Chin. Ann. Math. Ser. B
41(3), 2020, 383–396
DOI: 10.1007/s11401-020-0205-x

Chinese Annals of Mathematics, Series B

Explicit Meromorphic Solutions of a Certain Briot-Bouquet Differential Equations*

Xujie SHI[1] Liangwen LIAO[1] Song ZHANG[2]

Abstract This paper deals with the Briot-Bouquet differential equations with degree three. The previous result shows that all the meromorphic solutions belong to W. Here, by applying the Kowalevski-Gambier method, the authors give all the possible explicit meromorphic solutions. The result is more applicable. Also, this method can be used to deal with the more general Briot-Bouquet differential equations.

Keywords Movable poles, Differential equations, Meromorphic solutions
2010 MR Subject Classification 30D35, 33E30, 30D30

1 Introduction

As we all know, equations, mostly arising from geometry, physical, engineering and economic sources, are of importance in the exploration of nature, such as the NLS equation, the KdV equation. The Briot-Bouquet differential equation is also an important one. For example, in 2007, Zhang applied the real options to bank loan decision based on internal rate of return (IRR for short), and obtained a Briot-Bouquet differential equation (cf. [14]). This conclusion simplifies the previous results. In fact, Briot and Bouquet showed that every meromorphic solution of Briot-Bouquet differential equation belongs to W (cf. [3–4]), where we denote by W the class of meromorphic functions consisting of elliptic functions, and their degenerates, i.e., the rational functions of one exponential e^{az}, $a \in \mathbb{C}$ and rational functions. Other properties of Briot-Bouquet differential equation were also studied by Hille (cf. [8, 11]). Many results for higher-order Briot-Bouquet differential equation are also exhibited in a series of papers (cf. [2, 6–7, 9–10]). However, it is still difficult to find the explicit solutions.

In this paper, we shall use the Kowalevski-Gambier method (cf. [5]) to consider the cubic Briot-Bouquet differential equations

$$a_1 f'^3 + a_2 f'^2 f + a_3 f' f^2 + a_4 f^3 + a_5 f'^2 + a_6 f' f + a_7 f^2 + a_8 f' + a_9 f + a_{10} = 0, \tag{1.1}$$

where a_j $(j = 1, 2, \cdots, 10)$ are constants.

Manuscript received November 21, 2017.
[1]Department of Mathematics, Nanjing University, Nanjing 210093, China.
E-mail: yywn2006@sina.com maliao@nju.edu.cn
[2]Corresponding author. Department of Mathematics, Nanjing University, Nanjing 210093, China.
E-mail: dg1521017@smail.nju.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11671191).

The notations and the fundamental results are adopted as shown in [1, 12–13]. And in order to make our paper more readable, we first introduce the Kowalevski-Gambier method. Suppose that we are given an ODE written in the form

$$E = E(z, f^{(N)}, \cdots, f', f) \equiv 0. \tag{1.2}$$

If (1.2) has a meromorphic solution $f = f(z)$ which has a pole at $z = z_0$ of order $-p > 0$, then we can use a Laurent series to represent f locally, namely

$$f = \sum_{j=0}^{+\infty} u_j (z - z_0)^{j+p}, \qquad u_0 \neq 0, \quad p < 0.$$

When we put the Laurent series back into (1.2), we have the form

$$E = \sum_{j=0}^{+\infty} E_j (z - z_0)^{j+q} \equiv 0, \tag{1.3}$$

where q is the smallest integer among the list of leading powers and we shall call those terms involved in E_0 as dominant terms and we write them collectively as $\widehat{E}$. The key idea of this method is to express the necessary conditions for (1.3) to vanish identically, i.e.,

$$E_j \equiv 0, \quad \forall j \geq 0.$$

Firstly, there exists some negative integer p in (1.3) such that $u_0 \neq 0$. Secondly, notice that for $j = 1, 2, \cdots$, we have the linearized equation

$$E_j \equiv P(u_0, j) u_j + Q_j(\{u_l \mid l < j\}).$$

Thirdly, for each j, in order to have such an equation vanish identically, we would require either

(1) P vanishes for some indices j and Q_j vanish; or

(2) u_j is uniquely determined by P and Q_j.

For case 2, f will be uniquely determined by u_0. Here $P(u_0, j)$ is always a polynomial and $P(u_0, j) = 0$ is usually called the indicial equation of the given ODE. The indicial equation can be computed by the following formula.

Definition 1.1 (Indicial Equation) *The indicial equation of a system $E = 0$ is*

$$P(u_0, j) = \lim_{z \to z_0} (z - z_0)^{-j-q} \widehat{E}'(z, u_0(z - z_0)^p)(z - z_0)^{j+p} = 0,$$

where $\widehat{E}'(z, f)$ is defined as

$$\forall h, \quad \widehat{E}'(z, f)h = \lim_{t \to 0} \frac{\widehat{E}(z, f + th) - \widehat{E}(z, f)}{t}.$$

If the indicial equation has no positive integer solution, then there exists a Laurent series depending on the coefficients of the dominant terms, so one can represent the general solution by a locally singlevalued expression.

Using this method, we can get the following results.

Theorem 1.1 *The entire solutions of* (1.1) *must be in the following two forms*:

(1) $f(z) = \mathrm{e}^{naz}P(\mathrm{e}^{az})$, *where* P *is a polynomial with* $\deg P \leq 3$, $n \in \mathbb{Z}$, a *is a nonzero constant, and* $-\deg P \leq n \leq 0$ *if* $a_5a_9 \neq 0$ *or* $a_5a_9 = 0$, $a_6a_7 = 0$;

(2) f *is a polynomial with* $\deg f \leq 3$.

Theorem 1.2 *The meromorphic solutions with poles of* (1.1) *must be in the following three forms:*

(1) $f(z) = \frac{\mathrm{e}^{naz}P(\mathrm{e}^{az})}{Q(\mathrm{e}^{az})}$, *where* P *and* Q *are all polynomials,* $n \in \mathbb{Z}$, a *is a nonzero constant, and* Q *has only one zero* (*simple or double*), $\deg P \leq \deg Q$, *and* $0 \leq n \leq \deg Q$ *if* $a_3a_5 = 0$;

(2) $f(z) = -\frac{4a_5}{a_4}\wp(z-\eta_2) + C$, *where* $\wp$ *is the Weierstrass elliptic function,* C *and* η_2 *are both constants*;

(3) f *is a Möbus transformation or* $f(z) = -\frac{4a_5}{a_4}\frac{1}{(z-\zeta)^2} + \frac{a_6}{a_4}\frac{1}{z-\zeta} + C$, *where* ζ *and* C *are both constants.*

2 Proof of Theorem 1.1

Suppose that f is an entire solution of (1.1), then $f \in W$. When f is transcendental, if f' has no zeros, then $f'(z) = A_1a\mathrm{e}^{az}$, hence $f(z) = A_2 + A_1\mathrm{e}^{az}$, where A_1, A_2, a are constants. If f' has zeros, we may assume that f has at least one multiple zero. Otherwise, we use $f_1 = f - f(z_0)$ instead of f, where z_0 is a zero of f'. Since $f \in W$, f has infinitely many multiple zeros. Let $f = \frac{1}{g}$. Then g has infinitely many multiple poles and (1.1) becomes

$$a_1g'^3 - a_2g'^2g + a_3g'g^2 - a_4g^3 - a_5g'^2g^2 + a_6g'g^3 - a_7g^4 + a_8g'g^4 - a_9g^5 - a_{10}g^6 = 0. \quad (2.1)$$

In the following, our proof will be divided into serveral parts.

2.1 $a_1 \neq 0$

In this case, all the solutions of (1.1) are entire.

Case I g has infinitely many multiple poles whose multiplicities are at least three. Let z_3 be such a movable pole. By comparing the multiplicity of pole at z_3 of both sides of (2.1), we see that z_3 must be a triple pole, and $a_5 = a_6 = a_8 = a_9 = a_{10} = 0$, $a_7 \neq 0$. On this occasion, (2.1) becomes

$$a_1g'^3 - a_2g'^2g + a_3g'g^2 - a_4g^3 - a_7g^4 = 0. \quad (2.2)$$

In a neighbourhood of $z = z_3$, the Laurent series of g is in the form

$$g(z) = u_0(z-z_3)^{-3} + u_1(z-z_3)^{-2} + u_2(z-z_3)^{-1} + \cdots, \quad u_0 \neq 0. \quad (2.3)$$

Substituting (2.3) into (2.2) and balancing the leading terms, we must have the corresponding coefficients $E_j = 0$, $j = 0, 1, \cdots$. Particularly, $E_0 = -27a_1u_0^3 - a_7u_0^4 = 0$ means that u_0 takes only one value. We want to prove that there exists only one Laurent series at z_3. In order to do this, we shall compute the indicial equation of (2.2) by the dominant terms:

$$\widehat{E}(g) = a_1g'^3 - a_7g^4.$$

By the definition of indicial equation, we have

$$\widehat{E}' = 3a_1 g'^2 \partial_z - 4a_7 g^3.$$

Thus we get its indicial equation

$$P(u_0, j) = 27a_1 u_0^2 (j+1) = 0.$$

Hence, the Fuchs index $j = -1$. Because of the absence of any positive integer in the set of value of j, all other coefficients u_j are uniquely determined by the leading coefficient u_0. Therefore, we obtain that there is only one Laurent series at z_3 satisfying (2.2). We claim that g must be periodic. Now let ξ_j, $j = 1, 2, \cdots$ be the infinitely many poles of g. Then all $w_j(z) = g(z+\xi_j-z_3)$ are solutions of (2.2) with a pole at z_3. Since there exists only one meromorphic solution of (2.2) with poles at $z = z_3$, some of them must be equal. This implies that for some $j \neq i$, $g(z+\xi_j-z_3) \equiv g(z+\xi_i-z_3)$ and hence $g(z) \equiv g(z+\xi_j-\xi_i)$ in a neighborhood of $z = z_3$. Recalling that g is meromorphic, we can conclude that g is a periodic function with period $\xi_j - \xi_i$.

By a suitable rescaling, we may assume that g has a period of $2\pi\mathrm{i}$. Let $D = \{z : 0 \leq \operatorname{Im} z \leq 2\pi\}$. If g has more than one poles in D, then by the above argument, we can conclude that g is a periodic function in D and thus it must be an elliptic function, which contradicts the conclusion that $f = \frac{1}{g}$ is an entire function.

Now we know g has only one pole in D. Since $g \in W$, we have $g = R(\mathrm{e}^{az})$, where R has only one triple pole in $\mathbb{C} - \{0\}$. Thus, we can write g as

$$g(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{Z-X_1} + \frac{b_2}{(Z-X_1)^2} + \frac{b_3}{(Z-X_1)^3} + Q(Z), \tag{2.4}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_3 \neq 0$. Substituting (2.4) into (2.2) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{Z-X_1} + \frac{b_2}{(Z-X_1)^2} + \frac{b_3}{(Z-X_1)^3} + C$. Therefore, $f(z) = \frac{1}{g} = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Z_1)^3$, $n = 0, -1, -2, -3$. In this case and in the following discussion, the constants denoted by the same alphabet may be different in different cases.

Case II g has infinitely many double poles. Let z_2 be a movable double pole of g. By comparing the multiplicity of (2.1), we get that all the multiple poles of g must be double and $a_8 = a_{10} = 0$. Then, (2.1) becomes

$$a_1 g'^3 - a_2 g'^2 g + a_3 g' g^2 - a_4 g^3 - a_5 g'^2 g^2 + a_6 g' g^3 - a_7 g^4 - a_9 g^5 = 0. \tag{2.5}$$

Subcase i $a_5 a_9 \neq 0$. In a neighbourhood of $z = z_2$, the Laurent series of g is in the form

$$g(z) = u_0(z-z_2)^{-2} + u_1(z-z_2)^{-1} + u_2 + \cdots, \quad u_0 \neq 0. \tag{2.6}$$

Substituting (2.6) into (2.5) and balancing the leading terms, we obtain the condition $E_0 = -4a_5 u_0^6 - a_9 u_0^7 = 0$, which means that u_0 takes only one value. Now the dominant terms are

$$\widehat{E}(g) = -a_5 g'^2 g^2 - a_9 g^5.$$

Thus we get its indicial equation

$$P(u_0, j) = 4a_5u_0^3(j+1) = 0.$$

Hence, the Fuchs index $j = -1$. In a way similar to Case I, we obtain that there exists only one Laurent series at z_2. On this occasion, g may also have simple poles. By the same arguments in Case I, we can write g as

$$g(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{Z - Z_1} + \frac{b_2}{Z - Y_1} + \frac{b_3}{(Z - Y_1)^2} + Q(Z), \tag{2.7}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_3 \neq 0$. Substituting (2.7) into (2.5) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{Z-Z_1} + \frac{b_2}{Z-Y_1} + \frac{b_3}{(Z-Y_1)^2} + C$. Therefore, $f = A\mathrm{e}^{naz}(\mathrm{e}^{az} - X_1)(\mathrm{e}^{az} - Y_1)^2$, where $n = 0, -1, -2, -3$. If g only has double poles, then $g(z) = \frac{b_2}{Z-Y_1} + \frac{b_3}{(Z-Y_1)^2} + C$. Hence, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, where $n = 0, -1, -2$.

Subcase ii $a_5a_9 = 0$. Since g has double poles, we must have $a_5 = a_9 = 0$ and $a_6 \neq 0$ by (2.5). Then (2.5) becomes

$$a_1g'^3 - a_2g'^2g + a_3g'g^2 - a_4g^3 + a_6g'g^3 - a_7g^4 = 0. \tag{2.8}$$

Substituting (2.6) into (2.8) we get the condition $E_0 = -8a_1u_0^3 - 2a_6u_0^4 = 0$, which means that u_0 takes only one value. The dominant terms are

$$\widehat{E}(g) = a_1g'^3 + a_6g'g^3.$$

Thus we get its indicial equation

$$P(u_0, j) = 8a_1u_0^2(j+1) = 0.$$

Hence, the Fuchs index $j = -1$. In a way similar to Case I, we obtain that there is only one Laurent series at z_2. From (2.8), we see that g does not have simple poles, i.e., g only has double poles. Then we can write g as

$$g(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{(Z - Y_1)} + \frac{b_2}{(Z - Y_1)^2} + Q(Z), \tag{2.9}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_2 \neq 0$. Hence, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, where $n \in \mathbb{Z}$. Furthermore, if $a_7 = 0$, putting (2.9) into (2.8) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{(Z-Y_1)} + \frac{b_2}{(Z-Y_1)^2} + C$. Therefore, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, where $n = 0, -1, -2$.

2.2 $a_1 = 0$ but $a_2 \neq 0$

In this case, all the solutions of (1.1) are entire and (2.1) becomes

$$a_2g'^2 - a_3g'g + a_4g^2 + a_5g'^2g - a_6g'g^2 + a_7g^3 - a_8g'g^3 + a_9g^4 + a_{10}g^5 = 0. \tag{2.10}$$

First, from (2.10), we know that the multiple poles must be double and $a_8 = a_{10} = 0$. Then (2.10) becomes

$$a_2 g'^2 - a_3 g' g + a_4 g^2 + a_5 g'^2 g - a_6 g' g^2 + a_7 g^3 + a_9 g^4 = 0. \tag{2.11}$$

Case I $a_5 a_9 \neq 0$. By comparing the multiplicity of (2.11) at the pole of g, we get that g has no simple poles. In a way similar to Case I in Subsection 2.1, we have

$$g(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{(Z - Y_1)} + \frac{b_2}{(Z - Y_1)^2} + Q(Z), \tag{2.12}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_2 \neq 0$. Substituting (2.12) into (2.11) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{(Z-Y_1)} + \frac{b_2}{(Z-Y_1)^2} + C$. Therefore, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, where $n = 0, -1, -2$.

Case II $a_5 a_9 = 0$. Since g has double poles, we must have $a_5 = a_9 = 0$ and $a_6 = 0$, $a_7 \neq 0$. Then (2.11) becomes

$$a_2 g'^2 - a_3 g' g + a_4 g^2 + a_7 g^3 = 0. \tag{2.13}$$

Substituting (2.6) into (2.13), we can get $E_0 = 4a_2 u_0^2 + a_7 u_0^3 = 0$, which means that u_0 takes at most one value. The dominant terms are

$$\widehat{E}(g) = a_2 g'^2 + a_7 g^3.$$

Thus we get its indicial equation

$$P(u_0, j) = -4a_2 u_0 (j+1) = 0.$$

Hence, the Fuchs index $j = -1$. In a way similar to Case I in Subsection 2.1, we obtain that there exists only one Laurent series expansion at z_2. From (2.13), we see that g does not have simple poles. So we have (2.12). Substituting (2.12) into (2.13) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{Z-Y_1} + \frac{b_2}{(Z-Y_1)^2} + C$. Therefore, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, $n = 0, -1, -2$.

2.3 $a_1 = a_2 = 0$ but $a_3 \neq 0$

In this case, (2.1) becomes

$$a_3 g' - a_4 g - a_5 g'^2 + a_6 g' g - a_7 g^2 + a_8 g' g^2 - a_9 g^3 - a_{10} g^4 = 0. \tag{2.14}$$

Case I $a_5 = 0$. It is easy to see that the solutions of (1.1) are entire. Then (2.14) turns out to be

$$a_3 g' - a_4 g + a_6 g' g - a_7 g^2 + a_8 g' g^2 - a_9 g^3 - a_{10} g^4 = 0. \tag{2.15}$$

But from (2.15), we see that g can not have multiple poles.

Case II $a_5 \neq 0$. (2.14) may have meromorphic solutions f with poles, but we are not going to discuss it here, and it will be found in the proof of Theorem 1.2. Now we just investigate the entire solutions of (1.1). Since g has infinitely many poles, by comparing the multiplicity of (2.14) at z_2, which is a movable double pole of g, we have $a_8 = a_{10} = 0$ and $a_9 \neq 0$. Then (2.14) becomes

$$a_3 g' - a_4 g - a_5 g'^2 + a_6 g' g - a_7 g^2 - a_9 g^3 = 0. \tag{2.16}$$

In a way similar to Subcase i in Case II of Subsection 2.1, we have (2.12). Substituting (2.12) into (2.16) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{(Z-Y_1)} + \frac{b_2}{(Z-Y_1)^2} + C$. Therefore, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, where $n = 0, -1, -2$.

If $a_1 = a_2 = a_3 = a_4 = 0$, then (1.1) reduces to

$$a_5 f'^2 + a_6 f' f + a_7 f^2 + a_8 f' + a_9 f + a_{10} = 0. \tag{2.17}$$

2.4 $a_5 \neq 0$

Now (2.17) reduces to

$$a_5 g'^2 - a_6 g' g + a_7 g^2 - a_8 g' g^2 + a_9 g^3 + a_{10} g^4 = 0. \tag{2.18}$$

By comparing the multiplicity of (2.18) at a movable multiple pole of g, we obtain $a_8 = a_{10} = 0$ and these poles must be double. Then (2.18) becomes

$$a_5 g'^2 - a_6 g' g + a_7 g^2 + a_9 g^3 = 0. \tag{2.19}$$

Substituting (2.6) into (2.19), we obtain $E_0 = 4a_5 u_0^2 + a_9 u_0^3 = 0$, which means that u_0 takes only one value. The dominant terms are

$$\widehat{E}(g) = a_5 g'^2 g + a_9 g^3.$$

Thus we get its indicial equation

$$P(u_0, j) = -4a_5 u_0 (j+1) = 0.$$

Hence, the Fuchs index $j = -1$. In a way similar to Case I in Subsection 2.1, we obtain that there is only one Laurent series at z_2. From (2.19), we see that g does not have simple poles. Then we have

$$g(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{(Z-Y_1)} + \frac{b_2}{(Z-Y_1)^2} + Q(Z), \tag{2.20}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$. Substituting (2.20) into (2.19) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence $g(z) = \frac{b_1}{Z-Y_1} + \frac{b_2}{(Z-Y_1)^2} + C$. Therefore, $f(z) = A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2$, $n = 0, -1, -2$.

2.5 $a_5 = 0$ but $a_6 \neq 0$

In this case, the solutions of (1.1) also must be entire. Now (2.18) becomes

$$a_6 g' - a_7 g + a_8 g' g - a_9 g^2 - a_{10} g^3 = 0. \tag{2.21}$$

From (2.21), we obtain that g cannot have multiple poles.

2.6 Linear equation

When $a_5 = a_6 = a_7 = 0$, (2.16) reduces to

$$a_8 f' + a_9 f + a_{10} = 0. \tag{2.22}$$

It is easy to see that the solutions of (2.22) do not have multiple zeros.

To complete the proof, suppose that f is a polynomial solution of (1.1). Substituting it into (1.1), we can obtain that the degree of f is at most three.

3 Proof of Theorem 1.2

Let f be a meromorphic solution of (1.1) with poles. Since $f \in W$, we see that f must have infinitely many poles. From the proof of Theorem 1.1, we obtain $a_1 = a_2 = 0$.

3.1 $a_1 = a_2 = 0$ but $a_3 a_5 \neq 0$

We consider (1.1), i.e.,

$$a_3 f' f^2 + a_4 f^3 + a_5 f'^2 + a_6 f' f + a_7 f^2 + a_8 f' + a_9 f + a_{10} = 0. \tag{3.1}$$

Case I f is transcendental. From (3.1), we see that f can only have simple poles. Let η_1 be a movable pole of f. Then the Laurent series of f is in the form

$$f(z) = u_0 (z - \eta_1)^{-1} + u_1 + u_2 (z - \eta_1) + \cdots, \quad u_0 \neq 0. \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain $E_0 = a_3 u_0^3 + a_5 u_0^2 = 0$, which means that u_0 takes only one value. The dominant terms are

$$\widehat{E}(f) = a_3 f' f^2 + a_5 f'^2.$$

Thus we get its indicial equation

$$P(u_0, j) = -a_3 u_0^2 (j+1) = 0.$$

Hence, the Fuchs index $j = -1$. Similarly, we can obtain that there is only one Laurent series at η_1. Thus f has the form

$$f(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{(Z - S_1)} + Q(Z), \tag{3.3}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$.

In the following, we shall discuss $g = \frac{1}{f}$. By the uniqueness of f, we know that g also has only one Laurent series at its movable pole. It is easy to see that g can only have either simple poles or double poles from (2.14). If g has double poles, by Case I in Subsection 2.2 in the proof of Theorem 1.1, we have (2.12), i.e.,

$$g(z) = R(Z) = \frac{b_1}{(Z - Y_1)} + \frac{b_2}{(Z - Y_1)^2} + C, \quad b_2 \neq 0. \tag{3.4}$$

Combining (3.3)–(3.4) and $f = \frac{1}{g}$, we get $f = \frac{A\mathrm{e}^{naz}(\mathrm{e}^{az} - Y_1)^2}{\mathrm{e}^{az} - S_1}$, where $n = 0, -1$. If g has simple poles, then we have

$$g(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_3}{(Z - Y_1)} + Q(Z), \tag{3.5}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_3 \neq 0$. Combining (3.3), (3.5) and $f = \frac{1}{g}$, we get $f(z) = \frac{A\mathrm{e}^{naz}(\mathrm{e}^{az} - Z_1)}{\mathrm{e}^{az} - S_1}$, where $n \in \mathbb{Z}$. If g has no poles, then $f = \frac{A\mathrm{e}^{naz}}{\mathrm{e}^{az} - S_1}$, where $n \in \mathbb{Z}$.

Case II f is a rational function. From Case I, we know that all poles of f are simple with the residue $-\frac{a_5}{a_3}$. Hence $f = -\frac{a_5}{a_3}\frac{P_1'}{P_1} + P_2$, where P_1 is a polynomial which only has simple zeros with degree d_1 and P_2 is a polynomial. Substituting f into (1.1), we get $P_2 = C$, a constant. That is to say,

$$f = \frac{-a_5 P_1' + a_3 C P_1}{a_3 P_1}. \tag{3.6}$$

Substituting (3.6) into (1.1), we obtain

$$\begin{aligned} &a_3 a_5 (P_1'^2 - P_1'' P_1)(-a_5 P_1' + a_3 C P_1)^2 + a_4 P_1 (-a_5 P_1' + a_3 C P_1)^3 + a_3^2 a_5^2 (P_1'^2 - P_1'' P_1)^2 \\ &+ a_3 a_5 a_6 P_1 (P_1'^2 - P_1'' P_1)(-a_5 P_1' + a_3 C P_1) + a_3^2 a_7 P_1^2 (-a_5 P_1' + a_3 C P_1)^2 \\ &+ a_3^2 a_5 a_8 P_1^2 (P_1'^2 - P_1'' P_1) + a_3^2 a_9 P_1^3 (-a_5 P_1' + a_3 C P_1) + a_3^3 a_{10} P_1^4 = 0. \end{aligned} \tag{3.7}$$

By (3.7), the zeros of P_1 are the zeros of $(a_3 + a_5)P_1'^4$. If $\deg P > 1$, because P_1 only has simple zeros, then $a_3 = -a_5$. Thus, (3.7) becomes

$$\begin{aligned} &a_3 (P_1'' P_1 - P_1'^2)(2CP_1' + C^2 P_1 + P_1'') + a_4 (P_1' + CP_1)^3 \\ &- a_6 (P_1'^2 - P_1'' P_1)(P_1' + CP_1) + a_7 P_1 (P_1' + CP_1)^2 \\ &+ a_8 P_1 (P_1'^2 - P_1'' P_1) + a_9 P_1^2 (P_1' + CP_1)^2 + a_{10} P_1^3 = 0. \end{aligned} \tag{3.8}$$

Then the zeros of P_1 are zeros of $\left((-2a_3 C + a_4 - a_6)P_1' - a_3 P_1''\right) P_1'^2$. This is impossible. Therefore, f is a Möbius transformation.

3.2 $a_1 = a_2 = a_3 = 0$ but $a_4 \neq 0$

Consider (1.1), i.e.,

$$a_4 f^3 + a_5 f'^2 + a_6 f' f + a_7 f^2 + a_8 f' + a_9 f + a_{10} = 0. \tag{3.9}$$

Case I $a_5 \neq 0$. Let f be transcendental. Since $f \in W$, we get that f must have infinite poles. On this occasion, f only have double poles. Let η_2 be a movable double pole of f. Then in a neighbourhood of $z = \eta_2$, the Laurent series of f is in the form

$$f(z) = u_0(z - \eta_2)^{-2} + u_1(z - \eta_2)^{-1} + u_2 + \cdots, \quad u_0 \neq 0. \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain $E_0 = a_4 u_0^3 + 4a_5 u_0^2 = 0$, which means that u_0 takes at most one value. The dominant terms are

$$\widehat{E}(g) = a_4 f^3 + a_5 f'^2.$$

Thus we get its indicial equation

$$P(u_0, j) = -4a_5 u_0 (j + 1) = 0.$$

Hence, the Fuchs index $j = -1$. In a way similar to Case I in Subsection 2.1 in the proof of Theorem 1.1, we get that there exists only one Laurent series at η_2. From the discussion in the proof of Theorem 1.1, we know that f is an elliptic function or $f = R(\mathrm{e}^{az})$, where R is a rational function.

Subcase i f is an elliptic function. Since η_2 is the only pole in a fundamental parallelogram and it is a double pole, in a neighbourhood of origin, the Laurent series of the Weierstrass elliptic function $\wp(z)$ is in the form of

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) s_{2n+2}(L) z^{2n},$$

where $s_m(L) = \sum\limits_{\omega \neq 0} \frac{1}{\omega^n}$, $\omega \in L$, L is a set consisting of all the lattice points of all period parallelogram. Hence, $f(z) + \frac{4a_5}{a_4}\wp(z - \eta_2)$ must be an analytic elliptic function, then it is a constant. Therefore, we have $f(z) = -\frac{4a_5}{a_4}\wp(z - \eta_2) + C$, where C is a constant.

Subcase ii $f = R(\mathrm{e}^{az})$. In the following, we shall determine R. By the above explanation, we know that f can be written as

$$f(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{b_1}{(Z - T_1)} + \frac{b_2}{(Z - T_1)^2} + Q(Z), \tag{3.11}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$. Substituting (3.11) into (3.9) and letting Z tend to infinity, we can conclude that Q equals some constant C. Then letting Z tend to 0, we can deduce that $r_n = \cdots = r_1 = 0$. Hence

$$f(z) = \frac{b_1}{(Z - T_1)} + \frac{b_2}{(Z - T_1)^2} + C. \tag{3.12}$$

Next, we shall discuss $g = \frac{1}{f}$. Now (2.1) becomes

$$a_4 g + a_5 g'^2 - a_6 g' g + a_7 g^2 - a_8 g' g^2 + a_9 g^3 + a_{10} g^4 = 0. \tag{3.13}$$

By the uniqueness of f, we know that g has only one Laurent series at its poles. Considering g only has either simple poles or double poles from (3.13), we get

$$g(z)=\frac{r_{1,n}}{Z^n}+\cdots+\frac{r_{1,1}}{Z}+\frac{b_1}{(Z-Z_1)}+Q_1(Z), \tag{3.14}$$

or

$$g(z)=\frac{r_{2,n}}{Z^n}+\cdots+\frac{r_{2,1}}{Z}+\frac{b_2}{(Z-Y_1)}+\frac{b_3}{(Z-Y_1)^2}+Q_2(Z), \tag{3.15}$$

where Q_1 and Q_2 are polynomials in $Z=\mathrm{e}^{az}$. If (3.15) holds, then (3.13) reduces to

$$a_4g+a_5g'^2-a_6g'g+a_7g^2+a_9g^3=0. \tag{3.16}$$

Substituting (3.15) into (3.16) and letting Z tend to infinity, we can conclude that Q_2 equals some constant C. Then letting Z tend to 0, we can deduce that $r_{2,n}=\cdots=r_{2,1}=0$. Hence

$$g(z)=\frac{b_2}{(Z-Y_1)}+\frac{b_3}{(Z-Y_1)^2}+C. \tag{3.17}$$

Combining (3.12), (3.14), (3.17) and $f=\frac{1}{g}$, we get $f(z)=\frac{A(\mathrm{e}^{az}-Y_1)^2}{(\mathrm{e}^{az}-T_1)^2}$ or $f(z)=\frac{A\mathrm{e}^{naz}(\mathrm{e}^{az}-Z_1)}{(\mathrm{e}^{az}-T_1)^2}$, where $n=0,1$.

Now let f be a rational solution of (3.9). From the discussion above, we know that the poles of f can only be double. Firstly, by substituting (3.11) into (3.9), we get $u_0=-\frac{4a_5}{a_4}$ and $u_1=\frac{a_6}{a_4}$. Hence

$$f=\frac{4a_5}{a_4}\Big(\frac{P_1'}{P_1}\Big)'+\frac{a_6}{a_4}\frac{P_1'}{P_1}+P_2,$$

where P_1 is a polynomial which only has simple zeros with degree d_1 and P_2 is a polynomial. Substituting f into (3.9), by comparing the degree of the numerator, we immediately have $P_2=C$. That is,

$$f=\frac{4a_5P_1P_1''-4a_5P_1'^2+a_6P_1P_1'+Ca_4P_1^2}{a_4P_1^2}. \tag{3.18}$$

Substituting (3.18) into (3.9), we obtain that the zeros of P_1 are the zeros of $4a_5P_1''+a_6P_1'$. This means that P_1 is linear. Therefore,

$$f(z)=-\frac{4a_5}{a_4}\frac{1}{(z-\zeta)^2}+\frac{a_6}{a_4}\frac{1}{z-\zeta}+C,$$

where ζ and C are both constants.

Case II $a_5=0$. From (3.9), we get $a_6\neq 0$. Then (3.9) becomes

$$a_4f^3+a_6f'f+a_7f^2+a_8f'+a_9f+a_{10}=0, \tag{3.19}$$

and f only has simple poles. If f is transcendental, substituting (3.2) into (3.19), we obtain $E_0=a_4u_0^3-a_6u_0^2=0$, which means that u_0 takes only one value. The dominant terms are

$$\widehat{E}(g)=a_4f^3+a_6f'f.$$

Then its indicial equation is

$$P(u_0, j) = a_6 u_0 (j + 1) = 0.$$

Hence, the Fuchs index $j = -1$. Similarly to Subsection 2.1 in the proof of Theorem 1.1, we obtain that there is only one Laurent series at the pole η_1. Thus f has the form

$$f(z) = R(Z) = \frac{r_{1,n}}{Z^n} + \cdots + \frac{r_{1,1}}{Z} + \frac{b_1}{(Z - \eta_1)} + Q_1(Z), \tag{3.20}$$

where Q_1 is a polynomial in $Z = \mathrm{e}^{az}$. Substituting (3.20) into (3.19) and letting Z tend to infinity, we can conclude that Q_1 equals some constant C. Then letting Z tend to 0, we can deduce that $r_{1,n} = \cdots = r_{1,1} = 0$. Hence

$$f(z) = \frac{b_1}{Z - \eta_1} + C. \tag{3.21}$$

Let $g = \frac{1}{f}$. Then (3.13) becomes

$$a_4 g - a_6 g' g + a_7 g^2 - a_8 g' g^2 + a_9 g^3 + a_{10} g^4 = 0. \tag{3.22}$$

By the uniqueness of f, we know that g also has only one Laurent series at its poles. Thus

$$g(z) = \frac{r_{2,n}}{Z^n} + \cdots + \frac{r_{2,1}}{Z} + \frac{b_2}{(Z - \eta_1)} + Q_2(Z), \tag{3.23}$$

where Q_2 is a polynomial in $Z = \mathrm{e}^{az}$. Substituting (3.23) into (3.22) and letting Z tend to infinity, we can conclude that Q_2 equals some constant C. Then letting Z tend to 0, we can deduce that $r_{2,n} = \cdots = r_{2,1} = 0$. Hence

$$f(z) = \frac{b_2}{Z - \eta_1} + C. \tag{3.24}$$

Combining (3.21), (3.24) and $f = \frac{1}{g}$, we conclude that there are no meromorphic solutions in this case.

If $f = \frac{P1}{P_2}$ is a rational solution of (3.19), where P_1 and P_2 are two polynomials with degree d_1 and d_2 respectively. Then by substituting it into (3.19), we get

$$a_4 P_1^3 + a_6 P_1 P_1' P_2 - a_6 P_1^2 P_2' + a_7 P_1^2 P_2 + a_8 P_1' P_2^2 - a_8 P_1 P_2' P_2 + a_9 P_1 P_2^2 + a_{10} P_2^3 = 0. \tag{3.25}$$

From (3.25), the zeros of both P_1 and P_2 are simple. Furthermore, the zeros of P_1 are the zeros of $a_8 P_1' + a_{10} P_2$ and the zeros of P_2 are the zeros of $a_6 P_2' - a_4 P_1$. If $a_7 \neq 0$, $a_9 \neq 0$ or $a_{10} \neq 0$, then $d_1 = d_2$. Hence, $a_8 P_1' + a_{10} P_2 = c_1 P_1$ and $a_6 P_2' - a_4 P_1 = c_2 P_2$. Putting one equation above into another, we obtain P_1 and P_2 cannot be nonconstant polynomials. So $a_7 = a_9 = a_{10} = 0$ and $d_2 = d_1 + 1$. But the fact that zeros of P_2 are the zeros of $a_8 P_2' - a_7 P_1$ leads to $d_2 = 1$ and $d_1 = 0$. Therefore, $f(z) = \frac{c}{z-b}$.

3.3 $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$ but $a_7 \neq 0$

Consider (2.17), i.e.,

$$a_7 f^2 + a_8 f' + a_9 f + a_{10} = 0. \tag{3.26}$$

Case I f is transcendental. Since $f \in W$, we see that f must have infinite poles. Moreover, $a_8 \neq 0$ and f only has simple poles. Substituting (3.2) into (3.26), we obtain $E_0 = a_7u_0^2 - a_8u_0 = 0$, which means that u_0 takes only one value. The dominant terms are

$$\widehat{E}(g) = a_7f^2 + a_8f'.$$

Thus we get its indicial equation

$$P(j) = -a_8u_0(j+1) = 0.$$

Hence, the Fuchs index $j = -1$. In a way similar to Case I in Subsection 3.1 in the proof of Theorem 1.1, we obtain that there is only one Laurent series at one of its movable poles. Thus, we have

$$f(z) = R(Z) = \frac{r_{1,n}}{Z^n} + \cdots + \frac{r_{1,1}}{Z} + \frac{b_1}{(Z - S_1)} + Q_1(Z), \tag{3.27}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_1 \neq 0$. Substituting (3.27) into (3.26) and letting Z tend to infinity, we can conclude that Q_1 equals some constant C. Then letting Z tend to 0, we can deduce that $r_{1,n} = \cdots = r_{1,1} = 0$. Hence

$$f(z) = \frac{b_1}{Z - S_1} + C. \tag{3.28}$$

If f has no zeros, then $f(z) = \frac{A\mathrm{e}^{naz}}{\mathrm{e}^{az} - S_1}$, $n = 0, 1$. If f has zeros, by the uniqueness of f, we know that $g = \frac{1}{f}$ has only one Laurent series at its poles and satisfies

$$a_7 - a_8g' + a_9g + a_{10}g^2 = 0. \tag{3.29}$$

Thus g has the form

$$g(z) = \frac{r_{2,n}}{Z^n} + \cdots + \frac{r_{2,1}}{Z} + \frac{b_2}{(Z - S_1)} + Q_2(Z), \tag{3.30}$$

where Q is a polynomial in $Z = \mathrm{e}^{az}$ and $b_2 \neq 0$. Substituting (3.30) into (3.29) and letting Z tend to infinity, we can conclude that Q_2 equals some constant C_1. Then letting Z tend to 0, we can deduce that $r_{2,n} = \cdots = r_{2,1} = 0$. Hence

$$g(z) = \frac{b_2}{Z - S_1} + C_1. \tag{3.31}$$

Combining (3.28), (3.31) and $g = \frac{1}{f}$, we have $f(z) = \frac{A(\mathrm{e}^{az} - Z_1)}{\mathrm{e}^{az} - S_1}$.

Case II Let $f = \frac{P1}{P_2}$ be a rational solution of (3.26), where P_1 and P_2 are two polynomials with degree d_1 and d_2 respectively. Then by substituting it into (3.26), we get

$$a_7P_1^2 + a_8P_1'P_2 - a_8P_1P_2' + a_9P_1P_2 + a_{10}P_2^2 = 0. \tag{3.32}$$

From (3.32), the zeros of both P_1 and P_2 are simple. Furthermore, the zeros of P_1 are the zeros of $a_8P_1' + a_{10}P_2$ and the zeros of P_2 are the zeros of $a_8P_2' - a_7P_1$. If $a_9 \neq 0$ or $a_{10} \neq 0$, then $d_1 = d_2$. Hence, $a_8P_1' + a_{10}P_2 = c_1P_1$ and $a_8P_2' - a_7P_1 = c_2P_2$. Putting one equation above into another, we obtain that P_1 and P_2 cannot be nonconstant polynomials. So $a_9 = a_{10} = 0$ and $d_2 = d_1 + 1$. But the zeros of P_2 are the zeros of $a_8P_2' - a_7P_1$, we get $d_2 = 1$ and $d_1 = 0$. Therefore, $f(z) = \frac{c}{z-b}$, where b and c are constants.

References

[1] Akhiezer, N. I., Elements of the Theory of Elliptic Functions (Translation), Amer. Math. Soc., Providence, RI, 1990.

[2] Bank, S. and Kaufman, R., On Briot-Bouquet differential equations and a question of Einar Hille, *Math. Z,* **177**, 1981, 549–559.

[3] Briot, Ch. and Bouquet, J., Intégration deséquations différentielles au moyen de fonctions elliptiques, *J. École Polytechnique,* **21**, 1856, 199–254.

[4] Briot, Ch. and Bouquet, J., Théorie des Fonctions Doublement Périodiques et, en Particulier, des Fonctions Elliptiques, Mallet-Bachelier, Paris, 1859.

[5] Conte, R. and Musette, M., The Painlevé Handbook, Springer-Verlag, Berlin, 2008.

[6] Eremenko, A., Meromorphic solutions of equations of Briot-Bouquet type, *Teor. Funktsii, Funk. Anal. i Prilozh,* **38**, 1982, 48–56.

[7] Eremenko, A., Liao, L. W. and Ng, T. W., Meromorphic solutions of higher order Briot-Bouquet differential equations, *Math. Proc. Cambridge,* **146**(1), 2009, 197–206.

[8] Hille, E., Remarks on Briot-Bouquet differential equations, I, *Comment. Math.,* **1**, 1978, 119–132.

[9] Hille, E., Higher order Briot-Bouquet differential equations, *Ark. Mat.,* **16**, 1978, 271–286.

[10] Hille, E., Second-order Briot-Bouquet differential equations, *Acta Sci. Math.* (*Szeged*)*,* **40**, 1978, 63–72.

[11] Hille, E., Some remarks on Briot-Bouquet differential equations, II, *J. Math. Anal. Appl.,* **65**, 1978, 572–585.

[12] Laine. I., Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.

[13] Yang, C. C. and Yi, H. X., Uniqueness Theory of Meromorphic Functions, Science Press, Beijing, 2006.

[14] Zhang, X. Q., The study of enterprise loan investment decision on the stochastic interest rate —a real option model based on IRR, *The Journal of World Economy,* **5**, 2007, 65–73 (in Chinese).