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New Differential Harnack Inequalities for Nonlinear Heat Equations*

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Abstract This paper deals with constrained trace, matrix and constrained matrix Harnack inequalities for the nonlinear heat equation $\omega_t = \Delta\omega + a\omega \ln \omega$ on closed manifolds. A new interpolated Harnack inequality for $\omega_t = \Delta\omega - \omega \ln \omega + \varepsilon R\omega$ on closed surfaces under ε-Ricci flow is also derived. Finally, the author proves a new differential Harnack inequality for $\omega_t = \Delta\omega - \omega \ln \omega$ under Ricci flow without any curvature condition. Among these Harnack inequalities, the correction terms are all time-exponential functions, which are superior to time-polynomial functions.

Keywords Harnack inequality, Nonlinear heat equation, Ricci flow
2000 MR Subject Classification 53C44

1 Introduction

Recently, Cao, Fayyazuddin Ljungberg and Liu [4] improved gradient estimates of Ma [26] and Yang [33]. They proved a new differential Harnack inequality for any positive solution $\omega(x, t)$ to the nonlinear heat equation

$$\frac{\partial}{\partial t}\omega = \Delta\omega + a\,\omega \ln \omega, \tag{1.1}$$

where a is a nonzero real constant, on a complete smooth manifold.

Theorem A (see [4]) *Let (M^n, g) be an n-dimensional complete manifold without boundary with nonnegative Ricci curvature. Let $\omega(x, t)$ be a positive solution to* (1.1). *Then in any of the three cases*:

(i) *$a > 0$ and M is closed,*

(ii) *$a < 0$ and M is closed,*

(iii) *$a > 0$ and M is complete noncompact,*

the following inequality holds for all $x \in M^n$, $t > 0$,

$$\Delta \ln \omega + \frac{an}{2(1 - \mathrm{e}^{-at})} \geq 0. \tag{1.2}$$

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Since (1.1) is related to the gradient Ricci soliton (see [26]) and the logarithmic Sobolev constant (see [12]), the Harnack inequality (1.2) is useful in understanding these geometric invariants, even the singularities of Ricci flow. The essential idea of proving Theorem A is the parabolic maximum principle, which was ever used by Li and Yau [23] to prove differential Harnack estimates for the heat equation. One novel feature of Cao-Fayyazuddin Ljungberg-Liu Harnack inequalities is the correction term, which is an exponential function:

$$\frac{an}{2(1-\mathrm{e}^{-at})}. \tag{1.3}$$

This term is obviously different from the polynomial correction term $\frac{n}{2t}$, which appears in the following classical Li-Yau Harnack inequality.

Theorem B (see [23]) *Let (M^n, g) be a complete Riemannian manifold with nonnegative Ricci curvature. Let $\omega(x,t)$ be a positive solution to the linear heat equation. Then for all $x \in M^n$, $t > 0$,*

$$\Delta \ln \omega + \frac{n}{2t} \geq 0.$$

As we all know, Li-Yau Harnack inequality is sharp for the linear heat equation and the equality case holds for the fundamental solution

$$H(x,t) := \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\Big(-\frac{\|x\|^2}{4t}\Big)$$

of linear heat equation in Euclidean space. For (1.1), Cao, Fayyazuddin Ljungberg and Liu showed that (1.2) is sharp in case (iii) of Theorem A. That is, there exists a family of particular solutions of (1.1) on $\mathbb{R}^n$ (see [30]),

$$\omega(x,t) = -\frac{a\|x\|^2}{4(1-\mathrm{e}^{-at})} - \frac{n}{2}\mathrm{e}^{-at}\ln|1-\mathrm{e}^{-at}| + C\mathrm{e}^{at},$$

where $C \in \mathbb{R}$ is an arbitrary constant, such that (1.2) becomes an equality. The Harnack inequality (1.2) with new correction term (1.3) stimulates us to find more superior possible differential Harnack inequalities of (1.1) and its related equations.

In this paper, inspired by the work of Cao, Fayyazuddin Ljungberg and Liu [4], we can derive constrained trace Harnack inequalities, matrix Harnack inequalities and constrained matrix Harnack inequalities for the nonlinear heat equation

$$\omega_t = \Delta\omega + a\,\omega\ln\omega$$

on closed manifolds with fixed metric. We also can improve previous interpolated Harnack inequality in [31] for the nonlinear heat equation

$$\omega_t = \Delta\omega - \omega\ln\omega + \varepsilon R\omega, \quad \varepsilon \geq 0$$

on closed surfaces under the ε-Ricci flow. Finally we prove a new differential Harnack inequality for the nonlinear heat equation

$$\omega_t = \Delta\omega - \omega\ln\omega$$

on closed manifolds along the Ricci flow without any curvature assumption. Among our differential Harnack inequalities, the correction terms are all time-exponential functions, which are superior to time-polynomial functions.

The study of differential Harnack estimates for the heat equation originated in Li and Yau [23] (a precursory form appeared in [1]). This method was later brought into the study of the Ricci flow by Hamilton [18] and played an important role in the singularity analysis of the Ricci flow. Hamilton [17] also generalized the Li-Yau Harnack inequality to a matrix Harnack form on a class of manifolds. These results were furthermore extended to constrained, matrixed, and interpolated Harnack inequalities by Chow and Hamilton [10], Chow [9], Ni [27] and Li [24]. See [28] for excellent discussions on this subject.

Recently, differential Harnack inequalities for heat-type equations coupled with the Ricci flow have become an important object. This subject was ever explored by Chow and Hamilton [10], Chow and Knopf [11], etc. In particular, Perelman [29] discovered differential Harnack inequalities for the fundamental solution to the backward heat equation under the Ricci flow without any curvature assumption. This spectacular result is a crucial step in proving Poincaré Conjecture. Perelman's result was extended to all positive solutions by Cao [3] and independently by Kuang and Zhang [21], whereas scalar curvature is required to be nonnegative. For more work and progress in this direction, for example, see [2, 5–6, 8, 13–16, 20, 22, 25, 31–32, 34].

This paper is organized as follows. In Section 2, we will derive constrained trace, matrix and constrained matrix differential Harnack inequalities for (1.1). The proof relies on the parabolic maximum principle. In Section 3, we will prove an interpolated Harnack inequality for $\omega_t = \Delta\omega - \omega \ln \omega + \varepsilon R\omega$ on closed surfaces under the ε-Ricci flow. In Section 4, we will improve a previous Harnack inequality for $\omega_t = \Delta\omega - \omega \ln \omega$ on closed manifolds under the Ricci flow.

2 Constrained Trace, Matrix and Constrained Matrix Harnack Inequalities

In this section we will study various Harnack inequalities for the nonlinear heat equation

$$\frac{\partial}{\partial t}\omega = \Delta\omega + a\,\omega \ln \omega \tag{2.1}$$

for a nonzero real constant a, on a closed n-dimensional Riemannian manifold (M, g). Inspired by the work of Cao, Fayyazuddin Ljungberg and Liu [4], we can derive some new Harnack inequalities of this equation, such as constrained trace Harnack inequalities, matrix Harnack inequalities, constrained matrix Harnack inequalities. We first give constrained trace Harnack inequalities for (2.1).

Theorem 2.1 *Let (M, g) be a closed n-dimensional Riemannian manifold. Let φ and ψ be two solutions to* (2.1). *Then in any of the two cases*:

(i) $a > 0$, $0 < \varphi < \psi$ *and* $\mathrm{Ric}(M) \geq 0$,

(ii) *$a<0$, $0<c_0\psi<\varphi<\psi$, where $0<c_0<1$ is a free parameter, and* $\mathrm{Ric}(M)\geq -aK$ *for some constant* $K\geq -\frac{\ln c_0}{1-c_0^2}-\frac{1}{2}$,
the following inequality holds for all $x\in M$, $t>0$,

$$\frac{\partial}{\partial t}\ln\psi-|\nabla\ln\psi|^2-a\ln\psi+\frac{an}{2(1-\mathrm{e}^{-at})}=\Delta\ln\psi+\frac{an}{2(1-\mathrm{e}^{-at})}\geq\frac{|\nabla h|^2}{1-h^2},$$

where $h=\frac{\varphi}{\psi}$.

By integrating the above inequality in space-time we get a classical Harnack inequality.

Corollary 2.1 *Suppose that φ and ψ satisfy the condition of Theorem* 2.1. *Let $x_1,x_2\in M$ and $0<t_1<t_2$. Then we have*

$$\begin{aligned}&\mathrm{e}^{-at_1}\ln\psi(x_1,t_1)-\mathrm{e}^{-at_2}\ln\psi(x_2,t_2)\\ \leq\ &\frac{a}{4}\cdot\frac{\mathrm{d}(x_1,x_2)}{\mathrm{e}^{at_2}-\mathrm{e}^{at_1}}+\frac{n}{2}\cdot\ln\Big(\frac{1-\mathrm{e}^{-at_2}}{1-\mathrm{e}^{-at_1}}\Big)-\int_{t_1}^{t_2}\mathrm{e}^{-at}\Big(\frac{|\nabla h|^2}{1-h^2}\Big)\mathrm{d}t,\end{aligned}$$

where $h=\frac{\varphi}{\psi}$.

Theorem 2.1 can be regarded as a nonlinear version of a constrained trace Harnack inequality proved by Chow and Hamilton [10]. Due to an additional nonlinear term: $\omega\ln\omega$ in (2.1), the computations and estimates in our proof seems to be complicated but straight. In order to prove Theorem 2.1, we need some useful lemmas.

Let (M,g) be a closed n-dimensional manifold. Suppose that φ and ψ are two positive solutions to (2.1) satisfying $\varphi<\psi$, and let $h:=\frac{\varphi}{\psi}$. We set

$$P_{ij}:=\nabla_i\nabla_j\ln\psi-\frac{\nabla_ih\nabla_jh}{1-h^2}.$$

Then we have the following lemma.

Lemma 2.1 *Let $L=\ln\psi$. Then*

$$\begin{aligned}\frac{\partial}{\partial t}P_{ij}=\ &\Delta P_{ij}+2\nabla_lL\nabla_lP_{ij}+2P_{il}P_{lj}-R_{il}P_{lj}-R_{jl}P_{li}\\ &+\frac{2}{1-h^2}\Big(\nabla_i\nabla_lh+\frac{2h\nabla_ih\nabla_lh}{1-h^2}\Big)\Big(\nabla_j\nabla_lh+\frac{2h\nabla_jh\nabla_lh}{1-h^2}\Big)\\ &+2R_{ikjl}P_{kl}+2R_{ikjl}\frac{\nabla_kh\nabla_lh}{1-h^2}+2R_{ikjl}\nabla_kL\nabla_lL\\ &-(\nabla_iR_{jl}+\nabla_jR_{il}-\nabla_lR_{ij})\nabla_lL\\ &+aP_{ij}-\frac{a\nabla_ih\nabla_jh}{1-h^2}\Big(1+\frac{2\ln h}{1-h^2}\Big).\end{aligned}$$

Proof Since $L=\ln\psi$, we have

$$\frac{\partial}{\partial t}L=\Delta L+|\nabla L|^2+aL.$$

We directly compute that

$$\frac{\partial}{\partial t}\nabla_i\nabla_jL=\Delta\nabla_i\nabla_jL+2R_{ikjl}\nabla_k\nabla_lL-R_{il}\nabla_j\nabla_lL-R_{jl}\nabla_i\nabla_lL$$

$$
\begin{aligned}
&- (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})\nabla_l L \\
&+ 2\nabla_i\nabla_l L \cdot \nabla_j \nabla_l L + 2\nabla_l\nabla_i\nabla_j L \cdot \nabla_l L + 2R_{ikjl}\nabla_k L\nabla_l L \\
&+ a\nabla_i\nabla_j L.
\end{aligned}
\tag{2.2}
$$

Next we calculate the evolution of the term $\frac{\nabla_i h\nabla_j h}{1-h^2}$. Set $h = \frac{\varphi}{\psi}$. Then

$$
\frac{\partial}{\partial t}h = \Delta h + 2\langle \nabla L, \nabla h\rangle + ah\cdot \ln h
$$

and hence its gradient satisfies

$$
\begin{aligned}
\frac{\partial}{\partial t}(\nabla h) &= \nabla\Big(\frac{\partial}{\partial t}h\Big) \\
&= \nabla(\Delta h + 2\langle \nabla L, \nabla h\rangle + ah\cdot \ln h) \\
&= \Delta\nabla h + 2\langle \nabla\nabla L, \nabla h\rangle + 2\langle \nabla L, \nabla\nabla h\rangle - \mathrm{Ric}(\nabla h) + a(1+\ln h)\nabla h,
\end{aligned}
$$

which further implies

$$
\begin{aligned}
\frac{\partial}{\partial t}(\nabla_i h\nabla_j h) = {}& \Delta(\nabla_i h\nabla_j h) - 2\nabla_i\nabla_l h\nabla_j\nabla_l h + 2\nabla_i\nabla_l L\nabla_l h\nabla_j h \\
&+ 2\nabla_j\nabla_l L\nabla_l h\nabla_i h + 2\nabla_l L\nabla_l(\nabla_i h\nabla_j h) \\
&- R_{il}\nabla_l h\nabla_j h - R_{jl}\nabla_l h\nabla_i h \\
&+ 2a(1+\ln h)(\nabla_i h\nabla_j h).
\end{aligned}
$$

We also have

$$
\begin{aligned}
\frac{\partial}{\partial t}(1-h^2) &= -2h\frac{\partial}{\partial t}h = -2h(\Delta h + 2\langle \nabla L, \nabla h\rangle + ah\cdot\ln h) \\
&= \Delta(1-h^2) + 2\langle \nabla L, \nabla(1-h^2)\rangle + 2|\nabla h|^2 - 2ah^2\cdot \ln h.
\end{aligned}
$$

Using the above two evolution equations, we conclude that

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(\frac{\nabla_i h\nabla_j h}{1-h^2}\Big) = {}& \Delta\Big(\frac{\nabla_i h\nabla_j h}{1-h^2}\Big) + 2\nabla_l L\nabla_l\Big(\frac{\nabla_i h\nabla_j h}{1-h^2}\Big) - \frac{2\nabla_i h\nabla_j h}{(1-h^2)^2}|\nabla h|^2 \\
&+ \frac{1}{1-h^2}(-2\nabla_i\nabla_l h\nabla_j\nabla_l h + 2\nabla_i\nabla_l L\nabla_l h\nabla_j h \\
&+ 2\nabla_j\nabla_l L\nabla_l h\nabla_i h - R_{il}\nabla_l h\nabla_j h - R_{jl}\nabla_l h\nabla_i h) \\
&- \frac{4h\cdot\nabla_l h}{(1-h^2)^2}(\nabla_i\nabla_l h\nabla_j h + \nabla_i h\nabla_j\nabla_l h) - \frac{8h^2\nabla_i h\nabla_j h}{(1-h^2)^3}|\nabla h|^2 \\
&+ \frac{2a\nabla_i h\nabla_j h}{1-h^2}\Big(1 + \frac{\ln h}{1-h^2}\Big).
\end{aligned}
$$

Rearranging terms yields

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(\frac{\nabla_i h\nabla_j h}{1-h^2}\Big) = {}& \Delta\Big(\frac{\nabla_i h\nabla_j h}{1-h^2}\Big) + 2\nabla_l L\nabla_l\Big(\frac{\nabla_i h\nabla_j h}{1-h^2}\Big) \\
&- \frac{2}{1-h^2}\Big(\nabla_i\nabla_l h + \frac{2h\nabla_i h\nabla_l h}{1-h^2}\Big)\Big(\nabla_j\nabla_l h + \frac{2h\nabla_j h\nabla_l h}{1-h^2}\Big) \\
&+ \frac{1}{1-h^2}(2\nabla_i\nabla_l L\nabla_l h\nabla_j h + 2\nabla_j\nabla_l L\nabla_l h\nabla_i h
\end{aligned}
$$

$$\begin{aligned}&- R_{il}\nabla_l h\nabla_j h - R_{jl}\nabla_l h\nabla_i h)\\&-\frac{2\nabla_i h\nabla_j h}{(1-h^2)^2}|\nabla h|^2+\frac{2a\nabla_i h\nabla_j h}{1-h^2}\Big(1+\frac{\ln h}{1-h^2}\Big).\end{aligned}$$

Now we let

$$P_{ij}:=\nabla_i\nabla_j L-\frac{\nabla_i h\nabla_j h}{1-h^2}.$$

Combining the above computations, we have

$$\begin{aligned}\frac{\partial}{\partial t}P_{ij}&=\Delta P_{ij}+2\nabla_l L\nabla_l P_{ij}+2P_{il}P_{lj}-R_{il}P_{lj}-R_{jl}P_{li}\\&\quad+\frac{2}{1-h^2}\Big(\nabla_i\nabla_l h+\frac{2h\nabla_i h\nabla_l h}{1-h^2}\Big)\Big(\nabla_j\nabla_l h+\frac{2h\nabla_j h\nabla_l h}{1-h^2}\Big)\\&\quad+2R_{ikjl}P_{kl}+2R_{ikjl}\frac{\nabla_k h\nabla_l h}{1-h^2}+2R_{ikjl}\nabla_k L\nabla_l L\\&\quad-(\nabla_i R_{jl}+\nabla_j R_{il}-\nabla_l R_{ij})\nabla_l L\\&\quad+aP_{ij}-\frac{a\nabla_i h\nabla_j h}{1-h^2}\Big(1+\frac{2\ln h}{1-h^2}\Big).\end{aligned}$$

Note that we have used the second Bianchi identity in above evolution formula. The lemma then follows.

Tracing Lemma 2.1, we immediately get the following lemma.

Lemma 2.2 *If we let*

$$P=g^{ij}P_{ij}=\Delta L-\frac{|\nabla h|^2}{1-h^2},$$

then

$$\begin{aligned}\frac{\partial}{\partial t}P&=\Delta P+2\langle\nabla L,\nabla P\rangle+2\Big|\nabla\nabla L-\frac{\nabla h\nabla h}{1-h^2}\Big|^2\\&\quad+\frac{2}{(1-h^2)^3}|2h\nabla h\nabla h+(1-h^2)\nabla\nabla h|^2\\&\quad+2\mathrm{Ric}(\nabla L,\nabla L)+\frac{2}{1-h^2}\mathrm{Ric}(\nabla h,\nabla h)\\&\quad+aP-\frac{a|\nabla h|^2}{1-h^2}\Big(1+\frac{2\ln h}{1-h^2}\Big).\end{aligned}$$

We now prove Theorem 2.1 by Lemma 2.2.

Proof of Theorem 2.1 We first prove the complex case: $a<0$. By Lemma 2.2, using the curvature assumption $\mathrm{Ric}(M)\geq -aK$ for some constant $K\geq-\frac{\ln c_0}{1-c_0^2}-\frac12$, we obtain

$$\frac{\partial}{\partial t}P\geq\Delta P+2\langle\nabla L,\nabla P\rangle+\frac{2}{n}P^2+aP-\frac{a|\nabla h|^2}{1-h^2}\Big(2K+1+\frac{2\ln h}{1-h^2}\Big).\tag{2.3}$$

Here we have used an easy fact: $\mathrm{Ric}(\nabla L,\nabla L)\geq 0$ due to $K\geq-\frac{\ln c_0}{1-c_0^2}-\frac12>0$ and $a<0$ at this case.

In the following, we claim that the assumptions of theorem

$$K\geq-\frac{\ln c_0}{1-c_0^2}-\frac12>0,\quad 0<c_0<h<1$$

imply

$$2K + 1 + \frac{2\ln h}{1-h^2} > 0.$$

Indeed we only need to check that function $f(h) := \frac{\ln h}{1-h^2}$ is increasing on the interval $(c_0, 1)$. We compute its derivative

$$\begin{aligned} f'(h) &= \frac{\frac{1}{h}\cdot(1-h^2) - \ln h\cdot(-2h)}{(1-h^2)^2} \\ &= \frac{\frac{1}{h} - h + 2h\cdot\ln h}{(1-h^2)^2}. \end{aligned}$$

Let

$$g(h) := \frac{1}{h} - h + 2h\cdot\ln h.$$

Then $g(0+) = +\infty$, $g(1) = 0$ and

$$g'(h) = -\frac{1}{h^2} + 1 + 2\ln h < 0$$

for all $c_0 < h < 1$. So $g(h) > 0$ for all $c_0 < h < 1$. Hence we have $f'(h) > 0$ for all $c_0 < h < 1$. The claim follows.

Therefore (2.3) reduces to

$$\frac{\partial}{\partial t}P \geq \Delta P + 2\langle \nabla L, \nabla P\rangle + \frac{2}{n}P^2 + aP.$$

Let

$$\widetilde{P} := P + \frac{an}{2(1-\mathrm{e}^{-at})}.$$

Then

$$\frac{\partial}{\partial t}\widetilde{P} \geq \Delta\widetilde{P} + 2\langle \nabla L, \nabla\widetilde{P}\rangle + \frac{2}{n}\widetilde{P}\Big[P - \frac{an}{2(1-\mathrm{e}^{-at})}\Big] + a\widetilde{P} \tag{2.4}$$

and hence the theorem follows by applying the maximum principle to this equation. Indeed, for $t \to 0+$, since $a < 0$, we have

$$\frac{an}{2(1-\mathrm{e}^{-at})} \to +\infty.$$

Hence $\widetilde{P} \to +\infty$ as $t \to 0+$. In the following we will prove $\widetilde{P} \geq 0$ for all $t > 0$ in the closed manifold M.

Assume that there exists some space-time (x', t') such that $\widetilde{P} \leq 0$. Since M is closed, there must exist a first time $t_0 \leq t'$ and $x_0 \in M$ such that $\widetilde{P} < 0$, where $\widetilde{P}$ achieves its infimum. Then at (x_0, t_0), we have

$$\Delta\widetilde{P} \geq 0, \quad \nabla\widetilde{P} = 0, \quad \frac{\partial}{\partial t}\widetilde{P} \leq 0.$$

Therefore, combining the above inequalities with (2.4) at (x_0, t_0), we have

$$\frac{2}{n}\widetilde{P}\Big[P - \frac{an}{2(1-\mathrm{e}^{-at})}\Big] + a\widetilde{P} \leq 0. \tag{2.5}$$

However, indeed $a < 0$, $\widetilde{P}(x_0, t_0) < 0$ and

$$P(x_0, t_0) = \widetilde{P}(x_0, t_0) - \frac{an}{2(1 - \mathrm{e}^{-at_0})} < 0.$$

Hence (2.5) cannot hold and this is a contradiction. Therefore $\widetilde{P} \geq 0$ everywhere for all time $t > 0$.

The proof of the case $a > 0$ is similar to that of the case $a < 0$. Using Lemma 2.2, $a > 0$ and $\mathrm{Ric} \geq 0$, we have

$$\frac{\partial}{\partial t} P \geq \Delta P + 2\langle \nabla L, \nabla P\rangle + \frac{2}{n} P^2 + aP,$$

where we used the fact

$$1 + \frac{2 \ln h}{1 - h^2} < 0.$$

Let

$$\widetilde{P} := P + \frac{an}{2(1 - \mathrm{e}^{-at})}.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t} \widetilde{P} &= \frac{\partial}{\partial t} P - \frac{a^2\, n\, \mathrm{e}^{-at}}{2(1 - \mathrm{e}^{-at})^2} \\
&\geq \Delta \widetilde{P} + 2\langle \nabla L, \nabla \widetilde{P}\rangle + \frac{2}{n}\Big[\widetilde{P} - \frac{an}{2(1 - \mathrm{e}^{-at})}\Big]^2 \\
&\quad + a\Big[\widetilde{P} - \frac{an}{2(1 - \mathrm{e}^{-at})}\Big] - \frac{a^2\, n\, \mathrm{e}^{-at}}{2(1 - \mathrm{e}^{-at})^2} \\
&= \Delta \widetilde{P} + 2\langle \nabla L, \nabla \widetilde{P}\rangle + \frac{2}{n} \widetilde{P}^2 + a\widetilde{P}\Big(1 - \frac{2}{1 - \mathrm{e}^{-at}}\Big).
\end{aligned} \tag{2.6}$$

Similar to the above argument, $\widetilde{P} \geq 0$ follows by applying the maximum principle to this equation.

The classical Harnack inequality is obtained by integrating the differential Harnack inequality. The process is quite standard. We include it here for completeness.

Proof of Corollary 2.1 We pick a space-time path $\gamma(x, t)$ joining (x_1, t_1) and (x_2, t_2) with $t_2 > t_1 > 0$. Along γ, considering the one-parameter function $\psi(t) := \psi(\gamma(t), t)$, by Theorem 2.1 we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \ln \psi &= \frac{\partial}{\partial t} \ln \psi + \nabla \ln \psi \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}t} \\
&\geq |\nabla \ln \psi|^2 + a \ln \psi - \frac{an}{2(1 - \mathrm{e}^{-at})} + \frac{|\nabla h|^2}{1 - h^2} + \nabla \ln \psi \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}t} \\
&\geq -\frac{1}{4}\Big|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|^2 + a \ln \psi - \frac{an}{2(1 - \mathrm{e}^{-at})}.
\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-at} \ln \psi) \geq -\mathrm{e}^{-at}\Big(\frac{1}{4}\Big|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|^2 + \frac{an}{2(1 - \mathrm{e}^{-at})} - \frac{|\nabla h|^2}{1 - h^2}\Big).$$

Integrating this inequality from t_1 to t_2 yields

$$\mathrm{e}^{-at_1}\ln\psi(x_1,t_1)-\mathrm{e}^{-at_2}\ln\psi(x_2,t_2)\le\int_{t_1}^{t_2}\mathrm{e}^{-at}\Big(\frac{1}{4}\Big|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|^2+\frac{an}{2(1-\mathrm{e}^{-at})}-\frac{|\nabla h|^2}{1-h^2}\Big)\mathrm{d}t.$$

Notice the fact

$$\int_{t_1}^{t_2}\mathrm{e}^{-at}\Big(\Big|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|^2\Big)\mathrm{d}t\ge a\,\frac{d(x_1,x_2)^2}{\mathrm{e}^{at_2}-\mathrm{e}^{at_1}}$$

for any smooth path $\gamma:[t_1,t_2]\to M$ such that $\gamma(t_1)=x_1$ and $\gamma(t_2)=x_2$. Here the equality is attained when γ is a minimal geodesic from x_1 to x_2 with the speed

$$\Big|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|=a\,\mathrm{e}^{at}\cdot\frac{d(x_1,x_2)}{\mathrm{e}^{at_2}-\mathrm{e}^{at_1}}.$$

Then we finish the proof of Corollary 2.1.

Secondly, we can prove a new version of Chow-Hamilton matrix Harnack inequalities (see [10, Theorem 3.3]). The matrix Harnack inequalities were first considered by Hamilton [17–18] and further extended by Chow and Hamilton [10], Chow and Knopf [11], and Ni [27]. We remark that our heat-type equation is nonlinear and the evolution of Harnack quantity is more complicated.

Theorem 2.2 *Let (M,g) be a closed Riemannian manifold with the nonnegative sectional curvature and $\nabla\mathrm{Ric}=0$. If ψ is a positive solution to* (2.1)*, then for all $x\in M^n$, $t>0$,*

$$\nabla_i\nabla_j\ln\psi+\frac{ag_{ij}}{2(1-\mathrm{e}^{-at})}\ge 0.$$

Remark 2.1 If we trace the above Harnack inequality, Theorem 2.2 recovers the Cao-Fayyazuddin Ljungberg-Liu Harnack inequality (1.2).

Proof of Theorem 2.2 By (2.2) and the assumptions of theorem, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i\nabla_jL\ge{}&\Delta\nabla_i\nabla_jL+2\nabla_l\nabla_i\nabla_jL\cdot\nabla_lL+2\nabla_i\nabla_lL\cdot\nabla_j\nabla_lL\\&-R_{il}\nabla_j\nabla_lL-R_{jl}\nabla_i\nabla_lL+2R_{ikjl}\nabla_k\nabla_lL+a\nabla_i\nabla_jL.\end{aligned}\tag{2.7}$$

Letting

$$N_{ij}:=\nabla_i\nabla_jL+\frac{ag_{ij}}{2(1-\mathrm{e}^{-at})},$$

we have

$$\begin{aligned}\frac{\partial}{\partial t}N_{ij}\ge{}&\Delta N_{ij}+2\nabla_lL\nabla_lN_{ij}+2N_{il}\Big[\nabla_l\nabla_jL-\frac{ag_{lj}}{2(1-\mathrm{e}^{-at})}\Big]\\&-R_{il}N_{lj}-R_{jl}N_{li}+2R_{ikjl}N_{kl}+aN_{ij}.\end{aligned}$$

Using the tensor maximum principle yields the desired result.

Furthermore, we can prove constrained matrix Harnack inequalities for (2.1).

Theorem 2.3 *Let (M,g) be a closed Riemannian manifold. Let φ and ψ be two solutions to* (2.1). *Then in any of the two cases*:

(i) $a>0$, $0<\varphi<\psi$, $\nabla\mathrm{Ric}=0$ *and the curvature* $R_{ijkl}(M)\geq 0$,

(ii) $a<0$, $0<c_0\psi<\varphi<\psi$, *where* $0<c_0<1$ *is a free parameter*, $\nabla\mathrm{Ric}=0$ *and* $R_{ikjl}\geq -aK(g_{ij}g_{kl}-g_{il}g_{jk})$ *for some constant* $K\geq -\frac{\ln c_0}{1-c_0^2}-\frac{1}{2}>0$,

the following inequality holds for all $x\in M$, $t>0$,

$$\nabla_i\nabla_j\ln\psi+\frac{ag_{ij}}{2(1-\mathrm{e}^{-at})}\geq\frac{\nabla_i h\nabla_j h}{1-h^2},$$

where $h=\frac{\varphi}{\psi}$.

Proof We first discuss the case $a<0$. By Lemma 2.1, and using $a<0$ and $R_{ikjl}\geq -aK(g_{ij}g_{kl}-g_{il}g_{jk})$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}P_{ij}\geq{}&\Delta P_{ij}+2\nabla_l L\nabla_l P_{ij}+2P_{il}P_{lj}-R_{il}P_{lj}-R_{jl}P_{li}\\&+2R_{ikjl}P_{kl}-2aK|\nabla L|^2\gamma_{ij}+aP_{ij}-\frac{a|\nabla h|^2}{1-h^2}\Big(2K+1+\frac{2\ln h}{1-h^2}\Big)g_{ij}.\end{aligned}$$

Therefore, if

$$\widetilde{P}_{ij}:=P_{ij}+\frac{ag_{ij}}{2(1-\mathrm{e}^{-at})},$$

then

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{P}_{ij}\geq{}&\Delta\widetilde{P}_{ij}+2\nabla_l L\nabla_l\widetilde{P}_{ij}+2\widetilde{P}_{il}\Big[P_{lj}-\frac{ag_{lj}}{2(1-\mathrm{e}^{-at})}\Big]-R_{il}\widetilde{P}_{lj}-R_{jl}\widetilde{P}_{li}\\&+2R_{ikjl}\widetilde{P}_{kl}-2aK|\nabla L|^2g_{ij}+a\widetilde{P}_{ij}-\frac{a|\nabla h|^2}{1-h^2}\Big(2K+1+\frac{2\ln h}{1-h^2}\Big)g_{ij}.\end{aligned}$$

Since

$$K\geq-\frac{\ln c_0}{1-c_0^2}-\frac{1}{2}>0,\quad c_0<h<1,$$

we have

$$2K+1+\frac{2\ln h}{1-h^2}>0.$$

Then using the maximum principle for the above system, we have $\widetilde{P}_{ij}\geq 0$.

Now we prove the case $a>0$. By Lemma 2.1, and using $a>0$ and $R_{ikjl}\geq 0$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}P_{ij}\geq{}&\Delta P_{ij}+2\nabla_l L\nabla_l P_{ij}+2P_{il}P_{lj}-R_{il}P_{lj}-R_{jl}P_{li}\\&+2R_{ikjl}P_{kl}+aP_{ij}-\frac{a|\nabla h|^2}{1-h^2}\Big(1+\frac{2\ln h}{1-h^2}\Big)g_{ij}.\end{aligned}$$

Therefore, if

$$\widetilde{P}_{ij}:=P_{ij}+\frac{ag_{ij}}{2(1-\mathrm{e}^{-at})},$$

then

$$\frac{\partial}{\partial t}\widetilde{P}_{ij}\geq\Delta\widetilde{P}_{ij}+2\nabla_l L\nabla_l\widetilde{P}_{ij}+2\widetilde{P}_{il}\widetilde{P}_{lj}-R_{il}\widetilde{P}_{lj}-R_{jl}\widetilde{P}_{li}$$

$$+ 2R_{ikjl}\widetilde{P}_{kl} + a\widetilde{P}_{ij}\Big(1 - \frac{2}{1-\mathrm{e}^{-at}}\Big) - \frac{a|\nabla h|^2}{1-h^2}\Big(1 + \frac{2\ln h}{1-h^2}\Big)g_{ij}.$$

Since

$$1 + \frac{2\ln h}{1-h^2} < 0,$$

using the maximum principle for above tensor equation, we conclude that $\widetilde{P}_{ij} \geq 0$.

The above theorems also hold on complete noncompact Riemannian manifolds as long as the maximum principle can be used. We expect that our differential Harnack inequalities will be useful in understanding the Ricci solitons, as the potential function of Ricci solitons links with (2.1).

3 Interpolated Harnack Inequality

In [7], Cao and Zhang studied differential Harnack inequalities for nonlinear heat-type equation

$$\frac{\partial}{\partial t}\omega = \Delta\omega - \omega\ln\omega + R\omega \tag{3.1}$$

coupled with Ricci flow equation

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{3.2}$$

on a closed Riemannian manifold. They proved the following result.

Theorem C (see [7]) *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let f be a positive solution to the nonlinear heat equation* (3.1), *$u = -\ln f$ and*

$$H := 2\Delta u - |\nabla u|^2 - 3R - \frac{2n}{t}. \tag{3.3}$$

Then for all time $t \in [0, T)$,

$$H \leq \frac{n}{4}.$$

Theorem C generalizes the work of Cao and Hamilton [6] (see [21]) to nonlinear case. The motivation to study (3.1) under the Ricci flow comes from the study of expanding Ricci solitons, which has been nicely explained in [7]. Later, on a closed surface, Wu [31] improved their result as follows.

Theorem D (see [31]) *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the ε-Ricci flow $(\varepsilon \geq 0)$:*

$$\frac{\partial}{\partial t}g_{ij} = -\varepsilon R \cdot g_{ij} \tag{3.4}$$

on a closed surface with $R > 0$. Let f be a positive solution to the nonlinear heat equation

$$\frac{\partial}{\partial t}\omega = \Delta\omega - \omega\ln\omega + \varepsilon R\omega. \tag{3.5}$$

Then for all time $t \in (0, T)$,

$$\frac{\partial}{\partial t}\ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{t} = \Delta \ln f + \varepsilon R + \frac{1}{t} \geq 0.$$

Remark 3.1 In Theorem D, if $\varepsilon = 1$, then

$$\Delta \ln f + R + \frac{1}{t} \geq 0. \tag{3.6}$$

However (3.3) can be read as

$$2\Delta \ln f + \frac{|\nabla f|^2}{f^2} + 3R + \frac{4}{t} + \frac{n}{4} \geq 0,$$

which can be rewritten as

$$\Big(2\Delta \ln f + 2R + \frac{2}{t}\Big) + \Big(\frac{|\nabla f|^2}{f^2} + R + \frac{2}{t} + \frac{n}{4}\Big) \geq 0.$$

Compared this with (3.6), for the 2-dimensional surface, Theorem D is better than Theorem C.

Motivated by Theorem A, we can improve Theorem D by the following interpolated Harnack inequality.

Theorem 3.1 *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the ε-Ricci flow* (3.4) *on a closed surface with $R > 0$. Let f be a positive solution to the nonlinear parabolic equation* (3.5)*. Then for all time $t \in (0, T)$,*

$$\frac{\partial}{\partial t}\ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{\mathrm{e}^t - 1} = \Delta \ln f + \varepsilon R + \frac{1}{\mathrm{e}^t - 1} \geq 0.$$

As a consequence of Theorem 3.1, we have a classical Harnack inequality. Since the proof is standard, we only provide the following result.

Corollary 3.1 *Under the conditions of Theorem* 3.1*, assume that (x_1, t_1) and (x_2, t_2), $0 \leq t_1 < t_2 < T$, are two points in $M \times [0, T)$. Let*

$$\Gamma(x_1, t_1, x_2, t_2) := \frac{1}{4}\inf_{\gamma}\int_{t_1}^{t_2} \mathrm{e}^t \Big|\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\Big|^2 \mathrm{d}t,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2), and the norm $|\cdot|$ is calculated at time t. Then

$$\mathrm{e}^{t_1}\ln f(x_1, t_1) - \mathrm{e}^{t_2}\ln f(x_2, t_2) \leq \Gamma(x_1, t_1, x_2, t_2) + \ln\Big(\frac{1 - \mathrm{e}^{t_2}}{1 - \mathrm{e}^{t_1}}\Big).$$

Theorem 3.1 improves Theorem D because the exponential correction term $\frac{1}{\mathrm{e}^t - 1}$ is smaller than $\frac{1}{t}$ for all $t > 0$. If we take $\varepsilon = 0$, we can get the differential Harnack inequality of Cao, Fayyazuddin Ljungberg and Liu [4] on closed surfaces.

Corollary 3.2 (see [4]) *If $f : M \times [0, T) \to \mathbb{R}$ is a positive solution to the nonlinear heat equation*

$$\frac{\partial}{\partial t}\omega = \Delta\omega - \omega \ln \omega$$

on a closed surface (M, g) with $R > 0$, then for all time $t \in (0, T)$,

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{\mathrm{e}^t - 1} = \Delta \ln f + \frac{1}{\mathrm{e}^t - 1} \geq 0.$$

If set $\varepsilon = 1$ in Theorem 3.1, then we have the following result.

Corollary 3.3 *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the Ricci flow on a closed surface with $R > 0$. If f is a positive solution to (3.1), then for all time $t \in (0, T)$,*

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{\mathrm{e}^t - 1} = \Delta \ln f + R + \frac{1}{\mathrm{e}^t - 1} \geq 0.$$

Remark 3.2 Theorem 3.1 is a nonlinear version of the Chow's interpolated Harnack inequality (see [9]) which links Corollary 3.2 to Corollary 3.3.

Now we prove Theorem 3.1 via the maximum principle.

Proof of Theorem 3.1 Let $(M, g(t))$, $t \in [0, T)$ be a solution to the ε-Ricci flow (3.4) on a closed surface with $R > 0$. Let f be a positive solution to (3.5). By the maximum principle, we conclude that the solution will remain positive along the ε-Ricci flow when scalar curvature is positive. If

$$u = -\ln f,$$

then u satisfies

$$\frac{\partial}{\partial t}u = \Delta u - |\nabla u|^2 - \varepsilon R - u.$$

The proof involves a direct computation and the parabolic maximum principle.

Under the ε-Ricci flow (3.4) on a closed surface, we have

$$\frac{\partial R}{\partial t} = \varepsilon(\Delta R + R^2)$$

and

$$\frac{\partial}{\partial t}(\Delta) = \varepsilon R \Delta,$$

where the Laplacian Δ is acting on functions. Define the Harnack quantity

$$H_\varepsilon := \Delta u - \varepsilon R. \tag{3.7}$$

Using the evolution equations above, we first compute

$$\begin{aligned}\frac{\partial}{\partial t} H_\varepsilon &= \Delta\Big(\frac{\partial}{\partial t}u\Big) + \Big(\frac{\partial}{\partial t}\Delta\Big)u - \varepsilon\frac{\partial R}{\partial t} \\ &= \Delta(\Delta u - |\nabla u|^2 - \varepsilon R - u) + \varepsilon R \Delta u - \varepsilon \frac{\partial R}{\partial t} \\ &= \Delta H_\varepsilon - \Delta|\nabla u|^2 - \Delta u + \varepsilon R H_\varepsilon + \varepsilon^2 R^2 - \varepsilon\frac{\partial R}{\partial t}.\end{aligned}$$

Since

$$\Delta|\nabla u|^2 = 2|\nabla\nabla u|^2 + 2\nabla\Delta u\cdot\nabla u + R|\nabla u|^2$$

on a two-dimensional surface, we then have

$$\begin{aligned}\frac{\partial}{\partial t}H_\varepsilon &= \Delta H_\varepsilon - 2|\nabla\nabla u|^2 - 2\nabla\Delta u\cdot\nabla u - R|\nabla u|^2\\ &\quad + \varepsilon R H_\varepsilon + \varepsilon^2 R^2 - \varepsilon\frac{\partial R}{\partial t} - \Delta u\\ &= \Delta H_\varepsilon - 2|\nabla\nabla u|^2 - 2\nabla H_\varepsilon\cdot\nabla u - 2\varepsilon\nabla R\cdot\nabla u\\ &\quad - R|\nabla u|^2 + \varepsilon R H_\varepsilon + \varepsilon^2 R^2 - \varepsilon\frac{\partial R}{\partial t} - \Delta u\\ &= \Delta H_\varepsilon - 2\Big|\nabla_i\nabla_j u - \frac{\varepsilon}{2}Rg_{ij}\Big|^2 - 2\varepsilon R\Delta u - 2\nabla H_\varepsilon\cdot\nabla u\\ &\quad - 2\varepsilon\nabla R\cdot\nabla u - R|\nabla u|^2 + \varepsilon R H_\varepsilon + 2\varepsilon^2 R^2 - \varepsilon\frac{\partial R}{\partial t} - \Delta u.\end{aligned}$$

Since $\Delta u = H_\varepsilon + \varepsilon R$ by (3.7), these equalities become

$$\begin{aligned}\frac{\partial}{\partial t}H_\varepsilon = \Delta H_\varepsilon - 2\Big|\nabla_i\nabla_j u - \frac{\varepsilon}{2}Rg_{ij}\Big|^2 - \varepsilon R H_\varepsilon - 2\nabla H_\varepsilon\cdot\nabla u\\ - 2\varepsilon\nabla R\cdot\nabla u - R|\nabla u|^2 - \varepsilon\frac{\partial R}{\partial t} - \Delta u.\end{aligned}$$

Rearranging terms yields

$$\begin{aligned}\frac{\partial}{\partial t}H_\varepsilon &= \Delta H_\varepsilon - 2\Big|\nabla_i\nabla_j u - \frac{\varepsilon}{2}Rg_{ij}\Big|^2 - 2\nabla H_\varepsilon\cdot\nabla u - \varepsilon R H_\varepsilon\\ &\quad - R|\nabla u + \varepsilon\nabla\ln R|^2 - \varepsilon R\Big(\frac{\partial\ln R}{\partial t} - \varepsilon|\nabla\ln R|^2\Big) - \Delta u\\ &\le \Delta H_\varepsilon - H_\varepsilon^2 - 2\nabla H_\varepsilon\cdot\nabla u - (\varepsilon R+1)H_\varepsilon + \frac{\varepsilon}{t}R - \varepsilon R.\end{aligned}\tag{3.8}$$

The reason for this last inequality is that the trace Harnack inequality for the ε-Ricci flow on a closed surface proved in [9] states that

$$\frac{\partial\ln R}{\partial t} - \varepsilon|\nabla\ln R|^2 = \varepsilon(\Delta\ln R + R) \ge -\frac{1}{t},$$

since $g(t)$ has positive scalar curvature. Besides this, we also used (3.7) and the elementary inequality

$$\Big|\nabla_i\nabla_j u - \frac{\varepsilon}{2}Rg_{ij}\Big|^2 \ge \frac{1}{2}(\Delta u - \varepsilon R)^2 = \frac{1}{2}H_\varepsilon^2.$$

Adding $-\frac{1}{\mathrm{e}^t-1}$ to H_ε in (3.8) yields

$$\begin{aligned}\frac{\partial}{\partial t}\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big) &\le \Delta\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big) - 2\nabla\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big)\cdot\nabla u\\ &\quad - \Big(H_\varepsilon + \frac{1}{\mathrm{e}^t-1}\Big)\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big) - (\varepsilon R+1)\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big)\\ &\quad + \frac{\mathrm{e}^t}{(\mathrm{e}^t-1)^2} - \frac{1}{(\mathrm{e}^t-1)^2} - \frac{\varepsilon R+1}{\mathrm{e}^t-1} + \frac{\varepsilon R}{t} - \varepsilon R\\ &= \Delta\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big) - 2\nabla\Big(H_\varepsilon - \frac{1}{\mathrm{e}^t-1}\Big)\cdot\nabla u\end{aligned}$$

$$
\begin{aligned}
&-\Big(H_\varepsilon+\frac{1}{\mathrm{e}^t-1}\Big)\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big)-(\varepsilon R+1)\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big)\\
&-\varepsilon R\Big(\frac{1}{\mathrm{e}^t-1}+1-\frac{1}{t}\Big).
\end{aligned}\tag{3.9}
$$

Note that we claim

$$\frac{1}{\mathrm{e}^t-1}+1-\frac{1}{t}>0$$

for all $t>0$, which can be explained as follows. We first observe

$$\frac{1}{\mathrm{e}^t-1}+1-\frac{1}{t}=\frac{t\mathrm{e}^t-\mathrm{e}^t+1}{t(\mathrm{e}^t-1)}.$$

Since $t(\mathrm{e}^t-1)>0$, we only need to prove $t\mathrm{e}^t-\mathrm{e}^t+1>0$. This is obvious. Since

$$t\mathrm{e}^t-\mathrm{e}^t+1\,|_{t=0}=0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\mathrm{e}^t-\mathrm{e}^t+1)=t\mathrm{e}^t>0$$

for all $t>0$, the function $t\mathrm{e}^t-\mathrm{e}^t+1$ is increasing for $t\geq 0$. Therefore

$$t\mathrm{e}^t-\mathrm{e}^t+1>0$$

for $t>0$ and we prove that

$$\frac{1}{\mathrm{e}^t-1}+1-\frac{1}{t}>0$$

for all $t>0$. Thus (3.9) becomes

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big)\leq{}&\Delta\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big)-2\nabla\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big)\cdot\nabla u\\
&-\Big(H_\varepsilon+\frac{1}{\mathrm{e}^t-1}\Big)\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big)-(\varepsilon R+1)\Big(H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\Big).
\end{aligned}\tag{3.10}
$$

Clearly, for t small enough we have

$$H_\varepsilon-\frac{1}{\mathrm{e}^t-1}<0.$$

Since $R>0$, applying the maximum principle to the evolution formula (3.9) we conclude $H_\varepsilon-\frac{1}{\mathrm{e}^t-1}\leq 0$ for all positive time t, and the proof of this theorem is completed.

Remark 3.3 A question can be naturally posed: Can one improve Theorem C (high dimensional case) by considering the exponential correction term instead of the polynomial correction term in differential Harnack quantities?

4 New Differential Harnack Inequality Without Curvature Condition

In this section, we will study differential Harnack inequalities for a positive solution $f(x,t)<1$ to the nonlinear heat equation

$$\frac{\partial}{\partial t}\omega=\Delta\omega-\omega\ln\omega\tag{4.1}$$

with the metric evolved by the Ricci flow (3.2) on an n-dimensional closed manifold. This equation has been considered by Hsu [19] and Wu [31]. In [31], Wu proved the following result without any curvature assumption.

Theorem E (see [31]) *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the Ricci flow (3.2) on a closed manifold. Let $f < 1$ be a positive solution to (4.1) and $u = -\ln f$. Then for all $t \in (0, T)$,*

$$|\nabla u|^2 - \frac{u}{t} \le 0.$$

Theorem E can be also regarded as a nonlinear version of Cao and Hamilton's result (see [6, Theorem 5.1]). Now we can improve this result as follows.

Theorem 4.1 *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the Ricci flow on a closed manifold. Let $f < 1$ be a positive solution to (4.1) and $u = -\ln f$. Then for all $x \in M^n$, $t \in (0, T)$,*

$$|\nabla u|^2 - \frac{u}{\mathrm{e}^t - 1} \le 0.$$

We will prove Theorem 4.1 by the standard parabolic maximum principle. Let $f(x, t) < 1$ be a positive solution to nonlinear heat equation (4.1) under the Ricci flow (3.2) on a closed manifold M. If $u = -\ln f$, then $u > 0$ and u satisfies

$$\frac{\partial}{\partial t} u = \Delta u - |\nabla u|^2 - u.$$

Note that here $0 < f < 1$ is preserved under the Ricci flow by the maximum principle (see [31]).

Proof of Theorem 4.1 Following the arguments of [31], let

$$H := |\nabla u|^2 - \frac{u}{\mathrm{e}^t - 1}.$$

We first compute that $|\nabla u|^2$ satisfies

$$\frac{\partial}{\partial t} |\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla\nabla u|^2 - 2\nabla u \cdot \nabla(|\nabla u|^2) - 2|\nabla u|^2.$$

Then we also have

$$\frac{\partial}{\partial t}\Big(\frac{u}{\mathrm{e}^t - 1}\Big) = \Delta\Big(\frac{u}{\mathrm{e}^t - 1}\Big) - \frac{|\nabla u|^2 + u}{\mathrm{e}^t - 1} - \frac{u\mathrm{e}^t}{(\mathrm{e}^t - 1)^2}.$$

Combining above equations yields

$$\frac{\partial}{\partial t} H = \Delta H - 2\nabla u \cdot \nabla H - 2|\nabla\nabla u|^2 - \Big(2 + \frac{1}{\mathrm{e}^t - 1}\Big) H.$$

Notice that if t is small enough, then $H < 0$. Then applying the maximum principle to this equation, we obtain $H < 0$ for all $t > 0$.

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