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A Note on Gradient Estimates for Elliptic Equations with Discontinuous Coefficients*

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Abstract The authors will use a method in metric geometry to show an L^p-estimate for gradient of the weak solutions to elliptic equations with discontinuous coefficients, even the BMO semi-norms of the coefficients are not small. They also extend them to the weak solutions to parabolic equations.

Keywords L^p-Estimate, Elliptic equation, Discontinuous coefficients
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1 Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. We consider the elliptic equations in divergence form

$$Lu := \sum_{ij} \partial_i(a_{ij}(x)\partial_j u(x)) = \operatorname{div} F(x) = \sum_i \partial_i f_i, \tag{1.1}$$

where the matrix of coefficients $A(x) = (a_{ij}(x))_{n\times n}$ satisfies $a_{ij}(x) = a_{ji}(x)$, that each a_{ij} is measurable, and uniform ellipticity on Ω:

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{1.2}$$

for some $0 < \lambda \leq \Lambda < +\infty$. We say that $u \in W^{1,2}_{\rm loc}(\Omega)$ is a weak solution to (1.1) if

$$-\int_\Omega \sum_{ij} a_{ij}(x)\partial_j u(x)\partial_i\varphi(x)\mathrm{d}x = \int_\Omega F(x)\cdot\nabla\varphi(x)\mathrm{d}x \tag{1.3}$$

for all $\varphi \in \mathrm{Lip}_0(\Omega)$, the space of Lipschitz functions on Ω with compact support.

In [9], a point-wise Lipschitz estimate has been given. In this note, we are concerned with the L^p-estimate for the gradient of weak solutions to (1.1) in the form

$$\int_{B_r(x)} |\nabla u|^p \mathrm{d}x \leq C\int_{B_{2r}(x)} (|u|^p + |F|^p)\mathrm{d}x \quad \text{for all } B_{2r} \subset\subset \Omega. \tag{1.4}$$

This estimate (1.4) with constant coefficients was first proved by Calderón-Zygmund. It was extended to the case where the coefficients are continuous by De Giorgi and Companatto, the case where the coefficients are in VMO space (hence, may be discontinuous) (see [4]), and the

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case where the coefficients have small BMO semi-norms (see [5]). Let us recall the concept of BMO semi-norms of matrix A,

$$\|A\|_{BMO(\Omega)} := \sup_{x\in\Omega}\sup_{0<r<R} \fint_{B_r(x)} |A-\overline{A}_r|^2 \mathrm{d}x, \tag{1.5}$$

where

$$\overline{A}_r = \fint_{B_r(x)} A(x)\mathrm{d}x.$$

Theorem 1.1 (see [5, Theorem 1.5]) *Let Ω be an open subset of $\mathbb{R}^n$, $n\geq 2$ and $2\leq p<\infty$. There exists a number $\delta=\delta(p,\lambda,\Lambda,n)>0$ such that for all A which is uniformly elliptic on Ω with elliptic constants λ,Λ (see (1.2)) and if u is a weak solution to (1.1) such that $\|A\|_{BMO(\Omega)}\leq\delta$ and $F\in L^p(\Omega,\mathbb{R}^n)$, then $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ and (1.4) holds on any ball $B_r(x)$ with $B_{2r}(x)\subset\Omega$, where the constant C is independent of u and F.*

Before stating our main result, we recall the notation of semi-convex functions.

Definition 1.1 *Let Ω be an open subset of $\mathbb{R}^n$ and $a\in\mathbb{R}$, we say that a function f is a-convex on Ω if $f(x)-\frac{a}{2}\|x\|^2$ is convex on Ω.*

Remark 1.1 (1) If f is a-convex for some $a\in\mathbb{R}$, then $f\in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.
(2) If $f\in C^2$, then f is a-convex if and only if $\mathrm{Hess}\, f\geq a\cdot I$, where I is the identity matrix.

The main result of the note is the following.

Theorem 1.2 *Let Ω be an open subset of $\mathbb{R}^n$, $a\in\mathbb{R}$ and $2\leq p<\infty$. There exists a number $\delta=\delta(p,n,\lambda,\Lambda,a,M)>0$ such that for all A which is uniformly elliptic on Ω with elliptic constants λ,Λ and $F\in L^p(\Omega,\mathbb{R}^n)$, the following property holds*:

If $u\in W^{1,2}(\Omega)$ is a weak solution to (1.1) and if there exists an a-convex function Φ on Ω with $|\nabla\Phi|+|\nabla|\nabla\Phi||\leq M$ on Ω such that

$$\|A-A_0\|_{L^\infty(\Omega)}\leq\delta, \tag{1.6}$$

then $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ and the estimate (1.4) holds for all ball $B_r(x)$ with $B_{2r}(x)\subset\Omega$, where $A_0(x)=(a^0_{ij}(x))_{n\times n}$, is defined by

$$a^0_{ij}(x)=(1+|\nabla\Phi(x)|^2)^{\frac12}\Big(\delta_{ij}-\frac{\partial_i\Phi(x)\partial_j\Phi(x)}{1+|\nabla\Phi(x)|^2}\Big)\quad \text{a.e. } x\in\Omega. \tag{1.7}$$

We will extend this result to parabolic equations in Section 5.

To compare with Byun-Wang's result in [5], we consider the following example, the matrix of coefficients A_0 in Theorem 1.2 has no a small BMO semi-norm.

Example 1.1 Let $a_1,\cdots,a_n\in(0,+\infty)$ and $\Omega=B_1(0)$ and $\Phi(x)=\big(\sum\limits_i a_ix_i^2\big)^{\frac12}$ on $B_1(0)$. Then

$$\partial_i\Phi=\frac{a_ix_i}{\Phi}\quad \text{a.e. in } B_1(0) \tag{1.8}$$

and

$$a^0_{ij}(x)=\Big(1+\frac{\sum\limits_i a_i^2x_i^2}{\sum\limits_i a_ix_i^2}\Big)^{\frac12}\Big(\delta_{ij}-\frac{a_ia_jx_ix_j}{\big(1+\frac{\sum_i a_i^2x_i^2}{\sum_i a_ix_i^2}\big)\cdot\sum\limits_i a_ix_i^2}\Big). \tag{1.9}$$

In particular, when we take $a_1 = \cdots = a_n = c > 0$, then

$$a_{ij}^0(x) = \sqrt{1+c}\Big(\delta_{ij} - \frac{c}{1+c}\frac{x_i}{|x|}\frac{x_j}{|x|}\Big). \tag{1.10}$$

It is clear that a_{ij}^0 is not continuous at $x = 0$. Now we check that its BMO semi-norm is not small.

For each $i = 1, \cdots, n$, we have

$$a_{ii}^0(x) = \sqrt{1+c}(1 - \frac{c}{1+c}\frac{x_i^2}{|x|^2}),$$

then

$$\fint_{B_r(0)} a_{ii}^0(x)\mathrm{d}x = \sqrt{1+c}\fint_{B_r(0)} 1 - \frac{c}{1+c}\frac{x_i^2}{|x|^2}\mathrm{d}x = \sqrt{1+c}\cdot\Big(1 - \frac{c}{1+c}\frac{1}{n}\Big),$$

it implies

$$\Big|a_{ii}^0(x) - \fint_{B_r(0)} a_{ii}^0(x)\mathrm{d}x\Big| = \sqrt{1+c}\frac{c}{1+c}\Big|\frac{x_i^2}{|x|^2} - \frac{1}{n}\Big|.$$

Therefore,

$$\begin{aligned}
&\fint_{B_r(0)} \Big|a_{ii}^0(x) - \fint_{B_r(0)} a_{ii}^0(x)\Big|\mathrm{d}x \\
=&\sqrt{1+c}\frac{c}{1+c}\fint_{B_r(0)}\Big|\frac{x_i^2}{|x|^2} - \frac{1}{n}\Big|\mathrm{d}x \\
\geq&\sqrt{1+c}\frac{c}{1+c}\frac{1}{|B_r(0)|}\int_{\{|x_i|<\frac{|x|}{\sqrt{2n}}\}\cap B_r(0)}\Big|\frac{x_i^2}{|x|^2} - \frac{1}{n}\Big|\mathrm{d}x \\
\geq&\sqrt{1+c}\frac{c}{1+c}\frac{1}{2n}\frac{\big|\{|x_i| < \frac{|x|}{\sqrt{2n}}\}\cap B_r(0)\big|}{|B_r(0)|},
\end{aligned}$$

where we denote by $|A|$ the Lebegue's measure of $A \subset \mathbb{R}^n$. If we take the spherical coordinate $x_i = \rho\sin\theta$ with $\rho < r$ and $\theta \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. Then the BMO semi-norm of A_0 has a lower bound

$$\sqrt{1+c}\frac{c}{1+c}\frac{1}{2n}\frac{\big|\big\{\sin^2\theta < \frac{1}{2n}\big\}\cap B_1(0)\big|}{|B_1(0)|} > 0,$$

which goes to $+\infty$ as $c \to +\infty$.

2 Preliminaries on Alexandrov Spaces

Let (X, d) be a locally compact complete metric space. A curve $\gamma : [a, b] \to X$ is rectifiable if

$$L(\gamma) := \sup_{a=a_0\leq a_1\leq\cdots\leq a_n=b}\sum_{i=0}^{n-1} d(\gamma(a_i), \gamma(a_{i+1})) < +\infty,$$

where $a = a_0 \leq a_1 \leq \cdots \leq a_n = b$ is a partition of $[a, b]$.

Definition 2.1 *Let $k \in \mathbb{R}$. The metric space (X, d) is called to be an Alexandrov space with curvature $\geq k$ if it satisfies the following conditions*:

(1) *For any two points $p, q \in X$, there exists a curve γ joining p and q with $L(\gamma) = d(p, q)$, such a curve is called a geodesic*;

(2) *for any $p \in X$, there exists a neighborhood U of p such that if $x, y, z \in U \backslash \{p\}$, then*

$$\measuredangle_k xpy + \measuredangle_k ypz + \measuredangle_k zpx \leq 2\pi,$$

where, if $k > 0$,

$$\measuredangle_k xpy = \arccos\Big(\frac{\cos(d(x,y)\sqrt{k}) - \cos(d(x,p)\sqrt{k})\cos(d(p,y)\sqrt{k})}{\sin(d(x,p)\sqrt{k})\sin(d(p,y)\sqrt{k})}\Big)$$

(*with appropriate modification if $k \leq 0$*). *This makes sense if $k[d(p,x)+d(x,y)+d(y,p)]^2 \leq (2\pi)^2$. Otherwise, we put $\measuredangle_k xpy = -\infty$.*

It is well-known (see [2, Theorem 10.8.2, 3]) that the Hausdorff dimension of an Alexandrov space (X, d) is a nonnegative integer or $+\infty$. If the Hausdorff dimension $\dim_{\mathcal{H}}(X) = n < \infty$, we say that X is a n-dimensional Alexandrov space. And we denote its n-dimensional Hausdorff measure by $\mu = \mathcal{H}^n$. We refer the readers to [2–3] for the basic theory of Alexandrov geometry.

Let (X, d) be a n-dimensional Alexandrov space with curvature bounded below by k, and $n \geq 2$. There exists a decomposition (see [3, 11]):

$$X = X^* \cup \Sigma, \quad X^* \cap \Sigma = \emptyset,$$

where X^* is a convex open subset which is a Lipschitz manifold, and $\mu(\Sigma) = 0$. Moreover, there exists a $L^\infty(X^*)$-Riemannian metric $g = (g_{ij})_{n\times n}$ on X^* such that the distance d_g induced by g coincides with the original metric d, and μ is equal to the Riemannian volume, that is,

$$\mu = \sqrt{G}\mathrm{d}x_1 \cdots \mathrm{d}x_n, \quad G := \det(g_{ij}) \tag{2.1}$$

under a local coordinate system.

The Sobolev spaces $W^{1,2}(X, d, \mu)$ for the metric measure spaces (X, d, μ) were given in [1, 7]. Let $\Omega \subset X$ be a bounded domain, we say that a function $f \in W^{1,2}_{\mathrm{loc}}(\Omega, d, \mu)$ if $f \in W^{1,2}(\Omega', d, \mu)$ for any open subset $\Omega' \subset\subset \Omega$. It is well-known that, for any $f, h \in W^{1,2}(\Omega, d, \mu)$,

$$\langle \nabla_g f, \nabla_g h\rangle = g^{ij}\partial_i f \cdot \partial_j h \quad \mu-\text{a.e. } x \in \Omega, \tag{2.2}$$

where the matrix $(g^{ij})_{n\times n}$ is the inverse matrix of $(g_{ij})_{n\times n}$ and $\partial_i f$ is the weak derivative with respect to a local coordinate system $\{x_1, \cdots, x_n\}$.

Definition 2.2 (Measure-valued Laplacian) *Let (X, d) be a n-dimensional Alexandrov space with curvature bounded below, and let $\Omega \subset X$ be a bounded open subset. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, d, \mu)$. If there exists a Radon measure ν on Ω such that*

$$\int_\Omega \varphi \mathrm{d}\nu = -\int_\Omega \langle \nabla_g u, \nabla_g \varphi\rangle \mathrm{d}\mu, \quad \forall \varphi \in \mathrm{Lip}_0(\Omega), \tag{2.3}$$

then ν is unique, such a measure ν is called the measured Laplacian of f and is denoted by $\nu = \Delta u$.

The following Bochner formula is proved in [13, Theorem 1.2] and [14, Theorem 3.5].

Proposition 2.1 (Bochner formula) *Let (X, d) be a n-dimensional Alexandrov space with curvature bounded below by $k \in \mathbb{R}$, and let $\Omega \subset X$ be a bounded open subset. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega, d, \mu)$ and $\Delta u = f \cdot \mu$ with $f \in W^{1,2}(\Omega, d, \mu)$. Then*

$$|\nabla_g u|^2 \in L^\infty_{\mathrm{loc}}(\Omega, d, \mu) \cap W^{1,2}_{\mathrm{loc}}(\Omega, d, \mu)$$

and

$$-\int_\Omega \langle \nabla_g |\nabla_g u|^2, \nabla_g \varphi \rangle \mathrm{d}\mu \geq 2 \int_\Omega \Big(\frac{f^2}{n} + \langle \nabla_g u, \nabla_g f \rangle + K|\nabla_g u|^2 \Big) \varphi \mathrm{d}\mu \tag{2.4}$$

for all $\varphi \in \mathrm{Lip}_0(\Omega)$, $\varphi \geq 0$.

3 The Model Elliptic Operator

Let $n \geq 2$, $a \in \mathbb{R}$, $\Omega := B_2(0) \subset \mathbb{R}^n$, and let $\Phi : B_2(0) \to \mathbb{R}$ be an a-convex function on $B_2(0)$. Define its graph by

$$X := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \Omega,\ x_{n+1} = \Phi(x)\}. \tag{3.1}$$

We define the natural coordinate by $\Psi : \Omega \to X$ by

$$\Psi : (x_1, \cdots, x_n) \mapsto \sum_{i=1}^n x_i A_i + \Phi(x_1, \cdots, x_n) A_{n+1}, \tag{3.2}$$

where $A_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$ (1 in the i-th place). The induced Riemannian metric $g = (g_{ij})_{n\times n}$ on Ω is given by

$$g_{ij} = g(e_i, e_j) = \widehat{g}(A_i + \partial_i \Phi A_{n+1}, A_j + \partial_j \Phi A_{n+1}) = \delta_{ij} + \partial_i \Phi \partial_j \Phi, \tag{3.3}$$

where

$$e_i = \Phi^*(A_i + \partial_i \Phi A)$$

and $\widehat{g}$ is the induced Riemannian metric on graph X. Then the inverse matrix of g, denoted by $(g^{ij})_{n\times n}$, is given by

$$g^{ij} = \delta_{ij} - \frac{\partial_i \Phi \partial_j \Phi}{1 + |\nabla \Phi|^2}. \tag{3.4}$$

The normal vector is

$$\eta = \sum_{i=1}^n -\partial_i \Phi A_i + A_{n+1},$$

then the second fundamental form is

$$h_{ij} = \frac{\Phi_{ij}}{1 + |\nabla \Phi|^2}.$$

Lemma 3.1 *The metric space (Ω, d_g) is a n-dimensional Alexandrov space with curvature bounded below by $-C(n, a)$ for some constant $C(n, a) \geq 0$.*

Proof **Step 1** If Φ is C^3, then we can obtain the result by using the Gauss equation

$$R(e_i \wedge e_j, e_i \wedge e_j) = h_{ij} h_{ji} - h_{ii} h_{jj}.$$

Step 2 If Φ is a general a-convex function, then $\Phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. It is well-known that (see [12]) there is a sequence of a-convex functions $\Phi_j \in C^3(\Omega)$ such that Φ_j converges uniformly to Φ on each $\Omega' \subset\subset \Omega$ (see [12]). Therefore, for any $\Omega' \subset\subset \Omega$, the graph of Φ_j with metric $d_{\widehat{g_j}}$ Hausdorff converges to $(X, d_{\widehat{g}})$ on Ω'. By [2, Proposition 10.7.1], the limit space $(X, d_{\widehat{g}})$ is also a n-dimensional Alexandrov space with curvature bounded below by $-C(n, a)$.

Now let us consider the model operators. We set $A_0(x) = (a^0_{ij}(x))_{n\times n}$ as

$$a^0_{ij}(x) := (1 + |\nabla\Phi(x)|^2)^{\frac{1}{2}}\Big(\delta_{ij} - \frac{\partial_i\Phi(x)\partial_j\Phi(x)}{1 + |\nabla\Phi(x)|^2}\Big) \quad \text{a.e. } x \in B_2(0). \tag{3.5}$$

By a direct calculation, the set of its all eigenvalues is

$$(1 + |\nabla\Phi(x)|^2)^{\frac{1}{2}}\Big\{1, 1, \cdots, 1, \frac{1}{1 + |\nabla\Phi|^2}\Big\}. \tag{3.6}$$

Therefore, it is clear that the model elliptic operator

$$L^0u := \sum_{ij}\partial_i(a^0_{ij}\partial_j u) \tag{3.7}$$

is uniformly elliptic on each $\Omega' \subset\subset \Omega$. In fact, since $\Phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, there is $M > 0$ such that $\sup_{\Omega'}|\nabla\Phi| \le M$. This yields that all eigenvalues of $A_0(x)$ lie in $[(1 + M^2)^{-\frac{1}{2}}, (1 + M^2)^{\frac{1}{2}}]$ for all $x \in \Omega'$.

The relation between the operator L^0 and Δ on the Alexandrov space (Ω, d_g) is given in the following property.

Lemma 3.2 *A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (with respect to the Euclidean distance and Lebegue's measure $\mathcal{L}^n$) if and only if $u \in W^{1,2}_{\mathrm{loc}}(\Omega, d_g, \mu)$. Moreover, if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ then $\Delta u = L^0u \cdot \mathcal{L}^n$.*

Proof Since $\Phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, for each $\Omega' \subset\subset \Omega$, we have $|\nabla\Phi| \le M$ for all $x \in \Omega'$. This implies the coordinate map Ψ in (3.2) is bi-Lipschitz with

$$1 \le \frac{d_g(\Psi(x), \Psi(y))}{|x - y|} \le (1 + M^2)^{\frac{1}{2}}, \quad \forall\, x, y \in \Omega.$$

Therefore, we conclude that $u \in W^{1,2}(\Omega')$ if and only if $u \in W^{1,2}(\Omega', d_g, \mu)$. Moreover, for any $u \in\in W^{1,2}(\Omega')$

$$1 \le \frac{|\nabla_g u|}{|\nabla u|} \le (1 + M^2)^{\frac{1}{2}}, \quad \forall\, x \in \Omega. \tag{3.8}$$

For any $\varphi \in \mathrm{Lip}_0(\Omega)$, by Definition 2.2 and (2.1), we get

$$\Delta u(\varphi) = -\int_\Omega \langle\nabla_g u, \nabla_g\varphi\rangle \mathrm{d}\mu = -\int_\Omega g^{ij}\cdot\partial_i u\cdot\partial_j\varphi\cdot\sqrt{G}\mathrm{d}x$$
$$= \int_\Omega \partial_j(g^{ij}\sqrt{G}\partial_i u)\varphi\mathrm{d}x = \int_\Omega L^0u\varphi\mathrm{d}x.$$

The proof is finished.

From this lemma, one can obtain a Bochner-type formula for the operator L^0.

Lemma 3.3 *Let $\Omega' \subset\subset \Omega$. Suppose that $|\nabla\Phi| \le M_1$ and $|\nabla|\nabla\Phi|| \le M_2$ on Ω' for some $M_1, M_2 > 0$. If $u \in W^{1,2}(\Omega')$ with $L^0u \in W^{1,2}(\Omega')$, then we have $|\nabla u|^2 \in W^{1,2}_{\mathrm{loc}}(\Omega')$ and*

$$\frac{1}{2}L^0(|\nabla_g u|^2) \ge \langle\nabla_g u, \nabla_g(L^0u)\rangle - C|\nabla_g u|^2 \tag{3.9}$$

in the sense of distributions on Ω', where the constant $C = C_{n,a,M_1,M_2}$ depends only on n, a, M_1 and M_2.

Proof By applying Lemma 3.2 to u and (2.1), we have

$$\Delta u = L^0 u \cdot \mathcal{L}^n = \frac{L^0 u}{\sqrt{G}} \cdot \mu \quad \text{on } \Omega'.$$

From the assumption that

$$|\nabla \Phi| \in \mathrm{Lip}(\Omega')$$

and

$$\sqrt{G} = (1 + |\nabla \Phi|^2)^{\frac{1}{2}} \in [1, 1 + M],$$

we get $\frac{1}{\sqrt{G}} \in \mathrm{Lip}(\Omega')$. Since $L^0 u \in W^{1,2}(\Omega')$, then we have $\frac{L^0 u}{\sqrt{G}} \in W^{1,2}(\Omega')$. By using Proposition 2.1, we conclude that

$$|\nabla_g u|^2 \in W^{1,2}_{\mathrm{loc}}(\Omega', d_g, \mu)$$

and that

$$\frac{1}{2}\Delta(|\nabla_g u|^2) \geq \Big(\frac{\big(\frac{L^0 u}{\sqrt{G}}\big)^2}{n} + \Big\langle \nabla_g u, \nabla_g \Big(\frac{L^0 u}{\sqrt{G}} \Big) \Big\rangle - C(n,a)|\nabla_g u|^2 \Big) \cdot \mu$$

in the sense of distribution.

By applying Lemma 3.2 again to $|\nabla u|^2$, we have $|\nabla_g u|^2 \in W^{1,2}_{\mathrm{loc}}(\Omega')$ and for almost all $x \in \Omega'$,

$$\frac{1}{2} L^0(|\nabla_g u|^2) \geq \Big(\frac{(L^0 u)^2}{n \cdot G} + \Big\langle \nabla_g u, \nabla_g \Big(\frac{L^0 u}{\sqrt{G}} \Big) \Big\rangle - C(n,a)|\nabla_g u|^2 \Big) \sqrt{G}.$$

Therefore, it holds for almost all $x \in \Omega'$ that

$$\frac{1}{2} L^0(|\nabla_g u|^2) \geq \frac{(L^0 u)^2}{n \cdot \sqrt{G}} + \langle \nabla_g u, \nabla_g (L^0 u) \rangle + \Big\langle \nabla_g u, \nabla_g \Big(\frac{1}{\sqrt{G}} \Big) \Big\rangle L^0 u - C(n,a)|\nabla_g u|^2 \sqrt{G}.$$

Notice that for almost all $x \in \Omega$,

$$\begin{aligned} -\Big\langle \nabla_g u, \nabla_g \Big(\frac{1}{\sqrt{G}} \Big) \Big\rangle L^0 u &\leq |\nabla_g u| \cdot |L^0 u| \cdot |\nabla_g G| G^{-\frac{3}{2}} \leq 2M_1 M_2 |\nabla_g u| \cdot |L^0 u| \\ &\leq \frac{(L^0 u)^2}{n\sqrt{G}} + n(M_1 M_2)^2 |\nabla_g u|^2 \sqrt{G}. \end{aligned}$$

Instituting into the above inequality, one get

$$\frac{1}{2} L^0(|\nabla_g u|^2) \geq \langle \nabla_g u, \nabla_g (L^0 u) \rangle - (C(n,a) + n(M_1 M_2)^2)|\nabla_g u|^2 \sqrt{G}.$$

The proof is complete.

4 The Elliptic Case

Let $\Omega := B_2(0)$ be a domain in $\mathbb{R}^n$, $n \geq 2$. We consider the elliptic equations in divergence form

$$Lu := \sum_{ij} \partial_i (a_{ij}(x) \partial_j u(x)) = \operatorname{div} F(x) = \sum_i \partial_i f_i, \tag{4.1}$$

where the matrix of coefficients $A(x) = (a_{ij}(x))_{n \times n}$ satisfies $a_{ij}(x) = a_{ji}(x)$, that each a_{ij} is measurable, and uniform ellipticity on Ω with elliptic constants λ, Λ, and $F \in L^p(\Omega, \mathbb{R}^n)$.

Lemma 4.1 *Let $u \in W^{1,p}(B_2(0))$ be a weak solution to the equation $L^0 u = 0$ where $L^0 u$ defined in (3.7) by some a-convex function Φ on $B_2(0)$ with $|\nabla\Phi| + |\nabla|\nabla\Phi|| \le M$ for some $M > 0$ on $B_2(0)$, then*

$$\|\nabla u\|^2_{L^\infty(B_1(0))} \le C_{n,a,M} \fint_{B_2(0)} |\nabla u|^2 \mathrm{d}x. \tag{4.2}$$

Proof Since $L^0 u = 0$, by Lemma 3.3, we have

$$L^0(|\nabla_g u|^2) \ge -C|\nabla_g u|^2 \tag{4.3}$$

in the sense of distribution; therefore, by the De Giorgi-Nash-Moser theory for elliptic equations (see [8, Theorem 4.1]), we have

$$\|\nabla_g u\|^2_{L^\infty(B_1(0))} \le C \fint_{B_2(0)} |\nabla_g u|^2 \mathrm{d}x. \tag{4.4}$$

It follows the desired estimate by applying (3.8).

Proof of Theorem 1.2 By [6, Theorem A], comparing L with L^0, there exists a number $\delta = \delta(p, n, \lambda, \Lambda, a, M)$ such that if

$$\|a_{ij} - a^0_{ij}\|_{L^\infty(B_1(0))} \le \delta, \tag{4.5}$$

then $u \in W^{1,p}_{\mathrm{loc}}(B_2(0))$. The proof of Therem 1.2 is finished.

5 The Parabolic Case

Let $\Omega = B_2(0)$ be a bounded domain in $\mathbb{R}^n$. In this section, we consider the interior $W^{1,p}$ estimates for weak solution to parabolic equations in divergence form

$$u_t = \mathrm{div}(a(x,t,\nabla u)) + \mathrm{div}\, F, \quad x \in \Omega,\ t > 0, \tag{5.1}$$

where $a : \Omega \times (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $F \in L^p(\Omega \times (0,\infty), \mathbb{R}^n)$. We assume that $a(x,t,\xi)$ is a Caratheodory function (in the sense that is measurable in (x,t) and continuous with respect to ξ for each x) and satisfies the following conditions:

(1) $a(x,t,0) = 0$;

(2) $\langle a(x,t,\xi) - a(x,t,\eta), (\xi - \eta)\rangle \ge \gamma|\xi - \eta|^2$;

(3) $|a(x,t,\xi)| \le \Gamma|\xi|$;

(4) a is linear about ξ;

where γ and Γ are positive constants. If F is a constant, as in [10], the condition (4) can be removed.

Denote the following parabolic rectangles in $\mathbb{R}^{n+1}$:

(1) $R = \{(x,t) \in \mathbb{R}^{n+1} : |x_i| < \sqrt{2}, i = 1, 2, \cdots, n, t_0 - 2 < t < t_0\} \subset B_2(0) \times (0,\infty)$,

(2) $R' = \{(x,t) \in \mathbb{R}^{n+1} : |x_i| < \frac{\sqrt{2}}{2}, i = 1, 2, \cdots, n, t_0 - \frac{1}{2} < t < t_0\} \subset B_1(0) \times (0,\infty)$.

We define the parabolic boundary of R as

$$\partial_p R := \{(x,t) : |x_i| \le \sqrt{2}, t = t_0 - 2\} \cup \{(x,t) : |x_i| = \sqrt{2}, t_0 - 2 < t < t_0\}. \tag{5.2}$$

Definition 5.1 *We say that $u \in L^2(R)$, with $u_t, \nabla u \in L^2(R)$, is a weak solution to (5.1) if*

$$\int_R u\varphi_t \mathrm{d}x\mathrm{d}t = \int_R \langle a(x,t,\nabla u), \nabla\varphi\rangle \mathrm{d}x\mathrm{d}t - \int_R \langle F, \nabla\varphi\rangle \mathrm{d}x\mathrm{d}t \tag{5.3}$$

for all $\varphi \in \mathcal{W}^{1,2}_0(R)$, the compeletion of $C^\infty_0(R)$ with respect to the L^2-norm of the function and its gradient.

The reference equation is

$$u_t = \mathrm{div}(a_0(x,t,\nabla u)) = \mathrm{div}(A_0(\nabla u)), \quad x \in \Omega,\ 0 < t < \infty, \tag{5.4}$$

where $A_0(x,t) = (a_{ij}^0(x,t))_{n\times n}$, and $a_{ij}^0(x,t)$ is defined as in (3.5) for each fixed $t > 0$.

Lemma 5.1 *If w is a weak solution to* (5.4)*, then we have*

$$\|\nabla w\|^2_{L^\infty(R')} \le C \int_R |\nabla w|^2 \mathrm{d}x\mathrm{d}t, \tag{5.5}$$

where C is a constant depending only on n, a and M.

Proof By Lemma 3.3, we have

$$\frac{1}{2} L^0(|\nabla_g w|^2) \cdot \mathcal{L}^n \ge (\langle \nabla_g w, \nabla_g (L^0 w)\rangle - C|\nabla_g w|^2) \cdot \mathcal{L}^n \tag{5.6}$$

in the sense of distribution.

Since w satisfies the reference equation (5.4), we have

$$\Big(L^0 - \frac{\partial}{\partial t}\Big)|\nabla_g w|^2 \cdot \mathcal{L}^n \ge -C|\nabla_g w|^2 \cdot \mathcal{L}^n \tag{5.7}$$

in the sense of distribution. From this and the equivalence between the induced distance d_g and the standard Euclidean distance, we obtain the desired estimates.

Theorem 5.1 *Given $n \ge 2$, $p > 2$, $\Omega = B_2(0)$. There exists a constant $\delta = \delta(n,p,\lambda,\Lambda,a,M)$ such that if $F \in L^p(R;\mathbb{R}^n)$ and if u is a weak solution to* (5.1) *with*

$$|a(x,t,\xi) - A_0(\xi)| \le \delta|\xi| \tag{5.8}$$

uniformly in x and t. Then $|\nabla u| \in L^q_{\mathrm{loc}}(\Omega)$ for all $2 < q < p$.

Proof If F is a constant, by [10, Theorem 1.1], the proof is completed. Otherwise, we only need to modify [10, Lemma 3.3]. Let w be a weak solution to the reference equation (5.4) and satisfy $w|_{\partial_p R} = u$, we have

$$\begin{aligned}
&\gamma \fint_R |\nabla(u-w)|^2 \mathrm{d}x\mathrm{d}t \\
\le& \fint_R \langle A_0(\nabla u) - A_0(\nabla w), \nabla u - \nabla w\rangle \mathrm{d}x\mathrm{d}t \\
\le& \frac{1}{2}\fint_R |u(x,T) - w(x,T)|^2 \mathrm{d}x + \fint_R \langle A_0(\nabla u) - A_0(\nabla w), \nabla u - \nabla w\rangle \mathrm{d}x\mathrm{d}t \\
\le& \fint_R [(u-w)_t - (\mathrm{div}\, A_0(\nabla u) - \mathrm{div}\, A_0(\nabla w))](u-w)\mathrm{d}x\mathrm{d}t \\
=& \fint_R (u_t - \mathrm{div}\, A_0(\nabla u))(u-w)\mathrm{d}x\mathrm{d}t - \fint_R (w_t - \mathrm{div}\, A_0(\nabla w))(u-w)\mathrm{d}x\mathrm{d}t \\
=& \fint_R [u_t - \mathrm{div}(a(x,t,\nabla u))](u-w)\mathrm{d}x\mathrm{d}t \\
&+ \fint_R [\mathrm{div}(a(x,t,\nabla u)) - \mathrm{div}\, A_0(\nabla u)](u-w)\mathrm{d}x\mathrm{d}t \\
=& \fint_R [u_t - \mathrm{div}(a(x,t,\nabla u))](u-w)\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&-\fint_R \langle a(x,t,\nabla u)-A_0(\nabla u),\nabla(u-w)\rangle \mathrm{d}x\mathrm{d}t\\
&=\fint_R \langle F,\nabla(u-w)\rangle \mathrm{d}x\mathrm{d}t-\fint_R \langle a(x,t,\nabla u)-A_0(\nabla u),\nabla(u-w)\rangle \mathrm{d}x\mathrm{d}t\\
&\le \Big(\fint_R |F|^2\mathrm{d}x\mathrm{d}t\Big)^{\frac12}\Big(\fint_R |\nabla(u-w)|^2\mathrm{d}x\mathrm{d}t\Big)^{\frac12}\\
&\quad+\delta\Big(\fint_R |\nabla u|^2\mathrm{d}x dt\Big)^{\frac12}\Big(\fint_R |\nabla(u-w)|^2\mathrm{d}x dt\Big)^{\frac12},
\end{aligned}
\tag{5.9}
$$

where we have used the condition (5.8) and Cauchy-Schwarz inequality for the last inequality. So we obtain

$$
\fint_R |\nabla(u-w)|^2\mathrm{d}x dt\le C\Big(\delta^2\fint_R |\nabla u|^2\mathrm{d}x\mathrm{d}t+\fint_R |F|^2\mathrm{d}x\mathrm{d}t\Big). \tag{5.10}
$$

Since a is linear about ∇u, we can multiply (5.1) by a small constant such that $\|F\|_{L^p(R)}$ is small enough. We complete our proof.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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