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Generalized Ejiri's Rigidity Theorem for Submanifolds in Pinched Manifolds*

(*In memory of Professor Chaohao Gu on his 90th birthday*)

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Abstract Let $M^n (n \geq 4)$ be an oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional complete simply connected Riemannian manifold N^{n+p}. Then there exists a constant $\delta(n,p) \in (0,1)$ such that if the sectional curvature of N satisfies $\overline{K}_N \in [\delta(n,p), 1]$, and if M has a lower bound for Ricci curvature and an upper bound for scalar curvature, then N is isometric to S^{n+p}. Moreover, M is either a totally umbilic sphere $S^n\big(\frac{1}{\sqrt{1+H^2}}\big)$, a Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big) \times S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big)$ in the totally umbilic sphere $S^{n+1}\big(\frac{1}{\sqrt{1+H^2}}\big)$ with $n = 2m$, or $\mathbb{C}P^2\big(\frac{4}{3}(1+H^2)\big)$ in $S^7\big(\frac{1}{\sqrt{1+H^2}}\big)$. This is a generalization of Ejiri's rigidity theorem.

Keywords Minimal submanifold, Ejiri rigidity theorem, Ricci curvature, Mean curvature
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1 Introduction

The investigation of rigidity of submanifolds with parallel mean curvature attracts a lot of attention of differential geometers. After the pioneering work on compact minimal submanifolds in a sphere due to Simons [11], Lawson [3] and Chern-do Carmo-Kobayashi [1] obtained a classification theorem of n-dimensional oriented compact minimal submanifolds in S^{n+p}, whose squared norm of the second fundamental form satisfies $S \leq \frac{n}{\left(2-\frac{1}{p}\right)}$. In 1991, Li-Li [4] improved Simons' pinching constant for n-dimensional compact minimal submanifolds in S^{n+p} to $\max\big\{\frac{n}{2-\frac{1}{p}}, \frac{2}{3}n\big\}$.

This rigidity result was partially extended to submanifolds with parallel mean curvature in a sphere by Okumura [6–7], Yau [18–19] and others. In 1990s, Xu [12–13] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in spheres. When N is a positive pinched Riemannian manifold, Shiohama and Xu [10, 15] proved an interesting rigidity theorem for compact submanifolds with parallel

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mean curvature in N, which is an extension of the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem.

In 1979, Ejiri obtained the following rigidity theorem for $n(\geq 4)$-dimensional oriented compact simply connected minimal submanifolds with pinched Ricci curvatures in a sphere.

Theorem A (see [2]) *Let M be an $n(\geq 4)$-dimensional oriented compact simply connected minimal submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. If the Ricci curvature of M satisfies $\mathrm{Ric}_M \geq n-2$, then M is either the totally geodesic submanifold S^n, the Clifford torus $S^m\big(\sqrt{\frac{1}{2}}\big) \times S^m\big(\sqrt{\frac{1}{2}}\big)$ in S^{n+1} with $n=2m$, or $\mathbb{C}P^2\big(\frac{4}{3}\big)$ in S^7. Here $\mathbb{C}P^2\big(\frac{4}{3}\big)$ denotes the 2-dimensional complex projective space minimally immersed into S^7 with constant holomorphic sectional curvature $\frac{4}{3}$.*

In 1990s, Shen [9] and Li [5] extended Ejiri's rigidity theorem to the case of 3-dimensional compact minimal submanifolds in a sphere. In 2011, Xu and Tian [17] obtained a refined version of the Ejiri rigidity theorem without the assumption that M is simply connected. Recently, Xu and Gu [16] proved the following rigidity theorem for submanifolds with parallel mean curvature in space forms.

Theorem B (see [16]) *Let M be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in the space form $F^{n+p}(c)$ with $c+H^2>0$. If*

$$\mathrm{Ric}_M \geq (n-2)(c+H^2),$$

then M is either a totally umbilic sphere $S^n\big(\frac{1}{\sqrt{c+H^2}}\big)$, a Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2(c+H^2)}}\big) \times S^m\big(\frac{1}{\sqrt{2(c+H^2)}}\big)$ in the totally umbilic sphere $S^{n+1}\big(\frac{1}{\sqrt{c+H^2}}\big)$ with $n=2m$, or $\mathbb{C}P^2\big(\frac{4}{3}(c+H^2)\big)$ in $S^7\big(\frac{1}{\sqrt{c+H^2}}\big)$. Here $\mathbb{C}P^2\big(\frac{4}{3}(c+H^2)\big)$ denotes the 2-dimensional complex projective space minimally immersed into $S^7\big(\frac{1}{\sqrt{c+H^2}}\big)$ with constant holomorphic sectional curvature $\frac{4}{3}(c+H^2)$.

In this paper, motivated by Shiohama and Xu's work [10, 15], we will study the rigidity problem for submanifolds with parallel mean curvature under Ricci curvature pinching condition in a positive pinched Riemannian manifold, and prove the following theorem.

Main Theorem *Let M be an $n(\geq 4)$-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional complete simply connected Riemannian manifold N^{n+p}. Then there exists a constant $\delta(n,p) \in (0,1)$, such that if the sectional curvature of N satisfies $\overline{K}_N \in [\delta(n,p),1]$, and if*

$$\mathrm{Ric}_M \geq (n-2)(1+H^2) + A_1(n,p)(1-c) + A_2(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}},$$

$$R \leq n[(n-1)(1+H^2) - B_1(n,p)(1-c) - B_2(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}],$$

where $c := \inf \overline{K}_N$, then N^{n+p} is isometric to S^{n+p}. Moreover, M is either a totally umbilic sphere $S^n\big(\frac{1}{\sqrt{1+H^2}}\big)$, a Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big) \times S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big)$ in the totally umbilic sphere $S^{n+1}\big(\frac{1}{\sqrt{1+H^2}}\big)$ with $n=2m$, or $\mathbb{C}P^2\big(\frac{4}{3}(1+H^2)\big)$ in $S^7\big(\frac{1}{\sqrt{1+H^2}}\big)$. Here $\delta(n,p)$,

$A_2(n,p)$, $A_3(n,p)$, $B_2(n,p)$, $B_3(n,p)$ will be given in the proof, which are nonnegative constants depending on n and p.

Remark 1.1 When $c = 1$, the condition on the upper bound for scalar curvature in Main Theorem is automatically satisfied. Therefore, Main Theorem generalizes Theorems A and B.

Since the constant $\delta(n,p)$ satisfies $\delta(n,p) > \frac{1}{4}$, $\overline{K}_N \in [\delta(n,p), 1]$ implies that the ambient manifold N is diffeomorphic to S^{n+p}. Furthermore, we see that if N is not isometric to the standard sphere S^{n+p}, then there exists no submanifold with parallel mean curvature satisfying the pinching condition in Main Theorem.

2 Notation and Lemmas

Throughout this paper, let M be an $n(\geq 4)$-dimensional compact Riemannian manifold isometrically immersed into an $(n+p)$-dimensional complete and simply connected Riemannian manifold N^{n+p}. The following convention of indices are used throughout:

$$1 \leq A, B, C, \cdots \leq n+p, \quad 1 \leq i, j, k, \cdots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p.$$

Choose a local orthonormal frame field $\{e_A\}$ in N such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of N^{n+p}, respectively. Then we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha,$$

where h and ξ are the second fundamental form and the mean curvature field of M, respectively. Denote by $\overline{K}(\cdot)$, $\overline{R}_{ABCD}$ the sectional curvature and the curvature tensor of N. Let $a(x)$, $b(x)$ for $x \in N$ be the minimum and maximum of $\overline{K}_N$ at that point. Then by Berger's inequality, we obtain that

$$|\overline{R}_{ABCD}| \leq \frac{2}{3}(b-a) \tag{2.1}$$

for all distinct indices A, B, C, D, and

$$|\overline{R}_{ACBC}| \leq \frac{1}{2}(b-a) \tag{2.2}$$

for all distinct indices A, B, C. The curvature tensor and the normal curvature tensor of M are denoted by R_{ijkl} and $R_{\alpha\beta kl}$, respectively. Then we have

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.3}$$

$$R_{\alpha\beta kl} = \overline{R}_{\alpha\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{2.4}$$

Denote by $\mathrm{Ric}(u)$ the Ricci curvature of M in direction of $u \in UM$. From the Gauss equation, we get

$$\mathrm{Ric}(e_i) = \sum_j \overline{R}_{ijij} + \sum_{\alpha,j} \big[h^\alpha_{ii} h^\alpha_{jj} - (h^\alpha_{ij})^2\big]. \tag{2.5}$$

Set $\mathrm{Ric}_{\min}(x) = \min\limits_{u \in U_x M} \mathrm{Ric}(u)$. The scalar curvature R of M is given by

$$R = \sum_{i,j} \overline{R}_{ijij} + n^2 H^2 - S. \tag{2.6}$$

For an $(n \times n)$-matrix $A = (a_{ij})$, we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = \mathrm{tr}(AA^{\mathrm{T}}) = \sum_{i,j} a^2_{ij}.$$

We define

$$S = |h|^2, \quad H = |\xi|, \quad H_\alpha = (h^\alpha_{ij})_{n\times n}.$$

Definition 2.1 *M is called a submanifold with parallel mean curvature if ξ is parallel in the normal bundle of M. In particular, M is called minimal if $\xi = 0$.*

We assume that M admits a parallel mean curvature normal field and $H \neq 0$. We choose e_{n+1} such that $e_{n+1} \parallel \xi$, then $\mathrm{tr}\, H_{n+1} = nH$, and $\mathrm{tr}\, H_\beta = 0$ for $n+2 \le \beta \le n+p$. Set

$$S_H = \sum_{i,j} (h^{n+1}_{ij})^2, \quad S_I = \sum_{i,j,\beta\neq n+1} (h^\beta_{ij})^2.$$

Denoting the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl}, respectively. We have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha},$$
$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Hence

$$h^\alpha_{ijk} = h^\alpha_{ikj} - \overline{R}_{\alpha ijk}, \quad h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im} R_{mjkl} + \sum_m h^\alpha_{mj} R_{mikl} - \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}. \tag{2.7}$$

Since M^n is a submanifold with parallel mean curvature of N^{n+p}, $\mathrm{tr}\, H_\alpha$ is constant, i.e., $\sum\limits_i h^\alpha_{iikl} = 0$. Therefore

$$\triangle h^\alpha_{ij} = -\sum_k (\overline{R}_{\alpha kikj} + \overline{R}_{\alpha ijkk}) + \sum_{k,m} h^\alpha_{km} R_{mijk} + \sum_{k,m} h^\alpha_{mi} R_{mkjk} - \sum_{k,\beta} h^\beta_{ki} R_{\alpha\beta jk}. \tag{2.8}$$

The following lemma will be used in the proof of our main results.

Lemma 2.1 (see [18–19]) *If M^n is a submanifold with parallel mean curvature, then either $H \equiv 0$ or H is non-zero constant and $H_\alpha H_{n+1} = H_{n+1} H_\alpha + (\overline{R}_{n+1\alpha ij})_{n\times n}$ for $\alpha \neq n+1$.*

We also need the following lemma, which can be found in [8, 10] (also see [14]).

Lemma 2.2 *Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers satisfying $\sum_i a_i = \sum_i b_i = 0$, $\sum_i a_i^2 = a$ and $\sum_i b_i^2 = b$. Then*

$$\Big|\sum_i a_i b_i^2\Big| \le (n-2)[n(n-1)]^{-\frac{1}{2}} a^{\frac{1}{2}} b,$$

where equality holds if and only if either $ab = 0$, or at least $n-1$ pairs of numbers of (a_i, b_i)'s are the same.

3 Minimal Submanifolds

Let $M^n (n \ge 4)$ be an oriented compact minimal submanifold in N^{n+p}. We choose a frame $\{e_\alpha\}$ such that $\mathrm{tr}(H_\alpha H_\beta) = 0$ for $\alpha \ne \beta$. Then we get from (2.3), (2.5) and (2.8) that

$$\frac{1}{2}\triangle S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \mathrm{X}_1 + \mathrm{Y}_1 + \mathrm{Z}_1,$$

where

$$\begin{aligned}
\mathrm{X}_1 &:= -\sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_\alpha (\mathrm{tr}\, H_\alpha^2)^2, \\
\mathrm{Y}_1 &:= \sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik} + h_{mk}^\alpha h_{ij}^\alpha \overline{R}_{mijk}) - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta \overline{R}_{\alpha\beta jk}, \\
\mathrm{Z}_1 &:= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha \overline{R}_{\alpha kikj} + h_{ij}^\alpha \overline{R}_{\alpha ijkk}).
\end{aligned}$$

For fixed α, we choose the orthonormal frame fields $\{e_i\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, and have the following lemma.

Lemma 3.1 $\mathrm{X}_1 \ge n[\mathrm{Ric}_{\min} - (n-1)b]S$.

Proof (i) If $p = 1$, then it follows from (2.6) that

$$\mathrm{X}_1 = -S^2 \ge nS[\mathrm{Ric}_{\min} - (n-1)b]. \tag{3.1}$$

If $p \ge 2$, then for fixed e_α, let $\{e_i\}$ be a frame diagonalizing the matrix H_α such that $h_{ij}^\alpha = 0$ for $i \ne j$. So

$$(n-1)b - (h_{ii}^\alpha)^2 - \sum_{j,\beta\ne\alpha} (h_{ij}^\beta)^2 \ge \mathrm{Ric}(e_i) \ge \mathrm{Ric}_{\min}. \tag{3.2}$$

This implies that

$$\sum_{j,\beta\ne\alpha} (h_{ij}^\beta)^2 \le (n-1)b - (h_{ii}^\alpha)^2 - \mathrm{Ric}_{\min}. \tag{3.3}$$

On the other hand,

$$\sum_{\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) = \sum_{i,j,\beta\neq\alpha} (h_{ij}^{\beta})^2 (h_{ii}^{\alpha} - h_{jj}^{\alpha})^2. \tag{3.4}$$

This together with (3.3) and

$$(h_{ii}^{\alpha} - h_{jj}^{\alpha})^2 \leq 2[(h_{ii}^{\alpha})^2 + (h_{jj}^{\alpha})^2], \tag{3.5}$$

implies that

$$\begin{aligned}\sum_{\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) &\leq 4 \sum_{i,j,\beta\neq\alpha} (h_{ij}^{\beta})^2 (h_{ii}^{\alpha})^2 \\ &\leq 4 \sum_{i} \{[(n-1)b - (h_{ii}^{\alpha})^2 - \mathrm{Ric}_{\min}](h_{ii}^{\alpha})^2\} \\ &\leq 4[(n-1)b - \mathrm{Ric}_{\min}] \sum_{i} (h_{ii}^{\alpha})^2 - \frac{4}{n}(\mathrm{tr}\, H_\alpha^2)^2. \end{aligned} \tag{3.6}$$

Then we have

$$\sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) \leq 4[(n-1)b - \mathrm{Ric}_{\min}]S - \frac{4}{n} \sum_{\alpha} (\mathrm{tr}\, H_\alpha^2)^2. \tag{3.7}$$

Therefore, we obtain that

$$\begin{aligned}\mathrm{X}_1 &\geq 4[\mathrm{Ric}_{\min} - (n-1)b]S - \frac{n-4}{n} \sum_{\alpha} (\mathrm{tr}\, H_\alpha^2)^2 \\ &\geq 4[\mathrm{Ric}_{\min} - (n-1)b]S - \frac{n-4}{n} S^2 \\ &\geq n[\mathrm{Ric}_{\min} - (n-1)b]S. \end{aligned} \tag{3.8}$$

This completes the proof.

The estimates of Y_1 and Z_1 can be found in [15].

Lemma 3.2 (see [15]) (i) $\mathrm{Y}_1 \geq nbS - [n + \frac{2}{3}(p-1)(n-1)^{\frac{1}{2}}](b-a)S$;
(ii) $\int_M \mathrm{Z}_1 \mathrm{d}M \geq -\frac{1}{72} pn(n-1)(26n-25) \int_M (b-a)^2 \mathrm{d}M$.

Combing Lemmas 3.1–3.2, we get the following theorem.

Theorem 3.1 *Let $M^n (n \geq 4)$ be an oriented closed minimal submanifolds in a Riemannian manifolds N^{n+p}. Then*

$$\int_M \{nS[\mathrm{Ric}_{\min} - (n-2)b - G(n,p)(b-a)] - D(n,p)(b-a)^2\} \mathrm{d}M \leq 0.$$

Here

$$\begin{aligned} G(n,q) &:= 1 + \frac{2}{3n}(n-1)^{\frac{1}{2}}(q-1), \\ D(n,q) &:= \frac{1}{72} qn(n-1)(26n-25). \end{aligned}$$

Furthermore, we obtain the following rigidity theorem for minimal submanifolds.

Theorem 3.2 *Let $M^n (n \geq 4)$ be an oriented closed minimal submanifold in a complete simply connected Riemannian manifold N^{n+p}. Then there exists a constant $\theta_1(n,p) \in (0,1)$, such that if $\overline{K}_N \in [\theta_1(n,p), 1]$, and if*

$$\begin{aligned} \mathrm{Ric}_M &\geq n-2+\beta_1(n,p)(1-c), \\ R &\leq n(n-1)-\gamma_1(n,p)n(1-c), \end{aligned}$$

where $c := \inf \overline{K}_N$, then N^{n+p} is isometric to S^{n+p}. Moreover, M is either a totally geodesic sphere S^n, the Clifford torus $S^m\big(\sqrt{\frac{1}{2}}\big) \times S^m\big(\sqrt{\frac{1}{2}}\big)$ in S^{n+1} with $n = 2m$, or $\mathbb{C}\mathrm{P}^2\big(\frac{4}{3}\big)$ in S^7. Here

$$\begin{aligned} \beta_1(n,p) &= G(n,p) + D^{\frac{1}{2}}(n,p)n^{-1}, \\ \gamma_1(n,p) &= n-1+D^{\frac{1}{2}}(n,p)n^{-1}, \\ \theta_1(n,p) &= 1-[\beta_1(n,p)+\gamma_1(n,p)]^{-1}. \end{aligned}$$

Proof Since $c \leq a(x) \leq b(x) \leq 1$, it follows from Theorem 3.1 that

$$\int_M \{nS[\mathrm{Ric}_{\min} - (n-2) - G(n,p)(1-c)] - D(n,p)(1-c)^2\} \mathrm{d}M \leq 0. \tag{3.9}$$

From the assumption, we have

$$\theta_1(n,p) = 1-[\beta_1(n,p)+\gamma_1(n,p)]^{-1}.$$

Then

$$\begin{aligned} 1-c &\leq 1-\theta_1(n,p) \\ &= [\beta_1(n,p)+\gamma_1(n,p)]^{-1}. \end{aligned} \tag{3.10}$$

Therefore

$$\beta_1(n,p)(1-c) \leq 1-\gamma_1(n,p)(1-c). \tag{3.11}$$

It follows from (3.11) that the assumptions of the lower bound for the Ricci curvature and the upper bound for the scalar curvature are consistent. Then it is seen from (2.6) and the upper bound of R that

$$S \geq n[\gamma_1(n,p)-(n-1)](1-c). \tag{3.12}$$

This together with the definitions of $\beta_1(n,p)$, $\gamma_1(n,p)$ and the assumption

$$\mathrm{Ric}_M \geq n-2+\beta_1(n,p)(1-c),$$

implies that

$$nS[\mathrm{Ric}_{\min} - (n-2) - G(n,p)(1-c)]$$

$$\begin{aligned}&\geq n^2[\gamma_1(n,p)-(n-1)](1-c)[\beta_1(n,p)-G(n,p)](1-c)\\&=D(n,p)(1-c)^2.\end{aligned}\tag{3.13}$$

Hence, we obtain

$$\int_M\{nS[\mathrm{Ric}_{\min}-(n-2)-G(n,p)(1-c)]-D(n,p)(1-c)^2\}\mathrm{d}M\geq 0.\tag{3.14}$$

It is seen from (3.9) and (3.14) that the left side of (3.14) is equal to zero, which together with $c\leq a\leq b\leq 1$ implies that $a\equiv c$ and $b\equiv 1$. By a similar argument as in [15], we get $1-c=0$. Since N is complete and simply connected, we know that N is isometric to S^{n+p}. Moreover, it follows from Ejiri's theorem that M is either a totally geodesic sphere S^n, the Clifford torus $S^m\big(\sqrt{\frac{1}{2}}\big)\times S^m\big(\sqrt{\frac{1}{2}}\big)$ in S^{n+1} with $n=2m$, or $\mathbb{C}P^2\big(\frac{4}{3}\big)$ in S^7. This completes the proof.

4 Submanifolds with Parallel Mean Curvature

Let $M^n(n\geq 4)$ be an oriented compact submanifold with parallel mean curvature in N^{n+p} and $H\neq 0$, then it follows the same argument as in [10] that

$$\frac{1}{2}\triangle S_H=(h_{ijk}^{n+1})^2+\sum_{i,j}h_{ij}^{n+1}\triangle h_{ij}^{n+1}=\mathrm{X}_2+\mathrm{Y}_2,$$

where

$$\begin{aligned}\mathrm{X}_2:=\;&nH\mathrm{tr}\,H_{n+1}^3-(\mathrm{tr}\,H_{n+1}^2)^2-\sum_{\alpha\neq n+1}[\mathrm{tr}(H_{n+1}H_\alpha)]^2\\&+\sum_{i,j,k,m}(h_{ij}^{n+1}h_{mj}^{n+1}\overline{R}_{mkik}+h_{ij}^{n+1}h_{mk}^{n+1}\overline{R}_{mijk}),\\\mathrm{Y}_2:=\;&\sum_{i,j,k}(h_{ijk}^{n+1})^2-\sum_{i,j,k}(h_{ij}^{n+1}\overline{R}_{n+1kikj}+h_{ij}^{n+1}\overline{R}_{n+1ijkk})\\&+\sum_{\alpha\neq n+1}\mathrm{tr}(H_{n+1}H_\alpha)^2-\sum_{\alpha\neq n+1}\mathrm{tr}(H_{n+1}^2H_\alpha^2).\end{aligned}$$

Lemma 4.1 $\mathrm{X}_2\geq n(S_H-nH^2)[\mathrm{Ric}_{\min}-(n-2)(b+H^2)-(b-a)]$.

Proof We choose the orthonormal frame fields $\{e_i\}$ such that $h_{ij}^{n+1}=\lambda_i^{n+1}\delta_{ij}$. Letting

$$\begin{aligned}&f_k=\sum_i(\lambda_i^{n+1})^k,\\&\mu_i^{n+1}=H-\lambda_i^{n+1},\quad i=1,2,\cdots,n,\\&B_k=\sum_i(\mu_i^{n+1})^k,\end{aligned}$$

we have

$$\begin{aligned}&B_1=0,\quad B_2=S_H-nH^2,\\&B_3=3HS_H-2nH^3-f_3.\end{aligned}$$

Then

$$
\begin{aligned}
\mathrm{X}_2 &= -S_H^2 + nHf_3 - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2 \\
&\quad + \sum_{i,k}(\lambda_i^{n+1})^2\overline{R}_{ikik} + \sum_{i,k}\lambda_i^{n+1}\lambda_k^{n+1}\overline{R}_{kiik} \\
&= -S_H^2 + nH(3HS_H - 2nH^3 - B_3) - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2 \\
&\quad + \frac{1}{2}\sum_{i,k}(\lambda_i^{n+1}-\lambda_k^{n+1})^2\overline{R}_{ikik} \\
&\geq B_2[na + 2nH^2 - S_H] - nHB_3 - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2. \qquad (4.1)
\end{aligned}
$$

Since

$$
(n-1)b + nH\lambda_i^{n+1} - (\lambda_i^{n+1})^2 - \sum_{\alpha\neq n+1,j}(h_{ij}^{\alpha})^2 \geq \mathrm{Ric}(e_i) \geq \mathrm{Ric}_{\min}, \qquad (4.2)
$$

we have

$$
S - nH^2 \leq n[(n-1)(b+H^2) - \mathrm{Ric}_{\min}] \qquad (4.3)
$$

and

$$
H(\lambda_i^{n+1} - H) \geq \frac{(\lambda_i^{n+1}-H)^2}{n-2} + \frac{\sum\limits_{\alpha\neq n+1,j}(h_{ij}^{\alpha})^2}{n-2} + \frac{\mathrm{Ric}_{\min}}{n-2} - \frac{n-1}{n-2}(b+H^2). \qquad (4.4)
$$

It follows from (4.1) and (4.3)–(4.4) that

$$
\begin{aligned}
\mathrm{X}_2 &\geq B_2\Big\{na + 2nH^2 - S_H + \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]\Big\} \\
&\quad + \frac{n}{n-2}\sum_i(\mu_i^{n+1})^4 + \sum_{\alpha\neq n+1}\Big[\frac{n}{n-2}\sum_i(h_{ii}^{\alpha})^2(\mu_i^{n+1})^2 - \Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2\Big] \\
&\geq B_2\Big\{na + 2nH^2 - S_H + \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]\Big\} \\
&\quad + \frac{B_2^2}{n-2} - \frac{n-3}{n-2}\sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2 \\
&\geq B_2\Big\{na + nH^2 - \frac{n-3}{n-2}(S-nH^2) + \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]\Big\} \\
&\geq \frac{n}{n-2}B_2\{(n-2)(a+H^2) - (n-3)[(n-1)(b+H^2) - \mathrm{Ric}_{\min}] \\
&\quad + [\mathrm{Ric}_{\min} - (n-1)(b+H^2)]\} \\
&= n(S_H - nH^2)[\mathrm{Ric}_{\min} - (n-2)(b+H^2) - (b-a)]. \qquad (4.5)
\end{aligned}
$$

This complete the lemma.

The estimate of Y_2 can be found in [10].

Lemma 4.2 (see [10]) $\int_M \mathrm{Y}_2 \mathrm{d}M \geq -\frac{1}{72}n(n-1)(26n+16p-41)\int_M (b-a)^2 \mathrm{d}M$.

Combing Lemmas 4.1–4.2, we get the following theorem.

Theorem 4.1 *If $M^n (n \geq 4)$ is an oriented compact submanifold with parallel mean curvature in a Riemannian manifold N^{n+p} and $H \neq 0$. Then*

$$\int_M \{n(S_H - nH^2)[\mathrm{Ric}_{\min} - (n-2)(b+H^2) - (b-a)] - E_1(n,p)(b-a)^2\}\mathrm{d}M \leq 0.$$

Here

$$E_1(n,q) := \frac{1}{72}n(n-1)(26n+16q-41).$$

If $p \geq 2$, we choose a frame $\{e_\alpha\}$ such that $\mathrm{tr}(H_\alpha H_\beta) = 0$ for $\alpha \neq \beta,\ \alpha,\ \beta > n+1$. It follows from the same argument as in [10] that

$$\frac{1}{2}\triangle S_I = \sum_{i,j,k,\alpha\neq n+1} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha\neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha = \mathrm{X}_3 + \mathrm{Y}_3 + \mathrm{Z}_3,$$

where

$$\begin{aligned}
\mathrm{X}_3 :=& -\sum_{\alpha,\beta\neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha\neq n+1} (\mathrm{tr}\, H_\alpha^2)^2 \\
&+ \sum_{\alpha\neq n+1} \mathrm{tr}(H_\alpha^2 H_{n+1})\mathrm{tr}\, H_{n+1} - \sum_{\alpha\neq n+1} [\mathrm{tr}(H_\alpha H_{n+1})]^2, \\
\mathrm{Y}_3 :=& \sum_{i,j,k,m,\alpha\neq n+1} (h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik} + h_{mk}^\alpha h_{ij}^\alpha \overline{R}_{mijk}) - \sum_{i,j,k,\alpha,\beta\neq n+1} h_{ij}^\alpha h_{ki}^\beta \overline{R}_{\alpha\beta jk}, \\
\mathrm{Z}_3 :=& \sum_{i,j,k,\alpha\neq n+1} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha\neq n+1} (h_{ij}^\alpha \overline{R}_{\alpha kikj} + h_{ij}^\alpha \overline{R}_{\alpha ijkk}) \\
&- \sum_{\alpha\neq n+1} [\mathrm{tr}(H_\alpha^2 H_{n+1}^2) - \mathrm{tr}(H_\alpha H_{n+1})^2].
\end{aligned}$$

Lemma 4.3

$$\mathrm{X}_3 \geq nS_I\Big[H^2 + \mathrm{Ric}_{\min} - (n-1)(b+H^2) - \mathrm{sgn}(p-2)\frac{(3n-8)(n-2)}{n\sqrt{n(n-1)}}H\sqrt{S_H - nH^2}\Big].$$

Proof If $p = 2$, we choose an orthonormal frame fields $\{e_i\}$ such that $h_{ij}^{n+2} = 0$ for $i \neq j$. Then we have

$$(n-2)H(h_{ii}^{n+1} - H) \geq \mathrm{Ric}_{\min} - (n-1)(b+H^2) + (h_{ii}^{n+1} - H)^2 + (h_{ii}^{n+2})^2.$$

Hence we obtain

$$\begin{aligned}
&\mathrm{tr}(H_{n+2}^2 H_{n+1})\mathrm{tr}\, H_{n+1} - [\mathrm{tr}(H_{n+2}H_{n+1})]^2 \\
&= -[\mathrm{tr}(H_{n+1} - HI)H_{n+2}]^2 + nH\mathrm{tr}[(H_{n+1} - HI)H_{n+2}^2] + nH^2 S_I \\
&= nH\sum_i (h_{ii}^{n+1} - H)(h_{ii}^{n+2})^2 - \Big[\sum_i (h_{ii}^{n+1} - H)h_{ii}^{n+2}\Big]^2 + nH^2 S_I
\end{aligned}$$

$$\geq \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]S_I + \frac{1}{n-2}\Big[\Big(\sum_i (h_{ii}^{n+1} - H)h_{ii}^{n+2}\Big)^2$$
$$+ (\mathrm{tr}\, H_{n+2}^2)^2\Big] - \Big[\sum_i (h_{ii}^{n+1} - H)h_{ii}^{n+2}\Big]^2 + nH^2S_I$$
$$\geq \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]S_I$$
$$+ \frac{1}{n-2}(\mathrm{tr}\, H_{n+2}^2)^2 - \frac{n-3}{n-2}(S_H - nH^2)S_I + nH^2S_I. \tag{4.6}$$

Here I is a unit $(n \times n)$-matrix. This together with (4.3) implies that

$$\begin{aligned} \mathrm{X}_3 &\geq \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]S_I \\ &\quad - \frac{n-3}{n-2}(\mathrm{tr}\, H_{n+2}^2)^2 - \frac{n-3}{n-2}(S_H - nH^2)S_I + nH^2S_I \\ &= \frac{n}{n-2}[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]S_I \\ &\quad - \frac{n-3}{n-2}(S - nH^2)S_I + nH^2S_I \\ &\geq nS_I[H^2 + \mathrm{Ric}_{\min} - (n-1)(b+H^2)]. \end{aligned} \tag{4.7}$$

If $p \geq 3$, then for fixed $e_\alpha, \alpha \neq n+1$, we choose an orthonormal frame fields $\{e_i\}$ such that $h_{ij}^\alpha = 0$ for $i \neq j$. Hence we have

$$\begin{aligned} &(n-1)(b+H^2) + (n-2)H(h_{ii}^{n+1} - H) - (h_{ii}^{n+1} - H)^2 \\ &- (h_{ii}^\alpha)^2 - \sum_{j,\beta\neq\alpha,n+1} (h_{ij}^\beta)^2 \geq \mathrm{Ric}_{\min}. \end{aligned} \tag{4.8}$$

This implies that

$$\begin{aligned} \sum_{j,\beta\neq\alpha,n+1} (h_{ij}^\beta)^2 &\leq (n-1)(b+H^2) + (n-2)H(h_{ii}^{n+1} - H) \\ &\quad - (h_{ii}^{n+1} - H)^2 - (h_{ii}^\alpha)^2 - \mathrm{Ric}_{\min}. \end{aligned} \tag{4.9}$$

Combing (3.4)–(3.5) and (4.9), we get

$$\begin{aligned} &\sum_{\beta\neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) \\ &\leq 4\sum_i [(n-1)(b+H^2) - (h_{ii}^\alpha)^2 \\ &\quad + (n-2)H(h_{ii}^{n+1} - H) - (h_{ii}^{n+1} - H)^2 - \mathrm{Ric}_{\min}](h_{ii}^\alpha)^2 \\ &\leq 4[(n-1)(b+H^2) - \mathrm{Ric}_{\min}]\sum_i (h_{ii}^\alpha)^2 + 4(n-2)H\sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha)^2 \\ &\quad - \frac{4}{n}\Big[(\mathrm{tr}\, H_\alpha^2)^2 + \Big(\sum_i (h_{ii}^{n+1} - H)h_{ii}^\alpha\Big)^2\Big]. \end{aligned} \tag{4.10}$$

At the same time, we have

$$\mathrm{tr}(H_\alpha^2 H_{n+1})\mathrm{tr}\, H_{n+1} - [\mathrm{tr}(H_\alpha H_{n+1})]^2$$

$$= nH\sum_i (h_{ii}^{n+1}-H)(h_{ii}^{\alpha})^2 - \Big[\sum_i (h_{ii}^{n+1}-H)h_{ii}^{\alpha}\Big]^2 + nH^2\sum_i (h_{ii}^{\alpha})^2. \tag{4.11}$$

Using Lemma 2.2, we have

$$\sum_i (h_{ii}^{n+1}-H)(h_{ii}^{\alpha})^2 \le (n-2)[n(n-1)]^{-\frac{1}{2}}(S_H-nH^2)^{\frac{1}{2}}\operatorname{tr} H_\alpha^2. \tag{4.12}$$

From (4.3) and (4.10)–(4.12), we get

$$\begin{aligned}
&-\sum_{\beta\neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - (\operatorname{tr} H_\alpha^2)^2 + \operatorname{tr}(H_\alpha^2 H_{n+1})\operatorname{tr} H_{n+1} - [\operatorname{tr}(H_\alpha H_{n+1})]^2\\
&\ge 4[\mathrm{Ric}_{\min} - (n-1)(b+H^2)]\sum_i (h_{ii}^{\alpha})^2 - (3n-8)H\sum_i (h_{ii}^{n+1}-H)(h_{ii}^{\alpha})^2\\
&\quad - \frac{n-4}{n}\Big[(\operatorname{tr} H_\alpha^2)^2 + \Big(\sum_i (h_{ii}^{n+1}-H)h_{ii}^{\alpha}\Big)^2\Big] + nH^2\sum_i (h_{ii}^{\alpha})^2\\
&\ge \Big[nH^2 + 4\mathrm{Ric}_{\min} - 4(n-1)(b+H^2) - \frac{n-4}{n}(S-nH^2)\\
&\quad - \frac{(3n-8)(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_H-nH^2}\Big]\operatorname{tr} H_\alpha^2\\
&\ge n\Big[H^2 + \mathrm{Ric}_{\min} - (n-1)(b+H^2)\\
&\quad - \frac{(3n-8)(n-2)}{n\sqrt{n(n-1)}}H\sqrt{S_H-nH^2}\Big]\operatorname{tr} H_\alpha^2.
\end{aligned} \tag{4.13}$$

Then we obtain

$$\mathrm{X}_3 \ge nS_I\Big[H^2 + \mathrm{Ric}_{\min} - (n-1)(b+H^2) - \frac{(3n-8)(n-2)}{n\sqrt{n(n-1)}}H\sqrt{S_H-nH^2}\Big].$$

This proves the lemma.

The estimates of Y_3 and Z_3 can be found in [10].

Lemma 4.4 (see [10]) (i) $\mathrm{Y}_3 \ge naS_I - \frac{2}{3}(p-2)(n-1)^{\frac{1}{2}}(b-a)S_I$;
(ii) $\int_M \mathrm{Z}_3 \mathrm{d}M \ge -\frac{1}{72}(p-1)n(n-1)(26n-9)\int_M (b-a)^2 \mathrm{d}M$.

From Lemmas 4.3–4.4, we get the following theorem.

Theorem 4.2 *If $M^n (n\ge 4)$ is an oriented compact submanifold with parallel mean curvature in a Riemannian manifold N^{n+p} and $H\neq 0$. Then*

$$\int_M \{nS_I[\mathrm{Ric}_{\min} - (n-2)(b+H^2) - J(n,p)H(S_H-nH^2)^{\frac{1}{2}} - G(n,p-1)(b-a)] - E_2(n,p)(b-a)^2\}\mathrm{d}M \le 0.$$

Here $\mathrm{sgn}(\cdot)$ *is the standard sign function,*

$$E_2(n,q) := \frac{1}{72}(q-1)n(n-1)(26n-9),$$

$$G(n,q) := 1 + \frac{2}{3n}(n-1)^{\frac{1}{2}}(q-1),$$
$$J(n,q) := \mathrm{sgn}(q-2)(3n-8)(n-2)(n-1)^{-\frac{1}{2}}n^{-\frac{3}{2}}.$$

Let

$$E(n,q) := \frac{1}{72}n(n-1)(26qn+7q-32).$$

We have the following theorem.

Theorem 4.3 *Let $M^n (n \geq 4)$ be an oriented compact submanifold with parallel mean curvature in a complete simply connected Riemannian manifold N^{n+p}, $p \leq 2$ and $H \neq 0$. Then there exists a constant $\theta_2(n,p) \in (0,1)$, such that if $\overline{K}_N \in [\theta_2(n,p), 1]$, and if*

$$\mathrm{Ric}_M \geq (n-2)(1+H^2) + \beta_2(n,p)(1-c),$$
$$R \leq n(n-1)(1+H^2) - \gamma_2(n,p)n(1-c),$$

where $c := \inf \overline{K}_N$, then N^{n+p} is isometric to S^{n+p}. Moreover, M is either a totally umbilical sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, or the Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big) \times S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big)$ in $S^{n+1}\big(\frac{1}{\sqrt{1+H^2}}\big)(n=2m)$ for $p=2$. Here

$$\beta_2(n,p) = 1 + E^{\frac{1}{2}}(n,p)n^{-1},$$
$$\gamma_2(n,p) = n - 1 + E^{\frac{1}{2}}(n,p)n^{-1},$$
$$\theta_2(n,p) = 1 - [\beta_2(n,p) + \gamma_2(n,p)]^{-1}.$$

Proof Because $c \leq a(x) \leq b(x) \leq 1$ and $p \leq 2$, it follows from Theorems 4.1–4.2 that

$$\int_M \{n(S-nH^2)[\mathrm{Ric}_{\min} - (n-2)(1+H^2) - (1-c)] - E(n,p)(1-c)^2\}\mathrm{d}M \leq 0. \tag{4.14}$$

From the assumption

$$\theta_2(n,p) = 1 - [\beta_2(n,p) + \gamma_2(n,p)]^{-1},$$

we have

$$1 - c \leq 1 - \theta_2(n,p) = [\beta_2(n,p) + \gamma_2(n,p)]^{-1}. \tag{4.15}$$

So

$$\beta_2(n,p)(1-c) \leq 1 - \gamma_2(n,p)(1-c) \leq 1 + H^2 - \gamma_2(n,p)(1-c). \tag{4.16}$$

From (4.16), we see that the assumptions of the lower bound for the Ricci curvature and the upper bound for the scalar curvature are consistent. Then it follows from (2.6) and the assumption that

$$S - nH^2 \geq [\gamma_2(n,p) - (n-1)]n(1-c). \tag{4.17}$$

This together with the assumption implies

$$n(S-nH^2)[\mathrm{Ric}_{\min} - (n-2)(1+H^2) - (1-c)]$$

$$\begin{aligned}&\geq n^2[\gamma_2(n,p)-(n-1)][\beta_2(n,p)-1](1-c)^2\\&=E(n,p)(1-c)^2.\end{aligned}\tag{4.18}$$

Therefore

$$\int_M\{(S-nH^2)n[\mathrm{Ric}_{\min}-(n-2)(1+H^2)-(1-c)]-E(n,p)(1-c)^2\}\mathrm{d}M\geq 0.\tag{4.19}$$

From (4.14) and (4.19), we obtain the left side of (4.19) is equal to zero. This together with Theorems 4.1–4.2 and $c\leq a\leq b\leq 1$ implies that $a\equiv c$ and $b\equiv 1$. By the same argument as in [10] we have $1-c=0$. Since N is complete and simply connected, we get N is isometric to S^{n+p}. Moreover, it follows from Theorem B that M is totally umbilical sphere $S^n\big(\frac{1}{\sqrt{1+H^2}}\big)$, or $p=2$ and M is the Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big)\times S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big)$ in $S^{n+1}\big(\frac{1}{\sqrt{1+H^2}}\big)$ with $n=2m$. This completes the proof.

For the case $p\geq 3$, we need the following lemma.

Lemma 4.5 *Let $M^n(n\geq 4)$ be an oriented compact submanifold with parallel mean curvature in a Riemannian manifold N^{n+p}, $H\neq 0$. Let $a(x)$ and $b(x)$ for a point $x\in N$ be the minimum and maximum of $\overline{K}_N$ at the point x, respectively. If*

$$\mathrm{Ric}_M\geq (n-2)(d+H^2)+\beta_3(n,p)(d-c),$$

then $d=c$ or

$$\int_M(S_H-nH^2)\mathrm{d}M\leq \eta(n,p)\int_M(b-a)\mathrm{d}M.$$

Here $c:=\inf\overline{K}_N$, $d\geq\sup\overline{K}_N$, $\eta(n,p)=\frac{E_1(n,p)}{n[\beta_3(n,p)-1]}$.

Proof From $c\leq a\leq b\leq d$ and Theorem 4.1, we have

$$\int_M\{(S_H-nH^2)n[\mathrm{Ric}_{\min}-(n-2)(b+H^2)-(d-c)]-E_1(n,p)(b-a)(d-c)\}\mathrm{d}M\leq 0.$$

Then it is seen from the assumption that

$$\int_M\{(S_H-nH^2)n[\beta_3(n,p)-1](d-c)-E_1(n,p)(b-a)(d-c)\}\mathrm{d}M\leq 0.\tag{4.20}$$

Hence, we have $d=c$ or

$$\int_M(S_H-nH^2)\mathrm{d}M\leq\eta\int_M(b-a)\mathrm{d}M.$$

This completes the proof.

Theorem 4.4 *Let $M^n(n\geq 4)$ be an oriented compact submanifold with parallel mean curvature in a complete simply connected Riemannian manifold N^{n+p}, $p\geq 3$ and $H\neq 0$. Then there exists a constant $\theta_3(n,p)\in(0,1)$, such that if $\overline{K}_N\in[\theta_3(n,p),1]$, and if*

$$\mathrm{Ric}_M\geq (n-2)(1+H^2)+\beta_3(n,p)(1-c)+\beta_4(n,p)[H(1+H^2)]^{\frac12}(1-c)^{\frac14},$$

$$R \leq n[(n-1)(1+H^2) - \gamma_3(n,p)(1-c) - \gamma_4(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}],$$

where $c := \inf \overline{K}_N$, *then* N^{n+p} *is isometric to* S^{n+p}. *Moreover,* M *is either a totally umbilical sphere* $S^n(\frac{1}{\sqrt{1+H^2}})$, *the Clifford hypersurface* $S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big) \times S^m\big(\frac{1}{\sqrt{2(1+H^2)}}\big)$ *in* $S^{n+1}\big(\frac{1}{\sqrt{1+H^2}}\big)$ $(n=2m)$, *or* $\mathbb{C}\mathrm{P}^2\big(\frac{4}{3}(1+H^2)\big)$ *in* $S^7\big(\frac{1}{\sqrt{1+H^2}}\big)$. *Here* $\beta_3(n,p)$, $\beta_4(n,p)$, $\gamma_3(n,p)$, $\gamma_4(n,p)$ *will be given in the proof, and* $\theta_3(n,p) := 1 - [\beta_3(n,p) + \gamma_3(n,p) + \sqrt{2}\beta_4(n,p)]^{-4}$.

Remark 4.1 From the choice of $\theta_3(n,p)$, we see that the pinching condition of M makes sense.

Proof Assume that $c \neq 1$. It follows from Theorem 4.2 that

$$\begin{aligned}&\int_M \{nS_I[\mathrm{Ric}_{\min} - (n-2)(1+H^2) - J(n,p)H(S_H - nH^2)^{\frac{1}{2}} \\ &- G(n,p-1)(1-c)] - E_2(n,p)(1-c)^2\}\mathrm{d}M \leq 0. \end{aligned} \tag{4.21}$$

From the Gauss equation, the assumption $\mathrm{Ric}_M \geq (n-2)(1+H^2)$ and $S = S_H + S_I \geq nH^2 + S_I$, we obtain that

$$S_I \leq S - nH^2 \leq n(1+H^2). \tag{4.22}$$

Since

$$\mathrm{Ric}_M \geq (n-2)(1+H^2) + \beta_3(n,p)(1-c),$$

it is seen from the Schwarz inequality and Lemma 4.5 that

$$\begin{aligned}&\int_M HS_I(S_H - nH^2)^{\frac{1}{2}}\mathrm{d}M \\ &\leq H(\max S_I)\mathrm{vol}^{\frac{1}{2}}(M)\Big[\int_M (S_H - nH^2)\mathrm{d}M\Big]^{\frac{1}{2}} \\ &\leq \eta^{\frac{1}{2}}(n,p)nH(1+H^2)(1-c)^{\frac{1}{2}}\mathrm{vol}(M). \end{aligned} \tag{4.23}$$

Combing (4.21) and (4.23), we get

$$\begin{aligned}&\int_M \{nS_I[\mathrm{Ric}_{\min} - (n-2)(1+H^2) - G(n,p-1)(1-c)] - E_2(n,p)(1-c)^2 \\ &- n^2\eta^{\frac{1}{2}}(n,p)J(n,p)H(1+H^2)(1-c)^{\frac{1}{2}}\}\mathrm{d}M \leq 0. \end{aligned} \tag{4.24}$$

Let

$$\begin{aligned}&\beta_3(n,p) := G(n,p-1) + E_2^{\frac{1}{2}}(n,p)n^{-1}, \\ &\beta_4(n,p) := \eta^{\frac{1}{4}}(n,p)J^{\frac{1}{2}}(n,p). \end{aligned}$$

Because

$$\mathrm{Ric}_M \geq (n-2)(1+H^2) + \beta_3(n,p)(1-c) + \beta_4(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}},$$

we obtain

$$\int_M S_I\mathrm{d}M \leq [E_2^{\frac{1}{2}}(n,p)(1-c) + n\eta^{\frac{1}{4}}(n,p)J^{\frac{1}{2}}(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}]\mathrm{vol}(M). \tag{4.25}$$

This together with Lemma 4.5 implies

$$\int_M (S - nH^2)\mathrm{d}M \le \{[\eta(n,p) + E_2^{\frac{1}{2}}(n,p)](1-c) + \eta^{\frac{1}{4}}(n,p)nJ^{\frac{1}{2}}(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}\}\mathrm{vol}(M). \tag{4.26}$$

Here

$$\eta(n,p) = E_1(n,p)n^{-1}[\beta_3(n,p)-1]^{-1}.$$

On the other hand, it follows from the assumption that

$$S - nH^2 \ge [\gamma_3(n,p) - (n-1)]n(1-c) + \gamma_4(n,p)n[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}. \tag{4.27}$$

Let

$$\gamma_3(n,p) := n - 1 + [\eta(n,p) + E_2^{\frac{1}{2}}(n,p)]n^{-1},$$
$$\gamma_4(n,p) := \beta_4(n,p).$$

Then we have

$$S - nH^2 \equiv (\eta(n,p) + E_2^{\frac{1}{2}}(n,p))(1-c) + n\eta^{\frac{1}{4}}(n,p)J^{\frac{1}{2}}(n,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}. \tag{4.28}$$

Therefore, the inequalities above all become equalities and $1-c=b-a$. Since $c \le a \le b \le 1$, $a=c$, $b=1$. By a similar argument as in [10], we have $1=c$, contradicting to the assumption. Because N is complete and simply connected, we know that N is isometric to S^{n+p}. Moreover, it follows from Theorems 4.1–4.2 that

$$S = nH^2 \quad \text{or} \quad \mathrm{Ric}_{\min} = (n-2)(1+H^2).$$

This together with Theorem B implies the conclusion. This proves Theorem 4.4.

Proof of Main Theorem We define the pinching constants in the Main Theorem as follows:

$$\delta(n,p) = \begin{cases} \theta_1(n,p), & \text{if } H = 0, \\ \theta_2(n,p), & \text{if } p \le 2 \text{ and } H \ne 0, \\ \theta_3(n,p), & \text{if } p \ge 3 \text{ and } H \ne 0, \end{cases}$$

$$A_1(n,p) = \begin{cases} \beta_1(n,p), & \text{if } H = 0, \\ \beta_2(n,p), & \text{if } p \le 2 \text{ and } H \ne 0, \\ \beta_3(n,p), & \text{if } p \ge 3 \text{ and } H \ne 0, \end{cases}$$

$$A_2(n,p) = \begin{cases} \beta_4(n,p), & \text{if } p \ge 3 \text{ and } H \ne 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_1(n,p) = \begin{cases} \gamma_1(n,p), & \text{if } H = 0, \\ \gamma_2(n,p), & \text{if } p \le 2 \text{ and } H \ne 0, \\ \gamma_3(n,p), & \text{if } p \ge 3 \text{ and } H \ne 0, \end{cases}$$

$$B_2(n,p) = \begin{cases} \gamma_4(n,p), & \text{if } p \ge 3 \text{ and } H \ne 0, \\ 0, & \text{otherwise}. \end{cases}$$

When $H = 0$, the assertion follows from Theorem 3.2. When $H \neq 0$, we get the conclusion from Theorems 4.3–4.4. This proves the Main Theorem.

Motivated by Theorem B and the Main Theorem, we propose the following interesting problem.

Problem 4.1 Let M be a 3-dimensional oriented compact submanifold, with parallel mean curvature in a $(3+p)$-dimensional complete simply connected Riemannian manifold N^{3+p}. Does there exist constant $\delta(3,p) \in (0,1)$, such that if the sectional curvature of N satisfies $\overline{K}_N \in [\delta(3,p), 1]$, and if

$$\begin{aligned}&\mathrm{Ric}_M \geq 1 + H^2 + A_1(3,p)(1-c) + A_2(3,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}},\\ &R \leq 3[2(1+H^2) - B_1(3,p)(1-c) - B_2(3,p)[H(1+H^2)]^{\frac{1}{2}}(1-c)^{\frac{1}{4}}],\end{aligned}$$

where $c := \inf \overline{K}_N$, then N^{3+p} is isometric to S^{3+p}, and M is a totally umbilic sphere $S^3\big(\frac{1}{\sqrt{1+H^2}}\big)$?

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