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L^p $(p > 1)$ Solutions of BSDEs with Generators Satisfying Some Non-uniform Conditions in t and ω *

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Abstract This paper is devoted to the L^p $(p > 1)$ solutions of one-dimensional backward stochastic differential equations (BSDEs for short) with general time intervals and generators satisfying some non-uniform conditions in t and ω. An existence and uniqueness result, a comparison theorem and an existence result for the minimal solutions are respectively obtained, which considerably improve some known works. Some classical techniques used to deal with the existence and uniqueness of L^p $(p > 1)$ solutions of BSDEs with Lipschitz or linear-growth generators are also developed in this paper.

Keywords Backward stochastic differential equation, Existence and uniqueness, Comparison theorem, Minimal solution, Non-uniform condition in (t,ω)
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1 Introduction

Let us fix an extended real number $0 < T \le +\infty$, which can be finite or infinite. Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\ge 0}$ and $(\mathcal{F}_t)_{t\ge 0}$ be the natural σ-algebra generated by $(B_t)_{t\ge 0}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t\ge 0}$ is right-continuous and complete. In this paper, we are concerned with the following one-dimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \tag{1.1}$$

where the extended real number T is called the terminal time, ξ is an $\mathcal{F}_T$-measurable random variable called the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is $(\mathcal{F}_t)$-progressively measurable for each (y, z) called the generator of BSDE (1.1), and the solution $(y_t, z_t)_{t\in[0,T]}$ is a pair of $(\mathcal{F}_t)$-progressively measurable processes. The triple (ξ, T, g) is called the parameters of BSDE (1.1), and BSDE with parameters (ξ, T, g) is usually denoted by BSDE (ξ, T, g).

The nonlinear BSDEs were initially introduced by Pardoux and Peng [23]. In this pioneering paper, the authors established an existence and uniqueness result for the L^2 solutions of multidimensional BSDEs, where the generator g is Lipschitz continuous in (y, z) uniformly with respect to (t, ω), the terminal time T is finite, and both the terminal condition ξ and the

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process $\{g(t,0,0)\}_{t\in[0,T]}$ are square integrable. From then on, BSDEs have been extensively studied and many applications have been found in mathematical finance, stochastic control, partial differential equations and so on, see, for example, El Karoui, Peng and Quenez [11] and Morlais [22] for details.

Many papers have devoted to improving the existence and uniqueness result obtained in [23] by relaxing the uniform Lipschitz condition with respect to the generator g, changing the finite terminal time into the infinite case and studying the solutions of BSDE (1.1) under non-square integrable parameters. For instance, many works [1, 3–7, 9, 12, 16–17, 19–21, 26] (see also the references therein) weakened the uniform Lipschitz condition with respect to the generator g, and some of them investigated the L^p $(p > 1)$ solution of BSDE (1.1). Chen and Wang [8] first improved the result of Pardoux and Peng [23] to the infinite time interval case and proved an existence and uniqueness result for the L^2 solution of BSDE (1.1), where the generator g is Lipschitz continuous in (y, z) non-uniformly with respect to t. Fan and Jiang [13] and Fan, Jiang and Tian [15] further relaxed the Lipschitz condition of Chen and Wang [8] and established two existence and uniqueness results for the L^2 solutions of BSDE (1.1) with general time intervals, which also generalized the corresponding results obtained in [21] and [19].

We especially mention that El Karoui and Huang [10] first introduced a stochastic Lipschitz condition of the generator g in (y, z), where the Lipschitz constant depends also on (t, ω). In this paper, the authors investigate a general time interval BSDE driven by a general càdlàg martingale, and some stronger integrability conditions on the generator, the terminal condition and the solutions are required, which make it possible to replace the uniform Lipschitz condition by a stochastic one. Based on this idea, Bender and Kohlmann [2] and Wang, Ran and Chen [25] respectively proved an existence and uniqueness result for the L^2 solution and the L^p $(p > 1)$ solution of BSDE (1.1) with a general time horizon. After that, Briand and Confortola [3] introduced another stochastic Lipschitz condition involving a bounded mean oscillation martingale and investigated the L^p (for some certain $p > 1$) solution of an infinite dimensional BSDE, where some new higher order integrability conditions on the generator and the terminal condition (see their assumptions (A3) and (A4) for details) need to be satisfied.

Motivated by these results, in this paper we first put forward a new stochastic Lipschitz condition (see (H1) in Section 3) and prove an existence and uniqueness result of the L^p $(p > 1)$ solution of BSDE (1.1) with a general time interval (see Theorem 3.1). We do not impose any stronger integrability conditions to the parameters (ξ, g) and the solution (y, z) as made in [2, 10, 25], and the integrability condition (3.1) is the only requirement in (H1). By introducing an example, we also show that our stochastic Lipschitz condition is strictly weaker than the Lipschitz condition non-uniform in t used in [8] (see Example 3.1). And by using stopping times to subdivide the interval $[0, T]$, we successfully overcome a difficulty arisen naturally in our framework (see the proof of Theorem 3.1). Furthermore, in Section 4, by developing a method employed in [15, 20] we establish a general comparison theorem for the L^p $(p > 1)$ solutions of BSDEs when one of the generators satisfies a monotonicity condition in y and a uniform continuity condition in z, which are both non-uniform with respect to (t, ω) (see Theorem 4.1). Finally, in Section 5, we prove an existence result of the minimal L^p $(p > 1)$ solution for BSDE (1.1) when the generator g is continuous and has a linear growth in (y, z) non-uniform with respect to (t, ω) (see Theorem 5.1), by improving the method used in [18] to prove in a direct way that the sequence of solutions of the BSDEs approximated by Lipschitz generators is a Cauchy sequence in $S^p \times M^p$. And, based on Theorem 5.1 together with Theorem 4.1, we also give a new comparison theorem of the minimal L^p $(p > 1)$ solutions of BSDEs (see Theorem 5.2) and a general existence and uniqueness theorem of the L^p $(p > 1)$ solution of

BSDEs (see Theorem 5.3).

We would like to mention that the results of this paper improve some corresponding existing works including those obtained in [7–8, 15, 19, 23], etc. And, some classical techniques used to deal with the existence and uniqueness of L^p $(p>1)$ solutions of BSDEs with Lipschitz or linear-growth generators are also developed in this paper.

2 Notations and Lemmas

In this section, we introduce some basic notations and definitions, which will be used in this paper. First, we use $|\cdot|$ to denote the norm of Euclidean space $\mathbb{R}^d$, and for each subset $A\subset\Omega\times[0,T]$, let $\mathbb{1}_A=1$ in case of $(\omega,t)\in A$, otherwise, let $\mathbb{1}_A=0$. Then, for each real number $p>1$, let $L^p(\Omega,\mathcal{F}_T,P;\mathbb{R})$ be the set of all $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variables ξ such that $\mathbf{E}[|\xi|^p]<+\infty$, $S^p(0,T;\mathbb{R})$ (or S^p simply) be the set of $\mathbb{R}$-valued, adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{S^p}:=\Big(\mathbf{E}\Big[\sup_{t\in[0,T]}|Y_t|^p\Big]\Big)^{\frac{1}{p}}<+\infty,$$

and $M^p(0,T;\mathbb{R}^d)$ (or M^p simply) be the set of $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{M^p}:=\Big(\mathbf{E}\Big[\Big(\int_0^T|Z_t|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}}<+\infty.$$

Obviously, both S^p and M^p are Banach spaces for each $p>1$.

Finally, let $\mathbf{S}$ be the set of all nondecreasing continuous functions $\phi(\cdot)$: $\mathbb{R}^+\mapsto\mathbb{R}^+$ with $\phi(0)=0$ and $\phi(x)>0$ for all $x\in\mathbb{R}^+$, here and hereafter $\mathbb{R}^+:=[0,+\infty)$.

Definition 2.1 *A pair of processes $(y_t,z_t)_{t\in[0,T]}$ taking values in $\mathbb{R}\times\mathbb{R}^d$ is called an L^p solution of BSDE* (1.1) *for some $p>1$, if $(y_t,z_t)_{t\in[0,T]}\in S^p(0,T;\mathbb{R})\times M^p(0,T;\mathbb{R}^d)$ and* dP*-a.s., BSDE* (1.1) *holds true for each $t\in[0,T]$.*

Let us introduce the following Lemma 2.1, which will be used in Section 3 and Section 5.

Lemma 2.1 *Let $p>1$, $0<T\le+\infty$, and $(g_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-progressively measurable process such that $\int_0^T|g_t|\mathrm{d}t<+\infty$,* d$P$-a.s.. *If $(Y_t,Z_t)_{t\in[0,T]}$ is an L^p solution to the following BSDE*:

$$Y_t=Y_T+\int_t^T g_s\mathrm{d}s-\int_t^T Z_s\cdot\mathrm{d}B_s,\quad t\in[0,T],\tag{2.1}$$

then there exists a positive constant C_p depending only on p such that for each $t\in[0,T]$,

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big]\le C_p\mathbb{E}\Big[|Y_T|^p+\int_t^T(|Y_s|^{p-1}|g_s|)\mathrm{d}s\Big],\tag{2.2}$$

$$\mathbb{E}\Big[\Big(\int_t^T|Z_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\le C_p\Big\{\mathbb{E}\Big[|Y_T|^p+\Big(\int_t^T(|Y_s||g_s|)\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big]\Big\}.\tag{2.3}$$

Moreover, there exists a positive constant $\overline{C}_p$ depending only on p such that for each $t\in[0,T]$,

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big]+\mathbb{E}\Big[\Big(\int_t^T|Z_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\le\overline{C}_p\mathbb{E}\Big[|Y_T|^p+\Big(\int_t^T|g_s|\mathrm{d}s\Big)^p\Big].\tag{2.4}$$

Proof In the same way as Proposition 2.4 in [18], we can prove (2.2) and (2.3). It remains to show (2.4). In fact, by the inequality $2ab \le a^2 + b^2$ and Young's inequality we have, for each constant $\widetilde{C}_p > 0$,

$$\begin{aligned}\widetilde{C}_p\mathbb{E}\Big[\int_t^T (|Y_s|^{p-1}|g_s|)\mathrm{d}s\Big] &\le \widetilde{C}_p\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^{p-1}\int_t^T |g_s|\mathrm{d}s\Big]\\ &\le \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big] + \frac{1}{p}\Big(\frac{2(p-1)}{p}\widetilde{C}_p\Big)^p\mathbb{E}\Big[\Big(\int_t^T |g_s|\mathrm{d}s\Big)^p\Big] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_t^T (|Y_s||g_s|)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] &\le \mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^{\frac{p}{2}}\Big(\int_t^T |g_s|\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ &\le \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big] + \frac{1}{2}\mathbb{E}\Big[\Big(\int_t^T |g_s|\mathrm{d}s\Big)^p\Big]. \end{aligned} \tag{2.6}$$

Thus, (2.4) follows immediately from (2.2)–(2.3) and (2.5)–(2.6).

The following technical Lemma 2.2 comes from [14, Lemma 4], which will be used in Section 4. It gives a sequence of upper bounds for functions of linear growth.

Lemma 2.2 *Let $\Psi(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a nondecreasing function of linear growth, which means that $\Psi(x) \le K(x+1)$ $(K > 0)$ holds true for all $x \in \mathbb{R}^+$. Then for each $n \ge 1$,*

$$\Psi(x) \le (n+2K)x + \Psi\Big(\frac{2K}{n+2K}\Big)$$

holds true for each $x \in \mathbb{R}^+$.

3 An Existence and Uniqueness Result

In this section, we will use a stopping time technique involved in subdividing the time interval $[0,T]$ to prove a general existence and uniqueness result for the L^p $(p > 1)$ solution of BSDE (1.1), and introduce an example to show that our stochastic Lipschitz condition is strictly weaker than the Lipschitz condition non-uniform in t used in [8]. First, let us introduce the following assumptions with the generator g, where $0 < T \le +\infty$ and $p > 1$.

(H1) g is Lipschtiz continuous in (y,z) non-uniformly with respect to both t and ω , i.e., there exist two $(\mathcal{F}_t)$-progressively measurable nonnegative processes $\{u_t(\omega)\}_{t\in[0,T]}$ and $\{v_t(\omega)\}_{t\in[0,T]}$ satisfying

$$\int_0^T [u_t(\omega) + v_t^2(\omega)]\mathrm{d}t \le M, \quad \mathrm{d}P\text{-a.s.} \tag{3.1}$$

for some constant $M > 0$ such that $\mathrm{d}P \times \mathrm{d}t$-a.e., for each $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$|g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)| \le u_t(\omega)|y_1 - y_2| + v_t(\omega)|z_1 - z_2|;$$

(H2) $\mathbb{E}\big[\big(\int_0^T |g(\omega,t,0,0)|\mathrm{d}t\big)^p\big] < +\infty$.

Remark 3.1 We note that the above (3.1) is equivalent to $\big\|\int_0^T u_t(\omega) + v_t^2(\omega)\mathrm{d}t\big\|_\infty \le M$. For the sake of convenience, the ω in $u_t(\omega)$ and $v_t(\omega)$ is usually omitted without causing confusion.

The following Theorem 3.1 shows an existence and uniqueness result for L^p $(p>1)$ solutions of BSDEs under assumptions (H1) and (H2), which could be regarded as a generalization of the results obtained by Pardoux and Peng [23] and Chen and Wang [8], where u_t and v_t in (H1) do not depend on ω.

Theorem 3.1 *Assume that $p>1$, $0<T\le+\infty$ and the generator g satisfies assumptions* (H1)–(H2). *Then for each $\xi\in L^p(\Omega,\mathcal{F}_T,P;\mathbb{R})$, BSDE (ξ,T,g) admits a unique L^p solution.*

Proof Assume that $(y_t,z_t)_{t\in[0,T]}\in S^p(0,T;\mathbb{R})\times M^p(0,T;\mathbb{R}^d)$. It follows from (H1) that $|g(s,y_s,z_s)|\le|g(s,0,0)|+u_s|y_s|+v_s|z_s|$. Then from the inequality $(a+b+c)^p\le 3^p(a^p+b^p+c^p)$, Hölder's inequality and (H2), we have

$$\mathbb{E}\Big[\Big(\int_0^T|g(s,y_s,z_s)|\mathrm{d}s\Big)^p\Big]\le 3^p\mathbb{E}\Big[\Big(\int_0^T|g(s,0,0)|\mathrm{d}s\Big)^p\Big]+(3M)^p\mathbb{E}\Big[\sup_{s\in[0,T]}|y_s|^p\Big]$$
$$+3^pM^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_0^T|z_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]<+\infty.$$

As a result, the process $\big(\mathbb{E}\big[\xi+\int_0^Tg(s,y_s,z_s)\mathrm{d}s\big|\mathcal{F}_t\big]\big)_{0\le t\le T}$ is an L^p martingale. It then follows from the martingale representation theorem (see, for example, [24, Theorems 2.42 and 2.48]) that there exists a unique process $Z_\cdot\in M^p(0,T;\mathbb{R}^d)$ such that

$$\mathbb{E}\Big[\xi+\int_0^Tg(s,y_s,z_s)\mathrm{d}s\Big|\mathcal{F}_t\Big]=\mathbb{E}\Big[\xi+\int_0^Tg(s,y_s,z_s)\mathrm{d}s\Big]+\int_0^tZ_s\cdot\mathrm{d}B_s,\quad 0\le t\le T.$$

Let $Y_t:=\mathbb{E}\big[\xi+\int_t^Tg(s,y_s,z_s)\mathrm{d}s\big|\mathcal{F}_t\big]$, $0\le t\le T$. Obviously, $Y_\cdot\in S^p(0,T;\mathbb{R})$, and it is not difficult to verify that $(Y_t,Z_t)_{t\in[0,T]}$ is just the unique L^p solution to the following equation:

$$Y_t=\xi+\int_t^Tg(s,y_s,z_s)\mathrm{d}s-\int_t^TZ_s\cdot\mathrm{d}B_s,\quad t\in[0,T].\tag{3.2}$$

Thus, we have constructed a mapping from $S^p(0,T;\mathbb{R})\times M^p(0,T;\mathbb{R}^d)$ to itself. Denote this mapping by $I:(y_t,z_t)_{t\in[0,T]}\to(Y_t,Z_t)_{t\in[0,T]}$.

Now, suppose that $(y_t^i,z_t^i)_{t\in[0,T]}\in S^p(0,T;\mathbf{R})\times M^p(0,T;\mathbb{R}^d)$, and let $(Y_t^i,Z_t^i)_{t\in[0,T]}$ be the mapping of $(y_t^i,z_t^i)_{t\in[0,T]}$, $i=1,2$, that is, $I(y_t^i,z_t^i)_{t\in[0,T]}=(Y_t^i,Z_t^i)_{t\in[0,T]}$, $i=1,2$. We denote

$$\widehat{Y}_t:=Y_t^1-Y_t^2,\quad \widehat{Z}_t:=Z_t^1-Z_t^2,\quad \widehat{y}_t:=y_t^1-y_t^2,\quad \widehat{z}_t:=z_t^1-z_t^2,$$
$$\widehat{g}_t:=g(t,y_t^1,z_t^1)-g(t,y_t^2,z_t^2),\quad t\in[0,T].$$

Then $(\widehat{Y}_t,\widehat{Z}_t)_{t\in[0,T]}$ is an L^p solution of the following BSDE:

$$\widehat{Y}_t=\int_t^T\widehat{g}_s\mathrm{d}s-\int_t^T\widehat{Z}_s\cdot\mathrm{d}B_s,\quad t\in[0,T].$$

Furthermore, (2.4) of Lemma 2.1 yields the existence of a constant $c_p>0$ depending only on p such that for each $t\in[0,T]$,

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|\widehat{Y}_s|^p+\Big(\int_t^T|\widehat{Z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\le c_p\mathbb{E}\Big[\Big(\int_t^T|\widehat{g}_s|\mathrm{d}s\Big)^p\Big].$$

Thus, by virtue of (H1) and Hölder's inequality we can deduce that for each $t \in [0,T]$,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[t,T]}|\widehat{Y}_s|^p+\Big(\int_t^T|\widehat{Z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\&\le c_p\mathbb{E}\Big[\Big(\Big(\int_t^T u_s\mathrm{d}s\Big)^p+\Big(\int_t^T v_s^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big)\Big(\sup_{s\in[t,T]}|\widehat{y}_s|^p+\Big(\int_t^T|\widehat{z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big)\Big].\end{aligned}\tag{3.3}$$

In the sequel, we choose a sufficiently large number N such that

$$\frac{M}{N}\le\frac{1}{(4c_p)^{\frac{1}{p}}}\wedge\frac{1}{(4c_p)^{\frac{2}{p}}},$$

and subdivide the interval $[0,T]$ into some small stochastic intervals like $[T_{i-1},T_i], i=1,\cdots,N$, by defining the following $(\mathcal{F}_t)$-stopping times:

$$\begin{aligned}&T_0=0;\\&T_1=\inf\Big\{t\ge 0:\int_0^t(u_s+v_s^2)\mathrm{d}s\ge\frac{M}{N}\Big\}\wedge T;\\&\quad\vdots\\&T_i=\inf\Big\{t\ge T_{i-1}:\int_0^t(u_s+v_s^2)\mathrm{d}s\ge\frac{iM}{N}\Big\}\wedge T;\\&\quad\vdots\\&T_N=\inf\Big\{t\ge T_{N-1}:\int_0^t(u_s+v_s^2)\mathrm{d}s\ge\frac{NM}{N}\Big\}\wedge T=T.\end{aligned}$$

Thus, for any $[T_{i-1},T_i]\subset[0,T],\ i=1,\cdots,N$, it follows that

$$\Big(\int_{T_{i-1}}^{T_i}u_s\mathrm{d}s\Big)^p+\Big(\int_{T_{i-1}}^{T_i}v_s^2\mathrm{d}s\Big)^{\frac{p}{2}}\le\frac{1}{2c_p}.\tag{3.4}$$

Now, with the help of inequality (3.3) we have

$$\mathbb{E}\Big[\sup_{s\in[T_{N-1},T]}|\widehat{Y}_s|^p+\Big(\int_{T_{N-1}}^T|\widehat{Z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\le\frac{1}{2}\mathbb{E}\Big[\sup_{s\in[T_{N-1},T]}|\widehat{y}_s|^p+\Big(\int_{T_{N-1}}^T|\widehat{z}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big],$$

which means that I is a strict contraction from $S^p(T_{N-1},T;\mathbb{R})\times M^p(T_{N-1},T;\mathbb{R}^d)$ into itself. Then I admits a unique fixed point in this space. It follows that there exists a unique $(y_t,z_t)_{t\in[T_{N-1},T]}\in S^p(T_{N-1},T;\mathbb{R})\times M^p(T_{N-1},T;\mathbb{R}^d)$ satisfying BSDE (ξ,T,g) on $[T_{N-1},T]$. That is to say, BSDE (ξ,T,g) admits a unique L^p solution on $[T_{N-1},T]$.

Finally, note that (3.4) holds true for $i=N-1$. By replacing T_{N-1}, T and ξ by T_{N-2}, T_{N-1} and $y_{T_{N-1}}$ respectively in the above proof, we can obtain the existence and uniqueness for the L^p solution of BSDE (ξ,T,g) on $[T_{N-2},T_{N-1}]$. Furthermore, repeating the above procedure and making use of (3.4), we deduce the existence and uniqueness for the L^p solution of BSDE (ξ,T,g) on $[T_{N-3},T_{N-2}],\cdots,[0,T_1]$. The proof of Theorem 3.1 is then completed.

Remark 3.2 It is easy to see that Theorem 3.1 holds also true for multidimensional BSDEs.

The following example shows that the assumption (H1) is strictly weaker than the corresponding assumption in [8]. For readers' convenience, we list the assumption of Chen and Wang [8] as the following (H1′):

(H1$'$) g is Lipschitz continuous in (y,z) non-uniformly with respect to t, i.e., there exist two functions $\overline{u}(t),\ \overline{v}(t):[0,T]\mapsto\mathbb{R}^+$ satisfying

$$\int_0^T[\overline{u}(t)+\overline{v}^2(t)]\mathrm{d}t<+\infty$$

such that $\mathrm{d}P\times\mathrm{d}t$-a.e., for each $y_1,y_2\in\mathbb{R}$, $z_1,z_2\in\mathbb{R}^d$,

$$|g(\omega,t,y_1,z_1)-g(\omega,t,y_2,z_2)|\leq\overline{u}(t)|y_1-y_2|+\overline{v}(t)|z_1-z_2|.$$

Example 3.1 Let $0<T\leq+\infty$, and for each $t_0\in(0,T)$, define the following two stopping times:

$$\tau_1(\omega)=\inf\Big\{t>t_0:|B_{t_0}(\omega)|(t-t_0)\geq\frac{M}{2}\Big\}\wedge T,$$
$$\tau_2(\omega)=\inf\Big\{t>t_0:|B_{t_0}(\omega)|^2(t-t_0)\geq\frac{M}{2}\Big\}\wedge T.$$

Consider the generator $\widetilde{g}(\omega,t,y,z):=\widetilde{u}_t(\omega)|y|+\widetilde{v}_t(\omega)|z|$, where

$$\widetilde{u}_t(\omega)=|B_{t_0}(\omega)|\mathbb{1}_{((t_0,\tau_1(\omega)]]}(\omega,t),\quad \widetilde{v}_t(\omega)=|B_{t_0}|\mathbb{1}_{((t_0,\tau_2(\omega)]]}(\omega,t),\quad (t,\omega)\in[0,T]\times\Omega.$$

It is clear that $\widetilde{g}$ satisfies the assumptions (H1) and (H2) with $u_t=\widetilde{u}_t$ and $v(t)=\widetilde{v}_t$. Then, by Theorem 3.1 we know that for each $p>1$ and $\xi\in L^p(\Omega,\mathcal{F}_T,P;\mathbb{R})$, BSDE $(\xi,T,\widetilde{g})$ admits a unique L^p solution.

We especially mention that this $\widetilde{g}$ does not satisfy the above assumption (H1$'$). In fact, if the assumption (H1$'$) holds true for $\widetilde{g}$, then there exist two deterministic functions $\overline{u}(t),\ \overline{v}(t):[0,T]\mapsto\mathbb{R}^+$ such that

$$\widetilde{u}_t(\omega)\leq\overline{u}(t),\quad \widetilde{v}_t(\omega)\leq\overline{v}(t),\quad \mathrm{d}P\times\mathrm{d}t\text{-a.e.}\tag{3.5}$$

and

$$\int_0^T[\overline{u}(t)+\overline{v}^2(t)]\mathrm{d}t<+\infty.\tag{3.6}$$

This yields a contradiction which will be shown below. Note first that for each $t\in(t_0,T)$, we have

$$\begin{aligned}\{\omega:\widetilde{u}_t(\omega)>\overline{u}(t)\}&=\{\omega:t\leq\tau_1(\omega)\text{ and }|B_{t_0}(\omega)|>\overline{u}(t)\}\\&=\Big\{\omega:|B_{t_0}(\omega)|\leq\frac{M}{2(t-t_0)}\text{ and }|B_{t_0}(\omega)|>\overline{u}(t)\Big\},\end{aligned}$$

and note that $B_{t_0}(\omega)$ is a normal random variable with zero-expected value and t_0-variance values. If $\overline{u}(t)<\frac{M}{2(t-t_0)}$ for some $t\in(t_0,T)$, then $P(\{\omega:\widetilde{u}_t(\omega)>\overline{u}(t)\})>0$. Using this fact and (3.5) we can conclude that

$$\overline{u}_t\geq\frac{M}{2(t-t_0)},\quad \mathrm{d}t\text{-a.e. in }(t_0,T).$$

Thus,

$$\int_0^T\overline{u}(t)\mathrm{d}t\geq\frac{M}{2}\int_{t_0}^T\frac{1}{t-t_0}\mathrm{d}t=+\infty,$$

which contradicts (3.6).

Therefore, our assumption (H1) is strictly weaker than (H1$'$) used in [8].

4 A General Comparison Theorem

In this section, by developing a method employed in [15, 20] we will prove a general comparison theorem for L^p $(p > 1)$ solution of BSDE (1.1). Let us first introduce the following assumptions, where $0 < T \leq +\infty$.

(H3) g satisfies a monotonicity condition in y non-uniform with respect to both t and ω, i.e., there exists an $(\mathcal{F}_t)$-progressively measurable nonnegative process $\{u_t(\omega)\}_{t\in[0,T]}$ satisfying

$$\int_0^T u_t(\omega)\mathrm{d}t \leq M, \quad \mathrm{d}P\text{-a.s.}$$

for some constant $M > 0$ such that $\mathrm{d}P \times \mathrm{d}t$-a.e., for each $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$\mathrm{sgn}(y_1 - y_2)(g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \leq u_t(\omega)|y_1 - y_2|;$$

(H4) g satisfies a uniform continuity condition in z non-uniform with respect to both t and ω, i.e., there exists a linear-growth function $\phi(\cdot) \in \mathbf{S}$ and an $(\mathcal{F}_t)$-progressively measurable nonnegative process $\{v_t(\omega)\}_{t\in[0,T]}$ satisfying

$$\int_0^T v_t^2(\omega)\mathrm{d}t \leq M, \quad \mathrm{d}P\text{-a.s.}$$

such that $\mathrm{d}P \times \mathrm{d}t$-a.e., for each $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq v_t(\omega)\phi(|z_1 - z_2|).$$

Here and hereafter, we always assume that $0 \leq \phi(x) \leq ax + b$ for all $x \in \mathbb{R}^+$. Furthermore, when $b \neq 0$, we also assume that $\int_0^T v_t(\omega)\mathrm{d}t \leq M$, $\mathrm{d}P$-a.s., where M is defined in (H3).

Remark 4.1 Note that in case of $u_t(\omega) \equiv 0$, (H3) means that g is non-increasing in y.

The following Theorem 4.1 establishes a general comparison theorem for BSDEs under the assumptions (H3)–(H4), which generalizes partly Theorem 2 in [15], where $u_t(\omega)$ and $v_t(\omega)$ in (H3)–(H4) do not depend on ω and $p = 2$, and [20, Lemma 1], where $u_t(\omega)$ and $v_t(\omega)$ need to be bounded processes and $T < +\infty$.

Theorem 4.1 *Let $p > 1$, $0 < T \leq +\infty$, $\xi, \xi' \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R})$, g and g' be two generators of BSDEs, and $(y_t, z_t)_{t\in[0,T]}$ and $(y'_t, z'_t)_{t\in[0,T]}$ be, respectively, an L^p solution to BSDE (ξ, T, g) and BSDE (ξ', T, g'). If $\mathrm{d}P$-a.s., $\xi \leq \xi'$, g (respectively, g') satisfies* (H3) *and* (H4) *and $\mathrm{d}P \times \mathrm{d}t$-a.e., $g(t, y'_t, z'_t) \leq g'(t, y'_t, z'_t)$ (respectively, $g(t, y_t, z_t) \leq g'(t, y_t, z_t)$), then for each $t \in [0, T]$, we have*

$$\mathrm{d}P\text{-}a.s., \quad y_t \leq y'_t.$$

Proof Assume that $\mathrm{d}P$-a.s., $\xi \leq \xi'$, g satisfies (H3)–(H4) and $\mathrm{d}P \times \mathrm{d}t$-a.e., $g(t, y'_t, z'_t) \leq g'(t, y'_t, z'_t)$. Setting $\widehat{y}_t = y_t - y'_t$, $\widehat{z}_t = z_t - z'_t$, $\widehat{\xi} = \xi - \xi'$, since $g(s, y'_s, z'_s) - g'(s, y'_s, z'_s)$ is non-positive, we have

$$\begin{aligned} g(s, y_s, z_s) - g'(s, y'_s, z'_s) &= g(s, y_s, z_s) - g(s, y'_s, z'_s) + g(s, y'_s, z'_s) - g'(s, y'_s, z'_s) \\ &\leq g(s, y_s, z_s) - g(s, y'_s, z'_s) \\ &= g(s, y_s, z_s) - g(s, y'_s, z_s) + g(s, y'_s, z_s) - g(s, y'_s, z'_s), \end{aligned}$$

and we deduce, using (H3)–(H4), that

$$\mathbb{1}_{\widehat{y}_s>0}[g(s,y_s,z_s)-g'(s,y'_s,z'_s)]\le u_s\widehat{y}_s^+ + \mathbb{1}_{\widehat{y}_s>0}v_s\phi(|\widehat{z}_s|),\quad s\in[0,T]. \tag{4.1}$$

Then Tanaka's formula with (4.1) leads to the following inequality, with $A_t:=\int_0^t u_s\mathrm{d}s$,

$$\begin{aligned}\mathrm{e}^{A_t}\widehat{y}_t^+ &\le \mathrm{e}^{A_T}\widehat{\xi}^+ + \int_t^T \mathrm{e}^{A_s}\{\mathbb{1}_{\widehat{y}_s>0}[g(s,y_s,z_s)-g'(s,y'_s,z'_s)]-u_s\widehat{y}_s^+\}\mathrm{d}s - \int_t^T \mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}\widehat{z}_s\cdot\mathrm{d}B_s\\ &\le \int_t^T \mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}v_s\phi(|\widehat{z}_s|)\mathrm{d}s - \int_t^T \mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}\widehat{z}_s\cdot\mathrm{d}B_s,\quad t\in[0,T]. \end{aligned}\tag{4.2}$$

Furthermore, note that Lemma 2.2 with $\Psi(\cdot)=\phi(\cdot)$ and $K=c:=a+b$ yields that

$$\forall\ n\ge 1,\ x\in\mathbb{R}^+,\quad \phi(x)\le(n+2c)x+\mathbf{1}_{b\ne 0}\phi\Big(\frac{2c}{n+2c}\Big), \tag{4.3}$$

where $\mathbf{1}_{b\ne0}=1$ if $b\ne0$ and $\mathbf{1}_{b\ne0}=0$ if $b=0$. By (4.1)–(4.3) we get that for each $n\ge1$ and $t\in[0,T]$,

$$\begin{aligned}\mathrm{e}^{A_t}\widehat{y}_t^+ &\le a_n+\int_t^T[\mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}(n+2c)v_s|\widehat{z}_s|]\mathrm{d}s-\int_t^T \mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}\widehat{z}_s\cdot\mathrm{d}B_s\\ &=a_n-\int_t^T \mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}\widehat{z}_s\cdot\Big[-\frac{(n+2c)v_s\widehat{z}_s}{|\widehat{z}_s|}\mathbb{1}_{|\widehat{z}_s|\ne0}\mathrm{d}s+\mathrm{d}B_s\Big],\end{aligned}\tag{4.4}$$

where, by (H4),

$$a_n=\mathbf{1}_{b\ne0}\phi\Big(\frac{2c}{n+2c}\Big)\Big\|\int_0^T\mathrm{e}^{A_s}v_s\mathrm{d}s\Big\|_\infty\le\mathbf{1}_{b\ne0}\phi\Big(\frac{2c}{n+2c}\Big)M\mathrm{e}^M\to0\quad\text{as } n\to\infty. \tag{4.5}$$

Let P_n be the probability on $(\Omega,\mathcal{F})$ which is equivalent to P and defined by

$$\frac{\mathrm{d}P_n}{\mathrm{d}P}:=\exp\Big\{(n+2c)\int_0^T\frac{v_s\widehat{z}_s}{|\widehat{z}_s|}\mathbb{1}_{|\widehat{z}_s|\ne0}\cdot\mathrm{d}B_s-\frac12(n+2c)^2\int_0^T\mathbb{1}_{|\widehat{z}_s|\ne0}v_s^2\mathrm{d}s\Big\}.$$

It is worth noting that $\frac{\mathrm{d}P_n}{\mathrm{d}P}$ has moments of all orders since $\int_0^T v^2(s)\mathrm{d}s\le M$, $\mathrm{d}P$-a.s.. By Girsanov's theorem, under P_n the process

$$B_n(t)=B_t-\int_0^t\frac{(n+2c)v_s\widehat{z}_s}{|\widehat{z}_s|}\mathbb{1}_{|\widehat{z}_s|\ne0}\mathrm{d}s,\quad t\in[0,T]$$

is Brownian motion. Moreover, the process $\big(\int_0^t\mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}\widehat{z}_s\cdot\mathrm{d}B_n(s)\big)_{t\in[0,T]}$ is a $(\mathcal{F}_t,P_n)$-martingale. Indeed, let $\mathbb{E}_n[X|\mathcal{F}_t]$ represent the conditional expectation of random variable X with respect to $\mathcal{F}_t$ under P_n and let $\mathbb{E}_n[X]\widehat{=}\mathbb{E}_n[X|\mathcal{F}_0]$, it then follows from the Burkholder-Davis-Gundy (BDG for short) inequality and Hölder's inequality that

$$\begin{aligned}\mathbb{E}_n\Big[\sup_{0\le t\le T}\Big|\int_0^t\mathrm{e}^{A_s}\mathbb{1}_{\widehat{y}_s>0}\widehat{z}_s\cdot\mathrm{d}B_n(s)\Big|\Big]&\le4\mathrm{e}^M\mathbb{E}_n\Big[\sqrt{\int_0^T|\widehat{z}_s|^2\mathrm{d}s}\Big]\\ &\le4\mathrm{e}^M\mathbb{E}\Big[\Big(\frac{\mathrm{d}P_n}{\mathrm{d}P}\Big)^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}}\mathbb{E}\Big[\Big(\int_0^T|\widehat{z}_s|^2\mathrm{d}s\Big)^{\frac p2}\Big]^{\frac1p}<+\infty.\end{aligned}$$

Thus, by taking the conditional expectation with respect to $\mathcal{F}_t$ under P_n in (4.4), we obtain that for each $n \geq 1$ and $t \in [0,T]$,

$$e^{A_t}\widehat{y}_t^{+} \leq a_n, \quad dP\text{-a.s.} \tag{4.6}$$

And in view of (4.5), it follows that for each $t \in [0,T]$, dP-a.s., $y_t \leq y'_t$.

Now, let us assume that dP-a.s., $\xi \leq \xi'$, g' satisfies (H3)–(H4) and $dP \times dt$-a.e., $g(t, y_t, z_t) \leq g'(t, y_t, z_t)$. Then, since $g(s, y_s, z_s) - g'(s, y_s, z_s)$ is non-positive, we have

$$\begin{aligned} g(s, y_s, z_s) - g'(s, y'_s, z'_s) &= g(s, y_s, z_s) - g'(s, y_s, z_s) + g'(s, y_s, z_s) - g'(s, y'_s, z'_s) \\ &\leq g'(s, y_s, z_s) - g'(s, y'_s, z'_s) \\ &= g'(s, y_s, z_s) - g'(s, y'_s, z_s) + g'(s, y'_s, z_s) - g'(s, y'_s, z'_s), \end{aligned}$$

and using (H3)–(H4), we know that inequality (4.1) still holds. Therefore, the same proof as above yields that for each $t \in [0,T]$, dP-a.s., $y_t \leq y'_t$. Theorem 4.1 is proved.

From Theorem 4.1, the following corollary is immediate.

Corollary 4.1 *Let $p > 1$, $0 < T \leq +\infty$, $\xi, \xi' \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R})$, one of the generators g and g' satisfy assumptions (H3)–(H4), and $(y_t, z_t)_{t\in[0,T]}$ and $(y'_t, z'_t)_{t\in[0,T]}$ be, respectively, an L^p solution to BSDE (ξ, T, g) and BSDE (ξ', T, g'). If dP-a.s., $\xi \leq \xi'$, and $dP \times dt$-a.e., $g(t, y, z) \leq g'(t, y, z)$ for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, then for each $t \in [0,T]$, dP-a.s., $y_t \leq y'_t$.*

5 An Existence Result of the Minimal Solutions

In this section, we will put forward and prove an existence result of the minimal L^p $(p > 1)$ solution for BSDE (1.1) (see Theorem 5.1) by improving the method used in [18] to prove in a direct way that the sequence of solutions of the BSDEs approximated by the Lipschitz generators is a Cauchy sequence in $S^p \times M^p$. And, based on Theorem 5.1 together with Theorem 4.1, we will also give a new comparison theorem of the minimal L^p $(p > 1)$ solutions of BSDEs (see Theorem 5.2) and a general existence and uniqueness theorem of the L^p $(p > 1)$ solution of BSDEs (see Theorem 5.3). First, we introduce the following assumptions with respect to the generator g, where $0 < T \leq +\infty$.

(H5) g has a linear growth in (y, z) non-uniform with respect to both t and ω, i.e., there exist three $(\mathcal{F}_t)$-progressively measurable nonnegative processes $\{u_t(\omega)\}_{t\in[0,T]}$, $\{v_t(\omega)\}_{t\in[0,T]}$ and $\{f_t(\omega)\}_{t\in[0,T]}$ satisfying

$$\mathbb{E}\Big[\Big(\int_0^T f_t(\omega)dt\Big)^p\Big] < +\infty,$$

and

$$\int_0^T [u_t(\omega) + v_t^2(\omega)]dt \leq M, \quad dP\text{-a.s.},$$

for some constant $M > 0$ such that $dP \times dt$-a.e., for each $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$|g(\omega, t, y, z)| \leq f_t(\omega) + u_t(\omega)|y| + v_t(\omega)|z|;$$

(H6) $dP \times dt$-a.e., $g(\omega, t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is a continuous function.

The following Proposition 5.1 will play an important role in the proof of Theorem 5.1. Its proof is analogous to [19, Lemma 1], so we omit it here.

Proposition 5.1 *Assume that the generator g satisfies* (H5)–(H6). *Let g_n be the function defined as follows*:

$$g_n(\omega,t,y,z) := \inf_{(\overline{y},\overline{z})\in R^{1+d}} \{g(\omega,t,\overline{y},\overline{z}) + nu_t(\omega)|y-\overline{y}| + nv_t(\omega)|z-\overline{z}|\}.$$

Then the sequence of function g_n is well defined. For each $n \geq 1$, $g_n(\omega,t,y,z)$ is $(\mathcal{F}_t)$-progressively measurable for each $(y,z) \in \mathbb{R}\times\mathbb{R}^d$, and it satisfies, $\mathrm{d}P\times\mathrm{d}t$*-a.e.,*

(i) *Stochastic linear growth*: $\forall\ y,z,\ |g_n(\omega,t,y,z)| \leq f_t(\omega) + u_t(\omega)|y| + v_t(\omega)|z|$;

(ii) *Monotonicity in n*: $\forall\ y,z,\ g_n(\omega,t,y,z)$ *increases in* n;

(iii) *Stochastic Lipschitz condition*: $\forall\ y_1,y_2,z_1,z_2$, *we have*

$$|g_n(\omega,t,y_1,z_1) - g_n(\omega,t,y_2,z_2)| \leq nu_t(\omega)|y_1-y_2| + nv_t(\omega)|z_1-z_2|;$$

(iv) *Convergence*: *If $(y_n,z_n)\to(y,z)$, then $g_n(\omega,t,y_n,z_n)\to g(\omega,t,y,z)$, as $n\to\infty$.*

Now we state the main result of this section—Theorem 5.1. It improves [15, Theorem 1], where $u_t(\omega)$ and $v_t(\omega)$ in (H5) do not depend on ω and $p=2$, and [18, Theorem 3.3], where $u_t(\omega)$ and $v_t(\omega)$ need to be bounded processes and $T<+\infty$.

Theorem 5.1 *Assume that $p>1$, $0<T\leq+\infty$ and the generator g satisfies* (H5)–(H6). *Then for each $\xi\in L^p(\Omega,\mathcal{F}_T,P;\mathbb{R})$, BSDE (ξ,T,g) admits a minimal L^p solution $(y_t,z_t)_{t\in[0,T]}$, which means that if $(\overline{y}_t,\overline{z}_t)_{u\in[0,T]}$ is any L^p solution to BSDE (ξ,T,g), then for each $t\in[0,T]$,* $\mathrm{d}P$*-a.s., $y_t\leq\overline{y}_t$.*

Proof Let g_n be defined as in Proposition 5.1. In view of Proposition 5.1(i), for each $n\geq 1$ we have

$$\mathbb{E}\Big[\Big(\int_0^T |g_n(s,0,0)|\mathrm{d}s\Big)^p\Big] \leq \mathbb{E}\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big] < +\infty.$$

In view of Proposition 5.1(iii) and (H5), it follows from Theorem 3.1 that for each $n\geq 1$, BSDE (ξ,T,g_n) and BSDE (ξ,T,h) admit unique L^p solutions $(y_t^n,z_t^n)_{t\in[0,T]}$ and $(Y_t,Z_t)_{t\in[0,T]}$ respectively, where $h(\omega,t,y,z) := f_t(\omega)+u_t(\omega)|y|+v_t(\omega)|z|$ for each (ω,t,y,z). And in view of Proposition 5.1(ii), Corollary 4.1 yields that for each $n\geq 1$ and $t\in[0,T]$, $y_t^1(\omega)\leq y_t^n(\omega)\leq y_t^{n+1}(\omega)\leq Y_t(\omega)$, $\mathrm{d}P$-a.s.. Thus, there must exist an $(\mathcal{F}_t)$-progressively measurable process $(y_t)_{t\in[0,T]}$ satisfying that for each $t\in[0,T]$,

$$\lim_{n\to+\infty} y_t^n(\omega) = y_t(\omega), \quad \mathrm{d}P\text{-a.s.},$$

and for each $n\geq 1$,

$$|y_t^n(\omega)| \leq |y_t^1(\omega)| + |Y_t(\omega)|, \quad \mathrm{d}P\text{-a.s.}. \tag{5.1}$$

Now, let $G(\omega) = \sup_{t\in[0,T]}(|y_t^1(\omega)|+|Y_t(\omega)|)$. We have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|y_t|^p\Big] \leq \mathbb{E}[G^p] < +\infty. \tag{5.2}$$

Furthermore, it follows form (2.3) of Lemma 2.1 together with (5.1)–(5.2) that there exists a constant $C_p>0$ depending only on p such that for each $n\geq 1$,

$$\mathbb{E}\Big[\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \leq C_p\mathbb{E}\Big[|\xi|^p + \Big(\int_0^T (|y_s^n||g_n(s,y_s^n,z_s^n)|)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + C_p\mathbb{E}[G^p]. \tag{5.3}$$

On the other hand, in view of Proposition 5.1(i) and by inequalities $(a+b+c)^p \le 3^p(a^p+b^p+c^p)$, $ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ and Hölder's inequality we can deduce that for each $n \ge 1$ and $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T (|y_s^n||g_n(s,y_s^n,z_s^n)|)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\le 3^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_0^T (|y_s^n|f_s)\mathrm{d}s\Big)^{\frac{p}{2}} + \Big(\int_0^T (|y_s^n|^2 u_s)\mathrm{d}s\Big)^{\frac{p}{2}} + \Big(\int_0^T (|y_s^n||z_s^n|v_s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\le 3^{\frac{p}{2}}\Big\{\mathbb{E}\Big[\sup_{s\in[0,T]}|y_s^n|^p\Big] + \frac{1}{2}\mathbb{E}\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big] + \frac{1}{2}M^p + \Big(\frac{1}{\varepsilon}\Big)^{\frac{p}{2}} M^{\frac{p}{2}}\mathbb{E}\Big[\sup_{s\in[0,T]}|y_s^n|^p\Big]\Big\} \\
&\quad + (3\varepsilon)^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]. \qquad (5.4)
\end{aligned}$$

Now, by choosing $\varepsilon > 0$ such that $C_p(3\varepsilon)^{\frac{p}{2}} = \frac{1}{2}$, it follows from (5.3)–(5.4) together with (5.1)–(5.2) that

$$\sup_{n\ge1}\|z_\cdot^n\|_{M^p}^p = \sup_{n\ge1}\mathbb{E}\Big[\Big(\int_0^T |z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big] < +\infty. \qquad (5.5)$$

Next, we will show that $(y_t^n)_{t\in[0,T]}$ is a Cauchy sequence in space $S^p(0,T;\mathbb{R})$. Note that for each $m,n \ge 1$, $(y_\cdot^m - y_\cdot^n, z_\cdot^m - z_\cdot^n)$ satisfies the following equation:

$$y_t^m - y_t^n = \int_t^T [g_m(s,y_s^m,z_s^m) - g_n(s,y_s^n,z_s^n)]\mathrm{d}s - \int_t^T (z_s^m - z_s^n)\cdot \mathrm{d}B_s, \quad t\in[0,T].$$

In view of (H5) and (2.2) of Lemma 2.1, we obtain the existence of a constant c_p such that

$$\begin{aligned}
\|y_\cdot^m - y_\cdot^n\|_{S^p}^p &\le 2c_p\mathbb{E}\Big[\int_0^T [|y_s^m - y_s^n|^{p-1}f_s]\mathrm{d}s\Big] + c_p\mathbb{E}\Big[\int_0^T [|y_s^m - y_s^n|^{p-1}u_s(|y_s^m| + |y_s^n|)]\mathrm{d}s\Big] \\
&\quad + c_p\mathbb{E}\Big[\int_0^T [|y_s^m - y_s^n|^{p-1}v_s(|z_s^m| + |z_s^n|)]\mathrm{d}s\Big]. \qquad (5.6)
\end{aligned}$$

We can prove that the three terms of right-hand side of the previous inequality tend to zero as $m,n\to\infty$ respectively. Indeed, by (H5), Hölder's inequality and (5.2), note that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T (G^{p-1}f_s)\mathrm{d}s\Big] = \mathbb{E}\Big[G^{p-1}\int_0^T f_s\mathrm{d}s\Big] \le (\mathbb{E}[G^p])^{\frac{p-1}{p}}\Big(\mathbb{E}\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]\Big)^{\frac{1}{p}} < +\infty, \\
&\mathbb{E}\Big[\Big(\int_0^T (G^{p-1}u_s)\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big] = \mathbb{E}\Big[G^p\Big(\int_0^T u_s\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big] \le \mathbb{E}[G^p]M^{\frac{p}{p-1}} < +\infty, \\
&\mathbb{E}\Big[\Big(\int_0^T (G^{2p-2}v_s^2)\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big] = \mathbb{E}\Big[G^p\Big(\int_0^T v_s^2\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big] \le \mathbb{E}[G^p]M^{\frac{p}{2p-2}} < +\infty.
\end{aligned}$$

Since for each $m,n \ge 1$ and $s\in[0,T]$, $\mathrm{d}P$-a.s., $|y_s^m(\omega) - y_s^n(\omega)|^{p-1} \le 2^{p-1}G^{p-1}(\omega)$, and $\mathrm{d}P\times\mathrm{d}t$-a.e., $y_\cdot^n \to y_\cdot$ as $n\to+\infty$, by Lebesgue's dominated convergence theorem we deduce that as $m,n\to\infty$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T (|y_s^m - y_s^n|^{p-1}f_s)\mathrm{d}s\Big] \to 0, \\
&\mathbb{E}\Big[\Big(\int_0^T (|y_s^m - y_s^n|^{p-1}u_s)\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big] \to 0, \qquad (5.7) \\
&\mathbb{E}\Big[\Big(\int_0^T (|y_s^m - y_s^n|^{2p-2}v_s^2)\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big] \to 0.
\end{aligned}$$

Thus, in view of (5.1)–(5.2), (5.5) and (5.7), it follows from Hölder's inequality that as $m, n \to \infty$,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T [|y_s^m - y_s^n|^{p-1} u_s(|y_s^m| + |y_s^n|)]\mathrm{d}s\Big]\\ &\le 2(\mathbb{E}[G^p])^{\frac{1}{p}}\Big(\mathbb{E}\Big[\Big(\int_0^T (|y_s^m - y_s^n|^{p-1} u_s)\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big]\Big)^{\frac{p-1}{p}} \to 0 \end{aligned}\tag{5.8}$$

and

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T [|y_s^m - y_s^n|^{p-1} v_s(|z_s^m| + |z_s^n|)]\mathrm{d}s\Big]\\ &\le \mathbb{E}\Big[\Big(\int_0^T (|y_s^m - y_s^n|^{2p-2} v_s^2)\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^T (|z_s^m| + |z_s^n|)^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\ &\le \mathbb{E}\Big[\Big(\int_0^T (|y_s^m - y_s^n|^{2p-2} v_s^2)\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big]^{\frac{p-1}{p}}\mathbb{E}\Big[\Big(\int_0^T (|z_s^m| + |z_s^n|)^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \to 0. \end{aligned}\tag{5.9}$$

Consequently, combining (5.6)–(5.9) yields that

$$\lim_{n\to\infty} \|y_\cdot^n - y_\cdot\|_{S^p} = 0. \tag{5.10}$$

Furthermore, we prove that $(z_t^n)_{t\in[0,T]}$ is a Cauchy sequence in space $M^p(0,T;\mathbb{R}^d)$. In fact, by (2.3) of Lemma 2.1 we get the existence of a constant $\overline{C}_p$ depending only on p such that for each $m, n \ge 1$,

$$\begin{aligned}\|z_\cdot^m - z_\cdot^n\|_{M^p}^p \le{}& \overline{C}_p\mathbb{E}\Big[\Big(\int_0^T [|y_s^m - y_s^n||g_m(s, y_s^m, z_s^m) - g_n(s, y_s^n, z_s^n)|]\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ &+ \overline{C}_p\|y_\cdot^m - y_\cdot^n\|_{S^p}^p. \end{aligned}\tag{5.11}$$

On the other hand, it follows from (H5), inequality $(a+b+c)^p \le 3^p(a^p + b^p + c^p)$ and Hölder's inequality that

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_0^T [|y_s^m - y_s^n||g_m(s, y_s^m, z_s^m) - g_n(s, y_s^n, z_s^n)|]\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ &\le \mathbb{E}\Big[\Big(\int_0^T |y_s^m - y_s^n|(2f_s + u_s(|y_s^m| + |y_s^n|) + v_s(|z_s^m| + |z_s^n|))\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ &\le 3^{\frac{p}{2}}\|y_\cdot^m - y_\cdot^n\|_{S^p}^{\frac{p}{2}}\cdot\Big\{2^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]^{\frac{1}{2}} + 2^{\frac{p}{2}}\mathbb{E}[G^p]^{\frac{1}{2}}\cdot M^{\frac{p}{2}}\Big\}\\ &\quad + 3^{\frac{p}{2}}\|y_\cdot^m - y_\cdot^n\|_{S^p}^{\frac{p}{2}}\cdot\mathbb{E}\Big[\Big(\int_0^T (|z_s^m| + |z_s^n|)^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\cdot M^{\frac{p}{4}}. \end{aligned}\tag{5.12}$$

Thus, combining (5.5) and (5.10)–(5.12) yields the existence of a process $z_\cdot \in M^p(0,T;\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|z_\cdot^n - z_\cdot\|_{M^p} = 0. \tag{5.13}$$

Now, we can choose a subsequence of $\{z_\cdot^n\}$, still denoted by itself, such that $\|z_\cdot^n - z_\cdot\|_{M^p} \le \frac{1}{2^n}$

for each $n \geq 1$. Then

$$\Big\|\sup_n |z^n_\cdot|\Big\|_{M^p} \leq \Big\|\sup_n |z^n_\cdot - z_\cdot|\Big\|_{M^p} + \||z_\cdot\||_{M^p} \leq \Big\|\sum_{n=1}^{+\infty} |z^n_\cdot - z_\cdot|\Big\| + \||z_\cdot\||_{M^p}$$
$$\leq \sum_{n=1}^{+\infty} \||z^n_\cdot - z_\cdot\||_{M^p} + \||z_\cdot\||_{M^p} \leq 1 + \||z_\cdot\||_{M^p} < +\infty. \tag{5.14}$$

Denote $H_t(\omega) := f_t(\omega) + u_t(\omega)G(\omega) + v_t(\omega)\sup_n |z^n_t(\omega)|$. By (H5), (5.1)–(5.2) and Proposition 5.1(i) we know that for each $n \geq 1$, $\mathrm{d}P \times \mathrm{d}t$-a.e.,

$$|g_n(t, y^n_t, z^n_t) - g(t, y, z)| \leq 2H_t. \tag{5.15}$$

And, by Hölder's inequality together with (5.2) and (5.14) we have

$$\mathbb{E}\Big[\Big(\int_0^T |H_s|\mathrm{d}s\Big)^p\Big] \leq 3^p\mathbb{E}\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big] + 3^p\mathbb{E}[G^p]M^p$$
$$+ 3^p\mathbb{E}\Big[\Big(\int_0^T \sup_{n\geq 1} |z^n_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]M^{\frac{p}{2}} < +\infty. \tag{5.16}$$

On the other hand, in view of (5.10), (5.13) and Proposition 5.1(iv), we can assume that, choosing a subsequence if necessary, as $n \to \infty$,

$$g_n(t, y^n_t, z^n_t) \to g(t, y_t, z_t), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.e.} \tag{5.17}$$

Thus, by (5.15)–(5.17), it follows from Lesbesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big(\int_0^T |g_n(s, y^n_s, z^n_s) - g(s, y_s, z_s)|\mathrm{d}s\Big)^p\Big] = 0.$$

Finally, taking limits in BSDE (ξ, T, g_n) yields that $(y_t, z_t)_{t\in[0,T]}$ is an L^p solution of BSDE (ξ, T, g).

It remains to prove that $(y_\cdot, z_\cdot)$ is the minimal L^p solution of BSDE (ξ, T, g). Let $(\widehat{y}_t, \widehat{z}_t)_{t\in[0,T]}$ be any L^p solution of BSDE (ξ, T, g). In view of Proposition 5.1(ii)–(iii), by Corollary 4.1 we obtain that $\mathrm{d}P$-a.s., $y^n_t \leq \widehat{y}_t$ for each $t \in [0, T]$ and $n \geq 1$, from which and by letting $n \to \infty$ we get that for each $t \in [0, T]$, $\mathrm{d}P$-a.s., $y_t \leq \widehat{y}_t$. The proof of Theorem 5.1 is then complete.

Remark 5.1 In the same way as the proof of Theorem 5.1, we can prove the existence of the maximal L^p $(p > 1)$ solution of BSDE (1.1) under the assumptions (H5) and (H6).

By Theorem 4.1 and the proof of Theorem 5.1, we can easily get the following comparison theorem on the minimal (respectively, maximal) L^p solutions of BSDEs.

Theorem 5.2 *Assume that $p > 1$, $0 < T \leq +\infty$, $\xi, \xi' \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R})$, and both generators g and g' satisfy* (H5)–(H6). *Let $(y_\cdot, z_\cdot)$ and $(y'_\cdot, z'_\cdot)$ be, respectively, the minimal* (*respectively, maximal*) *L^p solution of BSDE (ξ, T, g) and BSDE (ξ', T, g')* (*recalling Theorem* 5.1 *and Remark* 5.1). *If $\mathrm{d}P$-a.s., $\xi \leq \xi'$ and $\mathrm{d}P \times \mathrm{d}t$-a.e., $g(\omega, t, y, z) \leq g'(\omega, t, y, z)$ for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, then for each $t \in [0, T]$,*

$$\mathrm{d}P\text{-a.s.}, \quad y_t \leq y'_t.$$

By Theorem 5.1 and Theorem 4.1, the following Theorem 5.3 follows immediately, which generalizes Theorem 3.1 in Section 3.

Theorem 5.3 *Assume that* $p > 1$, $0 < T \le +\infty$, *and the generator* g *satisfies the assumption* (H2) *and the following assumption* (H7):

(H7) g *is Lipschitz continuous in* y *and uniformly continuous in* z *non-uniformly with respect to both* t *and* ω, *i.e., there exists a linear-growth function* $\phi(\cdot) \in \mathbf{S}$ *and two* $(\mathcal{F}_t)$*-progressively measurable nonnegative processes* $\{u_t(\omega)\}_{t\in[0,T]}$ *and* $\{v_t(\omega)\}_{t\in[0,T]}$ *satisfying*

$$\int_0^T [u_t(\omega) + v_t^2(\omega)]\mathrm{d}t \le M, \quad \mathrm{d}P\text{-}a.s.$$

for some constant $M > 0$ *such that* $\mathrm{d}P \times \mathrm{d}t$*-a.e., for each* $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \le u_t(\omega)|y_1 - y_2| + v_t(\omega)\phi(|z_1 - z_2|).$$

Then for each $\xi \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R})$, *BSDE* (ξ, T, g) *admits a unique* L^p *solution.*

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