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Heat Kernel on Analytic Subvariety

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Abstract In this paper, the author extends Peter Li and Tian Gang's results on the heat kernel from projective varieties to analytic varieties. The author gets an upper bound of the heat kernel on analytic varieties and proves several properties. Moreover, the results are extended to vector bundles. The author also gets an upper bound of the heat operators of some Schröndinger type operators on vector bundles. As a corollary, an upper bound of the trace of the heat operators is obtained.

Keywords Heat kernel, Analytic subvariety, Schröndinger operator
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1 Introduction

Heat kernel is an important tool in geometric analysis. Many papers have been devoted to explore this subject. For instance, in [2], Cheeger and Yau developed the general theory of heat kernel on manifolds and proved a lower bound for the heat kernel under the condition when the Ricci curvature of the manifold is bounded from below. In [3], Cheng, Li and Yau derived an upper bound for the heat kernels of minimal submanifolds in $\mathcal{R}^{n+l}$, $\mathcal{H}^{n+l}$, and $\mathcal{S}^{n+l}$. Using the argument in [3], Li and Tian [9] proved that the heat kernel on projective varieties is square integrable and got an upper bound estimate.

In this paper, we extend the result of Li and Tian to the case of analytic subvariety and Schrödinger type operator of its vector bundles. Let N be a compact Kähler manifold of dimension n, and $M \subset N$ be an analytic subvariety of pure dimension m. We assume that the Riemanninan sectional curvature of N is bounded from above by b. For any $y \in N$, distance function of N to a fixed point $x \in M$ is denoted by $r_x(y)$. Our first main result is the following theorem.

Theorem 1.1 *The heat kernel $H(x,y,t)$ exists on M. H has an upper bound*

$$H(x,y,t) \le G(x,y,t). \tag{1.1}$$

Here $G(x,y,t) = \rho(r_x(y))\widehat{H}(r_x(y),t) + Bt$, $\widehat{H}(r_x(y),t)$ is the heat kernel of the $2m$-dimensional space form V_b^{2m} of constant curvature b, $\rho(x,y)$ is a C^2 function on M and B is a constant depending on the injectivity radius σ on N, the sectional curvature of N and the dimension of M. ρ and B will be constructed in Section 2. Moreover $H(x,y,t)$ belongs to $W^{1,2}(M)$, and

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satisfies the semi-group property and the conservation property:

$$\int_M H(z,y,t)H(y,x,s)\mathrm{d}y = H(z,x,t+s), \tag{1.2}$$

$$\int_M H(x,y,t)\mathrm{d}y = 1. \tag{1.3}$$

Now let $E \to M$ be a vector bundle endowed with a metric h. Let D be a metric connection on E, and the formal adjoint of D is denoted by D^t. Let L be a positive and symmetric bundle endomorphism on E. Assume that L is bounded in $L^2(M,E)$ and there exists a positive number l such that $(Ls_1,s_2)_h \geq l(s_1,s_2)_h$ for any $s_1,s_2 \in L^2(M,E)$. Consider the Schrödinger type operator $P = D^t \circ D + L$. We can get our second result.

Theorem 1.2 *Denote the L^2 closure of P by $\overline{P}$. Then $\overline{P}$ is a self-adjoint operator and we have the following domination of semi-groups*:

$$|\mathrm{e}^{-t\overline{P}}s|_h \leq \mathrm{e}^{-tl}|\mathrm{e}^{t\overline{\triangle}}s|_h \tag{1.4}$$

for any $s \in L^2(M,E)$.

As a corollary, we get an estimates of the trace class of the Schröndinger type operators P on E.

Corollary 1.1 *The heat operator $\mathrm{e}^{-t\overline{P}} : L^2(M,E) \to L^2(M,E)$ is a trace class operator and its trace class* Tr *satisfies the following inequality*:

$$\mathrm{Tr}(\mathrm{e}^{-t\overline{P}}) \leq r\mathrm{e}^{-tl}\mathrm{Tr}(\mathrm{e}^{t\triangle}), \tag{1.5}$$

where r is the rank of the vector bundle E. If H_p denotes the kernel function of $\mathrm{e}^{-t\overline{P}}$, then the following inequalities holds:

$$H_p(x,x,t) \leq r\mathrm{e}^{-tl}(\widehat{H}(0,t)+Bt), \tag{1.6}$$

$$\mathrm{Tr}(\mathrm{e}^{-t\overline{P}}) \leq r\mathrm{e}^{-tl}(\mathrm{vol}(M))(\widehat{H}(0,t)+Bt). \tag{1.7}$$

The paper is organised as follows. In Section 2, we prove an upper bound of $H(x,y,t)$ for the case when M is a submanifold of a compact Kähler manifold. It is well konwn that submanifolds of Kähler manifold are Kähler and minimal. So in order to get the upper bound, we first prove a Laplace comparison theorem for minimal submanifolds. After proving this, we then use the standard Duhamel's principle to derive the comparison theorem for the heat kernel $H(x,y,t)$.

In Section 3, we consider the case when M is an analytic subvariety. We use the result in Section 2 and method in [9] to prove Theorem 1.1. The ideal is as follows: Let M_{sing} denote the singular part of M, and let T_ε be the ε-tubular neighborhood of M_{sing}. The Dirichlet heat kernel on $M_\varepsilon = M\backslash T_\varepsilon$ is denoted by $H_\varepsilon(x,y,t)$. By the minimal property of Dirichlet heat kernel, H_ε is monotone increasing. On the other hand the comparsion theorem for the heat kernel in Section 2 implies that H_ε has a uniform upper bound and hence must converge to a function H. By the local gradient estimate of Li-Yau, it is the heat kernel on M. We will prove that H_ε converges in the $W^{1,2}(M)$-norm to H, and $H \in W^{1,2}(M)$. Moreover, H satisfies the semigroup property and conservation property. Hence H is the only heat kernel on M.

In Section 4, we consider the Schröndinger operator P on a vector bundle E. As mentioned above, P is of the form $P = D^t \circ D + L$. First we will prove the self-adjointness of $\overline{P}$. Then we will derive a Kato's type inequality for P. Using this inequality, we next prove that the heat kernel $\mathrm{e}^{-t\overline{P}}$ of $\overline{P}$ is bounded from above by the heat kernel $H(x, y, t)$ multiplying a constant. As a corollary, we get an upper bound of the trace class of $\mathrm{e}^{-t\overline{P}}$.

2 Comparison Theorem for Kähler Submanifold

Let N be an n-dimensional compact Riemannian manifold whose sectional curvature is not greater than b. Let $M \subset N$ be an m-dimensional minimal submanifold of N. Denote the distance function of N to a fixed point $x \in M$ by $r_x(y)$. Let σ be the injectivity radius of N, and $B(x, r)$ be a geodesic ball at x of radius $r \le \sigma$. Let V_b^m be the m-dimensional space form of constant curvature b, and $\widehat{B}(r)$ be a geodesic ball with radius r of V_b^m.

Suppose that $\widehat{H}$ is the heat kernel with Dirichlet boundary condition on $\widehat{B}(r)$. Then we know that $\widehat{H}$ is only a function of distance function r and time t. For a fixed point $x \in M$ and any $y \in M$, we can regard $\widehat{H}$ as a function of y by $\widehat{H}(x, y, t) = \widehat{H}(r_x(y), t)$. $\widehat{H}$ is smooth on $M \setminus C$, where C denotes the cut-locus of N. Define

$$s(r) = \begin{cases} 1 - \cos(\sqrt{b}r), & b > 0, \\ \dfrac{r^2}{2}, & b = 0, \\ \cosh(\sqrt{-b}r) - 1, & b < 0, \end{cases} \tag{2.1}$$

and rewrite $\widehat{H}{=}\widehat{H}(s(r), t)$. Then we have

$$\triangle_M \widehat{H}(x, y, t) = \triangle_M \widehat{H}(s(r_x(y)), t) = \widehat{H}_s \triangle_M s + \widehat{H}_{ss} |\nabla_M s|^2. \tag{2.2}$$

We denote the Laplace operator on V_b^m by $\widehat{\triangle}$.

Proposition 2.1 *On $B(x, r) \cap M$, we have $\triangle_M s {\ge} \widehat{\triangle} s$ with $r \le \sigma$.*

Proof Let $\{e_i\}$ $(1 \le i \le m)$ be a local orthonormal basis of TM, and $\left\{\widehat{e}_j,\ \frac{\partial}{\partial r}\right\}$ $(1 \le j \le m-1)$ be an orthnormal basis of TV_b^m. Note that M is a minimal submanifold of N. Then we have

$$\begin{aligned} \triangle_M s &= \sum_{i=1}^m (e_i e_i(s) - \nabla^M_{e_i} e_i(s)) \\ &= \sum_{i=1}^m (e_i e_i(s) - \nabla^N_{e_i} e_i(s)) \\ &= \frac{\partial^2 s}{\partial r^2} \sum_{i=1}^m (e_i(r))^2 + \frac{\partial s}{\partial r} \sum_{i=1}^m (e_i e_i(r) - \nabla^N_{e_i} e_i(r)) \\ &= \frac{\partial^2 s}{\partial r^2} \sum_{i=1}^m (e_i(r))^2 + \frac{\partial s}{\partial r} \sum_{i=1}^m H(r)(e_i, e_i). \end{aligned}$$

Here $H(r)$ means the Hessian bilinear form. Let $e_i' = \frac{e_i - e_i(r)\frac{\partial}{\partial r}}{\sqrt{1 - e_i(r)^2}}$. Then e_i' is orthnormal to $\frac{\partial}{\partial r}$ with norm 1, and it is easy to check that $H(r)(e_i',\ e_i') = \frac{1}{1 - (e_i(r))^2} H(e_i, e_i)$.

Using the Hessian comparison theorem for N and V_b^n, and because V_b^m can be regarded as a totally geodesic submanifold of $V_b^n(m<n)$, we have $H(r)(e_i',e_i')\geq \widehat{H}(r)(\widehat{e}_j,\widehat{e}_j)$, where $\widehat{H}(r)$ is the Hessian form on V_b^m.

So

$$\begin{aligned}
&\frac{\partial^2 s}{\partial r^2}\sum_{i=1}^m (e_i(r))^2+\frac{\partial s}{\partial r}\sum_{i=1}^m H(r)(e_i,e_i)\\
&=\frac{\partial^2 s}{\partial r^2}\sum_{i=1}^m (e_i(r))^2+\frac{\partial s}{\partial r}\sum_{i=1}^m (1-(e_i(r))^2)H(r)(e_i',e_i')\\
&\geq\frac{\partial^2 s}{\partial r^2}\sum_{i=1}^m (e_i(r))^2+\frac{\partial s}{\partial r}\sum_{i=1}^m (1-(e_i(r))^2)\Big(\frac{1}{m-1}\sum_{j=1}^{m-1}\widehat{H}(r)(\widehat{e}_j,\widehat{e}_j)\Big)\\
&=\frac{\partial^2 s}{\partial r^2}\sum_{i=1}^m (e_i(r))^2+\frac{\partial s}{\partial r}\sum_{i=1}^m (1-(e_i(r))^2)h\\
&=\Big(\frac{\partial^2 s}{\partial r^2}-\frac{\partial s}{\partial r}h\Big)\sum_{i=1}^m (e_i(r))^2+mh\frac{\partial s}{\partial r}\\
&=\widehat{\triangle}s,
\end{aligned}$$

where h denotes the mean curvature of geodesic sphere of radius r of V_b^m. The last equality follows from a direct calculation. For example, when $b>0$, $h=\sqrt{b}\cot(\sqrt{b}r)$, so $\frac{\partial^2 s}{\partial r^2}-\frac{\partial s}{\partial r}h=0$, and $\widehat{\triangle}s=\frac{\partial^2 s}{\partial r^2}+(m-1)h\frac{\partial s}{\partial r}=mh\frac{\partial s}{\partial r}$.

This theorem first appears in [11] without proof.

Here we give a complete proof.

According to [2], we have $\frac{\partial \widehat{H}}{\partial r}\leq 0$, so $\widehat{H}_s=\frac{\partial \widehat{H}}{\partial r}\frac{\partial r}{\partial s}\leq 0$, and by [3], we have $\widehat{H}_{ss}\geq 0$. Moreover $|\nabla_M s|^2\leq\big(\frac{\partial s}{\partial r}\big)^2$ for $|\nabla_M r|\leq 1$. So in $B(x,r)\cap M$ with $r\leq\sigma$ we have

$$\triangle_M\widehat{H}(x,y,t)=\widehat{H}_s\triangle_M s+\widehat{H}_{ss}|\nabla_M s|^2\leq\widehat{H}_s\triangle_{\widehat{M}}s+\widehat{H}_s\Big(\frac{\partial s}{\partial r}\Big)^2=\widehat{\triangle}\widehat{H}. \tag{2.3}$$

Now choose γ to be a C^2 function on $\mathbb{R}$ such that $\gamma(t)=1$ for $t\leq\frac{1}{2}$ and $\gamma(t)=0$ for $t\geq 1$. For fixed x and any $y\in M$, we define a C^2 function ρ on M by $\rho(r)=\rho(r_x(y))=\gamma\big(\frac{s(r)}{2}\big)$ if $b\geq 0$ and $\sigma\geq\frac{\pi}{\sqrt{b}}$, and $\rho(r)=\gamma\big(\frac{s(r)}{s(\sigma)}\big)$ otherwise.

Then ρ is a C^2 function on M satisfying

$$-C_1(\sigma)\leq\frac{\partial\rho}{\partial r}\leq 0,\quad \Big|\frac{\partial^2\rho}{\partial r^2}\Big|\leq C_2(\sigma). \tag{2.4}$$

Here $C_1(\rho)$ and $C_2(\rho)$ are constants depending on ρ. Set $G(x,y,t)=\rho(r_x(y))\widehat{H}(r_x(y),t)+tB$, where B is a constant to be determined.

Proposition 2.2 *There exists a constant B depending only on the injectivity radius σ on N, the sectional curvature of N and the dimension of M, such that*

$$\Big(\triangle-\frac{\partial}{\partial t}\Big)G(x,y,t)\leq 0 \tag{2.5}$$

on M.

Proof Since

$$\begin{aligned}\Big(\triangle-\frac{\partial}{\partial t}\Big)(\widehat{H}\rho+tB)&=\rho\triangle\widehat{H}+2(\nabla\widehat{H},\nabla\rho)+\widehat{H}\triangle\rho-\rho\frac{\partial}{\partial t}\widehat{H}-B\\&=\rho\Big(\triangle\widehat{H}-\frac{\partial}{\partial t}\Big)\widehat{H}+2(\nabla\widehat{H},\nabla\rho)+\widehat{H}\triangle\rho-B\\&\le 2(\nabla\widehat{H},\nabla\rho)+\widehat{H}\triangle\rho-B.\end{aligned}$$

In order to get (2.5), we must choose a positive constant B such that

$$B\ge 2(\nabla\widehat{H},\nabla\rho)+\widehat{H}\triangle\rho. \tag{2.6}$$

But

$$2(\nabla\widehat{H},\nabla\rho)+\widehat{H}\triangle\rho=2\rho'\frac{\partial s}{\partial r}\widehat{H}_r(\nabla r,\nabla r)+\widehat{H}\rho''\Big(\frac{\partial s}{\partial r}\Big)^2|\nabla r|^2+\widehat{H}\rho'\triangle s. \tag{2.7}$$

Now since $\rho'<0$, $\triangle s\ge\widehat{\triangle}s>0$, the third term is negative. Furthermore $\widehat{H}$ and $\widehat{H}_r$ are bounded on $[c,\sigma]\times[0,\infty)$ for some constant $c>0$, and by (2.4) we see that the first and second terms are bounded from above. So there exists a constant B which depends on the injective radius σ on N, the sectional curvature of N and the dimension of $M.B$ also satisfies (2.6).

Now we are ready to prove comparison theorem for the heat kernel of complex submanifold for minimal submanifold.

Theorem 2.1 *Let N be a compact Riemannian manifold with sectional curvature not greater than b, and $M\subset N$ be a minimal submanifold. Suppose that H is the Dirichlet heat kernel of M. For all $x,y\in M$ and $t\in[0,\infty)$, we have*

$$H(x,y,t)\le G(x,y,t). \tag{2.8}$$

Proof We have already proved that $\big(\triangle-\frac{\partial}{\partial t}\big)G(x,y,t)\le 0$. Using Duhamel's principle, we have

$$\begin{aligned}&G(x,y,t)-H(x,y,t)\\&=\int_0^t\frac{\partial}{\partial s}\Big(\int_M G(x,z,s)H(z,y,t-s)\mathrm{d}z\Big)\mathrm{d}s\\&=\int_0^t\int_M\frac{\partial G}{\partial s}(x,z,s)H(z,y,t-s)\mathrm{d}z\mathrm{d}s-\int_0^t\int_M G(x,z,s)\frac{\partial H}{\partial(t-s)}(z,y,t-s)\mathrm{d}z\mathrm{d}s\\&\ge\int_0^t\int_M\triangle G(x,z,s)H(z,y,t-s)\mathrm{d}z\mathrm{d}s-\int_0^t\int_M G(x,z,s)\triangle H(z,y,t-s)\mathrm{d}z\mathrm{d}s\\&=\int_0^t\int_{\partial M}\frac{\partial G}{\partial v_z}(x,z,s)H(z,y,t-s)\mathrm{d}z\mathrm{d}s-\int_0^t\int_{\partial M}G(x,z,s)\frac{\partial H}{\partial v_z}(z,y,t-s)\mathrm{d}z\mathrm{d}s\end{aligned}$$

for $x,\ y\in M$ and $t>0$. Since H satisfies the Dirichlet boundary condition, i.e., $H=0$ on ∂M, the first term of the right is zero. On the other hand H is positive on the interior of M, so $\frac{\partial H}{\partial v_z}(z,y,t-s)<0$. Because G is positive, we then conclude that $H(x,y,t)\le G(x,y,t)$.

Now if N is a compact Kähler manifold of complex dimesion n and M is a complex submanifold of N, then it is well known that M is Kähler and minimal. So we can derive directly from Theorem 2.1 the comparison theorem of the heat kernel for the Kähler case.

Corollary 2.1 *Let N be a compact Kähler manifold with Riemannian curvature not greater than b and M be a Kähler submanifold of N. Then the heat kernel $H(x,y,t)$ on M has an upper bound*

$$H(x,y,t) \le G(x,y,t). \tag{2.9}$$

3 Heat Kernel for Analytic Subvariety

In this section, let $M \subset N$ be an analytic subvariety of pure dimension m, where N is a compact Kähler manifold. Locally M is given by zero set of holomorphic functions of N. We denote the smooth part of M by $M_{\rm reg}$, and singular part of M by $M_{\rm sing}$. Then $M_{\rm reg}$ is an incomplete open Riemannian manifold, and the real codimension of $M_{\rm sing}$ is at least 2. The ε-neighborhood of the $M_{\rm sing}$ is defined by $T_\varepsilon = \{x \in M \mid {\rm dist}(x, M_{\rm sing}) \le \varepsilon\}$. Let $M_\varepsilon = M \setminus T_\varepsilon$ and let H_ε be the Dirichlet heat kernel on M_ε. Since Dirichlet heat kernel is minimal, when $\varepsilon' \le \varepsilon$, we have $H_{\varepsilon'}(x,y,t) \ge H_\varepsilon(x,y,t)$ for all $x,y \in M_\varepsilon$ and for all $t > 0$. By Corollary 2.1, we have $H_\varepsilon(x,y,t) \le G(x,y,t)$ for all ε. So on any compact subset of M, H_ε is monotonically increasing and bounded from above as ε tends to 0, and hence H_ε must converges to a function H. By the local gradient estimate of Li-Yau [10], H satisfies the heat equation on $M_{\rm reg}$.

Lemma 3.1 *H_ε converges in $W^{1,2}(M)$-norm to H on M, in particular $H \in W^{1,2}(M)$.*

Proof Firstly since $H(x,y,t) \le G(x,y,t)$, it is clear that $H(x,y,t) \in L^2(M)$. For any compact subset $K \subset M$, choose ε sufficiently small such that $K \subset M_{2\varepsilon}$. Take $0 < \varepsilon' < \varepsilon$ and ϕ a non negative truncation function supported on M_ε and $\phi = 1$ on $M_{2\varepsilon}$.

Using the local gradient estimate for $H_{\varepsilon'} - H_\varepsilon$ by [10] on M_ε, we have

$$|\nabla(H_{\varepsilon'} - H_\varepsilon)|^2 - \alpha(H_{\varepsilon'} - H_\varepsilon)\Big(\frac{\partial}{\partial t}(H_{\varepsilon'} - H_\varepsilon)\Big) \le C(H_{\varepsilon'} - H_\varepsilon)^2, \tag{3.1}$$

where $\alpha > 1$ is a constant. Then

$$\begin{aligned}\int_M \phi^2|\nabla(H_{\varepsilon'} - H_\varepsilon)|^2 &\le \alpha \int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)\Big(\frac{\partial}{\partial t}(H_{\varepsilon'} - H_\varepsilon)\Big) + C\int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)^2 \\ &\le \alpha \int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)(\triangle(H_{\varepsilon'} - H_\varepsilon)) + C\int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)^2 \\ &\le -\alpha \int_M \phi^2|\nabla(H_{\varepsilon'} - H_\varepsilon)|^2 - 2\int_M \phi(H_{\varepsilon'} - H_\varepsilon)(\nabla\phi, \nabla(H_{\varepsilon'} - H_\varepsilon)) \\ &\quad + C\int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)^2.\end{aligned}$$

Applying the Schwarz inequality

$$-2\int_M \phi(H_{\varepsilon'} - H_\varepsilon)(\nabla\phi, \nabla(H_{\varepsilon'} - H_\varepsilon)) \le \int_M \phi^2|\nabla(H_{\varepsilon'} - H_\varepsilon)|^2 + \int_M |\nabla\phi|^2(H_{\varepsilon'} - H_\varepsilon)^2,$$

we have

$$\int_M \phi^2|\nabla(H_{\varepsilon'} - H_\varepsilon)|^2 \le C\int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)^2 + \alpha\int_M |\nabla\phi|^2(H_{\varepsilon'} - H_\varepsilon)^2. \tag{3.2}$$

Thus

$$\int_K |\nabla(H_{\varepsilon'} - H_\varepsilon)|^2 \le C\int_M \phi^2(H_{\varepsilon'} - H_\varepsilon)^2 + \alpha\int_{M_{\varepsilon,2\varepsilon}} |\nabla\phi|^2(H_{\varepsilon'} - H_\varepsilon)^2. \tag{3.3}$$

Here $M_{\varepsilon,2\varepsilon}$ denotes $M_\varepsilon \setminus M_{2\varepsilon}$. If we choose the truncation function ϕ such that $|\nabla\phi| \le \frac{5}{\varepsilon}$, we have

$$\int_{M_{\varepsilon,2\varepsilon}} |\nabla\phi|^2 \le C_1\varepsilon^{-2}V(\varepsilon, 2\varepsilon), \tag{3.4}$$

where $V(\varepsilon, 2\varepsilon)$ is the volume of $M_{\varepsilon,2\varepsilon}$. The fact that the singular part of M is of at least real codimension 2 implies that

$$\int_{M_{\varepsilon,2\varepsilon}} |\nabla\phi|^2 \le C_1\varepsilon^{-2}V(\varepsilon, 2\varepsilon) \le C_2. \tag{3.5}$$

So the second term of right-hand side of (3.3) tends to 0 as $\varepsilon \to 0$, since H_ε uniformly converges to H. The first term is obviously tends 0, hence H_ε converges in the $W^{1,2}(M)$-norm to H on any compact subset of M.

If we using the local gradient estimate for H_ε, we have

$$\int_M \phi^2|\nabla H_\varepsilon|^2 \le \alpha\int_M \phi^2 H_\varepsilon\frac{\partial H_\varepsilon}{\partial t} + C\int_M \phi^2 H_\varepsilon^2 \le C\int_M \phi^2 H_\varepsilon^2 + \alpha\int_M |\nabla\phi|^2 H_\varepsilon^2. \tag{3.6}$$

By (3.5), we have an estimate of the second term

$$\int_M |\nabla\phi|^2 H_\varepsilon^2 = \int_{M_{\varepsilon,2\varepsilon}} |\nabla\phi|^2 H_\varepsilon^2 \le C(t). \tag{3.7}$$

The first term in (3.6) is clearly bounded, thus we have

$$\int_K |\nabla H_\varepsilon|^2 \le C(t).$$

Then H is uniformly bounded on any compact subset of M, hence $H \in W^{1,2}(M)$.

Lemma 3.2 *On M, H satisfies the semi-groups property.*

Proof Let $\varepsilon \ge \varepsilon'$,

$$\begin{aligned}\int_{M_\varepsilon} H_{\varepsilon'}(z,y,t)H_{\varepsilon'}(x,y,s)\mathrm{d}y &\le \int_{M_{\varepsilon'}} H_{\varepsilon'}(z,y,t)H_{\varepsilon'}(x,y,s)\mathrm{d}y \\ &= H_{\varepsilon'}(z,x,t+s) \le H(z,x,t+s).\end{aligned}$$

Let $\varepsilon' \to 0$, we have

$$\int_{M_\varepsilon} H(z,y,t)H(x,y,s)\mathrm{d}y \le H(z,x,t+s).$$

Let $\varepsilon \to 0$, we conclude that

$$\int_M H(z,y,t)H(x,y,s)\mathrm{d}y \le H(z,x,t+s).$$

On the other hand,

$$\begin{aligned}\int_M H(z,y,t)H(x,y,s)\mathrm{d}y &\ge \int_{M_{\varepsilon'}} H(z,y,t)H(x,y,s)\mathrm{d}y \\ &\ge \int_{M_{\varepsilon'}} H_{\varepsilon'}(z,y,t)H_{\varepsilon'}(x,y,s)\mathrm{d}y = H_{\varepsilon'}(z,x,t+s).\end{aligned}$$

Hence

$$\int_M H(z,y,t)H(x,y,s)\mathrm{d}y \geq H(z,x,t+s).$$

Therefore we have

$$\int_M H(z,y,t)H(x,y,s)\mathrm{d}y = H(z,x,t+s).$$

Proposition 3.1 *On M, H satisfies the conservation property*

$$\int_M H(x,y,t)\mathrm{d}y = 1. \tag{3.8}$$

Hence H is the only heat kernel on M satisfying this property.

Proof By Green formula we have

$$\frac{\partial}{\partial t}\int_M \phi H = \int_M \phi\triangle H = -\int_M (\nabla\phi, \nabla H),$$

where ϕ is the truncation function defined above. By the Schwarz inequality, we see that for any $0 < t_1 < t_2$,

$$\begin{aligned} &\Big|\int_M \phi(y)H(x,y,t_2)\mathrm{d}y - \int_M \phi(y)H(x,y,t_1)\mathrm{d}y\Big| \\ \leq &\int_{t_1}^{t_2}\Big|\frac{\partial}{\partial t}\int_M \phi(y)H(x,y,t)\mathrm{d}y\Big|\mathrm{d}t \\ \leq &\int_{t_1}^{t_2}\Big(\int_{M_{\varepsilon,2\varepsilon}}|\nabla\phi|^2(y)\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\int_{M_{\varepsilon,2\varepsilon}}|\nabla H|^2(x,y,t)\mathrm{d}y\Big)^{\frac{1}{2}}\mathrm{d}t. \end{aligned}$$

The first term is uniformly bounded as argued in Lemma 3.1, and the second term tends to 0 for $H \in W^{1,2}(M)$, hence

$$\int_M H(x,y,t_2)\mathrm{d}y = \int_M H(x,y,t_1)\mathrm{d}y.$$

Now $\lim\limits_{t_1\to 0}\int_M H(x,y,t_1)\mathrm{d}y = 1$, so the conservation property holds.

If (3.8) holds, the Neumann heat kernel on M must coincide with the Dirichlet heat kernel H, for instance see [8] or [9].

4 The Heat Kernel Operator on Vector Bundles

Let $M \subset N$ be an analytical subvariety, where N is a compact Kähler manifold. Now let $E \to M$ be a smooth complex vector bundle with Hermitian metric h, D be its metric connection and D^t be its formal adjoint. Let L be a smooth vector bundle endomorphism of E, which we would assume to be symmetric and positive. Moreover, we assume that L is bounded operator on $L^2(M,E)$ and satisfies $(Ls_1, s_2)_h \geq l(s_1, s_2)_h$, where l is a constant and $x, y \in L^2(M,E)$. Consider the following Schrödinger type operator:

$$P := D^t \circ D + L. \tag{4.1}$$

Then P is a (formally) self-adjoint operator on $C_0^\infty(M, E)$. Denote $\overline{P}$ as its closure in $L^2(M, E)$.

We will first consider the issue of self-adjointness of $\overline{P}$. Now we define domain of D to be the set $\mathcal{D}(D)$, where

$$\mathcal{D}(D) := \{s \mid s \in C^1(M_{\mathrm{reg}}, E) \cap L^2(M_{\mathrm{reg}}, E),\ Ds \in L^2(M_{\mathrm{reg}}, E \otimes T^*M)\}. \tag{4.2}$$

Similarly, we define

$$\mathcal{D}(D^t) := \{\omega \mid \omega \in L^2(M_{\mathrm{reg}}, E \otimes T^*M) \cap C^1(M_{\mathrm{reg}}, E \otimes T^*M),\ D^t\omega \in L^2(M_{\mathrm{reg}}, E)\}. \tag{4.3}$$

Now by a theorem of Gaffney (see [4]), if D and D^t are adjoint to each other with respect to their domains, then the closure of $D^t \circ D$ is self-adjoint. Therefore, to prove that $\overline{P}$ is self-adjoint we only need to prove that D and D^t are adjoint in their domains.

Proposition 4.1 *The closure $\overline{P}$ in $L^2(M, E)$ is self-adjoint.*

Proof D and D^t are adjoint if and only if the following statement is true:

$$(Ds, \omega) = (s, D^t\omega) \tag{4.4}$$

for all $s \in \mathcal{D}(D)$ and $\omega \in \mathcal{D}(D^t)$. We claim that if any section s in $\mathcal{D}(D)$ can be approximated by sections in $W_c^{1,2}(M_\varepsilon) = \{s \in W^{1,2}(M_\varepsilon, E) \mid s = 0 \text{ on } \partial M_\varepsilon\}$, then (4.4) holds. To see this, let $s_i \in W_c^{1,2}(M_{\varepsilon_i})$ and $s_i \to s$ in $W^{1,2}(M, E)$. Then

$$(Ds, \omega) = \lim_{i\to\infty}(Ds_i, \omega) = \lim_{i\to\infty}(s_i, D^t\omega) = (s, D^t\omega), \tag{4.5}$$

where the second equality holds since s_i satisfies the Dirichlet boundary condition.

To prove this equality we would like to show that the bounded sections are dense in $W^{1,2}(M, E)$ first. As suggested in [1], for any $s \in W^{1,2}(M, E)$, define

$$s_n := \frac{s}{\Big(\frac{|s|_h^2}{n} + 1\Big)^{\frac{1}{2}}}.$$

Clearly s_n is bounded and $|s_n|_h \le n^{\frac{1}{2}}$. We have

$$Ds_n = -\frac{1}{n}\Big(\frac{|s|_h^2}{n} + 1\Big)^{-\frac{3}{2}}\mathrm{Re}(h(Ds, s)) \otimes s + \Big(\frac{|s|_h^2}{n} + 1\Big)^{-\frac{1}{2}} Ds.$$

Now using the fact that $s \in W^{1,2}(M, E)$, it is easy to see that $Ds \in L^2(M, T^*M \otimes E)$. So it remains to show that $s_n \to s$ in the norm of $W^{1,2}(M, E)$, as $n \to \infty$. For $\|s - s_n\|^2_{L^2(M,E)}$, we have

$$\|s - s_n\|^2_{L^2(M,E)} = \int_M \Big(1 - \Big(\frac{|s|_h^2}{n} + 1\Big)^{-\frac{1}{2}}\Big)^2 |s|_h^2.$$

By the Lebesgue dominate convergence theorem, we get $\lim\limits_{n\to\infty} \|s - s_n\|^2_{L^2(M,E)} = 0$. For $\|Ds - Ds_n\|^2_{L^2(M,T^*M\otimes E)}$, we have

$$\begin{aligned}\|Ds - Ds_n\|_{L^2(M,T^*M\otimes E)} \le{}& \Big\| -\frac{1}{n}\Big(\frac{|s|_h^2}{n} + 1\Big)^{-\frac{3}{2}}\mathrm{Re}(h(Ds, s))\Big\|_{L^2(M,T^*M\otimes E)} \\ &+ \Big\|Ds - \Big(\frac{|s|_h^2}{n} + 1\Big)^{-\frac{1}{2}} Ds\Big\|_{L^2(M,T^*M\otimes E)}.\end{aligned}$$

For the first term, we have

$$\begin{aligned}&\Big\| -\frac{1}{n}\Big(\frac{|s|_h^2}{n}+1\Big)^{-\frac{3}{2}}\mathrm{Re}(h(Ds,s))\otimes s\Big\|^2_{L^2(M,T^*M\otimes E)}\\ =&\int_M\Big|-\frac{1}{n}\Big(\frac{|s|_h^2}{n}+1\Big)^{-\frac{3}{2}}\Big|\int_M\Big|-\frac{1}{n}\Big(\frac{|s|_h^2}{n}+1\Big)^{-\frac{3}{2}}\Big||\mathrm{Re}(h(Ds,s))\otimes s|^2_{\widetilde{h}}\\ \le&\int_M\frac{1}{n^2}\Big(\frac{|s|_h^2}{n}+1\Big)^{-3}|Ds|^2_{\widetilde{h}}|s|^4_h.\end{aligned}$$

Here $\widetilde{h}$ denotes the metric on $T^*M\otimes E$. Since

$$\Big(\frac{|s|_h^2}{n}+1\Big)^{-3}|s|_h^4\le n^2,$$

this implies that

$$\frac{1}{n^2}|Ds|^2_{\widetilde{h}}|s|^4_h\Big(\frac{|s_h|^2}{n}+1\Big)^{-3}\le|Ds|^2_{\widetilde{h}}.$$

So we can use the Lebesegue dominate convergence theorem to conclude that

$$\Big\| -\frac{1}{n}\Big(\frac{|s|_h^2}{n}+1\Big)^{-\frac{3}{2}}\mathrm{Re}(h(Ds,s))\otimes s\Big\|_{L^2(M,T^*M\otimes E)}\to 0.$$

For the second term, we have

$$\Big\|Ds-\Big(\frac{|s|_h^2}{n}+1\Big)^{-\frac{1}{2}}Ds\Big\|^2_{L^2(M,T^*M\otimes E)}=\int_M\Big(1-\Big(\frac{|s|_h^2}{n}+1\Big)^{\frac{1}{2}}\Big)^2|Ds|^2_{\widetilde{h}},$$

and using Lebesgue dominate convergence theorem again, it tends to 0. So we have $\int_M|Ds_n-Ds|_{\widetilde{h}}\to 0$. So the bounded sections are dense in $W^{1,2}(M,E)$.

From now on, we can assume that $s\in W^{1,2}(M,E)$ is bounded. For any $\varepsilon>0$, take $s'=\frac{1}{2}\exp(-\varepsilon^{-2})<\varepsilon$. Define $s_\varepsilon=\phi_\varepsilon s\in W^{1,2}_c(M_\varepsilon)$, where ϕ_ε is a cut-off function supported in M_ε and $\phi_\varepsilon=1$ on M_ε. Then $s_\varepsilon\in W^{1,2}_0(M_\varepsilon,E)$ and $s_\varepsilon\to s$ in $L^{1,2}(M,E)$, as $\varepsilon\to 0$. Now consider

$$\int_M|D(s-s_\varepsilon)|^2_{\widetilde{h}}=\int_{M_{\varepsilon',\varepsilon}}|D(s-s_\varepsilon)|^2_{\widetilde{h}}+\int_{T_\varepsilon}|Ds|^2_{\widetilde{h}},$$

where $M_{\varepsilon',\varepsilon}=M_\varepsilon\backslash M_{\varepsilon'}$, and $T_{\varepsilon'}$ is the ε'-tubular neighborhood of the singular set. Clearly, $\int_{T_{\varepsilon'}}|Ds|^2_{\widetilde{h}}\to 0$ as $\varepsilon'\to 0$, and

$$\begin{aligned}\int_{M_{\varepsilon',\varepsilon}}|D(s-s_\varepsilon)|^2_{\widetilde{h}}&=\int_{M_{\varepsilon',\varepsilon}}|Ds|^2_{\widetilde{h}}-2\int_{M_{\varepsilon',\varepsilon}}(Ds,Ds_\varepsilon)_{\widetilde{h}}+\int_{M_{\varepsilon',\varepsilon}}|Ds_\varepsilon|^2_{\widetilde{h}}\\ &\le 2\int_{M_{\varepsilon',\varepsilon}}|Ds|^2_{\widetilde{h}}+2\int_{M_{\varepsilon',\varepsilon}}|Ds_\varepsilon|^2_{\widetilde{h}}.\end{aligned}$$

Now it is easy to see that the first term tends to 0, hence we only need to consider the second term:

$$\begin{aligned}\int_{M_{\varepsilon',\varepsilon}}|Ds_\varepsilon|^2_{\widetilde{h}}=\int_{M_{\varepsilon',\varepsilon}}|D(\phi_\varepsilon s)|^2_{\widetilde{h}}&=\int_{M_{\varepsilon',\varepsilon}}(|\mathrm{d}\phi_\varepsilon|s+\phi_\varepsilon Ds)_{\widetilde{h}}\\ &=\int_{M_{\varepsilon',\varepsilon}}(\mathrm{d}\phi_\varepsilon s,\mathrm{d}\phi_\varepsilon s)_{\widetilde{h}}+2\int_{M_{\varepsilon',\varepsilon}}\phi_\varepsilon\mathrm{Re}(Ds,\mathrm{d}\phi_\varepsilon s)_{\widetilde{h}}+\int_{M_{\varepsilon',\varepsilon}}\phi_\varepsilon^2|Ds|^2_{\widetilde{h}}\\ &\le 2\int_{M_{\varepsilon',\varepsilon}}|s|^2_h|\mathrm{d}\phi_\varepsilon|^2+2\int_{M_{\varepsilon',\varepsilon}}\phi^2_\varepsilon|Ds|^2_{\widetilde{h}}.\end{aligned}$$

The second term tends to 0, so the problem is reduced to prove that the first term also tends to 0. By the assumption that $|s|_h$ is bounded, it suffices to show $\int_{M_{\varepsilon',\varepsilon}} |\mathrm{d}\phi_\varepsilon|^2 \to 0$.

Now as suggested in [9], we can choose

$$\phi_\varepsilon = \begin{cases} \Big(\dfrac{r}{\varepsilon}\Big)^{\varepsilon}, & 2\varepsilon' \le r \le \varepsilon, \\ \Big(\dfrac{2\varepsilon'}{\varepsilon}\Big)^{\varepsilon}\Big(\dfrac{r}{\varepsilon} - 1\Big), & \varepsilon \le r \le 2\varepsilon', \\ 0, & 0 \le r \le \varepsilon', \end{cases} \tag{4.6}$$

where r denotes the distance function to the singular set M_{sing}. Clearly,

$$|\mathrm{d}\phi_\varepsilon|^2 \le \begin{cases} \Big(\dfrac{r}{\varepsilon}\Big)^{2\varepsilon-2}, & 2\varepsilon' \le r \le \varepsilon, \\ \Big(\dfrac{2\varepsilon'}{\varepsilon}\Big)^{2\varepsilon}\Big(\dfrac{1}{(\varepsilon')^2}\Big), & \varepsilon \le r \le 2\varepsilon', \\ 0, & 0 \le r \le \varepsilon'. \end{cases} \tag{4.7}$$

Since the singular set is at most of real codimension 2, ∂T_r is at most linear in r, for sufficiently small r. So

$$\begin{aligned} \int_{M_{\varepsilon',\varepsilon}} |\mathrm{d}\phi_\varepsilon|^2 &\le C\int_{2\varepsilon'}^{\varepsilon} \Big(\frac{r}{\varepsilon}\Big)^{2\varepsilon-2} r\mathrm{d}r + C\int_{\varepsilon'} 2\varepsilon'\Big(\frac{2\varepsilon'}{\varepsilon}\Big)^{2\varepsilon}\Big(\frac{r}{(\varepsilon')^2}\Big)\mathrm{d}r \\ &= C\Big(\frac{\varepsilon}{2} + \frac{2^{2\varepsilon-2}\exp(-\varepsilon^{-2})}{\varepsilon^{2\varepsilon-1}} + \frac{3}{2}\Big(\frac{\exp(-2\varepsilon^{-1})}{\varepsilon^{2\varepsilon}}\Big)\Big) \to 0 \end{aligned}$$

as $\varepsilon \to 0$. This establishes the self-adjointness of $\overline{P}$.

Proposition 4.2 *On M, the Schrödinger type operator P satisfies the following inequality*:

$$\mathrm{Re}(Ps, s)_h \ge |s|_{h,\varepsilon}((-\triangle + l)|s|_{h,\varepsilon}) \tag{4.8}$$

for any $s \in C_c^\infty(M, E)$, where $|s|_{h,\varepsilon}^2 := (|s|_h^2 + \varepsilon^2)^{\frac{1}{2}}$.

When $\varepsilon \to 0$, we have the inequality between distributions:

$$\mathrm{Re}(Ps, \mathrm{sign}\, s)_h \ge (-\triangle + l)|s|_h \tag{4.9}$$

for any $s \in C_c^\infty(M, E)$, where

$$\mathrm{sign}\, s = \begin{cases} \dfrac{s}{|s|_h} & \textit{on } \mathrm{supp}\, s, \\ \xi & \textit{otherwise}, \end{cases} \tag{4.10}$$

where ξ is an arbitrary measurable section in the sphere bundle of E.

Proof Note that $\triangle = -\delta\mathrm{d} = *\mathrm{d}*\mathrm{d}$, where $*$ is the Hodge star operator. For P we have

$$\mathrm{Re}(Ps, s)_h = \mathrm{Re}(D^t \circ Ds, s)_h + (Ls, s)_h \ge \mathrm{Re}(D^t \circ Ds, s)_h + l(s, s)_h. \tag{4.11}$$

Now we claim that $\mathrm{Re}(D^t \circ Ds, s)_h \ge -\triangle|s|_{h,\varepsilon}$, and by this, the first part of the proposition will be proved. To prove this inequality, we calculate $\triangle|s|_{h,\varepsilon}^2$ in two different ways. Firstly,

$$\begin{aligned} \triangle|s|_{h,\varepsilon}^2 &= *\mathrm{d}*\mathrm{d}|s|_{h,\varepsilon}^2 = *\mathrm{d}*(2|s|_{h,\varepsilon}\mathrm{d}|s|_{h,\varepsilon}) \\ &= *\mathrm{d}(2|s|_{h,\varepsilon} * \mathrm{d}|s|_{h,\varepsilon}) = *(2\mathrm{d}|s|_{h,\varepsilon} \wedge *\mathrm{d}|s|_{h,\varepsilon} + 2|s|_{h,\varepsilon}\mathrm{d} * |s|_{h,\varepsilon}) \\ &= 2*(\mathrm{d}|s|_{h,\varepsilon} \wedge *\mathrm{d}|s|_{h,\varepsilon}) + 2|s|_{h,\varepsilon} * \mathrm{d} * |s|_{h,\varepsilon} \\ &= 2|\mathrm{d}|s|_{h,\varepsilon}|_g^2 + 2|s|_{h,\varepsilon}\triangle|s|_{h,\varepsilon}, \end{aligned}$$

where g denotes the metric on T^*M.

On the other hand,

$$\begin{aligned}\triangle|s|^2_{h,\varepsilon} &= *\mathrm{d}*\mathrm{d}((s,s)_h+\varepsilon^2)\\ &= *\mathrm{d}*((Ds,s)+(s,Ds))\\ &= *\mathrm{d}((*Ds,s)+(s,*Ds))\\ &= *((D*Ds,s)+(*Ds,Ds)+(Ds,*Ds)+(s,D*Ds))\\ &= -2\mathrm{Re}(D^t\circ Ds,s)+2(Ds,Ds).\end{aligned}$$

Next we prove that $|\mathrm{d}|s|_{h,\varepsilon}|^2_g \le |Ds|^2$. Now since $\mathrm{d}|s|^2_{h,\varepsilon}=2|s|_{h,\varepsilon}\mathrm{d}|s|_{h,\varepsilon}$ and $\mathrm{d}|s|^2_{h,\varepsilon}=2\mathrm{Re}(Ds,Ds)$, we have

$$|\mathrm{d}|s|_{h,\varepsilon}|^2=\Big|\frac{\mathrm{d}|s|^2_{h,\varepsilon}}{2|s|_{h,\varepsilon}}\Big|^2=\frac{\mathrm{Re}(Ds,s)}{|s|^2_{h,\varepsilon}}\le\frac{|Ds|^2|s|^2_h}{|s|^2_{h,\varepsilon}}\le|Ds|^2.$$

Combining above together, we prove the first part of the proposition. The second inequality follows easily from (4.8) by letting $\varepsilon\to 0$.

Theorem 4.1 *With the notation above, the operators $\overline{P}$ and $-\overline{\triangle}$ determine the heat operators $\mathrm{e}^{-t\overline{P}}$ and $\mathrm{e}^{t\overline{\triangle}}$. Then the following domination of semigroups holds*:

$$|\mathrm{e}^{-t\overline{P}}s|_h\le \mathrm{e}^{-tl}|\mathrm{e}^{t\overline{\triangle}}s|_h. \tag{4.12}$$

Proof Take $f_1\in C^\infty(M,E)\cap L^2(M,E)$, $g\in L^2(M)$, λ a positive constant. By [9, Lemma 5.2], for any $\lambda>0$, $(\lambda-\overline{\triangle})^{-1}:\ L^2(M)\to W^{1,2}(M)$ is a bounded operator. So for $\lambda>0$, $(\lambda-\overline{\triangle})^{-1}g\in L^2(M)$. Let ξ be an arbitrary measurable section in the sphere bundle on E. Define

$$\mathrm{sign}(f_1)=\begin{cases}\dfrac{f_1}{|f_1|}, & f_1\ne 0,\\ \xi, & f_1=0,\end{cases} \tag{4.13}$$

and define $f_2=(\lambda-\overline{\triangle})^{-1}g\ \mathrm{sign}(f_1)$. Then $f_2\in L^2(M,E)$, and it is easy to verify that

$$|f_2|_h=(\lambda-\overline{\triangle})^{-1}g \tag{4.14}$$

and

$$\int_M(f_1,f_2)_h=\int_M|f_1|_h|f_2|_h.$$

Now by Proposition 4.2, we have

$$\mathrm{Re}(Pf_1,f_1)_h\ge|f_1|_{h,\varepsilon}((-\triangle+l)|f_1|_{h,\varepsilon}),$$

so

$$\int_M\mathrm{Re}\Big(Pf_1,\frac{f_1}{|f_1|_{h,\varepsilon}}|f_2|_h\Big)\ge\int_M(-\triangle+l)|f_1|_{h,\varepsilon}|f_2|_h=\int_M(-\overline{\triangle}+l)|f_2|_h|f_1|_{h,\varepsilon}. \tag{4.15}$$

When $\varepsilon\to 0$, $\frac{f_1}{|f_1|_{h,\varepsilon}}\to\mathrm{sign}|f_1|$, and $\mathrm{sign}|f_1|=\mathrm{sign}|f_2|$, so $\frac{f_1}{|f_1|_{h,\varepsilon}}|f_2|\to f_2$,

$$\int_M\mathrm{Re}(Pf_1,f_2)_h\ge\int_M(-\triangle+l)|f_2|_h|f_1|_h.$$

Taking constant $\mu > \max(0, -l)$, we have

$$\int_M \mathrm{Re}((P+\mu)f_1, f_2)_h \geq \int_M (-\triangle + l + \mu)|f_2|_h |f_1|_h.$$

Taking $\lambda = l + \mu$, then we have

$$\int_M (\mathrm{Re}(P+\mu)f_1, f_2)_h \geq \int_M g|f_1|_h.$$

Now

$$\int_M |(P+\mu)f_1|_h |f_2|_h \geq \int_M \mathrm{Re}((P+\mu)f_1, f_2)_h.$$

Putting $s = (P+\mu)f_1$, then we get

$$\int_M |s|_h |f_2|_h \geq \int_M |(P+\mu)^{-1}s|_h g.$$

By (4.14) we get

$$\int_M |s|_h ((l+\mu-\triangle)^{-1})g \geq \int_M g|(P+\mu)^{-1}s|_h.$$

Using the self-adjointness of $\triangle$, we get

$$\int_M (l+\mu-\triangle)^{-1}|s|_h g \geq \int_M |(P+\mu)^{-1}s|_h g.$$

Now we get

$$(l+\mu-\triangle)^{-1}|s|_h \geq |(P+\mu)^{-1}s|_h. \tag{4.16}$$

Since $f_1 \in C^\infty(M,E) \cap L^2(M,E)$ is dense in $L^2(M,E)$, by the continuity of resolvent, we have

$$|(-\overline{\triangle} + l + \mu)^{-1}s|_h \geq |(\overline{P}+\mu)^{-1}s|_h$$

for any $s \in L^2(M,E)$. By a standard result in functional analysis (see [5] or [6, Theorem 2.15]), the domination of resolvent implies the domination of semigroup, i.e.,

$$|\mathrm{e}^{-t(-\overline{\triangle}+l)}s|_h \geq |\mathrm{e}^{-t\overline{P}}s|_h. \tag{4.17}$$

Finally, by the Trotter's product formula (see [12]), we get

$$|\mathrm{e}^{-t\overline{P}}s|_h \leq \mathrm{e}^{-tl}\mathrm{e}^{t\overline{\triangle}}|s|_h. \tag{4.18}$$

Corollary 4.1 *The heat operator* $\mathrm{e}^{-t\overline{P}} : L^2(M,E) \to L^2(M,E)$ *is a trace class operator and its trace class* Tr *satisfies the following inequality*:

$$\mathrm{Tr}(\mathrm{e}^{-t\overline{P}}) \leq r\mathrm{e}^{-tl}\mathrm{Tr}(\mathrm{e}^{t\overline{\triangle}}), \tag{4.19}$$

where r is the rank of the vector bundle E. If H_p denotes the kernel function of $\mathrm{e}^{-t\overline{P}}$, *then the following inequalities holds*:

$$H_p(x,x,t) \leq r\mathrm{e}^{-tl}(\widehat{H}(0,t) + Bt), \tag{4.20}$$

$$\mathrm{Tr}(\mathrm{e}^{-t\overline{P}}) \leq r\mathrm{e}^{-tl}(\mathrm{vol}(M)(\widehat{H}(0,t) + Bt), \tag{4.21}$$

where B is a constant chosen as in Proposition 2.2.

Proof By [7], (4.12) implies that for the pointwise operator norm $\|H_p(x,y,t)\|_{\rm op}$, the inequality $\|H_p(x,x,t)\|_{\rm op} \le {\rm e}^{-tl}H(x,y,t)$ holds. This implies that for the pointwise traces we have $H_p(x,x,t) \le r{\rm e}^{-tl}H(x,x,t)$. Now ${\rm e}^{-t\overline{P}}$ is of trace class as well as ${\rm e}^{t\overline{\triangle}}$, and its trace class must satisfy (4.19). Inequalities (4.20)–(4.21) follow from the construction of H and Corollary 2.1.

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