z \in U}\{(1-|z|^2)^2|S(f)(z)|\},$$

and $S(f)$ is defined by (1.6). Moreover, the sharp value of 6 is unknown.

Now, by the new Schwarzian derivative (1.7) and Schwarzian norm (3.1), we get the same result that if $f = h + \overline{g}$ is a harmonic mapping in the unit disk U, then $\|S_f\| \le 6$. We state this as a theorem.

Theorem 3.4 *Let f be a convex harmonic mapping in the unit disk, then*

$$\|S_f\| \le 6.$$

Note that, the sharp value of 6 is also unknown.

We say that f is a convex harmonic mapping in the domain Ω if f is a univalent harmonic mapping in Ω, and $f(\Omega)$ is convex.

The next lemma will play an important role in the proof of Theorem 3.4.

Lemma 3.1 *If f is a convex harmonic mapping in the unit disk U, then for each $\varepsilon \in \overline{U}$, the analytic function $\nu_\varepsilon = h + \varepsilon g$ is close-to-convex in U, thus univalent in $\overline{U}$.*

This result was proved by Clunie and Sheil-Small [4, Theorem 5.7]. Using this lemma, we can get a simple proof of Theorem 3.4.

Proof of Theorem 3.4 Since f is convex harmonic mapping, in terms of Lemma 3.1, there exist analytic harmonic functions $\nu_\varepsilon = h + \varepsilon g$ that are univalent for all $\varepsilon \in \overline{U}$. By the formula (1.1) for analytic function in the introduction, we obtain

$$\mathbb{S}_{\nu_\varepsilon} = \Big(\frac{\nu_\varepsilon''}{\nu_\varepsilon'}\Big)' - \frac{1}{2}\Big(\frac{\nu_\varepsilon''}{\nu_\varepsilon'}\Big)^2,$$

and in terms of $\omega = \frac{g'}{h'}$, a simple calculation gives

$$\mathbb{S}_{\nu_\varepsilon} = \mathbb{S}_h + \frac{\varepsilon}{1+\varepsilon\omega}\Big(\omega'' - \omega'\frac{h''}{h'}\Big) - \frac{3}{2}\Big(\frac{\varepsilon\omega'}{1+\varepsilon\omega}\Big)^2, \tag{3.4}$$

where $\mathbb{S}_h$ is defined by formula (1.1).

In particular, for an arbitrary point $z_0 \in U$, let $\varepsilon = \overline{\omega(z_0)}$, where ω is the dilatation of f, a simple calculation, we get

$$\mathbb{S}_{\nu_\varepsilon}(z_0) = \mathbb{S}_h(z_0) + \frac{\overline{\omega(z_0)}}{1+|\omega(z_0)|^2}\Big(\omega''(z_0) - \omega'(z_0)\frac{h''(z_0)}{h'(z_0)}\Big) - \frac{3}{2}\Big(\frac{\overline{\omega(z_0)}\omega'(z_0)}{1+|\omega(z_0)|^2}\Big)^2, \tag{3.5}$$

compare this formula (3.5) with (1.7), we can easily obtain

$$\mathbb{S}_{\nu_\varepsilon}(z_0) = S_f(z_0). \tag{3.6}$$

In other words,

$$\mathbb{S}_{h+\overline{\omega(z_0)}g}(z_0) = S_f(z_0). \tag{3.7}$$

Note that, by (3.7), we have also given the proof of Lemma 2.1 in this process.

Furthermore, as shown in Lemma 3.1, the analytic function ν_ε is univalent in $\overline{U}$. Therefore, in terms of Kraus' theorem (see [8]), which claims that any function f is analytic and univalent in the unit disk has $\|\mathbb{S}_f\| \le 6$. So that

$$(1-|z_0|^2)^2|\mathbb{S}_{\nu_\varepsilon}(z_0)| \le 6$$

for z_0 is arbitrary in U. Thus, for all $z \in U$, there is

$$(1-|z|^2)^2|\mathbb{S}_{\nu_\varepsilon}(z)| \le 6,$$

and by (3.6), we get

$$(1-|z|^2)^2|S_f(z)| \le 6.$$

Thus

$$\|S_f\| \le 6. \tag{3.8}$$

This proves the theorem. We do not know if the constant 6 in this theorem is sharp.

Now, we can see that the definitions (1.6)–(1.7) have the same upper bound 6 of the Schwarzian norm for convex harmonic mappings, which is in agreement with the upper bound of classical definition of Schwarzian derivative (1.1) for analytic functions. Indeed, if f is analytic, then both (1.6)–(1.7) is equivalent to (1.1), of course, have the same sharp upper bound of Schwarzian norm of mapping f.

In the next, we will discuss the Schwarzian norm of special convex harmonic mapping f, which maps the unit disk U onto the region inside a regular n-gon inscribed in the unit circle with positive integer $n \geq 3$. Moreover, by a simple proof, we also get that for any convex harmonic mapping f, which maps the unit disk onto an regular n-gon has Schwarzian norm $\|S_f\| \leq \frac{8}{3}$.

Theorem 3.5 *Let $f = h+\overline{g}$ be harmonic mapping in the unit disk U, and $f(U)$ is a regular n-gon (where integer $n \geq 3$) inscribed in the unit circle, then*

$$\|S_f\| \leq \frac{8}{3}.$$

Proof In [6], Duren has given the explicit formula of a canonical harmonic mapping $f = h + \overline{g}$, which maps the unit disk onto the domain inside a regular n-gon (where integer $n \geq 3$) with vertices at the n-th roots of unity. The formula is

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{k=0}^{n-1} \alpha^k \Big\{ \log \frac{z-\beta^{2k+1}}{z-\beta^{2k-1}} - \overline{\log \frac{z-\beta^{2k+1}}{z-\beta^{2k-1}}} \Big\}, \tag{3.9}$$

where $\alpha = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$ $(n \geq 3)$ is the nth roots of unity, and $\beta = \sqrt{\alpha} = \mathrm{e}^{\frac{\pi \mathrm{i}}{n}}$.

After a simple calculation, we obtain

$$h'(z) = \frac{1-\alpha}{2\pi \mathrm{i}} \frac{n\beta^{n-1}}{1+z^n}, \quad g'(z) = \frac{1-\overline{\alpha}}{2\pi \mathrm{i}} \frac{n\beta z^{n-2}}{1+z^n}. \tag{3.10}$$

So the dilatation of f has the form $\omega(z) = z^{n-2}$.

Now, let us calculate the Schwarzian derivative and Schwarzian norm of f.

First, as the new Schwarzian derivative formula (1.7) shows

$$S_f = \mathbb{S}_h + \frac{\overline{\omega}}{1+|\omega|^2}\Big(\omega'' - \omega' \frac{h''}{h'}\Big) - \frac{3}{2}\Big(\frac{\omega' \overline{\omega}}{1+|\omega|^2}\Big)^2, \tag{3.11}$$

where

$$\mathbb{S}_h = \Big(\frac{h''}{h'}\Big)' - \frac{1}{2}\Big(\frac{h''}{h'}\Big)^2.$$

By (3.10) and $\omega(z) = z^{n-2}$, a calculation then yields

$$\mathbb{S}_h = \frac{-2n(n-1)z^{n-2} + (2n-n^2)z^{2n-2}}{2(1+z^n)^2},$$

and

$$\frac{\overline{\omega}}{1+|\omega|^2}\Big(\omega'' - \omega' \frac{h''}{h'}\Big) = \frac{(n^2-5n+6)|z|^{2n-8}\overline{z}^2 + (2n^2-7n+6)|z|^{2n-4}z^{n-2}}{(1+|z|^{2n-4})(1+z^n)},$$

and

$$\frac{3}{2}\Big(\frac{\omega' \overline{\omega}}{1+|\omega|^2}\Big)^2 = \frac{3(n-2)^2|z|^{4n-12}\overline{z}^2}{2(1+|z|^{2n-4})^2}.$$

Thus

$$\begin{aligned}S_f(z)=&\frac{-2n(n-1)z^{n-2}+(2n-n^2)z^{2n-2}}{2(1+z^n)^2}\\&+\frac{(n^2-5n+6)|z|^{2n-8}\overline{z}^2+(2n^2-7n+6)|z|^{2n-4}z^{n-2}}{(1+|z|^{2n-4})(1+z^n)}\\&-\frac{3(n-2)^2|z|^{4n-12}\overline{z}^2}{2(1+|z|^{2n-4})^2}.\end{aligned}$$

Next, let $F(z)=(1-|z|^2)^2|S_f(z)|$, then

$$\begin{aligned}F(z)=&\frac{(1-|z|^2)^2}{2(1+|z|^{2n-4})^2(1+z^n)^2}|(2n-2n^2)z^{n-2}+(2n-2n^2)|z|^{4n-8}z^{n-2}+(2n-n^2)z^{2n-2}\\&+(2n^2-20n+24)|z|^{2n-4}z^{n-2}+(2n^2-10n+12)|z|^{2n-4}z^{2n-2}\\&+2(n^2-5n+6)|z|^{2n-8}\overline{z}^2+(2n-n^2)|z|^{4n-12}\overline{z}^2|.\end{aligned}$$

Note that the zeros of the function $\psi(z)=1+z^n$ may be the discontinuous points of function $F(z)$. Indeed, a simple calculation yields

$$z_k=\mathrm{e}^{(\frac{\pi}{n}+\frac{2k\pi}{n})\mathrm{i}},\quad k=0,1,2,\cdots,n-1$$

is the nth roots of function $\psi(z)=1+z^n$, and for any $k=0,1,2,\cdots,n-1$, a simple proof shows

$$\prod_{j\neq k}|z_k-z_j|=n,$$

where $j=0,1,2,\cdots,n-1$. We can just consider the case $k=0$ for the other cases it keeps the same result under circle rotation.

It is not difficult to see

$$\prod_{j=1}^{n-1}|z_0-z_j|=\prod_{i=1}^{n-1}|1-\alpha^i|,$$

where $\alpha=\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$, and function

$$f_1(x)=(x-\alpha)(x-\alpha^2)(x-\alpha^3)\cdots(x-\alpha^{n-1}),$$

then

$$\prod_{i=1}^{n-1}|1-\alpha^i|=|f_1(1)|.$$

Furthermore, polynomial

$$g_1(x)=x^{n-1}+x^{n-2}+\cdots+x+1$$

has the same roots with $f_1(x)$, so

$$|f_1(1)|=|g_1(1)|=n.$$

Thus

$$\prod_{j=1}^{n-1}|z_0-z_j|=n$$

and

$$\prod_{j\neq k}|z_k - z_j| = n,$$

where $k, j = 0, 1, 2, \cdots, n-1$. So that, let z approach to z_k by the Stolz angle, we obtain

$$\lim_{z\to z_k}\frac{1-|z|^2}{|1+z^n|} = \lim_{z\to z_k}\frac{1-|z|}{z-z_k}\frac{1+|z|}{\prod\limits_{j\neq k}|z_k-z_j|} = \frac{2}{n}. \tag{3.12}$$

Therefore, $z_k (k = 0, 1, 2, \cdots, n-1)$ are not the discontinuous points of $F(z)$. In other words, $F(z)$ is continuous and bounded in the unit disk. Then

$$\begin{aligned}\lim_{z\to z_k} F(z) &= \lim_{z\to z_k}\frac{1}{2n^2}|(24-2n^2-16n)z^{n-2} + (12-8n+n^2)z^{2n-2} + (n^2-8n+12)\overline{z}^2| \\ &\leq \frac{n^2+8n-12+|n^2-8n+12|}{n^2}, \quad n \geq 3. \end{aligned} \tag{3.13}$$

It is not difficult to verify that if $n \geq 6$, then

$$n^2 - 8n + 12 \geq 0.$$

Thus

$$\lim_{z\to z_k} F(z) \leq \frac{2n^2}{n^2} = 2,$$

and if $3 \leq n \leq 5$, then

$$n^2 - 8n + 12 \leq 0.$$

Therefore

$$\lim_{z\to z_k} F(z) \leq \frac{16n-24}{n^2},$$

and it is easy to see that function $\varphi(n) = \frac{16n-24}{n^2}$ is strictly decreasing for integer $n \geq 3$. Thus, if $3 \leq n \leq 5$, we have

$$\lim_{z\to z_k} F(z) \leq \frac{8}{3}.$$

Finally, in terms of the maximum principle of analytic functions, we obtain that for any integer $n \geq 3$, it follows that

$$\|S_f\| = \sup_{z\in U}\{(1-|z|^2)^2|S_f(z)|\} \leq \frac{8}{3}. \tag{3.14}$$

This proves the theorem.

Next, for any harmonic f which maps the unit disk onto a regular n-gon ($n \geq 3$), we also have the same result about the Schwarzian norm $\|S_f\|$.

Theorem 3.6 *Let $f = h + \overline{g}$ be harmonic mapping in the unit disk U, and $f(U)$ is a regular n-gon $(n \geq 3)$, then $\|S_f\| \leq \frac{8}{3}$.*

Proof Since the Schwarzian derivative of a locally univalent harmonic mapping f is p-reserved by composition with translation, there is no loss of generality, we can assume that

$f(0) = 0$. Then, by a pre-composition with a conformal affine mapping A, which maps $f(U)$ to a regular n-gon inscribed in the unit circle. Since $S_{A\circ f} = S_f$, we have $\|S_f\| = \|S_{A\circ f}\|$, and by Theorem 3.5, it is not difficult to find that $\|S_{A\circ f}\| \leq \frac{8}{3}$. Therefore, $\|S_f\| \leq \frac{8}{3}$. This proves the theorem.

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