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Gradient Estimates for p-Laplacian Lichnerowicz Equation on Noncompact Metric Measure Space*

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Abstract In this paper, the authors obtain the gradient estimates for positive solutions to the weighted p-Laplacian Lichnerowicz equation

$$\triangle_{p,f} u + cu^{\sigma} = 0$$

on noncompact smooth metric measure space, where c is a nonnegative constant, and p, σ $(1 < p \leq 2,\ \sigma \leq p - 1)$ are real constants. Moreover, by the gradient estimate, they can get the corresponding Liouville theorem and Harnack inequality.

Keywords p-Laplacian, Positive solutions, Liouville theorem
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1 Introduction

Recently, there has been an active interest in the study of gradient estimates for the partial differential equation on manifolds. Kotschwar and Ni [7] studied p-heat equation from a variational point and obtained some interesting results about gradient estimates and entropy formula. Later, combing the usual maximum principle approach with some geometric techniques including the use of a nonlinear Bochner formula and curvature-dimension inequality, Wang, Yang and Chen [15] got the sharp estimate for weighted p-heat equation. For doubly nonlinear diffusion equation (DNDE for short) on complete Riemannian manifolds with nonnegative Ricci curvature, Wang and Chen [13] proved the sharp global Li-Yau type gradient estimates and Chen and Xiong [3] obtained some other types of gradient estimates for this equation. Moreover, for the weighted p-Laplace case, Wang and Li [14] derived the lower bounds of its first nonzero eigenvalue on compact smooth metric measure spaces.

The aim of this paper is to give some existence results for positive solutions to the Lichnerowicz type equations on smooth metric measure space. We know a smooth metric measure space is a triple $(M, g, \mathrm{d}\mu)$, where (M, g) is a complete n-dimensional Riemannian manifold and $\mathrm{d}\mu := \mathrm{e}^{-f}\mathrm{d}v$ with f a fixed smooth real-valued function on M. Denote by $\nabla, \triangle$ and Hess the gradient, Laplace and Hessian operators, and by $\mathrm{d}v$ the Riemannian volume measure.

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The smooth metric measure space carries a natural analog of the Ricci curvature, the so-called m-Bakry-Émery Ricci curvature, which is defined as

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \mathrm{Hess}\, f - \frac{\nabla f \otimes \nabla f}{m-n}, \quad n < m \leq \infty.$$

In particular, when $m = \infty, \mathrm{Ric}_f^\infty := \mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess} f$ is the classical Bakry-Émery Ricci curvature, which is introduced by Bakry and Émery [1] in the study of diffusion processes, and then it has been extensively investigated in the theory of Ricci flow. The case where $m = n$ is only defined when f is a constant function. There is also a analog of the p-Laplacian, that is, the weighted p-Laplacian, which is defined by

$$\triangle_{p,f} := \mathrm{e}^f \mathrm{div}(\mathrm{e}^{-f} |\nabla u|^{p-2} \nabla u).$$

It is also understood in distribution sense.

Here we recall the Lichnerowicz equation on manifolds. Given a smooth symmetric 2-tensor σ, a smooth vector field W, and a triple data (π, τ, φ) of smooth functions on M. Set $c_n = \frac{n-2}{4(n-1)}$, $p = \frac{2n}{n-2}$, and let

$$R_{\gamma,\varphi} = c_n(R(\gamma) - |\nabla \varphi|_\gamma^2), \quad A_{\gamma,W,\pi} = c_n(|\sigma + DW|_\gamma^2) + \pi^2,$$

$B(\tau, \varphi) = c_n\big(\frac{n-1}{n}\tau^2 - V(\varphi)\big)$, where $V : R \to R$ is a given smooth function and $R(\gamma)$ is the scalar curvature function of γ. Then the Lichnerowicz equation for the Einstein-scalar conformal data $(\pi, \sigma, \pi, \tau, \varphi)$ with the given vector field W is

$$\triangle_\gamma u - R_{\gamma,\varphi} u + A_{\gamma,W,\pi} u^{-p-1} - B(\tau, \varphi) u^{p-1} = 0, \quad u > 0.$$

In this paper, we first consider the local gradient estimate for the positive solutions to the p-Laplacian Lichnerowicz equation

$$\triangle_{p,f} u + c u^\sigma = 0 \tag{1.1}$$

on noncompact smooth metric measure space. Here c is a nonnegative constant and $1 < p \leq 2, \sigma \leq p - 1$ are real constants. This equation can be seen as a simple version of Lichnerowicz equation which arises from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity (see [4, 6, 12] and the references therein). When $p = 2$, Ma [8–11] studied the existence and stability of positive solutions to Lichnerowicz equation and the first author of the paper proved some gradient estimates for this equation which can be referred to [16–18]. However, if $p > 1$, the p-Laplacian Lichnerowicz equation is referred to as the generalized scalar curvature type equation, it is an extension of the equation of prescribed scalar curvature. The problem of positive solutions to p-Laplacian Lichnerowicz equation was considered in [5] in case of compact manifold, and then Benalili and Maliki [2] extended the corresponding results to the complete Riemannian manifolds.

In this paper, we will follow the methods in [7] to establish the local and global gradient estimates for positive solutions to equation (1.1) on smooth metric measure space.

Theorem 1.1 *Let $(M, g, \mathrm{d}\mu)$ be a n-dimensional $(n \geq 2)$ noncompact smooth metric measure space and $\mathrm{d}\mu = \mathrm{e}^{-f}\mathrm{d}v$ with $|\nabla f| \leq C_1$. Suppose that u is a positive solution to* (1.1) *on the* *ball $B(x_0,R)$, and that on the ball sectional curvature $K_M \geq -K_1, \mathrm{Ric}_f^m(M) \geq -K_2, K_1, K_2$ are nonnegative constants. Then for any positive constant $\varepsilon \leq \min\left\{\frac{4}{(p-m)^2}, \frac{4}{((p-2)m+p)^2}\right\}$, we have*

$$(p-1)\frac{|\nabla u|^2}{u^2} \leq \frac{4(m-1)}{4-\varepsilon(p-m)^2}K_2 + \frac{\widetilde{C}}{R^2},$$

where

$$\begin{aligned}\widetilde{C} = \frac{2(m-1)}{4-\varepsilon(p-m)^2}\Big\{&10(2-p-\alpha) + \frac{20}{\varepsilon(m-1)}\\ &+ 10p(2-p) + 20(n+p-2)(1+K_1R) + 10 + \sqrt{10}C_1R\Big\}\end{aligned}$$

and $\alpha = \min\left\{2, 1+\frac{(p-1)^2}{m-1}\right\}$.

From the above Theorem 1.1, let $R\to\infty$, and $\varepsilon \to 0$, we can get the global gradient estimate for the positive solutions to equation (1.1).

Theorem 1.2 *Let $(M, g, \mathrm{d}\mu)$ be a noncompact smooth metric measure space and $\mathrm{d}\mu = \mathrm{e}^{-f}\mathrm{d}v$ with $|\nabla f| \leq C_1$. Suppose that u is a positive solution to (1.1) and that sectional curvature $K_M \geq -K_1, \mathrm{Ric}_f^m(M) \geq -K_2, K_1, K_2$ are nonnegative constants. Then we have in the region $|\nabla u| > 0$,*

$$(p-1)\frac{|\nabla u|^2}{u^2} \leq (m-1)K_2.$$

As an application of the Theorem 1.2, we can obtain the corresponding Liouville theorem and Harnack inequality for the positive solutions to equation (1.1).

Theorem 1.3 *Let $(M, g, \mathrm{d}\mu)$ be a noncompact smooth metric measure space and $\mathrm{d}\mu = \mathrm{e}^{-f}\mathrm{d}v$ with $|\nabla f| \leq C_1$. Suppose that u is a positive solution to (1.1) and that sectional curvature $K_M \geq -K_1, \mathrm{Ric}_f^m(M) \geq 0$. Then $u \equiv$ constant.*

Theorem 1.4 *Under the same conditions as in the Theorem 1.2. For any $x, y \in M$, and let $\gamma(s)$ be a minimal geodesic $\gamma(s) : [0,1] \to M, \gamma(0) = x, \gamma(1) = y$. The following inequality holds:*

$$u(x) \leq u(y)\mathrm{e}^{\rho(x,y)\left[\frac{(m-1)K_2}{(p-1)}\right]^{\frac{1}{2}}},$$

where $\rho = \rho(x,y)$ denotes the geodesic distance x and y.

2 Proof of Theorem 1.1

Assume that u is a positive solution to (1.1), let $v = (p-1)\log u$, then we can obtain

$$\begin{aligned}\triangle_{p,f}u + cu^\sigma &= \mathrm{e}^f \mathrm{div}(\mathrm{e}^{-f}|\nabla \mathrm{e}^{\frac{v}{p-1}}|^{p-2}\nabla \mathrm{e}^{\frac{v}{p-1}}) + c\mathrm{e}^{\frac{v\sigma}{p-1}}\\ &= (p-1)^{1-p}\mathrm{e}^v(|\nabla v|^p + \triangle_{p,f}v) + c\mathrm{e}^{\frac{v\sigma}{p-1}}\\ &= 0.\end{aligned}$$

That is to say,

$$\triangle_{p,f}v = -c(p-1)^{p-1}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v} - |\nabla v|^p. \tag{2.1}$$

The linearized operator of the weighted p-Laplacian at points $\{\nabla v > 0\}$ is given by

$$\mathcal{L}_f(\psi) = \mathrm{e}^f \mathrm{div}(\mathrm{e}^{-f}|\nabla v|^{p-2}A(\nabla\psi)),$$

where

$$A_{ij} = g_{ij} + (p-2)\frac{\nabla v \otimes \nabla v}{|\nabla v|^2}.$$

We can get the following lemma through direct computation.

Lemma 2.1 *Let* $w = |\nabla v|^2$,

$$\begin{aligned}\mathcal{L}_f(w) &= 2w^{\frac{p-1}{2}}(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)) - pw^{\frac{p}{2}-1}\langle \nabla v, \nabla w\rangle \\ &\quad - 2c(p-1)^{p-1}\Big(\frac{\sigma}{p-1} - 1\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}w + \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}|\nabla w|^2.\end{aligned}$$

Proof

$$\begin{aligned}\mathcal{L}_f(w) &= \mathrm{e}^f \mathrm{div}[\mathrm{e}^{-f}(w^{\frac{p}{2}-1}\nabla w + (p-2)w^{\frac{p}{2}-2}\langle \nabla v, \nabla w\rangle \nabla v)] \\ &= -w^{\frac{p}{2}-1}\langle \nabla f, \nabla w\rangle - (p-2)w^{\frac{p}{2}-2}\langle \nabla v, \nabla w\rangle\langle \nabla v, \nabla f\rangle + \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}|\nabla w|^2 \\ &\quad + w^{\frac{p}{2}-1}\triangle w + \Big(\frac{p}{2}-2\Big)(p-2)w^{\frac{p}{2}-3}|\langle \nabla v, \nabla w\rangle|^2 + (p-2)w^{\frac{p}{2}-2}(\langle \nabla\nabla v, \nabla w\rangle \\ &\quad + \langle \nabla v, \nabla\nabla w\rangle)\nabla v + (p-2)w^{\frac{p}{2}-2}\langle \nabla v, \nabla w\rangle \triangle v \\ &= w^{\frac{p}{2}-1}\triangle_f w + (p-2)w^{\frac{p}{2}-2}\langle \nabla v, \nabla w\rangle\triangle_f v + \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}|\nabla w|^2 + w^{\frac{p}{2}-1}\triangle w \\ &\quad + \Big(\frac{p}{2}-2\Big)(p-2)w^{\frac{p}{2}-3}|\langle \nabla v, \nabla w\rangle|^2 + (p-2)w^{\frac{p}{2}-2}(\langle \nabla\nabla v, \nabla w\rangle + \langle \nabla v, \nabla\nabla w\rangle)\nabla v \\ &= w^{\frac{p}{2}-1}(2|\nabla\nabla v|^2 + 2\langle \nabla\triangle_f v, \nabla v\rangle + 2\mathrm{Ric}_f(\nabla v, \nabla v)) \\ &\quad + (p-2)w^{\frac{p}{2}-2}\langle \nabla v, \nabla w\rangle\triangle_f v + \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}|\nabla w|^2 \\ &\quad + \Big(\frac{p}{2}-2\Big)(p-2)w^{\frac{p}{2}-3}|\langle \nabla v, \nabla w\rangle|^2 \\ &\quad + (p-2)w^{\frac{p}{2}-2}(\langle \nabla\nabla v, \nabla w\rangle + \langle \nabla v, \nabla\nabla w\rangle)\nabla v,\end{aligned}$$

where we used the weighted Bochner formula

$$\triangle_f w = 2|\nabla\nabla v|^2 + 2\langle \nabla\triangle_f v, \nabla v\rangle + 2\mathrm{Ric}_f(\nabla v, \nabla v).$$

Note that in terms of w,

$$\begin{aligned}\triangle_{p,f} v &= \mathrm{e}^f \mathrm{div}(\mathrm{e}^{-f}w^{\frac{p-2}{2}}\nabla v) \\ &= w^{\frac{p-2}{2}}\triangle_f v + \frac{p-2}{2}\langle \nabla w, \nabla v\rangle.\end{aligned}$$

Therefore, (2.1) has the equivalent form

$$w^{\frac{p-2}{2}}\triangle_f v + \frac{p-2}{2}w^{\frac{p-4}{2}}\langle \nabla w, \nabla v\rangle = -c(p-1)^{p-1}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v} - w^{\frac{p}{2}}. \tag{2.2}$$

Taking the gradient of both sides of (2.2) and computing its product with ∇v, we have that

$$\begin{aligned}&\Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}\langle \nabla v, \nabla w\rangle\triangle_f v + \Big(\frac{p}{2}-1\Big)\Big(\frac{p}{2}-2\Big)w^{\frac{p}{2}-3}|\langle \nabla v, \nabla w\rangle|^2 \\ &+ \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}(\langle \nabla\nabla v, \nabla w\rangle + \langle \nabla v, \nabla\nabla w\rangle)\nabla v + \frac{p}{2}w^{(\frac{p}{2}-1)}\langle \nabla v, \nabla w\rangle \\ &+ c(p-1)^{p-1}\Big(\frac{\sigma}{p-1} - 1\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}w \\ = {}& 0.\end{aligned}$$

Combing the above identities, we prove the Lemma 2.1.

Now let $\eta(x) = \theta = \left(\frac{r(x)}{R}\right)$, where $\theta(t)$ is a cut-off function such that $\theta(t) \equiv 1$ for $0 \le t \le \frac{1}{2}$ and $\theta(t) \equiv 0$ for $t \ge 1$. Furthermore, take the derivatives of θ to satisfy $\frac{(\theta')^2}{\theta} \le 10$ and $\theta'' \ge -10\theta \ge -10$. Here $r(x)$ denotes the distance from some fixed x_0. Let $Q = \eta w$, which vanishes outside $B(x_0, R)$. At the maximum point of Q, it is easy to see that

$$\nabla Q = (\nabla\eta)w + (\nabla w)\eta = 0 \tag{2.3}$$

and

$$0 \ge \mathcal{L}_f(Q).$$

At the maximal point,

$$\begin{aligned}
\mathcal{L}_f(Q) &= \mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} w^{\frac{p}{2}-1}(\eta A(\nabla w) + wA(\nabla\eta))) \\
&= \eta \mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} w^{\frac{p}{2}-1} A(\nabla w)) + w^{\frac{p}{2}-1}\langle \nabla\eta, A(\nabla w)\rangle + \mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} w^{\frac{p}{2}} A(\nabla\eta)) \\
&= \eta\mathcal{L}_f(w) - \Big(\frac{p}{2}+1\Big) w^{\frac{p}{2}} \frac{\langle A(\nabla\eta), \nabla\eta\rangle}{\eta} + w^{\frac{p}{2}} \mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} A(\nabla\eta)).
\end{aligned}$$

Since we know,

$$\begin{aligned}
\mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} A(\nabla\eta)) &= \mathrm{e}^f \operatorname{div}\Big(\mathrm{e}^{-f}\Big(\nabla\eta + (p-2)\frac{\langle\nabla v, \nabla\eta\rangle}{|\nabla v|^2}\nabla v\Big)\Big) \\
&= \triangle_f \eta + (p-2)\frac{\langle\nabla v, \nabla\eta\rangle}{|\nabla v|^2}\triangle_f v \\
&\quad + (p-2)\Big(\frac{\langle\nabla\nabla v, \nabla\eta\rangle}{|\nabla v|^2} + \frac{\langle\nabla v, \nabla\nabla\eta\rangle}{|\nabla v|^2}\Big)\nabla v \\
&\quad - (p-2)\frac{\langle\nabla v, \nabla\eta\rangle\langle\nabla|\nabla v|^2, \nabla v\rangle}{|\nabla v|^4}.
\end{aligned}$$

From (2.2),

$$\begin{aligned}
\triangle_f v &= -\Big(\frac{p}{2}-1\Big)\frac{\langle\nabla v, \nabla w\rangle}{w} - chw^{\frac{2-p}{2}} - w \\
&= \Big(\frac{p}{2}-1\Big)\frac{\langle\nabla v, \nabla\eta\rangle}{\eta} - chw^{\frac{2-p}{2}} - w,
\end{aligned}$$

where $h = (p-1)^{p-1}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}$.

Therefore, we get

$$\begin{aligned}
\mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} A(\nabla\eta)) = \triangle_f\eta &+ \frac{p}{2}(p-2)\frac{|\langle\nabla v, \nabla\eta\rangle|^2}{\eta w} - c(p-2)hw^{-\frac{p}{2}}\langle\nabla v, \nabla\eta\rangle \\
&- \Big(\frac{p}{2}-1\Big)\frac{|\nabla\eta|^2}{\eta} + (p-2)\frac{\eta_{ij}v_i v_j}{w} - (p-2)\langle\nabla v, \nabla\eta\rangle.
\end{aligned}$$

From Kotschwar and Ni [7], we note that

$$\triangle\eta + (p-2)\frac{\eta_{ij}v_i v_j}{w} \ge -20(n+p-2)\frac{1+KR}{R^2} - \frac{10}{R^2},$$

since $|\nabla f| \le C_1$, so we deduce $|\langle \nabla f, \nabla \eta\rangle| \ge -C_1|\nabla \eta| \ge -\frac{\sqrt{10}C_1}{R}$. At last, we get

$$\begin{aligned}
\mathrm{e}^f \mathrm{div}(\mathrm{e}^{-f} A(\nabla \eta)) \ge& -20(n+p-2)\frac{1+KR}{R^2} - \frac{10}{R^2} - \frac{\sqrt{10}C_1}{R} - c(p-2)hw^{-\frac{p}{2}}\langle \nabla v, \nabla \eta\rangle \\
&- \Big(\frac{p}{2}-1\Big)\frac{|\nabla \eta|^2}{\eta} + (p-2)\frac{\eta_{ij}v_i v_j}{w} - (p-2)\langle \nabla v, \nabla \eta\rangle \\
&+ \frac{p}{2}(p-2)\frac{|\langle \nabla v, \nabla \eta\rangle|^2}{\eta w}.
\end{aligned}$$

Now we are at the position to estimate $|\nabla\nabla v|^2$. We only need to estimate it over the point where $w > 0$. Choose a local orthonormal frame $\{e_i\}_{i=1}^n$ near any such a given point so that $\nabla v = |\nabla v| e_1$. Then $w = v_1^2, w_1 = 2v_{i1}v_i = 2v_{11}v_1$, and for $j \ge 2, w_j = 2v_{j1}v_1$. Hence $2v_{j1} = \frac{w_j}{w^{\frac{1}{2}}}$.

From (2.2), we immediately deduce that

$$\begin{aligned}
\sum_{j=2}^n v_{jj} &= -chw^{1-\frac{p}{2}} - \Big(\frac{p}{2}-1\Big)\frac{w_1 v_1}{w} - v_{11} + f_1 v_1 - w \\
&= -chw^{1-\frac{p}{2}} - (p-1)v_{11} + f_1 v_1 - w.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\sum_{i,j=1}^n v_{ij}^2 &\ge v_{11}^2 + 2\sum_{j=2}^n v_{j1}^2 + \sum_{j=2}^n v_{jj}^2 \\
&\ge v_{11}^2 + 2\sum_{j=2}^n v_{j1}^2 + \frac{1}{n-1}(chw^{1-\frac{p}{2}} + (p-1)v_{11} + w - f_1 v_1)^2 \\
&\ge v_{11}^2 + 2\sum_{j=2}^n v_{j1}^2 + \frac{1}{m-1}(chw^{1-\frac{p}{2}} + (p-1)v_{11} + w)^2 - \frac{(f_1 v_1)^2}{m-n} \\
&\ge \alpha\sum_{j=1}^n v_{j1}^2 + \frac{1}{m-1}(chw^{1-\frac{p}{2}} + w)^2 + \frac{2(p-1)v_{11}}{m-1}(chw^{1-\frac{p}{2}} + w) - \frac{(f_1 v_1)^2}{m-n},
\end{aligned}$$

where $\alpha = \min\big\{2, 1 + \frac{(p-1)^2}{m-1}\big\}$, and we applied the inequality $(a-b)^2 \ge \frac{a^2}{1+\delta} - \frac{b^2}{\delta}$ with $\delta = \frac{m-n}{n-1} > 0$. Substituting the identities,

$$2wv_{11} = \langle \nabla v, \nabla w\rangle, \quad \sum_{j=1}^n v_{j1}^2 = \frac{1}{4}\frac{|\nabla w|^2}{w},$$

we can obtain

$$|\mathrm{Hess}\ v|^2 \ge \frac{\alpha}{4}\frac{|\nabla w|^2}{w} + \frac{w^2}{m-1}(chw^{-\frac{p}{2}}+1)^2 + \frac{p-1}{m-1}(1 + chw^{-\frac{p}{2}})\langle \nabla v, \nabla w\rangle - \frac{(f_1 v_1)^2}{m-n},$$

which yields

$$\begin{aligned}
\mathcal{L}_f(w) \ge& 2w^{\frac{p}{2}-1}\Big[\frac{\alpha}{4}\frac{|\nabla w|^2}{w} + \frac{w^2}{m-1}(1+chw^{-\frac{p}{2}})^2 + \frac{p-1}{m-1}(1+chw^{-\frac{p}{2}})\langle \nabla v, \nabla w\rangle \\
&+ \mathrm{Ric}_f^m(\nabla v, \nabla v)\Big] + \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}|\nabla w|^2 - pw^{\frac{p}{2}-1}\langle \nabla v, \nabla w\rangle - 2ch\Big(\frac{\sigma}{p-1} - 1\Big)w.
\end{aligned}$$

Therefore, the following inequalities holds

$$
\begin{aligned}
& w^{\frac{p}{2}-1}\mathcal{L}_f(w) \\
\geq\ & \frac{\alpha}{2}w^{p-3}|\nabla w|^2 + \frac{2w^p}{m-1}(1+chw^{-\frac{p}{2}})^2 + 2\frac{p-1}{m-1}w^{p-2}(1+chw^{-\frac{p}{2}})\langle\nabla v,\nabla w\rangle \\
& -2w^{p-1}K_2 + \Big(\frac{p}{2}-1\Big)w^{p-3}|\nabla w|^2 - pw^{p-2}\langle\nabla v,\nabla w\rangle - 2ch\Big(\frac{\sigma}{p-1}-1\Big)w^{\frac{p}{2}} \\
\geq\ & \frac{\alpha}{2}w^{p-1}\frac{|\nabla\eta|^2}{\eta^2} + \frac{2w^p}{m-1}(1+chw^{-\frac{p}{2}})^2 - 2\frac{p-1}{m-1}w^{p-1}(1+chw^{-\frac{p}{2}})\frac{\langle\nabla v,\nabla\eta\rangle}{\eta} \\
& -2w^{p-1}K_2 + \Big(\frac{p}{2}-1\Big)w^{p-1}\frac{|\nabla\eta|^2}{\eta} + pw^{p-1}\frac{\langle\nabla v,\nabla\eta\rangle}{\eta} - 2ch\Big(\frac{\sigma}{p-1}-1\Big)w^{\frac{p}{2}}.
\end{aligned}
$$

Now combing the previous estimates, we have

$$
\begin{aligned}
0 \geq\ & w^{\frac{p}{2}-1}\eta^{p-1}\mathcal{L}_f(Q) \\
\geq\ & w^{\frac{p}{2}-1}\eta^{p}\mathcal{L}_f(w) - w^{\frac{p}{2}-1}\eta^{p-1}\Big(\frac{p}{2}+1\Big)w^{\frac{p}{2}}\frac{\langle A(\nabla\eta),\nabla\eta\rangle}{\eta} + w^{p-1}\eta^{p-1}\mathrm{e}^{f}\mathrm{div}(\mathrm{e}^{-f}A(\nabla\eta)) \\
\geq\ & \frac{\alpha}{2}Q^{p-1}\frac{|\nabla\eta|^2}{\eta} + \frac{2Q^p}{m-1}(1+chw^{-\frac{p}{2}})^2 - 2\frac{p-1}{m-1}Q^{p-1}(1+chw^{-\frac{p}{2}})\langle\nabla v,\nabla\eta\rangle \\
& -2\eta^p w^{p-1}K_2 + \Big(\frac{p}{2}-1\Big)Q^{p-1}\frac{|\nabla\eta|^2}{\eta} + pQ^{p-1}\langle\nabla v,\nabla\eta\rangle - 2ch\Big(\frac{\sigma}{p-1}-1\Big)\eta^p w^{\frac{p}{2}} \\
& -\Big(\frac{p}{2}+1\Big)Q^{p-1}\frac{\langle A(\nabla\eta),\nabla\eta\rangle}{\eta} + Q^{p-1}\Big[-20(n+p-2)\frac{1+K_1R}{R^2} - \frac{10}{R^2} - \frac{\sqrt{10}C_1}{R} \\
& -c(p-2)hw^{-\frac{p}{2}}\langle\nabla v,\nabla\eta\rangle - \Big(\frac{p}{2}-1\Big)\frac{|\nabla\eta|^2}{\eta} + (p-2)\frac{\eta_{ij}u_iu_j}{w} - (p-2)\langle\nabla v,\nabla\eta\rangle\Big] \\
& + \frac{p(p-2)}{2}\langle\nabla v,\nabla\eta\rangle^2 Q^{p-2} \\
\geq\ & \frac{\alpha+p-2}{2}Q^{p-1}\frac{|\nabla\eta|^2}{\eta} - \Big[2\frac{p-m}{m-1} + \Big(p-2+\frac{2p-2}{m-1}\Big)chw^{-\frac{p}{2}}\Big]Q^{p-1}\langle\nabla v,\nabla\eta\rangle \\
& + \frac{2Q^p}{m-1}(1+chw^{-\frac{p}{2}})^2 - 2ch\Big(\frac{\sigma}{p-1}-1\Big)\eta^p w^{\frac{p}{2}} - \Big[2K_2 + 20(n+p-2)\frac{1+K_1R}{R^2} \\
& + \frac{10}{R^2} + \frac{\sqrt{10}C_1}{R}\Big]Q^{p-1} + \frac{p(p-2)}{2}\langle\nabla v,\nabla\eta\rangle^2 Q^{p-2}.
\end{aligned}
$$

Since

$$
\frac{p(p-2)}{2}\langle\nabla v,\nabla\eta\rangle^2 Q^{p-2} \geq \frac{p(p-2)}{2}\frac{|\nabla\eta|^2}{\eta}Q^{p-1},
$$

let $b = 2\frac{p-m}{m-1} + \big(p-2+\frac{2p-2}{m-1}\big)chw^{-\frac{p}{2}}$, then we have

$$
-b\langle\nabla v,\nabla\eta\rangle Q^{p-1} \geq \frac{-b^2(m-1)\varepsilon}{8}Q^p - \frac{2}{(m-1)\varepsilon}Q^{p-1}\frac{|\nabla\eta|^2}{\eta}.
$$

Under the above inequalities, we have

$$
\begin{aligned}
0 \geq & \Big(\frac{2Q^p}{m-1}(1+chw^{-\frac{p}{2}})^2 - \frac{\varepsilon(m-1)}{8}b^2\Big)Q^p \\
& - \Big\{\frac{10(2-p-\alpha)}{2R^2} + \frac{10p(2-p)}{2R^2} + \frac{20}{(m-1)\varepsilon R^2} \\
& + 2K_2 + 20(n+p-2)\frac{1+K_1R}{R^2} + \frac{10}{R^2} + \frac{\sqrt{10}C_1}{R}\Big\}Q^{p-1} \\
= & \Big\{\Big(\frac{2}{m-1} - \frac{\varepsilon(p-m)^2}{2(m-1)}\Big) + \Big(\frac{4}{m-1} - \frac{\varepsilon(p-m)(p+(p-2)m)}{2(m-1)}\Big)chw^{-\frac{p}{2}} \\
& + \Big(\frac{2}{m-1} - \frac{\varepsilon((p-2)m+p)^2}{8(m-1)}\Big)(chw^{-\frac{p}{2}})^2\Big\}Q^p \\
& - \Big\{\frac{10(2-p-\alpha)}{2R^2} + \frac{10p(2-p)}{2R^2} + \frac{20}{(m-1)\varepsilon R^2} \\
& + 2K_2 + 20(n+p-2)\frac{1+K_1R}{R^2} + \frac{10}{R^2} + \frac{\sqrt{10}C_1}{R}\Big\}Q^{p-1}.
\end{aligned}
$$

Since $1 < p \leq 2, m > n \geq 2, m \geq p$ we know

$$\frac{4}{m-1} - \frac{\varepsilon(p-m)(p+(p-2)m)}{2(m-1)} \geq 0.$$

We can choose the constant $\varepsilon \leq \min\big\{\frac{4}{(p-m)^2}, \frac{4}{((p-2)m+p)^2}\big\}$ to make

$$\frac{2}{m-1} - \frac{\varepsilon(p-m)^2}{2(m-1)} \geq 0$$

and

$$\frac{2}{m-1} - \frac{\varepsilon((p-2)m+p)^2}{8(m-1)} \geq 0.$$

At last, we arrive at

$$
\begin{aligned}
0 \geq & \Big(\frac{2}{m-1} - \frac{\varepsilon(p-m)^2}{2(m-1)}\Big)Q^p \\
& - \Big\{\frac{10(2-p-\alpha)}{2R^2} + \frac{10p(2-p)}{2R^2} + \frac{20}{(m-1)\varepsilon R^2} \\
& + 2K_2 + 20(n+p-2)\frac{1+K_1R}{R^2} + \frac{10}{R^2} + \frac{\sqrt{10}C_1}{R}\Big\}Q^{p-1}.
\end{aligned}
$$

That is to say,

$$Q \leq \frac{4(m-1)}{4-\varepsilon(p-m)^2}K_2 + \frac{\widetilde{C}}{R^2},$$

where

$$
\begin{aligned}
\widetilde{C} = & \frac{2(m-1)}{4-\varepsilon(p-m)^2}\Big\{10(2-p-\alpha) + \frac{20}{\varepsilon(m-1)} \\
& + (10p(2-p)) + 20(n+p-2)(1+K_1R) + 10 + \sqrt{10}C_1R\Big\},
\end{aligned}
$$

which implies

$$(p-1)\frac{|\nabla u|^2}{u^2} \leq \frac{4(m-1)}{4-\varepsilon(p-m)^2}K_2 + \frac{\widetilde{C}}{R^2}.$$

We have proved the Theorem 1.1.

Proof of Theorem 1.4 Under the same conditions as in the Theorem 1.2. Let $\gamma(s)$ be a minimal geodesic joining x_1 and x_2 in M, $\gamma(s) : [0,1] \to M, \gamma(0) = x, \gamma(1) = y$.

$$\begin{aligned}
\ln \frac{u(x)}{u(y)} &= \int_0^1 \frac{\mathrm{d} \ln(u(\gamma(s)))}{\mathrm{d}s} \mathrm{d}s \\
&= \int_0^1 \frac{\gamma' \nabla u}{u(\gamma(s))} \mathrm{d}s \\
&\le \int_0^1 |\gamma'| \frac{\nabla u}{u} \mathrm{d}s \\
&= \rho(x,y) \int_0^1 \frac{\nabla u}{u} \mathrm{d}s \\
&= \rho(x,y) \Big[\frac{m-1}{p-1} K_2 \Big]^{\frac{1}{2}},
\end{aligned}$$

which implies

$$u(x) \le u(y) \mathrm{e}^{\rho(x,y)[\frac{(m-1)K_2}{(p-1)}]^{\frac{1}{2}}}.$$

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