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Schwarz Lemma at the Boundary on the Classical Domain of Type $\mathcal{III}$*

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Abstract Let $\mathcal{R}_{\mathcal{III}}(n)$ be the classical domain of type $\mathcal{III}$ with $n \geq 2$. This article is devoted to a deep study of the Schwarz lemma on $\mathcal{R}_{\mathcal{III}}(n)$ via not only exploring the smooth boundary points of $\mathcal{R}_{\mathcal{III}}(n)$ but also proving the Schwarz lemma at the smooth boundary point for holomorphic self-mappings of $\mathcal{R}_{\mathcal{III}}(n)$.

Keywords Holomorphic mapping, Schwarz lemma at the boundary, The classical domain of type $\mathcal{III}$
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1 Introduction

Schwarz lemma is one of the most important results in the classical complex analysis. A great deal of work has been devoted to generalizations of Schwarz lemma to more general settings. We refer to [1–8] for a more complete insight on the Schwarz lemma.

In the case of several complex variables, the Schwarz lemma originated from the work of Cartan. In [9], Cartan obtained the following rigidity theorem for holomorphic mappings.

Theorem 1.1 (cf. [9]) *Let Ω be a bounded domain in $\mathbb{C}^n$. If $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z) = z + o(\|z - z_0\|)$ as $z \to z_0$ for some $z_0 \in \Omega$, then $f(z) \equiv z$.*

On the other hand, Look first considered the properties of the Jacobian matrix of holomorphic mapping in [10].

Theorem 1.2 (cf. [10]) *Let Ω be a bounded domain in $\mathbb{C}^n$, and let f be a holomorphic self-mapping of Ω which fixes a point $p \in \Omega$. Then the eigenvalues of $J_f(p)$ all have modulus not exceeding 1 and $|\det J_f(p)| \leq 1$. Moreover, if $|\det J_f(p)| = 1$, then f is a biholomorphism of Ω.*

It is natural to explore the high-dimensional versions of the Schwarz lemma at the boundary. Motivated by Theorem 1.1, Burns and Krantz first studied the boundary Schwarz lemma and

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the rigidity problem for holomorphic mappings in [11]. See [12–16] for more on these matters. Motivated by Theorem 1.2, we focused on the characterizations of the Jacobian matrix of holomorphic mapping at the boundary point of some domains in $\mathbb{C}^n$, and established the boundary Schwarz lemmas (see [17–18]).

These results are widely applied in many fields. By the classical Schwarz lemma at the boundary, Bonk improved the Bloch constant in [19], and Liu, Ren, Gong and Zhang obtained the growth, covering and distortion theorems for biholomorphic convex mappings or quasi-convex mappings on some domains in [20–22]. Recently, using the Schwarz lemma at the boundary of the unit ball, we gave a new and simple proof of the distortion theorem of determinants for biholomorphic convex mappings in [18], and established the distortion theorem of determinants and the distortion theorem of matrices at extreme points for biholomorphic starlike mappings in [23].

Let $\mathcal{R}_{\mathcal{I}}(m,n)$ be the classical domain of type $\mathcal{I}$ in $\mathbb{C}^{m\times n}$ with $1 \le m \le n$. And let $\mathcal{R}_{\mathcal{II}}(n)$ be the classical domain of type $\mathcal{II}$. More recently, we investigated the Schwarz lemmas at the boundary on $\mathcal{R}_{\mathcal{I}}(m,n)$ and $\mathcal{R}_{\mathcal{II}}(n)$ in [24–25], respectively. In this paper, we consider the case of the classical domain of type $\mathcal{III}$. We first characterize the properties of the smooth boundary points of $\mathcal{R}_{\mathcal{III}}(n)$, and then prove the Schwarz lemma at the boundary. Namely, we obtain the optimal estimates of the eigenvalues of the Fréchet derivative of holomorphic self-mapping at the smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$.

Remark 1.1 Notice that the corresponding inner products are different on these classical domains, which means that the methods and techniques of matrix are completely different on these classical domains. We need to find a different approach for such a study. For instance, because there are multiple roots for the characteristic polynomial of $Z\overline{Z}'$ on $\mathcal{R}_{\mathcal{III}}(n)$ we can not apply directly the implicit function existence theorem to study the smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$ by the methods and techniques similar to $\mathcal{R}_{\mathcal{I}}(m,n)$ and $\mathcal{R}_{\mathcal{II}}(n)$.

Remark 1.2 Although $\mathcal{R}_{\mathcal{III}}(n)$ is a convex domain, $\mathcal{R}_{\mathcal{III}}(n)$ is not a strongly pseudoconvex domain and $\partial\mathcal{R}_{\mathcal{III}}(n)$ is not smooth. Therefore, we can not apply the similar method in [17] to prove the main result of this paper.

Remark 1.3 The Carathéodory metric and Kobayashi metric of $\mathcal{R}_{\mathcal{III}}(n)$ are difficult to characterize explicitly. We must find some new approaches to prove the Schwarz lemma at the boundary of $\mathcal{R}_{\mathcal{III}}(n)$, which means that the proof is completely different from that of [18] in the unit ball.

The rest of the article is organized as follows. In Section 2, we develop some properties of the smooth boundary points of $\mathcal{R}_{\mathcal{III}}(n)$. In Section 3, we present some lemmas. In Section 4, we give the main result of the article and its proof.

2 Smooth Boundary Points of $\mathcal{R}_{\mathcal{III}}(n)$

In this section, we present some characterizations of the smooth boundary points of $\mathcal{R}_{\mathcal{III}}(n)$, which will be used in the subsequent sections.

We first introduce some notations and definitions. Let $\mathbb{C}^{n\times n}$ be the set of all complex square matrices of order n. For $n \geq 2$, let

$$\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}} = \{Z \in \mathbb{C}^{n\times n} : Z' = -Z\}$$

be the family of all anti-symmetric complex square matrices of order n. Throughout this paper, Z' and $\overline{Z}$ represent the transpose and the complex conjugate of Z, respectively. For any $Z, W \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$, the inner product and the corresponding norm are given by

$$\langle Z, W\rangle = \sum_{1\leq i<j\leq n} z_{ij}\overline{w}_{ij} = \frac{1}{2}tr(Z\overline{W}'), \quad \|Z\| = \langle Z, Z\rangle^{\frac{1}{2}},$$

where $Z = (z_{ij})_{n\times n}$ and $W = (w_{ij})_{n\times n}$. It is well known that $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ is an $\frac{n(n-1)}{2}$-dimensional Hilbert space and $\|\cdot\|$ is a Euclidean norm. As real vectors in $\mathbb{R}^{n(n-1)}$, Z and W are orthogonal if and only if $\Re\langle Z, W\rangle = 0$.

The classical domain of type $\mathcal{III}$, denoted by $\mathcal{R}_{\mathcal{III}}(n)$, is defined as

$$\mathcal{R}_{\mathcal{III}}(n) = \{Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}} : \mathrm{I}_n - Z\overline{Z}' > 0\},$$

where I_n is the unit square matrix of order n, and the inequality sign means that the left-hand side is positive definite. Let $\partial\mathcal{R}_{\mathcal{III}}(n)$ be the boundary of $\mathcal{R}_{\mathcal{III}}(n)$, and write $\mathbb{C}^{1\times n} = \mathbb{C}^n$. Let $B^n \subset \mathbb{C}^n$ be the open unit ball under the Euclidean metric. The Minkowski functional $\rho(Z)$ of $\mathcal{R}_{\mathcal{III}}(n)$ is defined by

$$\rho(Z) = \max\{\|\alpha Z\| : \alpha \in \partial B^n\}, \quad Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}.$$

By [26], it is easy to see that $\rho(Z)$ is a Banach norm of $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$, $(\rho(Z))^2$ is the largest eigenvalue of $Z\overline{Z}'$, $\mathcal{R}_{\mathcal{III}}(n) = \{Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}} : \rho(Z) < 1\}$, and $\mathcal{R}_{\mathcal{III}}(n)$ is a bounded convex circular domain in $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$. In particular, $\mathcal{R}_{\mathcal{III}}(2)$ is just the open unit disk $\triangle$ in the complex plane $\mathbb{C}$, and $\mathcal{R}_{\mathcal{III}}(3) = B^3$. For the unitary square matrix U of order n, it is clear that

$$Z \in \mathcal{R}_{\mathcal{III}}(n) \Leftrightarrow UZU' \in \mathcal{R}_{\mathcal{III}}(n), \quad Z \in \partial\mathcal{R}_{\mathcal{III}}(n) \Leftrightarrow UZU' \in \partial\mathcal{R}_{\mathcal{III}}(n).$$

Note that $Z \in \mathcal{R}_{\mathcal{III}}(n)$ shows that the elements in the principal diagonal of $\mathrm{I}_n - Z\overline{Z}'$ are positive. So we have $|z_{ij}| < 1$ for $i, j = 1, \cdots, n$. We also get $\rho(UZU') = \rho(Z)$ for each $Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$.

For $\mathring{Z} \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$, according to [4], $\mathring{Z}$ has the following polar decompositions:

$$\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} \end{pmatrix} U'$$

or

$$\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} & & & & 0 \\ & \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} & \\ 0 & & & & 0 \end{pmatrix} U',$$

where $r_1 \geq r_2 \geq \cdots \geq r_p \geq 0$ and U is a unitary square matrix of order n. For our later use we denote by $[x]$, for $x \in \mathbb{R}$, the greatest integer not greater than x. Notice that $\partial\mathcal{R}_{\mathcal{III}}(2) = \partial\triangle$ and $\partial\mathcal{R}_{\mathcal{III}}(3) = \partial B^3$ are smooth. Then from now on, we always assume that $n \geq 4$ for $\mathcal{R}_{\mathcal{III}}(n)$.

Theorem 2.1 *Let $\mathring{Z} \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ be the polar decomposition above. Then $\mathring{Z}$ is a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$ if and only if $1 = r_1 > r_2 \geq \cdots \geq r_p \geq 0$. Furthermore, $\rho(Z)$ is holomorphic about Z and $\overline{Z}$ near $\mathring{Z}$, and the gradient of ρ at $\mathring{Z}$*

$$\nabla\rho(\mathring{Z}) = U \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} U'$$

is a unit outward normal vector to $\partial\mathcal{R}_{\mathcal{III}}(n)$ at $\mathring{Z}$ with $\langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1$.

Proof It is easy to see that $\mathring{Z} \in \partial\mathcal{R}_{\mathcal{III}}(n)$ if and only if $r_1 = 1$. Suppose that $1 = r_1 > r_2 \geq \cdots \geq r_p \geq 0$. For $Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$, let

$$Z = U(Z) \begin{pmatrix} \begin{pmatrix} 0 & r_1(Z) \\ -r_1(Z) & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & r_2(Z) \\ -r_2(Z) & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & r_p(Z) \\ -r_p(Z) & 0 \end{pmatrix} \end{pmatrix} U(Z)'$$

or

$$Z = U(Z) \begin{pmatrix} \begin{pmatrix} 0 & r_1(Z) \\ -r_1(Z) & 0 \end{pmatrix} & & & & 0 \\ & \begin{pmatrix} 0 & r_2(Z) \\ -r_2(Z) & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} 0 & r_p(Z) \\ -r_p(Z) & 0 \end{pmatrix} & \\ 0 & & & & 0 \end{pmatrix} U(Z)',$$

where $\rho(Z) = r_1(Z) \geq r_2(Z) \geq \cdots \geq r_p(Z) \geq 0$, $U(Z)$ is a unitary square matrix of order n, $r_1(\mathring{Z}) = 1, r_2(\mathring{Z}) = r_2, \cdots, r_p(\mathring{Z}) = r_p$ and $U(\mathring{Z}) = U$. Then the characteristic polynomial of $Z\overline{Z}'$ is $\det(x\mathrm{I}_n - Z\overline{Z}') = x^n - tr(Z\overline{Z}')x^{n-1} + \cdots + (-1)^n \det(Z\overline{Z}') = \prod_{k=1}^{p}(x - r_k^2(Z))^2$ or $\det(x\mathrm{I}_n - Z\overline{Z}') = x\prod_{k=1}^{p}(x - r_k^2(Z))^2$. Write

$$\Phi(x, Z) = \prod_{k=1}^{p}(x - r_k^2(Z)) \quad \text{or} \quad \Phi(x, Z) = \sqrt{x}\prod_{k=1}^{p}(x - r_k^2(Z)).$$

Then $\Phi[(\rho(Z))^2, Z] \equiv 0$. Because

$$\Phi(1, \mathring{Z}) = 0, \quad \frac{\partial\Phi}{\partial x}(1, \mathring{Z}) = \prod_{k=2}^{p}(1 - r_k^2) > 0, \tag{2.1}$$

by the implicit function existence theorem we know that $(\rho(Z))^2$ is a holomorphic function about Z and $\overline{Z}$ near $\mathring{Z}$, and satisfies $(\rho(\mathring{Z}))^2 = 1$. Therefore, $\rho(Z)$ is also a holomorphic function about Z and $\overline{Z}$ near $\mathring{Z}$. Now, we compute $\nabla\rho(\mathring{Z})$. Since $\Phi[(\rho(Z))^2, Z] = (\det[(\rho(Z))^2\mathrm{I}_n - Z\overline{Z}'])^{\frac{1}{2}} \equiv 0$ near $\mathring{Z}$, we get

$$\frac{\partial\Phi}{\partial x}(1, \mathring{Z})2\rho(\mathring{Z})\frac{\partial\rho}{\partial\overline{z}_{ij}}(\mathring{Z}) + \frac{\partial\Phi}{\partial\overline{z}_{ij}}(1, \mathring{Z}) = 0, \quad 1 \le i < j \le n.$$

This, together with (2.1), gives

$$\prod_{k=2}^{n}(1 - r_k^2)(\nabla\rho(\mathring{Z}))_{ij} + \frac{\partial\Phi}{\partial\overline{z}_{ij}}(1, \mathring{Z}) = 0. \tag{2.2}$$

Notice that when $\varepsilon \in (0, 1)$, we have $\det(x\mathrm{I}_n - \varepsilon^2 Z\overline{Z}')|_{(x,Z)=(1,\mathring{Z})} = \det(\mathrm{I}_n - \varepsilon^2\overline{U}'Z\overline{Z}'U)|_{Z=\mathring{Z}}$, and

$$\mathrm{I}_n - \varepsilon^2\overline{U}'\mathring{Z}\overline{\mathring{Z}}'U = \begin{pmatrix} (1-\varepsilon^2 r_1^2)\mathrm{I}_2 & & & 0 \\ & (1-\varepsilon^2 r_2^2)\mathrm{I}_2 & & \\ & & \ddots & \\ 0 & & & (1-\varepsilon^2 r_p^2)\mathrm{I}_2 \end{pmatrix}$$

or

$$\mathrm{I}_n - \varepsilon^2\overline{U}'\mathring{Z}\overline{\mathring{Z}}'U = \begin{pmatrix} (1-\varepsilon^2 r_1^2)\mathrm{I}_2 & & & & 0 \\ & (1-\varepsilon^2 r_2^2)\mathrm{I}_2 & & & \\ & & \ddots & & \\ & & & (1-\varepsilon^2 r_p^2)\mathrm{I}_2 & \\ 0 & & & & 1 \end{pmatrix}.$$

Then the algebraic cofactor of the element at s-th row and t-th column for $\det(\mathrm{I}_n - \varepsilon^2\overline{U}'\mathring{Z}\overline{\mathring{Z}}'U)$ is

$$J_{st} = \begin{cases} (1-\varepsilon^2 r^2_{[\frac{s+1}{2}]}) \displaystyle\prod_{k\neq[\frac{s+1}{2}],k=1}^{p}(1-\varepsilon^2 r_k^2)^2, & 1 \le s = t \le n, \\ 0, & s \neq t, \end{cases}$$

or

$$J_{st} = \begin{cases} (1-\varepsilon^2 r^2_{[\frac{s+1}{2}]}) \displaystyle\prod_{k\neq[\frac{s+1}{2}],k=1}^{p}(1-\varepsilon^2 r_k^2)^2, & 1 \le s = t \le n-1, \\ 0, & s \neq t, \\ \displaystyle\prod_{k=1}^{p}(1-\varepsilon^2 r_k^2)^2, & s = t = n. \end{cases}$$

On the other hand,

$$\det(\mathrm{I}_n-\varepsilon^2\overline{U}'Z\overline{Z}'U)|_{Z=\mathring{Z}}=\prod_{k=1}^{p}(1-\varepsilon^2r_k^2)^2,\quad \Phi(1,\varepsilon Z)=[\det(\mathrm{I}_n-\varepsilon^2\overline{U}'Z\overline{Z}'U)]^{\frac{1}{2}}.$$

Thus, we obtain

$$\begin{aligned}\frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\varepsilon\mathring{Z})&=\frac{1}{2[\det(\mathrm{I}_n-\varepsilon^2\overline{U}'\mathring{Z}\overline{\mathring{Z}}'U)]^{\frac{1}{2}}}\frac{\partial}{\partial\overline{z}_{ij}}[\det(\mathrm{I}_n-\varepsilon^2\overline{U}'Z\overline{Z}'U)]|_{Z=\mathring{Z}}\\&=\frac{\varepsilon^2}{2\prod\limits_{k=1}^{p}(1-\varepsilon^2r_k^2)}\Big(-\sum_{s,t=1}^{n}\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{st}|_{Z=\mathring{Z}}J_{st}\Big).\end{aligned}$$

It follows that

$$\frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\varepsilon\mathring{Z})=-\frac{\varepsilon^2}{2}\sum_{s=1}^{n}\prod_{k\neq[\frac{s+1}{2}],k=1}^{p}(1-\varepsilon^2r_k^2)\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{ss}|_{Z=\mathring{Z}},$$

or

$$\begin{aligned}&\frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\varepsilon\mathring{Z})\\&=-\frac{\varepsilon^2}{2}\Big[\sum_{s=1}^{n-1}\prod_{k\neq[\frac{s+1}{2}],k=1}^{p}(1-\varepsilon^2r_k^2)\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{ss}|_{Z=\mathring{Z}}+\prod_{k=1}^{p}(1-\varepsilon^2r_k^2)\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{nn}|_{Z=\mathring{Z}}\Big].\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\mathring{Z})&=-\frac{1}{2}\prod_{k=2}^{p}(1-r_k^2)\sum_{s=1}^{2}\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{ss}|_{Z=\mathring{Z}}\\&=-\frac{1}{2}\prod_{k=2}^{p}(1-r_k^2)\sum_{s=1}^{2}\frac{\partial}{\partial\overline{z}_{ij}}\Big(\sum_{l,m,t=1}^{n}\overline{u}_{ls}z_{lm}\overline{z}_{tm}u_{ts}\Big)\Big|_{Z=\mathring{Z}}\\&=-\frac{1}{2}\prod_{k=2}^{p}(1-r_k^2)\sum_{s=1}^{2}\sum_{l=1}^{n}(\overline{u}_{ls}\mathring{z}_{lj}u_{is}-\overline{u}_{ls}\mathring{z}_{li}u_{js})\quad(1\le i<j\le n)\\&=-\frac{1}{2}\prod_{k=2}^{p}(1-r_k^2)[(\overline{U}'\mathring{Z})_{1j}u_{i1}-(\overline{U}'\mathring{Z})_{1i}u_{j1}+(\overline{U}'\mathring{Z})_{2j}u_{i2}-(\overline{U}'\mathring{Z})_{2i}u_{j2}]\\&=-\frac{1}{2}\prod_{k=2}^{p}(1-r_k^2)(u_{j2}u_{i1}-u_{i2}u_{j1}-u_{j1}u_{i2}+u_{i1}u_{j2})\\&=-\prod_{k=2}^{p}(1-r_k^2)(u_{i1}u_{j2}-u_{i2}u_{j1}),\quad 1\le i<j\le n,\end{aligned}$$

where $U=(u_{ij})_{n\times n}$ and $\mathring{Z}=(\mathring{z}_{ij})_{n\times n}$. This, together with (2.2), shows

$$(\nabla\rho(\mathring{Z}))_{ij}=u_{i1}u_{j2}-u_{i2}u_{j1},\quad 1\le i<j\le n.$$

Therefore

$$\nabla\rho(\mathring{Z}) = U \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} U' \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$$

is a nonzero vector. This means that $\partial\mathcal{R}_{\mathcal{III}}(n)$ is smooth near $\mathring{Z}$. Moreover, utilizing

$$\langle W, \nabla\rho(\mathring{Z})\rangle = \frac{1}{2} tr \left(W\overline{U} \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \overline{U}' \right)$$

for any $W \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ we find

$$\langle \nabla\rho(\mathring{Z}), \nabla\rho(\mathring{Z})\rangle = 1, \quad \langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1.$$

Conversely, suppose that $\mathring{Z}$ is a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$. Assume

$$1 = r_1 = r_2 \geq \cdots \geq r_p \geq 0.$$

Then any two nonzero outward normal vectors to $\partial\mathcal{R}_{\mathcal{III}}(n)$ at $\mathring{Z}$ have the same direction. We discuss the following two different $[n(n-1)-1]$-dimensional real affine spaces through $\mathring{Z}$ in $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$:

$$\Sigma_1 = \{\mathring{Z} + U\alpha U' : \alpha \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}},\ \Re\alpha_{12} = 0\}, \quad \Sigma_2 = \{\mathring{Z} + U\alpha U' : \alpha \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}},\ \Re\alpha_{34} = 0\}.$$

To simplify our notations, set

$$T_1 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & 0 & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}.$$

Then T_1 and T_2 are the unit vectors in $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$. On the one hand, for any $\mathring{Z} + U\alpha U' \in \Sigma_1$ we obtain

$$\Re\langle U\alpha U', UT_1U'\rangle = \frac{1}{2}\Re tr(U\alpha U'\overline{U}\ \overline{T_1}'\overline{U}') = \Re\alpha_{12} = 0.$$

Hence, UT_1U' is a normal vector to Σ_1 at $\mathring{Z}$. See the following figure 1.

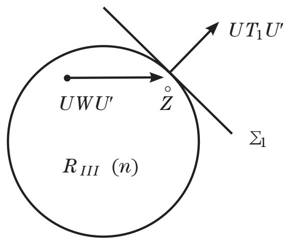

Figure 1 $\mathcal{R}_{\mathcal{III}}(n)$ and real affine space.

Similarly, UT_2U' is a normal vector to Σ_2 at $\mathring{Z}$. On the other hand, for each $UWU' \in \mathcal{R}_{\mathcal{III}}(n)$ we get

$$\Re\langle \mathring{Z} - UWU', UT_1U'\rangle = 1 - \frac{1}{2}\Re tr(UWU'\overline{U}\ \overline{T_1}'\overline{U}') = 1 - \Re w_{12} > 0.$$

This shows that $\mathcal{R}_{\mathcal{III}}(n)$ is located on one side of Σ_1. That is, Σ_1 is an affine tangent space to $\partial\mathcal{R}_{\mathcal{III}}(n)$ at $\mathring{Z}$. Similar to the proof above, we know that Σ_2 is also an affine tangent space to $\partial\mathcal{R}_{\mathcal{III}}(n)$ at $\mathring{Z}$. Since $\mathring{Z}$ is a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$, this contradicts with $\Sigma_1 \neq \Sigma_2$. Thus, we have $1 = r_1 > r_2 \geq \cdots \geq r_m \geq 0$. The proof is complete.

3 Some Lemmas

In this section, we exhibit some notations and collect several lemmas, which will be used in the subsequent section.

Let $f : \mathcal{R}_{\mathcal{III}}(n) \to \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ be a holomorphic mapping. The Fréchet derivative of f at $a \in \mathcal{R}_{\mathcal{III}}(n)$ is defined by

$$(Df(a)(W))_{ij} = \sum_{1\leq s<t\leq n} \frac{\partial f_{ij}}{\partial z_{st}}(a) w_{st}, \quad W \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}.$$

It is easy to see that $Df(a)$ is a linear transformation from $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ to $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ and $\mathrm{d}f(Z)|_{Z=a} = Df(a)(\mathrm{d}Z)$. Let $D^*f(a)$ be the adjoint transformation of $Df(a)$ with respect to the inner product $\langle\cdot,\cdot\rangle$. That is,

$$\langle D^*f(a)(Z), W\rangle = \langle Z, Df(a)(W)\rangle, \quad Z, W \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}.$$

$D^*f(a)$ is also a linear transformation from $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ to $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$. Specifically,

$$(D^*f(a)(Z))_{ij} = \sum_{1\leq s<t\leq n} \frac{\partial \overline{f}_{st}}{\partial \overline{z}_{ij}}(a) z_{st}, \quad Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}.$$

In fact, suppose that $e_{ij} \in \mathbb{C}^{n\times n}$ is a square matrix which has 1 at i-th row and j-th column, and 0s elsewhere. Then when $i < j$ we have

$$\begin{aligned}(D^*f(a)(Z))_{ij} &= \langle D^*f(a)(Z), e_{ij} - e_{ji}\rangle = \langle Z, Df(a)(e_{ij} - e_{ji})\rangle \\ &= \Big\langle Z, \frac{\partial f}{\partial z_{ij}}(a)\Big\rangle = \sum_{1\leq s<t\leq n} \frac{\partial \overline{f}_{st}}{\partial \overline{z}_{ij}}(a) z_{st}.\end{aligned}$$

It is clear that λ is an eigenvalue of $Df(a)$ if and only if $\overline{\lambda}$ is an eigenvalue of $D^*f(a)$.

Let $\triangle$ be the open unit disk in the complex plane $\mathbb{C}$. There is the classical boundary Schwarz lemma as follows.

Lemma 3.1 (cf. [3]) *Let $f : \triangle \to \triangle$ be a holomorphic function. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$. Moreover, the inequality is sharp.*

If the condition $f(0) = 0$ is removed, then by applying Lemma 3.1 to $g(z) = \frac{1-\overline{f(0)}}{1-f(0)} \frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}$, one has the following estimate instead:

$$f'(1) \geq \frac{|1 - \overline{f(0)}|^2}{1 - |f(0)|^2} > 0. \tag{3.1}$$

Lemma 3.2 (cf. [26]) *Let*

$$a = A \begin{pmatrix} \begin{pmatrix} 0 & l_1 \\ -l_1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & l_p \\ -l_p & 0 \end{pmatrix} \end{pmatrix} A' \in \mathcal{R}_{\mathcal{III}}(n)$$

or

$$a = A \begin{pmatrix} \begin{pmatrix} 0 & l_1 \\ -l_1 & 0 \end{pmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{pmatrix} 0 & l_p \\ -l_p & 0 \end{pmatrix} & \\ 0 & & & 0 \end{pmatrix} A' \in \mathcal{R}_{\mathcal{III}}(n).$$

Write

$$Q = A \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-l_1^2}} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{I}_2}{\sqrt{1-l_p^2}} \end{pmatrix} \overline{A}' \quad or \quad Q = A \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-l_1^2}} & & & 0 \\ & \ddots & & \\ & & \frac{\mathrm{I}_2}{\sqrt{1-l_p^2}} & \\ 0 & & & 1 \end{pmatrix} \overline{A}',$$

where $1 > l_1 \geq \cdots \geq l_p \geq 0$ and A is a unitary square matrix of order n. For any $Z \in \overline{\mathcal{R}_{\mathcal{III}}(n)}$, define

$$\varphi_a(Z) = Q^{-1}(\mathrm{I}_n - Z\overline{a}')^{-1}(a - Z)\overline{Q}.$$

Then the following statements hold:

(1) *$\varphi_a(Z)$ is a holomorphic automorphism of $\mathcal{R}_{\mathcal{III}}(n)$, and $\varphi_a(Z)$ is biholomorphic in a neighborhood of $\overline{\mathcal{R}_{\mathcal{III}}(n)}$*;

(2) $\varphi_a(0) = a, \varphi_a(a) = 0, \varphi_a^{-1} = \varphi_a$;

(3) $\mathrm{d}\varphi_a(Z)|_{Z=a} = -Q\mathrm{d}Z\overline{Q}, \quad \mathrm{d}\varphi_a(Z)|_{Z=0} = -Q^{-1}\mathrm{d}Z\overline{Q}^{-1}$.

In what follows, we always denote by $F(Z, \xi)$ the infinitesimal form of Carathéodory metric or Kobayashi metric on $\mathcal{R}_{\mathcal{III}}(n)$, where $Z \in \mathcal{R}_{\mathcal{III}}(n)$ and $\xi \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ (see [27] for details).

Lemma 3.3 *Let $\rho(Z)$ be the Minkowski functional of $\mathcal{R}_{\mathcal{III}}(n)$. Then under the notations of Lemma 3.2, for any $\xi \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$,*

$$F(a,\xi) = \rho(Q\xi\overline{Q}).$$

Proof Note that $\varphi_a(Z)$ is a holomorphic automorphism of $\mathcal{R}_{\mathcal{III}}(n)$, and $F(Z,\xi)$ is a biholomorphically invariant metric on $\mathcal{R}_{\mathcal{III}}(n)$. It follows that $F(a,\xi) = F(0, D\varphi_a(a)(\xi))$. This, together with $D\varphi_a(a)(\mathrm{d}Z) = \mathrm{d}\varphi_a(Z)|_{Z=a}$ and Lemma 3.2, implies

$$F(a,\xi) = F(0, D\varphi_a(a)(\xi)) = F(0, -Q\xi\overline{Q}) = F(0, Q\xi\overline{Q}).$$

Hence, by Lemma 3.2 in [20], we obtain $F(a,\xi) = F(0, Q\xi\overline{Q}) = \rho(Q\xi\overline{Q})$. The proof is complete.

Lemma 3.4 *Let $\mathring{Z}$ be a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$. Then for each $W \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$,*

$$|\langle W, \nabla\rho(\mathring{Z})\rangle| \le \rho(W).$$

Proof Without loss of generality, we may assume $W \neq 0$. Then $\frac{W}{\rho(W)} \in \partial\mathcal{R}_{\mathcal{III}}(n)$. Since $\mathcal{R}_{\mathcal{III}}(n)$ is a bounded convex circular domain, we have

$$\Re\Big\langle \mathring{Z} - \mathrm{e}^{\mathrm{i}\theta}\frac{W}{\rho(W)}, \nabla\rho(\mathring{Z})\Big\rangle \ge 0$$

for any $\theta \in \mathbb{R}$. It follows from this and Theorem 2.1 that

$$\Re\frac{\mathrm{e}^{\mathrm{i}\theta}}{\rho(W)}\langle W, \nabla\rho(\mathring{Z})\rangle \le \Re\langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1.$$

This gives $|\langle W, \nabla\rho(\mathring{Z})\rangle| \le \rho(W)$. The proof is complete.

Lemma 3.5 (cf. [28]) *Let $f : \mathcal{R}_{\mathcal{III}}(n) \to \mathcal{R}_{\mathcal{III}}(n)$ be a holomorphic mapping and let $f(0) = 0$. Then for any $Z \in \mathcal{R}_{\mathcal{III}}(n)$,*

$$\rho(f(Z)) \le \rho(Z).$$

4 Schwarz Lemma at the Boundary

In this section, we establish the Schwarz lemma at the smooth boundary point for holomorphic self-mappings of $\mathcal{R}_{\mathcal{III}}(n)$.

Let

$$\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} \end{pmatrix} U'$$

or

$$\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} & \\ 0 & & & 0 \end{pmatrix} U'$$

be a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$, where $1 = r_1 > r_2 \geq \cdots \geq r_p \geq 0$ and U is a unitary square matrix of order n. Then by Theorem 2.1, $\langle U\alpha U', \nabla\rho(\mathring{Z})\rangle = \alpha_{12}$ for any $\alpha \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$. This shows that the tangent space $T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n))$ to $\partial\mathcal{R}_{\mathcal{III}}(n)$ at $\mathring{Z}$ is

$$T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n)) = \left\{\beta \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}} : \Re\langle\beta, \nabla\rho(\mathring{Z})\rangle = 0\right\} = \left\{U\alpha U' : \alpha \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}, \Re\alpha_{12} = 0\right\},$$

and the holomorphic tangent space $T_{\mathring{Z}}^{1,0}(\partial\mathcal{R}_{\mathcal{III}}(n))$ to $\partial\mathcal{R}_{\mathcal{III}}(n)$ at $\mathring{Z}$ is

$$T_{\mathring{Z}}^{1,0}(\partial\mathcal{R}_{\mathcal{III}}(n)) = \left\{\beta \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}} : \langle\beta, \nabla\rho(\mathring{Z})\rangle = 0\right\} = \left\{U\alpha U' : \alpha \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}, \alpha_{12} = 0\right\}.$$

Theorem 4.1 *Let $f : \mathcal{R}_{\mathcal{III}}(n) \to \mathcal{R}_{\mathcal{III}}(n)$ be a holomorphic mapping with $f(0) = a$, and let*

$$\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} \end{pmatrix} U'$$

or

$$\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} & \\ 0 & & & 0 \end{pmatrix} U'$$

be a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$, where $1 = r_1 > r_2 \geq \cdots \geq r_p \geq 0$ and U is a unitary square matrix of order n. If f is holomorphic at $\mathring{Z}$ and $f(\mathring{Z}) = \mathring{Z}$, then all the eigenvalues $\lambda, \mu_i (i = 1, \cdots, 2(n-2))$ and $\nu_i (i = 1, \cdots, \frac{(n-2)(n-3)}{2})$ of the linear transformation $Df(\mathring{Z})$ on $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$ have the following properties.

(1) *$\nabla\rho(\mathring{Z})$ is an eigenvector of $D^*f(\mathring{Z})$ and the corresponding eigenvalue is a real number λ that we just mentioned above. That is, $D^*f(\mathring{Z})(\nabla\rho(\mathring{Z})) = \lambda\nabla\rho(\mathring{Z})$.*

(2) *$\lambda \geq \frac{1-\rho(a)}{1+\rho(a)} > 0$.*

(3) *$T_{\mathring{Z}}^{1,0}(\partial\mathcal{R}_{\mathcal{III}}(n)) = M \oplus N$, where $N = \left\{U\alpha U' : \alpha \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}, \alpha_{12} = 0, \begin{pmatrix} \alpha_{13} & \cdots & \alpha_{1n} \\ \alpha_{23} & \cdots & \alpha_{2n} \end{pmatrix} = 0\right\}$ is an $\frac{(n-2)(n-3)}{2}$-dimensional invariant subspace of $Df(\mathring{Z})$, and M is a $2(n-2)$-dimensional invariant subspace of $Df(\mathring{Z})$. Moreover, the eigenvalues μ_i of $Df(\mathring{Z})$, which is a linear transformation on M, satisfy*

$$|\mu_i| \leq \sqrt{\lambda}, \quad i = 1, \cdots, 2(n-2);$$

and the eigenvalues ν_i of $Df(\mathring{Z})$, which is a linear transformation on N, satisfy

$$|\nu_i| \leq 1, \quad i = 1, \cdots, \frac{(n-2)(n-3)}{2}.$$

(4) *$|\det Df(\mathring{Z})| \leq \lambda^{n-1}$, $|trDf(\mathring{Z})| \leq \lambda + 2\sqrt{\lambda}(n-2) + \frac{(n-2)(n-3)}{2}$.*

Moreover, the inequalities in (2)–(4) are sharp.

Proof Without loss of generality, we may assume that $n = 2p$ is an even number. When $n = 2p + 1$ is an odd number, the proof is similar.

(1) For any $\beta \in T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n))$, we have $Df(\mathring{Z})(\beta) \in T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n))$. Then

$$\Re\langle Df(\mathring{Z})(\beta), \nabla\rho(\mathring{Z})\rangle = \Re\langle \beta, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle = 0.$$

Hence, there is $\lambda \in \mathbb{R}$ such that

$$D^*f(\mathring{Z})(\nabla\rho(\mathring{Z})) = \lambda\nabla\rho(\mathring{Z}).$$

That is, λ is an eigenvalue of $D^*f(\mathring{Z})$ and $\nabla\rho(\mathring{Z})$ is an eigenvector of $D^*f(\mathring{Z})$ with respect to λ. Since $\lambda \in \mathbb{R}$, we know that λ is also an eigenvalue of $Df(\mathring{Z})$. The proof of (1) is complete.

(2) The proof of (2) is divided into two cases.

Case 1 $f(0) = a = 0$. For each $t \in (0, 1)$, by Lemma 3.5 we obtain

$$\rho(f(t\mathring{Z})) \le \rho(t\mathring{Z}) = t.$$

This, together with Lemma 3.4, yields

$$\Re\langle f(t\mathring{Z}), \nabla\rho(\mathring{Z})\rangle \le \rho(f(t\mathring{Z})) \le t. \tag{4.1}$$

By Theorem 2.1, $\langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1$. Thus, combine $f(t\mathring{Z}) = \mathring{Z} - (1-t)Df(\mathring{Z})(\mathring{Z}) + O(|t-1|^2)(t \to 1^-)$ and (4.1) to get

$$1 - (1-t)\Re\langle Df(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{Z})\rangle + O(|t-1|^2) \le t.$$

This implies

$$\Re\langle \mathring{Z}, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle + O(|t-1|) \ge 1.$$

It follows from $D^*f(\mathring{Z})(\nabla\rho(\mathring{Z})) = \lambda\nabla\rho(\mathring{Z})$ and $\langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1$ that

$$\lambda + O(|t-1|) \ge 1.$$

Taking $t \to 1^-$, we have $\lambda \ge 1$.

Case 2 $f(0) = a \ne 0$. Suppose that

$$a = A\begin{pmatrix} \begin{pmatrix} 0 & l_1 \\ -l_1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & l_p \\ -l_p & 0 \end{pmatrix} \end{pmatrix} A' \in \mathcal{R}_{\mathcal{III}}(n)$$

and

$$Q = A\begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-l_1^2}} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{I}_2}{\sqrt{1-l_p^2}} \end{pmatrix} \overline{A}',$$

where $1 > l_1 \geq \cdots \geq l_p \geq 0$ and A is a unitary square matrix of order n. By Lemma 3.2, $g = \varphi_a \circ f : \mathcal{R}_{\mathcal{III}}(n) \to \mathcal{R}_{\mathcal{III}}(n)$ is a holomorphic mapping, $g(0) = 0$ and g is holomorphic at $\mathring{Z}$. Moreover,

$$\mathring{W} = g(\mathring{Z}) = \varphi_a(\mathring{Z}) = Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}(a - \mathring{Z})\overline{Q}$$

is also a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$. Notice that $D\varphi_a(\mathring{Z})(\beta) \in T_{\mathring{W}}(\partial\mathcal{R}_{\mathcal{III}}(n))$ for each $\beta \in T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n))$. Then

$$\Re\langle D\varphi_a(\mathring{Z})(\beta), \nabla\rho(\mathring{W})\rangle = 0, \quad \Re\langle \beta, D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))\rangle = 0.$$

It follows that there exists $\mu \in \mathbb{R}$ such that

$$D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W})) = \mu\nabla\rho(\mathring{Z}). \tag{4.2}$$

Take

$$h_1(\zeta) = \langle \varphi_a(\zeta\mathring{Z}), \nabla\rho(\mathring{W})\rangle, \quad \zeta \in \triangle.$$

Then $h_1 : \triangle \to \triangle$ is a holomorphic function, and h_1 is holomorphic at $\zeta = 1$ with $h_1(1) = \langle \mathring{W}, \nabla\rho(\mathring{W})\rangle = 1$. This, together with (3.1) and (4.2), shows

$$\mu = \langle \mathring{Z}, \mu\nabla\rho(\mathring{Z})\rangle = \langle \mathring{Z}, D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))\rangle = \langle D\varphi_a(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle = h_1'(1) > 0.$$

Set

$$h_2(\zeta) = \langle g(\zeta\mathring{Z}), \nabla\rho(\mathring{W})\rangle, \quad \zeta \in \triangle.$$

Then $h_2 : \triangle \to \triangle$ is a holomorphic function, and h_2 is holomorphic at $\zeta = 1$ with $h_2(0) = 0$ and $h_2(1) = 1$. It follows from Lemma 3.1, (4.2) and (1) that

$$\begin{aligned} 1 &\leq h_2'(1) \\ &= \langle Dg(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle \\ &= \langle D\varphi_a(\mathring{Z})(Df(\mathring{Z})(\mathring{Z})), \nabla\rho(\mathring{W})\rangle \\ &= \langle Df(\mathring{Z})(\mathring{Z}), D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))\rangle \\ &= \mu\langle Df(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{Z})\rangle \\ &= \mu\langle \mathring{Z}, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle \\ &= \lambda\mu. \end{aligned}$$

This gives

$$\lambda \geq \frac{1}{\mu}.$$

Now, we estimate $\mu = \langle D\varphi_a(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle$. For $X \in \mathbb{C}^{n\times n}$, let $\rho_n(X) = \max\{\|\alpha X\| : \alpha \in \partial B^n\}$ be the matrix norm of X. Then $\rho_n(XY) \leq \rho_n(X)\rho_n(Y)$ for each $X, Y \in \mathbb{C}^{n\times n}$ and $\rho_n(Z) = \rho(Z)$ for any $Z \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$. Notice that

$$\begin{aligned} &D\varphi_a(\mathring{Z})(\mathring{Z}) \\ &= Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}\mathring{Z}\overline{a}'(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}(a - \mathring{Z})\overline{Q} - Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}\mathring{Z}\overline{Q} \\ &= Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}\mathring{Z}\overline{a}'Q\mathring{W} + \mathring{W} - Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}a\overline{Q} \end{aligned}$$

$$= Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}(\mathring{Z}\overline{a}' - \mathrm{I}_n)Q\mathring{W} + Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}Q\mathring{W} + \mathring{W} - Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}Qa$$
$$= Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}Q(\mathring{W} - a)$$

and

$$\begin{aligned}
&Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}(a - \mathring{Z})\overline{Q}\overline{a}' - \mathrm{I}_n = \mathring{W}\overline{a}' - \mathrm{I}_n,\\
&Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}\left[(a - \mathring{Z})\overline{Q}\overline{a}' - (\mathrm{I}_n - \mathring{Z}\overline{a}')Q\right] = \mathring{W}\overline{a}' - \mathrm{I}_n,\\
&Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}(Q - a\overline{Q}\overline{a}') = \mathrm{I}_n - \mathring{W}\overline{a}',\\
&Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1}Q^{-1} = \mathrm{I}_n - \mathring{W}\overline{a}',\\
&Q^{-1}(\mathrm{I}_n - \mathring{Z}\overline{a}')^{-1} = (\mathrm{I}_n - \mathring{W}\overline{a}')Q.
\end{aligned}$$

Then

$$D\varphi_a(\mathring{Z})(\mathring{Z}) = (\mathrm{I}_n - \mathring{W}\overline{a}')Q^2(\mathring{W} - a).$$

This, together with Lemma 3.4, yields

$$\begin{aligned}
\mu &= \langle D\varphi_a(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle\\
&\le \rho[D\varphi_a(\mathring{Z})(\mathring{Z})]\\
&\le [\rho_n(\mathrm{I}_n) + \rho(\mathring{W})\rho(\overline{a}')][\rho_n(Q)]^2[\rho(\mathring{W}) + \rho(a)]\\
&= [1 + \rho(a)]^2[1 - (\rho(a))^2]^{-1}\\
&= \frac{1 + \rho(a)}{1 - \rho(a)}.
\end{aligned}$$

Hence, we obtain

$$\lambda \ge \frac{1}{\mu} \ge \frac{1 - \rho(a)}{1 + \rho(a)}.$$

The proof of (2) is complete.

(3) It is well known that the $\left[\frac{n(n-1)}{2} - 1\right]$-dimensional space $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n)) = \{U\alpha U' : \alpha \in \mathbb{C}^{\frac{n(n-1)}{2}}_{\mathcal{III}}, \alpha_{12} = 0\}$ is an invariant subspace of $Df(\mathring{Z})$. That means

$$(\overline{U}' Df(\mathring{Z})(\beta)\overline{U})_{12} = 0$$

for any $\beta \in T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n))$. Now, we claim that

$$N = \left\{U\alpha U' : \alpha \in \mathbb{C}^{\frac{n(n-1)}{2}}_{\mathcal{III}}, \alpha_{12} = 0, \begin{pmatrix}\alpha_{13} & \cdots & \alpha_{1n}\\ \alpha_{23} & \cdots & \alpha_{2n}\end{pmatrix} = 0\right\}$$

is an invariant subspace of $Df(\mathring{Z})$. We only need to prove that for each

$$\beta = U\begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 0 & \alpha_{34} & \cdots & \alpha_{3(n-1)} & \alpha_{3n}\\ -\alpha_{34} & 0 & \cdots & \alpha_{4(n-1)} & \alpha_{4n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -\alpha_{3(n-1)} & -\alpha_{4(n-1)} & \cdots & 0 & \alpha_{(n-1)n}\\ -\alpha_{3n} & -\alpha_{4n} & \cdots & -\alpha_{(n-1)n} & 0 \end{pmatrix}\end{pmatrix}U' \in N,$$

if we set $\varepsilon = \overline{U}' Df(\mathring{Z})(\beta)\overline{U} \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$, then $\varepsilon_{12} = 0$ and $\begin{pmatrix} \varepsilon_{13} & \cdots & \varepsilon_{1n} \\ \varepsilon_{23} & \cdots & \varepsilon_{2n} \end{pmatrix} = 0$.

According to $Df(\mathring{Z})(\beta) \in T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n))$, we have $\varepsilon_{12} = 0$. For $t \in (0,1)$, write the polar decompositions of $t\mathring{Z}$ and $f(t\mathring{Z})$ as

$$t\mathring{Z} = U \begin{pmatrix} \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & tr_p \\ -tr_p & 0 \end{pmatrix} \end{pmatrix} U'$$

and

$$f(t\mathring{Z}) = U(t) \begin{pmatrix} \begin{pmatrix} 0 & r_1(t) \\ -r_1(t) & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & r_p(t) \\ -r_p(t) & 0 \end{pmatrix} \end{pmatrix} U(t)',$$

respectively, where $1 > r_1(t) \geq r_2(t) \geq \cdots \geq r_p(t) \geq 0$ and $U(t)$ is a unitary square matrix of order n. By Lemma 3.2, corresponding to $a = t\mathring{Z}$ and $a = f(t\mathring{Z})$, take

$$Q = U \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-t^2}} & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-t^2r_p^2}} \end{pmatrix} \overline{U}'$$

and

$$Q(t) = U(t) \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} \overline{U(t)}'.$$

Since $\lim_{t\to 1^-} f(t\mathring{Z}) = \mathring{Z}$, we obtain

$$\lim_{t\to 1^-} r_1(t) = 1, \quad \lim_{t\to 1^-} r_2(t) = r_2, \quad \cdots, \quad \lim_{t\to 1^-} r_p(t) = r_p.$$

Meanwhile, we get

$$U(t) = U + O(|t-1|), \quad Df(t\mathring{Z})(\beta) = Df(\mathring{Z})(\beta) + O(|t-1|)$$

as $t \to 1^-$. Moreover, it follows from $f(t\mathring{Z}) = \mathring{Z} - (1-t)Df(\mathring{Z})(\mathring{Z}) + O(|t-1|^2)$ that

$$\begin{aligned} r_1(t) &= \rho(f(t\mathring{Z})) \\ &= 1 - (1-t)2\Re \sum_{1\leq i<j\leq n} \frac{\partial\rho}{\partial z_{ij}}(\mathring{Z}) Df_{ij}(\mathring{Z})(\mathring{Z}) + O(|t-1|^2) \\ &= 1 - (1-t)\Re\langle Df(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{Z})\rangle + O(|t-1|^2) \\ &= 1 - (1-t)\Re\langle \mathring{Z}, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle + O(|t-1|^2) \end{aligned}$$

$$= 1 - \lambda(1-t) + O(|t-1|^2)$$

as $t \to 1^-$. This implies

$$\sqrt{1 - r_1^2(t)} = \sqrt{1 - [1 - \lambda(1-t) + O(|t-1|^2)]^2} = \sqrt{2\lambda(1-t) + O(|t-1|^2)} \tag{4.3}$$

as $t \to 1^-$. By Lemma 3.3,

$$\begin{aligned}
&F(t\mathring{Z}, \beta)\\
&= \rho \left[U \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-t^2}} & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} \overline{U}' \beta \overline{U} \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-t^2}} & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} U' \right]\\
&= \rho \left[\begin{pmatrix} \mathrm{I}_2 & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} \right.\\
&\left. \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 0 & \alpha_{34} & \cdots & \alpha_{3(n-1)} & \alpha_{3n} \\ -\alpha_{34} & 0 & \cdots & \alpha_{4(n-1)} & \alpha_{4n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{3(n-1)} & -\alpha_{4(n-1)} & \cdots & 0 & \alpha_{(n-1)n} \\ -\alpha_{3n} & -\alpha_{4n} & \cdots & -\alpha_{(n-1)n} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} \right].
\end{aligned}$$

This gives

$$\lim_{t \to 1^-} \sqrt{1-t^2} F(t\mathring{Z}, \beta) = 0. \tag{4.4}$$

Similarly, we have

$$\begin{aligned}
&F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta)]\\
&= \rho \left[U(t) \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} \overline{U(t)}' Df(t\mathring{Z})(\beta) \overline{U(t)} \right.\\
&\left. \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} U(t)' \right].
\end{aligned}$$

Notice that

$$\overline{U(t)}' Df(t\mathring{Z})(\beta) \overline{U(t)} = \overline{U}' Df(\mathring{Z})(\beta) \overline{U} + O(|t-1|) = \varepsilon + O(|t-1|)$$

as $t \to 1^-$. This, together with (4.3), shows

$$\begin{aligned}
&\lim_{t\to 1^-}\sqrt{1-r_1^2(t)}F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta)]\\
&=\lim_{t\to 1^-}\sqrt{1-r_1^2(t)}\rho\left[\begin{pmatrix} \mathrm{I}_2 & & & 0\\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}}\end{pmatrix}\right.\\
&\quad\begin{pmatrix}
\frac{O(|t-1|)}{1-r_1^2(t)} & \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{\varepsilon_{13}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{\varepsilon_{1n}+O(|t-1|)}{\sqrt{1-r_1^2(t)}}\\
\frac{O(|t-1|)}{1-r_1^2(t)} & \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{\varepsilon_{23}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{\varepsilon_{2n}+O(|t-1|)}{\sqrt{1-r_1^2(t)}}\\
\frac{-\varepsilon_{13}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \frac{-\varepsilon_{23}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & O(|t-1|) & \cdots & \varepsilon_{3n}+O(|t-1|)\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{-\varepsilon_{1n}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \frac{-\varepsilon_{2n}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & -\varepsilon_{3n}+O(|t-1|) & \cdots & O(|t-1|)
\end{pmatrix}\\
&\quad\left.\begin{pmatrix} \mathrm{I}_2 & & & 0\\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}}\end{pmatrix}\right]\\
&=\rho\left[\begin{pmatrix}
0 & 0 & \frac{\varepsilon_{13}}{\sqrt{1-r_2^2}} & \cdots & \frac{\varepsilon_{1n}}{\sqrt{1-r_p^2}}\\
0 & 0 & \frac{\varepsilon_{23}}{\sqrt{1-r_2^2}} & \cdots & \frac{\varepsilon_{2n}}{\sqrt{1-r_p^2}}\\
\frac{-\varepsilon_{13}}{\sqrt{1-r_2^2}} & \frac{-\varepsilon_{23}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{-\varepsilon_{1n}}{\sqrt{1-r_p^2}} & \frac{-\varepsilon_{2n}}{\sqrt{1-r_p^2}} & 0 & \cdots & 0
\end{pmatrix}\right].
\end{aligned}$$

By the contraction property of the Kobayashi metric, we get

$$F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta)] \le F(t\mathring{Z},\beta). \tag{4.5}$$

Thus, by (4.3)–(4.5), we obtain

$$\begin{aligned}
&\rho\left[\begin{pmatrix}
0 & 0 & \frac{\varepsilon_{13}}{\sqrt{1-r_2^2}} & \cdots & \frac{\varepsilon_{1n}}{\sqrt{1-r_p^2}}\\
0 & 0 & \frac{\varepsilon_{23}}{\sqrt{1-r_2^2}} & \cdots & \frac{\varepsilon_{2n}}{\sqrt{1-r_p^2}}\\
\frac{-\varepsilon_{13}}{\sqrt{1-r_2^2}} & \frac{-\varepsilon_{23}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{-\varepsilon_{1n}}{\sqrt{1-r_p^2}} & \frac{-\varepsilon_{2n}}{\sqrt{1-r_p^2}} & 0 & \cdots & 0
\end{pmatrix}\right]\\
&=\lim_{t\to 1^-}\sqrt{1-r_1^2(t)}F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta)]\\
&\le \lim_{t\to 1^-}\frac{\sqrt{1-r_1^2(t)}}{\sqrt{1-t^2}}\sqrt{1-t^2}F(t\mathring{Z},\beta)\\
&=\sqrt{\lambda}\lim_{t\to 1^-}\sqrt{1-t^2}F(t\mathring{Z},\beta)=0.
\end{aligned}$$

That means $\begin{pmatrix}\varepsilon_{13} & \cdots & \varepsilon_{1n}\\ \varepsilon_{23} & \cdots & \varepsilon_{2n}\end{pmatrix}=0$. It follows that N is an $\frac{(n-2)(n-3)}{2}$-dimensional invariant subspace of $Df(\mathring{Z})$. Hence, there is a $2(n-2)$-dimensional invariant subspace M of $Df(\mathring{Z})$ such that $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n)) = M\oplus N$. Because $M\cap N=\{0\}$, we have $\begin{pmatrix}\alpha_{13} & \cdots & \alpha_{1n}\\ \alpha_{23} & \cdots & \alpha_{2n}\end{pmatrix}\neq 0$ for any $\beta = U\alpha U'\in M\setminus\{0\}$.

For each eigenvalue μ_i of $Df(\mathring{Z})$ on M, suppose that $\beta^{(i)}=U\alpha^{(i)}U'\in M\setminus\{0\}$ is a nonzero eigenvector with respect to μ_i. Here

$$\begin{pmatrix}\alpha^{(i)}_{13} & \cdots & \alpha^{(i)}_{1n}\\ \alpha^{(i)}_{23} & \cdots & \alpha^{(i)}_{2n}\end{pmatrix}\neq 0,\quad \overline{U}'Df(\mathring{Z})(\beta^{(i)})\overline{U}=\mu_i\alpha^{(i)},\quad i=1,\cdots,2(n-2).$$

By Lemma 3.3, we get

$$\begin{aligned}
F(t\mathring{Z},\beta^{(i)}) &= \rho\left[U\begin{pmatrix}\frac{I_2}{\sqrt{1-t^2}} & & & 0\\ & \frac{I_2}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2r_p^2}}\end{pmatrix}\overline{U}'\beta^{(i)}\overline{U}\right.\\
&\qquad\left.\begin{pmatrix}\frac{I_2}{\sqrt{1-t^2}} & & & 0\\ & \frac{I_2}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2r_p^2}}\end{pmatrix}U'\right]\\
&=\rho\left[\begin{pmatrix}I_2 & & & 0\\ & \frac{I_2}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2r_p^2}}\end{pmatrix}\begin{pmatrix}0 & 0 & \frac{\alpha^{(i)}_{13}}{\sqrt{1-t^2}} & \cdots & \frac{\alpha^{(i)}_{1n}}{\sqrt{1-t^2}}\\ 0 & 0 & \frac{\alpha^{(i)}_{23}}{\sqrt{1-t^2}} & \cdots & \frac{\alpha^{(i)}_{2n}}{\sqrt{1-t^2}}\\ \frac{-\alpha^{(i)}_{13}}{\sqrt{1-t^2}} & \frac{-\alpha^{(i)}_{23}}{\sqrt{1-t^2}} & 0 & \cdots & \alpha^{(i)}_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{-\alpha^{(i)}_{1n}}{\sqrt{1-t^2}} & \frac{-\alpha^{(i)}_{2n}}{\sqrt{1-t^2}} & -\alpha^{(i)}_{3n} & \cdots & 0\end{pmatrix}\right.\\
&\qquad\left.\begin{pmatrix}I_2 & & & 0\\ & \frac{I_2}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2r_p^2}}\end{pmatrix}\right].
\end{aligned}$$

This yields

$$\lim_{t\to 1^-}\sqrt{1-t^2}F(t\mathring{Z},\beta^{(i)})=\rho\left[\begin{pmatrix}0 & 0 & \frac{\alpha^{(i)}_{13}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha^{(i)}_{1n}}{\sqrt{1-r_p^2}}\\ 0 & 0 & \frac{\alpha^{(i)}_{23}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha^{(i)}_{2n}}{\sqrt{1-r_p^2}}\\ \frac{-\alpha^{(i)}_{13}}{\sqrt{1-r_2^2}} & \frac{-\alpha^{(i)}_{23}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{-\alpha^{(i)}_{1n}}{\sqrt{1-r_p^2}} & \frac{-\alpha^{(i)}_{2n}}{\sqrt{1-r_p^2}} & 0 & \cdots & 0\end{pmatrix}\right]\neq 0.$$

On the other hand,

$$F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})] = \rho \left[U(t) \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-r_1^2(t)}} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} \overline{U(t)}' Df(t\mathring{Z})(\beta^{(i)}) \overline{U(t)} \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-r_1^2(t)}} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} U(t)' \right].$$

Notice that $\overline{U(t)}' Df(t\mathring{Z})(\beta^{(i)})\overline{U(t)} = \overline{U}' Df(\mathring{Z})(\beta^{(i)})\overline{U} + O(|t-1|) = \mu_i \alpha^{(i)} + O(|t-1|)$ as $t \to 1^-$ and $\alpha_{12}^{(i)} = 0$. Then

$$\begin{aligned}
&\lim_{t\to 1^-} \sqrt{1-r_1^2(t)} F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})] \\
&= \lim_{t\to 1^-} \sqrt{1-r_1^2(t)} \rho \left[\begin{pmatrix} \mathrm{I}_2 & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} \right. \\
&\quad \begin{pmatrix} \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{\mu_i\alpha_{13}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{\mu_i\alpha_{1n}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} \\ \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{\mu_i\alpha_{23}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{\mu_i\alpha_{2n}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} \\ \frac{-\mu_i\alpha_{13}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \frac{-\mu_i\alpha_{23}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & O(|t-1|) & \cdots & \mu_i\alpha_{3n}^{(i)}+O(|t-1|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\mu_i\alpha_{1n}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \frac{-\mu_i\alpha_{2n}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & -\mu_i\alpha_{3n}^{(i)}+O(|t-1|) & \cdots & O(|t-1|) \end{pmatrix} \\
&\quad \left. \begin{pmatrix} \mathrm{I}_2 & & & 0 \\ & \frac{\mathrm{I}_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{\mathrm{I}_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix} \right] \\
&= |\mu_i| \rho \left[\begin{pmatrix} 0 & 0 & \frac{\alpha_{13}^{(i)}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha_{1n}^{(i)}}{\sqrt{1-r_p^2}} \\ 0 & 0 & \frac{\alpha_{23}^{(i)}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha_{2n}^{(i)}}{\sqrt{1-r_p^2}} \\ \frac{-\alpha_{13}^{(i)}}{\sqrt{1-r_2^2}} & \frac{-\alpha_{23}^{(i)}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\alpha_{1n}^{(i)}}{\sqrt{1-r_p^2}} & \frac{-\alpha_{2n}^{(i)}}{\sqrt{1-r_p^2}} & 0 & \cdots & 0 \end{pmatrix} \right] \\
&= |\mu_i| \lim_{t\to 1^-} \sqrt{1-t^2} F(t\mathring{Z}, \beta^{(i)}).
\end{aligned}$$

It follows from this and (4.3) that

$$1 \geq \lim_{t\to 1^-} \frac{F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})]}{F(t\mathring{Z}, \beta^{(i)})}$$

$$= \lim_{t\to 1^-} \frac{\sqrt{1-t^2}}{\sqrt{1-r_1^2(t)}} \frac{\sqrt{1-r_1^2(t)}}{\sqrt{1-t^2}} \frac{F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})]}{F(t\mathring{Z}, \beta^{(i)})} = \frac{|\mu_i|}{\sqrt{\lambda}}.$$

This implies

$$|\mu_i| \le \sqrt{\lambda}, \quad i = 1, \cdots, 2(n-2).$$

For any eigenvalue ν_i of $Df(\mathring{Z})$ on N, suppose that $\beta^{(i)} = U\alpha^{(i)}U' \in N \setminus \{0\}$ is a nonzero eigenvector with respect to ν_i. Here

$$\begin{pmatrix} \alpha_{13}^{(i)} & \cdots & \alpha_{1n}^{(i)} \\ \alpha_{23}^{(i)} & \cdots & \alpha_{2n}^{(i)} \end{pmatrix} = 0, \quad \begin{pmatrix} 0 & \alpha_{34}^{(i)} & \cdots & \alpha_{3(n-1)}^{(i)} & \alpha_{3n}^{(i)} \\ -\alpha_{34}^{(i)} & 0 & \cdots & \alpha_{4(n-1)}^{(i)} & \alpha_{4n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{3(n-1)}^{(i)} & -\alpha_{4(n-1)}^{(i)} & \cdots & 0 & \alpha_{(n-1)n}^{(i)} \\ -\alpha_{3n}^{(i)} & -\alpha_{4n}^{(i)} & \cdots & -\alpha_{(n-1)n}^{(i)} & 0 \end{pmatrix} \ne 0$$

and $\overline{U}' Df(\mathring{Z})(\beta^{(i)})\overline{U} = \nu_i \alpha^{(i)}$ for $i = 1, \cdots, \frac{(n-2)(n-3)}{2}$. Then by Lemma 3.3, we have

$$\begin{aligned} & F(t\mathring{Z}, \beta^{(i)}) \\ & = \rho \left[U \begin{pmatrix} \frac{I_2}{\sqrt{1-t^2}} & & & 0 \\ & \frac{I_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} \right. \\ & \quad \left. \overline{U}'\beta^{(i)}\overline{U} \begin{pmatrix} \frac{I_2}{\sqrt{1-t^2}} & & & 0 \\ & \frac{I_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} U' \right] \\ & = \rho \left[\begin{pmatrix} I_2 & & & 0 \\ & \frac{I_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 0 & \alpha_{34}^{(i)} & \cdots & \alpha_{3(n-1)}^{(i)} & \alpha_{3n}^{(i)} \\ -\alpha_{34}^{(i)} & 0 & \cdots & \alpha_{4(n-1)}^{(i)} & \alpha_{4n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{3(n-1)}^{(i)} & -\alpha_{4(n-1)}^{(i)} & \cdots & 0 & \alpha_{(n-1)n}^{(i)} \\ -\alpha_{3n}^{(i)} & -\alpha_{4n}^{(i)} & \cdots & -\alpha_{(n-1)n}^{(i)} & 0 \end{pmatrix} \end{pmatrix} \right. \\ & \quad \left. \begin{pmatrix} I_2 & & & 0 \\ & \frac{I_2}{\sqrt{1-t^2 r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-t^2 r_p^2}} \end{pmatrix} \right]. \end{aligned}$$

Hence

$$\lim_{t\to 1^-} F(t\mathring{Z},\beta^{(i)}) = \rho\left[\begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 0 & \frac{\alpha_{34}^{(i)}}{1-r_2^2} & \cdots & \frac{\alpha_{3(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{\alpha_{3n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} \\ \frac{-\alpha_{34}^{(i)}}{1-r_2^2} & 0 & \cdots & \frac{\alpha_{4(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{\alpha_{4n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-\alpha_{3(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{-\alpha_{4(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \cdots & 0 & \frac{\alpha_{(n-1)n}^{(i)}}{1-r_p^2} \\ \frac{-\alpha_{3n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{-\alpha_{4n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \cdots & \frac{-\alpha_{(n-1)n}^{(i)}}{1-r_p^2} & 0 \end{pmatrix}\end{pmatrix}\right]$$
$$\neq 0.$$

On the other hand,

$$F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})] = \rho\left[U(t)\begin{pmatrix} \frac{I_2}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{I_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix}\overline{U(t)}'Df(t\mathring{Z})(\beta^{(i)})\overline{U(t)}\right.$$
$$\left.\begin{pmatrix} \frac{I_2}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{I_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix}U(t)'\right].$$

Thus we utilize $\overline{U(t)}'Df(t\mathring{Z})(\beta^{(i)})\overline{U(t)} = \overline{U}'Df(\mathring{Z})(\beta^{(i)})\overline{U} + O(|t-1|) = \nu_i\alpha^{(i)} + O(|t-1|)$ as $t\to 1^-$ and (4.3) to achieve

$$\begin{aligned}
&\lim_{t\to 1^-} F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})] \\
&= \lim_{t\to 1^-}\rho\left[\begin{pmatrix} I_2 & & & 0 \\ & \frac{I_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix}\begin{pmatrix}\begin{pmatrix} 0 & \frac{O(|t-1|)}{1-r_1^2(t)} \\ \frac{O(|t-1|)}{1-r_1^2(t)} & 0 \end{pmatrix} & \frac{O(|t-1|)}{\sqrt{1-r_1^2(t)}} \\ \frac{O(|t-1|)}{\sqrt{1-r_1^2(t)}} & B \end{pmatrix}\right. \\
&\qquad\left.\begin{pmatrix} I_2 & & & 0 \\ & \frac{I_2}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{I_2}{\sqrt{1-r_p^2(t)}} \end{pmatrix}\right] \\
&= \rho\left[\begin{pmatrix}\begin{pmatrix} 0 & b_{12} \\ -b_{12} & 0 \end{pmatrix} & 0 \\ 0 & \widetilde{B}\end{pmatrix}\right] \\
&\geq |\nu_i|\lim_{t\to 1^-} F(t\mathring{Z},\beta^{(i)}),
\end{aligned}$$

where

$$b_{12} = \lim_{t\to 1^-} \frac{\left(\overline{U(t)}' Df(t\mathring{Z})(\beta^{(i)})\overline{U(t)}\right)_{12}}{1-r_1^2(t)},$$

$B=$

$$\begin{pmatrix} 0 & \nu_i\alpha_{34}^{(i)}+O(|t-1|) & \cdots & \nu_i\alpha_{3(n-1)}^{(i)}+O(|t-1|) & \nu_i\alpha_{3n}^{(i)}+O(|t-1|) \\ -\nu_i\alpha_{34}^{(i)}+O(|t-1|) & 0 & \cdots & \nu_i\alpha_{4(n-1)}^{(i)}+O(|t-1|) & \nu_i\alpha_{4n}^{(i)}+O(|t-1|) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\nu_i\alpha_{3(n-1)}^{(i)}+O(|t-1|) & -\nu_i\alpha_{4(n-1)}^{(i)}+O(|t-1|) & \cdots & 0 & \nu_i\alpha_{(n-1)n}^{(i)}+O(|t-1|) \\ -\nu_i\alpha_{3n}^{(i)}+O(|t-1|) & -\nu_i\alpha_{4n}^{(i)}+O(|t-1|) & \cdots & -\nu_i\alpha_{(n-1)n}^{(i)}+O(|t-1|) & 0 \end{pmatrix}$$

and

$$\widetilde{B} = \begin{pmatrix} 0 & \frac{\nu_i\alpha_{34}^{(i)}}{1-r_2^2} & \cdots & \frac{\nu_i\alpha_{3(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{\nu_i\alpha_{3n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} \\ \frac{-\nu_i\alpha_{34}^{(i)}}{1-r_2^2} & 0 & \cdots & \frac{\nu_i\alpha_{4(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{\nu_i\alpha_{4n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-\nu_i\alpha_{3(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{-\nu_i\alpha_{4(n-1)}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \cdots & 0 & \frac{\nu_i\alpha_{(n-1)n}^{(i)}}{1-r_p^2} \\ \frac{-\nu_i\alpha_{3n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \frac{-\nu_i\alpha_{4n}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_p^2}} & \cdots & \frac{-\nu_i\alpha_{(n-1)n}^{(i)}}{1-r_p^2} & 0 \end{pmatrix}.$$

It follows that

$$1 \geq \lim_{t\to 1^-} \frac{F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})]}{F(t\mathring{Z}, \beta^{(i)})} \geq |\nu_i|.$$

This shows

$$|\nu_i| \leq 1, \quad i = 1, \cdots, \frac{(n-2)(n-3)}{2}.$$

The proof of (3) is complete.

(4) Note that $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n)) = \left\{U\alpha U' : \alpha \in \mathbb{C}^{\frac{n(n-1)}{2}}_{\mathcal{III}}, \alpha_{12} = 0\right\} = M \oplus N$ is an $\left[\frac{n(n-1)}{2}-1\right]$-dimensional invariant subspace of $Df(\mathring{Z})$. So, there is a one-dimensional invariant subspace L of $Df(\mathring{Z})$ such that

$$\mathbb{C}^{\frac{n(n-1)}{2}}_{\mathcal{III}} = L \oplus M \oplus N.$$

Since $L \cap T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{III}}(n)) = \{0\}$ we have $\alpha_{12} \neq 0$ for any $\beta = U\alpha U' \in L\setminus\{0\}$. Now, we prove that λ is just the eigenvalue of $Df(\mathring{Z})$ on L. Suppose that $\widetilde{\lambda}$ is an eigenvalue of $Df(\mathring{Z})$ on L, and $\beta = U\alpha U' \in L\setminus\{0\}$ is a nonzero eigenvector of $Df(\mathring{Z})$ with respect to $\widetilde{\lambda}$. Then Theorem 2.1 is utilized to derive

$$\begin{aligned}\langle Df(\mathring{Z})(\beta), \nabla\rho(\mathring{Z})\rangle &= \widetilde{\lambda}\langle\beta, \nabla\rho(\mathring{Z})\rangle \\ &= \frac{1}{2}\widetilde{\lambda} tr\left(U\alpha U'\overline{U}\begin{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & 0\end{pmatrix}\overline{U}'\right) = \widetilde{\lambda}\alpha_{12}.\end{aligned}$$

Meanwhile

$$\langle Df(\mathring{Z})(\beta), \nabla\rho(\mathring{Z})\rangle = \langle \beta, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle = \lambda\langle\beta, \nabla\rho(\mathring{Z})\rangle = \lambda\alpha_{12}.$$

This, together with $\alpha_{12} \neq 0$, gives $\widetilde{\lambda} = \lambda$. Therefore λ, μ_i $(i = 1, \cdots, 2(n-2))$ and ν_i $\big(i = 1, \cdots, \frac{(n-2)(n-3)}{2}\big)$ are all the eigenvalues of the linear transformation $Df(\mathring{Z})$ on $\mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$. This implies

$$|\det Df(\mathring{Z})| \leq \lambda^{n-1}, \quad |trDf(\mathring{Z})| \leq \lambda + 2\sqrt{\lambda}(n-2) + \frac{(n-2)(n-3)}{2}.$$

The proof of (4) is complete.

Remark 4.1 From the view of geometry, N is an invariant subspace of $Df(\mathring{Z})$ perhaps because the Levi form of ρ at $\mathring{Z}$ is positive semi-definite and not positive definite on N. We get the same conclusions of $|\mu_i| \leq \sqrt{\lambda}$ $(i = 1, \cdots, 2(n-2))$ with Theorem 3.1 in [17] perhaps because the Levi form of ρ at $\mathring{Z}$ is positive definite on M.

Remark 4.2 From the proof of Theorem 4.1, it is clear that we need only to assume that the mapping f is C^1 up to the boundary of $\mathcal{R}_{\mathcal{III}}(n)$ near $\mathring{Z}$.

Remark 4.3 When $n = 2, f(0) = 0$ and $\mathcal{R}_{\mathcal{III}}(2) = \triangle$, Theorem 4.1 is just Lemma 3.1. And when $\mathcal{R}_{\mathcal{III}}(3) = B^3$, Theorem 4.1 is just Theorem 3.1 in [18].

Finally, we give the following example to show that the inequalities in (2)–(4) of Theorem 4.1 are sharp.

Example 4.1 Let

$$a = \begin{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & 0 \end{pmatrix} \in \mathcal{R}_{\mathcal{III}}(n)$$

and $0 < \varepsilon < 1$. Write $e_{ij} \in \mathbb{C}^{n\times n}$ as a square matrix, which has 1 at i-th row and j-th column, and 0s elsewhere. According to Lemma 3.2, take

$$Q = \begin{pmatrix} \frac{1}{\sqrt{1-\varepsilon^2}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix}.$$

Let $\mathring{Z} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} \end{pmatrix}$ or $\mathring{Z} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} & \\ 0 & & & 0 \end{pmatrix}$ be a smooth boundary point of $\mathcal{R}_{\mathcal{III}}(n)$, where $1 > r_2 \geq \cdots \geq r_p \geq 0$. Define

$$f(Z) = -\varphi_{-a}(Z) = Q^{-1}(\mathrm{I}_n + Z\overline{a}')^{-1}(a + Z)\overline{Q}, \quad Z \in \overline{\mathcal{R}_{\mathcal{III}}(n)}.$$

Then $f : \mathcal{R}_{\mathcal{III}}(n) \to \mathcal{R}_{\mathcal{III}}(n)$ is a holomorphic mapping with $f(0) = a$, and f is holomorphic at $\mathring{Z}$. Moreover, f has the following properties.

(1) $f(\mathring{Z}) = \mathring{Z}$.

(2) For any $\beta \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$,

$$Df(\mathring{Z})(\beta) = \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \beta \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix}.$$

(3) $Df(\mathring{Z})(e_{12} - e_{21}) = \frac{1-\rho(a)}{1+\rho(a)}(e_{12} - e_{21})$. This shows that one of eigenvalues of $Df(\mathring{Z})$ is $\frac{1-\rho(a)}{1+\rho(a)}$.

(4) $Df(\mathring{Z})(e_{ij} - e_{ji}) = \sqrt{\frac{1-\rho(a)}{1+\rho(a)}}(e_{ij} - e_{ji})$ $(i = 1, 2; 3 \le j \le n)$. This shows that the $2(n-2)$ eigenvalues of $Df(\mathring{Z})$ are all $\sqrt{\frac{1-\rho(a)}{1+\rho(a)}}$.

(5) $Df(\mathring{Z})(e_{ij} - e_{ji}) = e_{ij} - e_{ji}(3 \le i < j \le n)$. This shows that the $\frac{(n-2)(n-3)}{2}$ eigenvalues of $Df(\mathring{Z})$ are all 1.

Proof By Lemma 3.2, it is clear that $f : \mathcal{R}_{\mathcal{III}}(n) \to \mathcal{R}_{\mathcal{III}}(n)$ is a holomorphic mapping with $f(0) = a$, and f is holomorphic at $\mathring{Z}$. Without loss of generality, we may assume that $n = 2p$ is an even number.

(1) It is obvious that $\rho(a) = \varepsilon$. Since

$$\mathring{Z}\overline{a}' = \begin{pmatrix} \varepsilon\mathrm{I}_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{I}_n + \mathring{Z}\overline{a}' = \begin{pmatrix} (1+\varepsilon)\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix}$$

and

$$a + \mathring{Z} = \begin{pmatrix} \begin{pmatrix} 0 & 1+\varepsilon \\ -(1+\varepsilon) & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} \end{pmatrix},$$

we have

$$\begin{aligned} & f(\mathring{Z}) \\ &= Q^{-1}(\mathrm{I}_n + \mathring{Z}\overline{a}')^{-1}(a + \mathring{Z})\overline{Q} \\ &= \begin{pmatrix} \sqrt{1-\varepsilon^2}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{I}_2}{1+\varepsilon} & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \\ & \quad \begin{pmatrix} \begin{pmatrix} 0 & 1+\varepsilon \\ -(1+\varepsilon) & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & r_p \\ -r_p & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{I}_2}{\sqrt{1-\varepsilon^2}} & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \\ &= \mathring{Z}. \end{aligned}$$

(2) For any $\beta \in \mathbb{C}_{\mathcal{III}}^{\frac{n(n-1)}{2}}$, we get

$$\begin{aligned} Df(\mathring{Z})(\beta) &= Q^{-1}(\mathrm{I}_n + \mathring{Z}\overline{a}')^{-1}\beta\overline{Q} - Q^{-1}(\mathrm{I}_n + \mathring{Z}\overline{a}')^{-1}\beta\overline{a}'(\mathrm{I}_n + \mathring{Z}\overline{a}')^{-1}(a + \mathring{Z})\overline{Q} \\ &= Q^{-1}(\mathrm{I}_n + \mathring{Z}\overline{a}')^{-1}\beta\overline{Q} - Q^{-1}(\mathrm{I}_n + \mathring{Z}\overline{a}')^{-1}\beta\overline{a}'Qf(\mathring{Z}) \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \beta(\overline{Q} - \overline{a}' Q \mathring{Z}) \\
&= \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \beta \left[\begin{pmatrix} \frac{1}{\sqrt{1-\varepsilon^2}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} - \begin{pmatrix} \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\mathrm{I}_2 & 0 \\ 0 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \beta \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix} \beta \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}}\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_{n-2} \end{pmatrix}. \qquad (4.6)
\end{aligned}
$$

(3)–(5) By (4.6) and a straightforward calculation, we can obtain (3)–(5) at once. The proof is complete.

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