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Sequential Propagation of Chaos for Mean-Field BSDE Systems*

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Abstract A new class of backward particle systems with sequential interaction is proposed to approximate the mean-field backward stochastic differential equations. It is proven that the weighted empirical measure of this particle system converges to the law of the McKean-Vlasov system as the number of particles grows. Based on the Wasserstein metric, quantitative propagation of chaos results are obtained for both linear and quadratic growth conditions. Finally, numerical experiments are conducted to validate our theoretical results.

Keywords Backward propagation of chaos, Particle system, Sequential interaction, McKean–Vlasov BSDE, Convergence rate
2000 MR Subject Classification 65C35, 82C22, 60J60, 60B10

1 Introduction

The McKean-Vlasov stochastic differential equations (MV-SDEs for short), also known as mean-field or distribution dependent SDEs, originated from the work of [1] which provided a probabilistic interpretation for nonlinear Vlasov equations. Since then, MV-SDEs have found a wide range of applications in various fields such as finance, control theory and statistical physics (see [2–6] and references therein). The theory of propagation of chaos (PoC for short) was initially introduced by [7] to investigate particle system approximations of non-local partial differential equations (PDEs for short) that arise in thermodynamics. PoC for large interacting particle systems has become a crucial theory in many areas of applied mathematics (see [8–9]). Recent developments and applications of PoC can be found in works such as [6, 10–11]. Furthermore, the numerical simulation of MV-SDE is an important issue, and the PoC property is widely applied in computational problems (see [2, 12–13], etc.).

Since the work of [14], backward stochastic differential equations (BSDEs for short) have been used widely in a variety of areas (see [15–17]). Mean-field backward stochastic differential equations (MF-BSDEs for short), also called McKean-Vlasov BSDEs (MV-BSDEs for short), were introduced by [18–19]. Similar to the forward SDE, we also have backward PoC property

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for McKean-Vlasov BSDEs (see [20]). Since most BSDEs cannot be solved analytically, numerical methods must be applied to approximate their solutions. Various numerical methods have been proposed over the past few decades (see [21–27]). As for numerical methods for MV-BSDEs, a variety of methods can be found in the literature for mean-field games (see [28–31]). Since [32–33] proposed the Deep BSDE method, deep learning based methods have also been used to solve McKean-Vlasov BSDEs (see [34–38]).

Recently, [39] propose a new class of particle systems with sequential interaction to approximate the McKean–Vlasov SDEs. They use a recursive form to compute McKean-Vlasov SDEs particle-by-particle (or batch-by-batch), resulting in a great reduction of the computational burden. Specifically, they derive estimates for recursive inequalities, and get a new estimate for the convergence rate of weighted empirical measures of an i.i.d. sequence in the Wasserstein distance. The results for classical empirical measures can be founded in [40–42] and references therein. More significantly, as more and more particles are added (without affecting existing particles), the approximation accuracy continuously improves until it reaches the desired level. In other words, there is no need to specify the required number of particles in advance.

In this paper, we adopt the approach of [39] to approximate McKean-Vlasov BSDE by a recursive form. We model backward particles as solutions of backward stochastic differential equations with sequential interaction and investigate the asymptotic behavior of the BSDE system. The major feature of the system is that each particle is influenced solely by the particles with smaller ordinal numbers. Specifically, the n-th particle process Y^n $(n \geq 1)$ is recursively determined by

$$\begin{cases} Y_t^n = \xi^n + \displaystyle\int_t^T f_s(Y_s^n, \mu_s^{n-1}, Z_s^n)\,\mathrm{d}s - \int_t^T Z_s^n\,\mathrm{d}W_s^n, \quad t \in [0,T], \\ \mu_t^n = \mu_t^{n-1} + \alpha_n(\delta_{Y_t^n} - \mu_t^{n-1}), \end{cases} \tag{1.1}$$

where $\mu_t^0 \equiv \delta_0$. Here, the update rate sequence $\{\alpha_n\}_{n\geq 1}$ is decreasing and positive, with $\alpha_1 = 1$. The multidimensional Brownian motions W^n are independent and the terminal data Y_T^n are i.i.d. $\mathbb{R}^d$-valued random variables independent of $\{W^n\}_{n\geq 1}$. It is evident that μ_t^n is a weighted empirical measure of $Y_t^1, \cdots, Y_t^n$. If we set $\alpha_n = \frac{1}{n}$, μ_t^n becomes classical empirical measure. As the interaction is asymmetric and heterogeneous, the model (1.1) stands in sharp contrast to the corresponding mean-field interacting system $\{Y^{n,N} : n = 1, \cdots, N\}$ given by

$$\begin{cases} Y_t^{n,N} = \xi^n + \displaystyle\int_t^T f_s(Y_s^{n,N}, \mu_s^N, Z_s^{n,n,N})\,\mathrm{d}s - \sum_{k=1}^N \int_t^T Z_s^{n,k,N}\,\mathrm{d}W_s^k, \quad t \in [0,T], \\ \mu_t^N = \dfrac{1}{N}\displaystyle\sum_{n=1}^N \delta_{Y_t^{n,N}}. \end{cases} \tag{1.2}$$

It is known from the theory of backward propagation of chaos (see [20, 43]) that, as N grows, the empirical measure of the system (1.2) may converge to the law of a McKean–Vlasov BSDE described by

$$\begin{cases} \overline{Y}_t = \xi + \displaystyle\int_t^T f_s(\overline{Y}_s, \mu_s, \overline{Z}_s)\,\mathrm{d}s - \int_t^T \overline{Z}_s\,\mathrm{d}W_s, \quad t \in [0,T], \\ \mu_t = \mathcal{L}(Y_t) := \mathrm{Law}(Y_t). \end{cases} \tag{1.3}$$

This paper establishes a similar property for the system (1.1): As n tends to infinity, both the weighted empirical measure μ_t^n and the law of Y_t^n converge in a very general setting to μ_t defined in (1.3).

The main results of this paper give quantitative PoC estimates for the system (1.1) using the Wasserstein distance. The proofs rely on synchronous coupling and the biggest challenge arises from the special interaction mechanism of our model. To overcome this challenge, we utilize new estimates for recursive inequalities introduced in [39].

Finally, we also consider the sequential propagation of chaos in one-dimensional mean-field BSDEs with quadratic growth (QBSDEs for short). In 2000, [44] proved the existence and uniqueness of one-dimensional BSDEs when the generator has a quadratic growth in Z and the terminal value is bounded. The case of one-dimensional BSDEs with an unbounded terminal value was obtained by [45–47]. The multi-dimensional case was investigated by [48–50] and the mean-field case was considered by [43, 51].

The remainder of this paper is organized as follows. Section 2 presents some preliminary notations and main results. Section 3 is devoted to the proofs of the BSDE and decoupled FBSDE cases. The proofs of the QBSDE case are postponed to Section 4. Numerical examples are provided in Section 5. Auxiliary lemmas are proven in Appendix A.

2 Main Results

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion W_t is defined, where $\mathbb{F} = \{\mathscr{F}_t\}_{t=0}^{\infty}$ is the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathscr{F}$. Denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the inner product, respectively, in the Euclidean spaces, and by $\|\cdot\|$ the Frobenius norm of a matrix. Let $\mathscr{P}(E)$ be the space of all Borel probability measures on a normed space $(E, \|\cdot\|_E)$. The p-Wasserstein distance between $\mu, \nu \in \mathscr{P}(E)$ is defined as

$$\mathcal{W}_p(\mu, \nu) := \inf\{(\mathbb{E}[\|\xi - \eta\|_E^p])^{\frac{1}{p}} : \mathcal{L}(\xi) = \mu,\ \mathcal{L}(\eta) = \nu\}.$$

Denote by $\mathscr{P}_p$ the metric space of all probability measures $\mu \in \mathscr{P} := \mathscr{P}(\mathbb{R}^d)$ with $\|\mu\|_p := [\int |x|^p \mu(\mathrm{d}x)]^{\frac{1}{p}} < \infty$, equipped with the p-Wasserstein distance.

Throughout this paper, we fix a decreasing positive sequence $\{\alpha_n\}_{n\geq 1}$ with $\alpha_1 = 1$. α_n is the update rate of our weighted empirical measures. We denote

$$\alpha_\infty := \lim_{n\to\infty} \alpha_n, \quad \underline{\alpha} := \liminf_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2}, \quad \overline{\alpha} := \limsup_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2}.$$

In the following, we will frequently encounter the weighted sum s_n recursively given by

$$s_n = s_{n-1} + \alpha_n(x_n - s_{n-1}).$$

This gives

$$s_n = \frac{\sum\limits_{i=1}^{n} w_i x_i}{\sum\limits_{i=1}^{n} w_i},$$

where $w_1 = 1$ and $w_n = \alpha_n \prod_{i=2}^{n}(1-\alpha_i)^{-1}$ for $n \geq 2$. For simplicity, we denote such a weighted sum s_n by $\mathcal{K}_n(x)$, i.e.,

$$\mathcal{K}_n(x) := \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}.$$

2.1 BSDE

In this subsection, we investigate the asymptotic behavior of a backward particle system with sequential interaction. Specifically, we will provide the convergence of the backward particle systems and their rate of convergence.

Let $\{\xi^n; n \geq 1\}$ denote independent copies of ξ, and $\{W^n; n \geq 1\}$ denote independent d-dimensional Brownian motions. A key feature of this system is that each particle is only influenced by particles with smaller ordinal numbers. Specifically, the particle process of the sequential BSDE system $\{Y_t^n, n \in \mathbb{Z}^+\}$ is recursively determined by

$$\begin{cases} Y_t^n = \xi^n + \int_t^T f_s(Y_s^n, \mu_s^{n-1}, Z_s^n)\,\mathrm{d}s - \int_t^T Z_s^n\,\mathrm{d}W_s^n, \quad t \in [0,T], \\ \mu_t^n = \mu_t^{n-1} + \alpha_n(\delta_{Y_t^n} - \mu_t^{n-1}); \end{cases} \tag{2.1}$$

where $Y_t^0 = 0$ and $\mu_t^0 = \delta_{Y_t^0}$.

To apply the synchronous coupling method, we introduce a sequence of i.i.d. McKean–Vlasov BSDEs defined by

$$\overline{Y}_t^n = \xi^n + \int_t^T f_s(\overline{Y}_s^n, \mathcal{L}(\overline{Y}_s^n), \overline{Z}_s^n)\,\mathrm{d}s - \int_t^T \overline{Z}_s^n\,\mathrm{d}W_s^n, \quad t \in [0,T]. \tag{2.2}$$

Assumption 2.1 There exist constants $L_1 \in \mathbb{R}$ and $L_0, L_2 \geq 0$ such that for all $t \in [0,T]$, $y, y_1, y_2 \in \mathbb{R}^d$, $z, z_1, z_2 \in \mathbb{R}^{d\times m}$ and $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, we have

(1) $\langle y_1 - y_2, f(t, y_1, \mu, z) - f(t, y_2, \mu, z)\rangle \leq L_1|y_1 - y_2|^2$,

(2) $|f(t, y, \mu, z_1) - f(t, y, \nu, z_2)| \leq L_2[\mathcal{W}_2(\mu, \nu) + \|z_1 - z_2\|]$,

(3) $|f(t, 0, \mu)|^2 \leq L_0[1 + \|\mu\|_2^2]$.

Theorem 2.1 *Let $p \geq 1$, $\mu_T = \mathcal{L}(\xi) \in \mathscr{P}_r$ with $r > 2p + (p-1)d$, and $\gamma := \frac{1}{2+\left(1-\frac{1}{p}\right)d}$. Let (Y^n, Z^n, μ^n) and $(\overline{Y}^n, \overline{Z}^n)$ be the solutions of* (2.1) *and* (2.2)*, respectively. We denote $\delta Y_t^n := Y_t^n - \overline{Y}_t^n$, $\delta Z_t^n := Z_t^n - \overline{Z}_t^n$ and $\mu_t = \mathcal{L}(\overline{Y}_t)$ is given by McKean–Vlasov BSDE* (1.3)*. Suppose Assumption* 2.1 *holds, then*

(1) *if $\overline{\alpha} < (2 - \alpha_\infty)$, we have*

$$\sup_{0\leq t\leq T} \mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t) + \mathcal{W}_2^{2p}(\mathcal{L}(Y_t^n), \mu_t)] \leq C\,\mathrm{e}^{CT}\alpha_n^{\gamma p};$$

$$\mathbb{E}\Big[\Big(\int_0^T \|\delta Z_t^n\|^2\,\mathrm{d}t\Big)^p\Big] \leq C\,\mathrm{e}^{CT}\alpha_n^{\gamma p}.$$

(2) *If $\overline{\alpha} \geq 2$, for any $\delta < 1 \wedge \gamma\underline{\alpha} \wedge 2\gamma$, we have*

$$\sup_{0 \le t \le T} \mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t) + \mathcal{W}_2^{2p}(\mathcal{L}(Y_t^n), \mu_t)] \le C\,\mathrm{e}^{CT}\mathrm{e}^{-p\delta \sum_{i=1}^{n} \alpha_i};$$

$$\mathbb{E}\Big[\Big(\int_0^T \|\delta Z_t^n\|^2 \,\mathrm{d}t\Big)^p\Big] \le C\,\mathrm{e}^{CT}\mathrm{e}^{-p\delta \sum_{i=1}^{n} \alpha_i},$$

where the constant $C > 0$ depends only on d, p, L_0, L_1, L_2, $\alpha(\cdot)$ and $\|\mu_T\|_r$.

Remark 2.1 The case $2 - \alpha_\infty \le \overline{\alpha} < 2$ never appears because $\overline{\alpha} = 0$ as long as $\alpha_\infty > 0$. So $2 - \alpha_\infty \le \overline{\alpha}$ means $\alpha_\infty = 0$ and $\overline{\alpha} \ge 2$. A typical choice of step size is $\alpha_n \sim n^{-r}$ with some $r \in (0, 1]$. Then we have

$$\mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t)] \lesssim n^{-\frac{rp}{2+(1-\frac{1}{p})d}}.$$

Remark 2.2 Combining the two cases, we know that for $\delta < 1 \wedge \gamma\underline{\alpha} \wedge 2\gamma$, we have

$$\sup_{0 \le t \le T} \mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t)] \lesssim \alpha_n^{\gamma p} + \mathrm{e}^{-p\delta \sum_{i=1}^{n} \alpha_i}.$$

This estimate provides an insightful analysis of the impact of step size on the asymptotic behavior of the system. The empirical measure μ_t^n converges to μ_t as long as $\alpha_\infty = 0$ and $\sum \alpha_n = \infty$. However, if $\alpha_\infty > 0$ or $\sum \alpha_n < \infty$, the convergence of μ_t^n may fail. Even so, we still give an upper bound for the quantities concerned. Intuitively, $\alpha_\infty > 0$ (or $\sum \alpha_n < \infty$) means that the particles with large (or small) ordinals get too many weights in μ_t^n. The following example may demonstrate this point: Define a sequence of measures μ_n recursively as $\mu_n = \mu_{n-1} + \alpha_n(\delta_{w_n} - \mu_{n-1})$ and $\mu_0 = \delta_0$, where $w_n \sim \mathcal{N}(0, 1)$ are i.i.d. The expected limit of μ_n is the standard Gaussian measure $\big($e.g., taking $\alpha = \frac{1}{n}\big)$. However, for $\xi_n = \int x\,\mu_n(\mathrm{d}x)$, by a simple computation one can see that $\mathbb{E}\xi_n = 0$ but $\liminf_{n\to 0} \mathbb{E}|\xi_n|^2 > 0$ whenever $\alpha_\infty > 0$ or $\sum \alpha_n < \infty$. In other words, ξ_n never converges (in L^2) to zero, thus μ_n does not converge to the standard Gaussian measure.

[52] proposed a probabilistic representation of parabolic PDEs on the Wasserstein space, which establishes a connection between decoupled McKean Vlasov FBSDEs and parabolic PDEs. Next, let's consider the following PDE:

$$\begin{cases} \partial_t V(t,x,\mu) = -f(x, V(t,x,\mu), \sigma^{\mathrm{T}}(x,\mu)\partial_x V(t,x,\mu), \mu, \nu) \\ \qquad\qquad - b(x,\mu)\partial_x V(t,x,\mu) - \dfrac{1}{2}\mathrm{tr}(\partial_{xx} V(t,x,\mu) a(x,\mu)) \\ \qquad\qquad - \displaystyle\int_{\mathbb{R}^d} b(y,\mu) \cdot \partial_\mu V(t,x,\mu)(y)\,\mathrm{d}\mu(y) \\ \qquad\qquad - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d} \mathrm{tr}(\partial_y \partial_\mu V(t,x,\mu)(y) a(x,\mu))\,\mathrm{d}\mu(y), \\ V(T,x,\mu) = h(x,\mu), \end{cases} \tag{2.3}$$

where $a = \sigma\sigma^{\mathrm{T}}$, $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d)$, and ν is the law of $V(t,\xi,\mu)$ with $\mathcal{L}(\xi) = \mu$. Under some appropriate regularity conditions, we can know from [20, 52] that the PDE (2.3) is well-posed and that its solution V satisfies

$$V(t, \overline{X}_t, \mathcal{L}(\overline{X}_t)) = \overline{Y}_t,$$

where $(\overline{X}_t, \overline{Y}_t)$ solves the decoupled FBSDE

$$\begin{cases} \overline{X}_t = x_0 + \int_0^t b(\overline{X}_s, \mathcal{L}(\overline{X}_s))\,\mathrm{d}s + \int_0^t \sigma(\overline{X}_s, \mathcal{L}(\overline{X}_s))\,\mathrm{d}W_s, \quad t \in [0,T], \\ \overline{Y}_t = h(\overline{X}_T, \mathcal{L}(\overline{X}_T)) + \int_t^T f(\overline{X}_s, \overline{Y}_s, \overline{Z}_s, \mathcal{L}(\overline{X}_s), \mathcal{L}(\overline{Y}_s))\,\mathrm{d}s - \int_t^T \overline{Z}_s\,\mathrm{d}W_s. \end{cases} \tag{2.4}$$

We aim to demonstrate that the solution V to the PDE (2.3), defined on the infinite dimensional space $[0,T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d)$ can be approximated by a sequential FBSDE system. In fact, we consider the particles in the sequential FBSDE system $\{Y_t^n, n \in \mathbb{Z}^+\}$ which are recursively determined by

$$\begin{cases} X_t^n = x_0 + \int_0^t b\big(X_s^n, \mu_s^n\big)\,\mathrm{d}s + \int_0^t \sigma\big(X_s^n, \mu_s^n\big)\,\mathrm{d}W_s^n, \quad t \in [0,T], \\ Y_t^n = h\big(X_T^n, \mu_T^n\big) + \int_t^T f\big(X_s^n, Y_s^n, Z_s^n, \mu_s^n, \nu_s^n\big)\,\mathrm{d}s - \int_t^T Z_s^n\,\mathrm{d}W_s^n, \\ \mu_t^n = \mu_t^{n-1} + \alpha_n(\delta_{X_t^n} - \mu_t^{n-1}), \\ \nu_t^n = \nu_t^{n-1} + \alpha_n(\delta_{Y_t^n} - \nu_t^{n-1}). \end{cases} \tag{2.5}$$

Correspondingly, we consider a sequence of i.i.d. McKean–Vlasov FBSDE Y^n defined by

$$\begin{cases} \overline{X}_t^n = x_0 + \int_0^t b\big(\overline{X}_s^n, \mathcal{L}(\overline{X}_s^n)\big)\,\mathrm{d}s + \int_0^t \sigma\big(\overline{X}_s^n, \mathcal{L}(\overline{X}_s^n)\big)\,\mathrm{d}W_s^n, \quad t \in [0,T], \\ \overline{Y}_t^n = h\big(\overline{X}_T^n, \mathcal{L}(\overline{X}_T^n)\big) + \int_t^T f\big(\overline{X}_s^n, \overline{Y}_s^n, \overline{Z}_s^n, \mathcal{L}(\overline{X}_s^n), \mathcal{L}(\overline{Y}_s^n)\big)\,\mathrm{d}s - \int_t^T \overline{Z}_s^n\,\mathrm{d}W_s^n. \end{cases} \tag{2.6}$$

Then, we have the following estimate.

Corollary 2.1 *Let $p \geq 2$, $\mu_T \in \mathscr{P}_r$ with $r > 2p + (p-1)d$, and $\gamma := \frac{1}{2+(1-\frac{1}{p})d}$. Let $(X^n, Y^n, Z^n, \mu^n, \nu^n)$* and *$(\overline{X}^n, \overline{Y}^n, \overline{Z}^n)$ be the solutions of* (2.5) *and* (2.6)*, respectively. $\mu_t = \mathcal{L}(\overline{X}_t)$ and $\nu_t = \mathcal{L}(\overline{Y}_t)$ are given by McKean–Vlasov FBSDE* (2.4)*. Suppose Assumption* 2.1 *holds and b, σ are Lipschitz continuous, then for any $\delta < 1 \wedge \gamma p \underline{\alpha}$, we have*

$$\sup_{0 \le t \le T} \mathbb{E}[\mathcal{W}_2^{2p}(\nu_t^n, \nu_t)] \le C\,\mathrm{e}^{CT}\big(\alpha_n^{\gamma p} + \mathrm{e}^{-\delta \sum_{i=1}^n \alpha_i}\big);$$

where the constant $C > 0$ depends only on d, p, L_0, L_1, L_2, $\alpha(\cdot)$ and $\|\mu_T\|_r$.

Remark 2.3 If $\alpha_n = \frac{1}{n}$, then $\alpha_\infty = 0$, $\overline{\alpha} = \underline{\alpha} = 1$. According to Corollary 2.1, we have $\sup_{0 \le t \le T} \mathbb{E}[\mathcal{W}_2^{2p}(\nu_t^n, \nu_t)] \lesssim n^{-\delta}$ for some $\delta < 1$. While in Theorem 2.1, we have $\sup_{0 \le t \le T} \mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t)] \lesssim n^{-\frac{p}{2+(1-\frac{1}{p})d}}$.

2.2 Quadratic BSDE

In this section, we study the sequential particle systems for the mean-field BSDE (2.1) with quadratic growth. The convergence of the particle systems and the rate of convergence will be given.

Let M be a continuous local martingale, denote $\mathcal{E}(M)_0^t = \mathrm{e}^{M_t - \frac{1}{2}\langle M\rangle_t}$ for $0 \le t < \infty$. In addition, for any $p \ge 1$, $t \in [0,T)$ and Euclidean space $\mathbb{H}$, we introduce the following spaces:

$$\mathcal{Z}_{\mathbb{F}}^2(t,T;\mathbb{H}) = \Big\{ Z \in L_{\mathbb{F}}^2(t,T;\mathbb{H}) \mid \|Z\|_{\mathcal{Z}_{\mathbb{F}}^2(t,T)} \triangleq \sup_{\tau} \Big\| \mathbb{E}_\tau \Big[\int_\tau^T |Z_s|^2 \,\mathrm{d}s \Big] \Big\|_\infty^{\frac{1}{2}} < \infty \Big\},$$

$$\mathcal{S}_{\mathbb{F}}^\infty(t,T;\mathbb{H}) = \Big\{ \varphi \in \Omega \times [t,T] \to \mathbb{H} \mid \varphi \text{ is } \mathbb{F}-\text{adapted, continous,}$$
$$\|\varphi\|_{\mathcal{S}_{\mathbb{F}}^\infty(t,T)} \triangleq \operatorname*{esssup}_{(\omega,s)\in\Omega\times[t,T]} |\varphi(\omega,s)| < \infty \Big\}.$$

We denote by $\mathbb{E}^{\mathbb{P}}$ the expectation operator with respect to the probability measure $\mathbb{P}$. Moreover, let $M = (M_t, \mathscr{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$, and we set

$$\|M\|_{BMO_p(\mathbb{P})} \triangleq \sup_\tau \big\| \mathbb{E}_\tau \big[\langle M\rangle_\infty - \langle M\rangle_\tau^{\frac{p}{2}} \big]^{\frac{1}{p}} \big\|_\infty.$$

The class $\{M : \|M\|_{BMO_p(\mathbb{P})} < \infty\}$ is denoted by $BMO_p(\mathbb{P})$. Note that $BMO_p(\mathbb{P})$ is a Banach space under the norm $\|\cdot\|_{BMO_p(\mathbb{P})}$ and

$$\|Z \cdot W\|_{BMO_2(\mathbb{P})} \equiv \|Z\|_{\mathcal{Z}_{\mathbb{F}}^2(0,T)}.$$

For the QBSDE case, we consider the following assumption instead of Assumption 2.1.

Assumption 2.2 For all $n \ge 1$, $t \in [0,T]$, $y, y_1, y_2 \in \mathbb{R}$, $z, z_1, z_2 \in \mathbb{R}^d$ and $\mu, \nu \in \mathscr{P}_2(\mathbb{R})$, the terminal value $\xi^n : \Omega \to \mathbb{R}$ and the generator $f : [0,T] \times \mathbb{R} \times \mathscr{P}(\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}$ satisfy the following conditions:

(1) There exists a constant K such that

$$\max_{n\ge 1} \|\xi^n\|_\infty \le K.$$

(2) There exist constants L_0, γ such that

$$f(t,y,\mu,z) \le L_0(1 + |y| + \|\mu\|_2) + \frac{\gamma}{2}|z|^2.$$

(3) There exists a constant L_1 and a non-decreasing continuous function $\phi : [0,\infty) \to [0,\infty)$ such that

$$f(t,y_1,\mu_1,z_1) - f(t,y_2,\mu_2,z_2) \le L_1(|y_1 - y_2| + \mathcal{W}(\mu_1,\mu_2))$$
$$+ \phi(|y_1| \vee |y_2| \vee \|\mu_1\|_2 \vee \|\mu_2\|_2)(1 + |z_1| + |z_2|)|z_1 - z_2|.$$

Theorem 2.2 *Let Assumption* 2.2 *holds and we use the same notation as in Theorem* 2.1. *Then for $p \ge 1$, there exists a positive constant $q > 1$ and a positive constant C depending on T, K, L_0, L_1, p such that*

(1) *if $\overline{\alpha} < (2 - \alpha_\infty)$, we have*

$$\sup_{0\le t\le T} \mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t) + \mathcal{W}_2^{2p}(\mathcal{L}(Y_t^n), \mu_t)] \le C\,\mathrm{e}^{CT} \alpha_n^{\gamma_q p};$$

(2) *if $\overline{\alpha} \ge 2$, we have, for any $\delta < 1 \wedge \underline{\alpha}\gamma_q \wedge 2\gamma_q$,*

$$\sup_{0\le t\le T} \mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t) + \mathcal{W}_2^{2p}(\mathcal{L}(Y_t^n), \mu_t)] \le C\,\mathrm{e}^{CT} \mathrm{e}^{-p\delta \sum\limits_{i=1}^n \alpha_i},$$

where $\gamma_q := \frac{1}{2 + (1 - \frac{1}{pq})d}$.

Remark 2.4 Comparing the convergence rate of Theorems 2.1–2.2, we find that there is an additional $q>1$ in QBSDE case. This is because we use the reverse Hölder inequality (see Proposition 4.1) when estimating BMO martingale.

Remark 2.5 If $\alpha_n=\frac{1}{n}$, then $\alpha_\infty=0$, $\overline{\alpha}=\underline{\alpha}=1$. According to Theorem 2.2, we have $\sup\limits_{0\le t\le T}\mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n,\mu_t)]\lesssim\alpha_n^{\gamma p}\lesssim n^{-\frac{p}{3-\frac{1}{pq}}}$.

3 Proofs for BSDEs and FBSDEs

Proposition 3.1 *Let* (Y^n,Z^n,μ^n) and $(\overline{Y}^n,\overline{Z}^n)$ *be the solutions of* (2.1) *and* (2.2), *respectively. We denote* $\delta Y_t^n:=Y_t^n-\overline{Y}_t^n$, $\delta Z_t^n:=Z_t^n-\overline{Z}_t^n$ *and* $\mu_t=\mathcal{L}(\overline{Y}_t)$ *is given by McKean–Vlasov BSDE* (1.3). *Suppose* $p\ge1$ *and Assumption* 2.1 *holds. Let* ε *such that* $0<\varepsilon<1$ *and* $2\lambda\ge\frac{L_2^2}{\varepsilon}+2L_1$. *Then there exists a constant* $C(p,\varepsilon)$ *which depends only on* p,ε,L_2 *such that*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\mathrm{e}^{2\lambda pt}|\delta Y_t^n|^{2p}+\Big(\int_0^T\mathrm{e}^{2\lambda t}\|\delta Z_t^n\|^2\,\mathrm{d}t\Big)^p\Big]\\&\le C_{p,\varepsilon}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p}+\Big(\int_0^T\mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1},\mu_t)\,\mathrm{d}t\Big)^{2p}\Big].\end{aligned}\tag{3.1}$$

Moreover, if $C_\lambda:=2\lambda-\frac{L_2^2}{\varepsilon}-2L_1>0$, *we have*

$$\mathbb{E}\Big[\Big(\int_0^T\mathrm{e}^{2\lambda t}|\delta Y_t^n|^2\,\mathrm{d}t\Big)^p\Big]\le\frac{C_{p,\varepsilon}}{C_\lambda^p}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p}+\Big(\int_0^T\mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1},\mu_t)\,\mathrm{d}t\Big)^{2p}\Big].\tag{3.2}$$

3.1 Proof of Proposition 3.1

Proof of Proposition 3.1 We use Itô's formula to get

$$\begin{aligned}\mathrm{e}^{2\lambda t}|\delta Y_t^n|^2+\int_t^T\mathrm{e}^{2\lambda s}\|\delta Z_s^n\|^2\,\mathrm{d}s=\mathrm{e}^{2\lambda T}|\delta Y_T^n|^2-2\lambda\int_t^T\mathrm{e}^{2\lambda s}|\delta Y_s^n|^2\,\mathrm{d}s\\+2\int_t^T\mathrm{e}^{2\lambda s}\delta Y_s^n\cdot(f_s(Y_s^n,\mu_s^{n-1},Z_s^n)-f_s(\overline{Y}_s^n,\mu_s,\overline{Z}_s^n))\,\mathrm{d}s\\-2\int_t^T\mathrm{e}^{2\lambda s}\delta Y_s^n\cdot\delta Z_s^n\,\mathrm{d}W_s^n.\end{aligned}$$

By Assumption 2.1, we know that

$$\begin{aligned}&2\delta Y_s^n\cdot\big(f_s(Y_s^n,\mu_s^{n-1},Z_s^n)-f_s(\overline{Y}_s^n,\mu_s,\overline{Z}_s^n)\big)\\&\le2L_1|\delta Y_s^n|^2+2L_2|\delta Y_s^n|\big[\mathcal{W}_2(\mu_s^{n-1},\mu_s)+\|\delta Z_s^n\|\big]\\&\le\Big(2L_1+\frac{L_2^2}{\varepsilon}\Big)|\delta Y_s^n|^2+\varepsilon\|\delta Z_s^n\|+2L_2|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s).\end{aligned}$$

Thus

$$\begin{aligned}&\mathrm{e}^{2\lambda t}|\delta Y_t^n|^2+(1-\varepsilon)\int_t^T\mathrm{e}^{2\lambda s}\|\delta Z_s^n\|^2\,\mathrm{d}s+\Big(2\lambda-2L_1-\frac{L_2^2}{\varepsilon}\Big)\int_t^T\mathrm{e}^{2\lambda s}|\delta Y_s^n|^2\,\mathrm{d}s\\&\le\mathrm{e}^{2\lambda T}|\delta Y_T^n|^2-2\int_t^T\mathrm{e}^{2\lambda s}\delta Y_s^n\cdot\delta Z_s^n\,\mathrm{d}W_s^n+2L_2\int_t^T\mathrm{e}^{2\lambda s}|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s.\end{aligned}\tag{3.3}$$

We take the condition expectation with respect to $\mathcal{F}_t$ to obtain

$$\mathrm{e}^{2\lambda t}|\delta Y_t^n|^2 \le \mathbb{E}\Big[\mathrm{e}^{2\lambda T}|\delta Y_T^n|^2 + 2L_2\int_t^T \mathrm{e}^{2\lambda s}|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s \mid \mathcal{F}_t\Big].$$

After taking the p-th power, we know from Doob's inequality that

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\mathrm{e}^{2\lambda pt}|\delta Y_t^n|^{2p}\Big] &\le C_{p,L_2}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p} + \Big(\int_0^T \mathrm{e}^{2\lambda s}|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\Big)^p\Big]\\ &\le C_{p,L_2}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p} + \Big(\int_0^T \mathrm{e}^{\lambda s}\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\Big)^{2p}\Big]\\ &\quad + \frac{1}{2}\mathbb{E}\Big[\sup_{t\in[0,T]}\mathrm{e}^{2\lambda pt}|\delta Y_t^n|^{2p}\Big].\end{aligned}$$

So far, we have proved the first part of (3.1). Then we consider the $\|\delta Z_s^n\|$ term. We again use the estimation (3.3) to obtain

$$\begin{aligned}(1-\varepsilon)\int_t^T \mathrm{e}^{2\lambda s}\|\delta Z_s^n\|^2\,\mathrm{d}s &\le \mathrm{e}^{2\lambda T}|\delta Y_T^n|^2 + 2L_2\int_t^T \mathrm{e}^{2\lambda s}|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\\ &\quad - 2\int_t^T \mathrm{e}^{2\lambda s}\delta Y_s^n\cdot\delta Z_s^n\,\mathrm{d}W_s^n.\end{aligned}$$

By BDG's inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_t^T \mathrm{e}^{2\lambda s}\|\delta Z_s^n\|^2\,\mathrm{d}s\Big)^p\Big] &\le C_{p,L_2,\varepsilon}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p} + \Big(\int_0^T \mathrm{e}^{2\lambda s}|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\Big)^p\\ &\quad + \Big(\int_0^T \mathrm{e}^{4\lambda s}|\delta Y_s^n|^2\|\delta Z_s^n\|^2\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\ &\le C_{p,L_2,\varepsilon}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p} + \sup_{t\in[0,T]}\mathrm{e}^{2\lambda pt}|\delta Y_t^n|^{2p}\\ &\quad + \Big(\int_0^T \mathrm{e}^{\lambda s}\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\Big)^{2p} + \frac{1}{2}\Big(\int_0^T \mathrm{e}^{2\lambda s}\|\delta Z_s^n\|^2\,\mathrm{d}s\Big)^p\Big].\end{aligned}$$

Combining the estimates of $\mathbb{E}\big[\sup_{t\in[0,T]}\mathrm{e}^{2\lambda pt}|\delta Y_t^n|^{2p}\big]$, we have

$$\mathbb{E}\Big[\Big(\int_t^T \mathrm{e}^{2\lambda s}\|\delta Z_s^n\|^2\,\mathrm{d}s\Big)^p\Big] \le C_{p,L_2,\varepsilon}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}|\delta Y_T^n|^{2p} + \Big(\int_0^T \mathrm{e}^{\lambda s}\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\Big)^p\Big].$$

Finally, we use once again the estimate from (3.3) to obtain

$$\Big(2\lambda - 2L_1 - \frac{L_2^2}{\varepsilon}\Big)\mathbb{E}\int_t^T \mathrm{e}^{2\lambda s}|\delta Y_s^n|^2\,\mathrm{d}s \le \mathbb{E}\Big[\mathrm{e}^{2\lambda T}|\delta Y_T^n|^2 + 2L_2\int_t^T \mathrm{e}^{2\lambda s}|\delta Y_s^n|\mathcal{W}_2(\mu_s^{n-1},\mu_s)\,\mathrm{d}s\Big].$$

This completes the proof of the estimate in (3.2) and thus the proof of the theorem.

3.2 Proof of Theorem 2.1

Proof of Theorem 2.1 Since (3.2) and triangle inequality in $\mathcal{W}_2$ space, we can conclude that

$$\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}|\delta Y_t^n|^2\,\mathrm{d}t\Big)^p\Big] \le \frac{C_{p,\varepsilon}}{C_\lambda^p}\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}[\mathcal{W}_2(\mu_t^{n-1},\widehat{\mu}_t^{n-1}) + \mathcal{W}_2(\widehat{\mu}_t^{n-1},\mu_t)]\,\mathrm{d}t\Big)^{2p}\Big], \quad (3.4)$$

where $\widehat{\mu}_t^n := \widehat{\mu}_t^{n-1} + \alpha_n(\delta_{\overline{Y}_t^n} - \widehat{\mu}_t^{n-1})$. We can use the inequality $(x+y)^{2p} \leq 2^{2p-1}(x^{2p}+y^{2p})$ and the Hölder inequality to obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\big[\mathcal{W}_2(\mu_t^{n-1},\widehat{\mu}_t^{n-1}) + \mathcal{W}_2(\widehat{\mu}_t^{n-1},\mu_t)\big]\,\mathrm{d}t\Big)^{2p}\Big] \\
&\leq 2^{2p-1}\Big(\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1},\widehat{\mu}_t^{n-1})\,\mathrm{d}t\Big)^{2p}\Big] + \mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\widehat{\mu}_t^{n-1},\mu_t)\,\mathrm{d}t\Big)^{2p}\Big]\Big) \\
&\leq 2^{2p-1}\Big(T^p\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}\mathcal{W}_2^2(\mu_t^{n-1},\widehat{\mu}_t^{n-1})\,\mathrm{d}t\Big)^{p}\Big] \\
&\quad + T^{2p-1}\mathbb{E}\Big[\int_0^T \mathrm{e}^{2p\lambda t}\mathcal{W}_2^{2p}(\widehat{\mu}_t^{n-1},\mu_t)\,\mathrm{d}t\Big]\Big). \qquad (3.5)
\end{aligned}
$$

We choose λ to be large enough such that $(1-\varepsilon)^p C_\lambda^p = 2^{2p-1}T^p C_{p,\varepsilon}$. Recall Lemma A.3 that for i.i.d. random variables ξ_n and $m_n := \frac{\sum_{i=1}^n w_i\delta_{\xi_i}}{\sum_{i=1}^n w_i}$, we have

$$\mathbb{E}\mathcal{W}_r^{rp}(m_n,m) \leq C\theta_n^{\gamma_{r,p}}$$

with

$$\theta_n = \frac{\sum_{i=1}^n w_i^2}{\big(\sum_{i=1}^n w_i\big)^2}, \quad \gamma_{r,p} = \frac{rp}{2r+2\big(1-\frac{1}{p}\big)d}.$$

So it follows from Lemma A.3 that $\mathbb{E}[\mathcal{W}_2(\widehat{\mu}_t^n,\mu_t)^{2p}] \lesssim \theta_n^{\gamma p}$ where $\gamma = \frac{1}{2+(1-\frac{1}{p})d}$. Now, going back to (3.4), we have

$$\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}|\delta Y_t^n|^2\,\mathrm{d}t\Big)^p\Big] \leq (1-\varepsilon)^p\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}\mathcal{W}_2^2(\mu_t^{n-1},\widehat{\mu}_t^{n-1})\,\mathrm{d}t\Big)^p\Big] + C\theta_n^{\gamma p}, \qquad (3.6)$$

where we use the notation C to represent a constant that depends only on p, T, ε, and whose value may change from line to line. It follows from the property of the Wasserstein distance that

$$\mathcal{W}_2^2(\mu_t^{n-1},\widehat{\mu}_t^{n-1}) \leq \mathcal{K}_{n-1}(|\delta Y_t|^2) := \frac{\sum_{i=1}^n w_i|\delta Y_t^i|^2}{\sum_{i=1}^n w_i}.$$

Thus

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}\mathcal{W}_2^2(\mu_t^{n-1},\widehat{\mu}_t^{n-1})\,\mathrm{d}t\Big)^p\Big] &\leq \mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}\mathcal{K}_{n-1}(|\delta Y_t|^2)\,\mathrm{d}t\Big)^p\Big] \\
&= \mathbb{E}\Big[\Big(\mathcal{K}_{n-1}\Big(\int_0^T \mathrm{e}^{2\lambda t}|\delta Y_t|^2\,\mathrm{d}t\Big)\Big)^p\Big] \\
&\leq \Big(\mathcal{K}_{n-1}\Big[\Big(\mathbb{E}\Big(\int_0^T \mathrm{e}^{2\lambda t}|\delta Y_t|^2\,\mathrm{d}t\Big)^p\Big)^{\frac{1}{p}}\Big]\Big)^p,
\end{aligned}
$$

where the third equality is due to Minkovski's inequality.

If we denote $y_n := \big[\mathbb{E}\big(\int_0^T e^{2\lambda t}|\delta Y_t|^2\,dt\big)^p\big]^{\frac{1}{p}}$ and $s_n := \mathcal{K}_n(y)$, then we have

$$\begin{aligned} y_n^p = \mathbb{E}\Big[\Big(\int_0^T e^{2\lambda t}|\delta Y_t^n|^2\,dt\Big)^p\Big] &\le (1-\varepsilon)^p\Big(\mathcal{K}_{n-1}\Big[\Big(\mathbb{E}\Big(\int_0^T e^{2\lambda t}|\delta Y_t|^2\,dt\Big)^p\Big)^{\frac{1}{p}}\Big]\Big)^p + C\theta_n^{\gamma p} \\ &= (1-\varepsilon)^p s_{n-1}^p + C\theta_n^{\gamma p}. \end{aligned}$$

Thus

$$\begin{aligned} y_n &\le ((1-\varepsilon)^p s_{n-1}^p + C\theta_n^{\gamma p})^{\frac{1}{p}} \\ &\le (1-\varepsilon)s_{n-1} + C\theta_n^{\gamma}. \end{aligned}$$

According to the definition of s_n, we can write out the recursive formula

$$s_n := s_{n-1} + \alpha_n(y_n - s_{n-1}).$$

Combining the above inequality, we obtain

$$\begin{aligned} s_n &\le s_{n-1} + \alpha_n[(1-\varepsilon)s_{n-1} + C\theta_n^{\gamma}] - \alpha_n s_{n-1} \\ &\le (1-\varepsilon\alpha_n)s_{n-1} + \alpha_n C\theta_n^{\gamma}. \end{aligned}$$

Since $\theta_n = \frac{\sum_{i=1}^n w_i^2}{\big(\sum_{i=1}^n w_i\big)^2}$ and Remark A.3, for all $\delta_1 < \underline{\alpha} \wedge 2$, one has

$$\theta_n^{\gamma} \le C \begin{cases} e^{-\delta_1\gamma\sum_{i=1}^n \alpha_i} & \text{if } \overline{\alpha} \ge 2, \\ \alpha_n^{\gamma} & \text{if } 2 - \alpha_\infty > \overline{\alpha}. \end{cases}$$

In the case of $\overline{\alpha} \ge 2$, we have

$$\lim_{n\to\infty}\Big[-\frac{e^{-\delta_1\gamma\sum_{i=1}^{n+1}\alpha_i} - e^{-\delta_1\gamma\sum_{i=1}^{n}\alpha_i}}{\alpha_n\, e^{-\delta_1\gamma\sum_{i=1}^{n}\alpha_i}}\Big] = \delta_1\gamma.$$

By Corollary A.1, we have

(1) if $\varepsilon < \delta_1\gamma$, we have

$$s_n \le C\, e^{-\varepsilon\sum_{i=1}^n \alpha_i}.$$

(2) If $\varepsilon = \delta_1\gamma$, we have, for any $\delta < \delta_1\gamma$,

$$s_n \le C\, e^{-\delta\sum_{i=1}^n \alpha_i}.$$

(3) If $\varepsilon > \delta_1\gamma$, we have

$$s_n \le C\, e^{-\delta_1\gamma\sum_{i=1}^n \alpha_i}.$$

Recall that $\delta_1 < \underline{\alpha} \wedge 2$, we simplify the above inequality to

$$s_n \le C\, e^{-\delta\sum_{i=1}^n \alpha_i}, \quad \forall \delta < 1 \wedge \underline{\alpha}\gamma \wedge 2\gamma.$$

In the case of $2-\alpha_\infty > \overline{\alpha}$, we have

$$\limsup_{n\to\infty}\Big[-\frac{\alpha_{n+1}^\gamma-\alpha_n^\gamma}{\alpha_n\alpha_n^\gamma}\Big]=\gamma\overline{\alpha},\quad \liminf_{n\to\infty}\Big[-\frac{\alpha_{n+1}^\gamma-\alpha_n^\gamma}{\alpha_n\alpha_n^\gamma}\Big]=\gamma\underline{\alpha}.$$

By Corollary A.1,

(1) if $\varepsilon<\gamma\underline{\alpha}$, we have

$$s_n\le C\,\mathrm{e}^{-\varepsilon\sum\limits_{i=1}^n\alpha_i}.$$

(2) If $\gamma\underline{\alpha}\le\varepsilon\le\gamma\overline{\alpha}$, we have, for any $\delta<\gamma\underline{\alpha}$,

$$s_n\le C\,\mathrm{e}^{-\delta\sum\limits_{i=1}^n\alpha_i}.$$

(3) If $\varepsilon>\gamma\overline{\alpha}$, we have

$$s_n\le C\alpha_n^\gamma.$$

Combining the first two cases, we have

$$s_n\le C\begin{cases}\mathrm{e}^{-\delta\sum\limits_{i=1}^n\alpha_i}, & \forall\delta<1\wedge\gamma\underline{\alpha}, \quad \text{if } \gamma\overline{\alpha}\ge 1,\\ \alpha_n^\gamma, & \qquad\qquad\qquad\quad\ \ \text{if } \gamma\overline{\alpha}<1.\end{cases}$$

Summarizing the above discussion, we know that

$$s_n\le C\begin{cases}\mathrm{e}^{-\delta\sum\limits_{i=1}^n\alpha_i}, & \forall\delta<1\wedge\gamma\underline{\alpha}\wedge 2\gamma, \quad \text{if } \overline{\alpha}\ge 2,\\ \mathrm{e}^{-\delta\sum\limits_{i=1}^n\alpha_i}, & \forall\delta<1\wedge\gamma\underline{\alpha}, \qquad\quad\ \text{if } \gamma^{-1}\le\overline{\alpha}<2-\alpha_\infty,\\ \alpha_n^\gamma, & \qquad\qquad\qquad\qquad\qquad\ \text{if } \overline{\alpha}<\gamma^{-1}\wedge(2-\alpha_\infty).\end{cases}$$

Since $\gamma=\frac{1}{2+(1-\frac{1}{p})d}\le\frac{1}{2}$, we can deduce that the case $2-\alpha_\infty>\overline{\alpha}\ge\gamma^{-1}\ge 2$ cannot happen. Therefore, we can simplify the above equation to

$$s_n\le C\begin{cases}\mathrm{e}^{-\delta\sum\limits_{i=1}^n\alpha_i}, & \forall\delta<1\wedge\gamma\underline{\alpha}\wedge 2\gamma, \quad \text{if } \overline{\alpha}\ge 2,\\ \alpha_n^\gamma, & \qquad\qquad\qquad\qquad\qquad\quad \text{if } \overline{\alpha}<(2-\alpha_\infty).\end{cases}$$

Together with (3.6), this implies that

$$\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}|\delta Y_t^n|^2\,\mathrm{d}t\Big)^p\Big]\le Cs_n^p$$

and

$$\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1},\mu_t)\,\mathrm{d}t\Big)^{2p}\Big]\le Cs_n^p.$$

By plugging the above estimates into (3.1), we can conclude the proof of Theorem 2.1.

Remark 3.1 In SDE case (see [39]), they denote $x_n:=\int_0^T\mathbb{E}|\Delta_t^n|^{2p}\,\mathrm{d}t$ and use the estimation

$$\int_0^T\mathbb{E}|\Delta_t^n|^{2p}\,\mathrm{d}t\le(1-\varepsilon)\int_0^T\mathbb{E}[\mathcal{K}_{n-1}(|\Delta_t|^{2p})]+C_p\theta_n^{\gamma p}.$$

Then they obtain

$$x_n \le (1-\varepsilon)s^x_{n-1} + C_p\theta_n^{\gamma p}.$$

While in BSDE case, we denote $y_n := \big[\mathbb{E}\big(\int_0^T |\Delta_t^n|^2\,\mathrm{d}t\big)^p\big]^{\frac{1}{p}}$ and use the estimation

$$\Big[\mathbb{E}\Big(\int_0^T |\Delta_t^n|^2\,\mathrm{d}t\Big)^p\Big]^{\frac{1}{p}} \le (1-\varepsilon)\mathcal{K}_{n-1}\Big[\Big(\mathbb{E}\Big(\int_0^T |\Delta_t|^2\,\mathrm{d}t\Big)^p\Big)^{\frac{1}{p}}\Big] + C\theta_n^{\gamma}.$$

And we have

$$y_n \le (1-\varepsilon)s^y_{n-1} + C_p\theta_n^{\gamma}.$$

3.3 Proof of Theorem 2.1

Proof of Theorem 2.1 We denote $\theta_t^n := |X_t^n - \overline{X}_t^n|$ and $\pi_t^n := \mathcal{W}_2\big(\mu_t^n, \mu_t\big)$. Then by Proposition 3.1, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} \mathrm{e}^{2\lambda p t}|\delta Y_t^n|^{2p} + \Big(\int_0^T \mathrm{e}^{2\lambda t}\|\delta Z_t^n\|^2\,\mathrm{d}t\Big)^p\Big] &(3.7)\\
&\le C_{p,\varepsilon}\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}(|\theta_t^n|^{2p} + |\pi_t^n|^{2p}) + \Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1}, \mu_t)\,\mathrm{d}t\Big)^{2p}\\
&\quad + \Big(\int_0^T \mathrm{e}^{\lambda t}(|\theta_t^n| + |\pi_t^n|)\,\mathrm{d}t\Big)^{2p}\Big]. &(3.8)
\end{aligned}$$

Therefore, there is a constant C depending only on p, ε and T such that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} \mathrm{e}^{2\lambda p t}|\delta Y_t^n|^{2p}\Big]\\
&\le C\mathbb{E}\Big[\mathrm{e}^{2\lambda pT}\Big(\sup_{t\in[0,T]}|\theta_t^n|^{2p} + \sup_{t\in[0,T]}|\pi_t^n|^{2p}\Big) + \Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1}, \mu_t)\,\mathrm{d}t\Big)^{2p}\Big].
\end{aligned}$$

Moreover, by the theory of (forward) sequential propagation of chaos, see e.g. [39], for any $\delta < 1 \wedge \underline{\alpha}\gamma p \wedge 2\gamma p$, it holds that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\theta_t^n|^{2p} + \sup_{t\in[0,T]}|\pi_t^n|^{2p}\Big] \le C\,\mathrm{e}^{CT}(\alpha_n^{\gamma p} + \mathrm{e}^{-\delta\sum_{i=1}^n \alpha_i}).$$

Similar to the proof of Theorem 2.1, we can get the conclusion.

4 Proofs for Quadratic BSDEs

First, we recall two propositions of BMO martingales in [53].

Define $\psi(s) := \big[1 + \frac{1}{x^2}\log\frac{2x-1}{2(x-1)}\big]^{\frac{1}{2}} - 1$ for $x > 1$. It is clearly continuous and decreasing, satisfying $\psi(1+) = +\infty$ and $\psi(+\infty) = 0$.

Proposition 4.1 *Let $p \in (1,\infty)$ and M be a one-dimensional continuous BMO martingale. If $\|M\|_{BMO(\mathbb{P})} < \psi(p)$, then $\mathcal{E}(M)$ satisfies the reverse Hölder inequality:*

$$\mathbb{E}_\tau[\mathscr{E}(M)_\tau^\infty]^p \le c_p$$

for any stopping time τ, with a positive constant c_p depending only on p.

Proposition 4.2 *For $K>0$, there are constants $c_1>0$ and $c_2>0$ depending on K such that for any BMO martingale M and any one-dimensional BMO martingale N such that $\|N\|_{BMO(\mathbb{P})}\le K$, we have*

$$c_1\|M\|_{BMO(\mathbb{P})}\le \|\widetilde{M}\|_{BMO(\widetilde{\mathbb{P}})}\le c_2\|M\|_{BMO(\mathbb{P})},$$

where $\widetilde{M}\triangleq M-\langle M,N\rangle$ and $\mathrm{d}\widetilde{\mathbb{P}}\triangleq \mathscr{E}(N)_0^\infty \mathrm{d}\mathbb{P}$.

Proposition 4.3 *Under Assumption 2.2, there exists a positive constant C depending on T,K,L_0,L_1, such that the solution (Y^n,Z^n) of BSDE (2.1) admits the following estimate:*

$$\|Y^n\|_{\mathcal{S}^\infty_{\mathbb{F}}(0,T)}\le C,\quad \|Z^n\|_{\mathcal{Z}^2_{\mathbb{F}}(0,T)}\le C.$$

Proof Since Assumption 2.2 holds, we can estimate BSDE (2.1) as follows:

$$\begin{aligned}Y_t^n&=\xi^n+\int_t^T f_s(Y_s^n,\mu_s^{n-1},Z_s^n)\,\mathrm{d}s-\int_t^T Z_s^n\,\mathrm{d}W_s^n\\&=\xi^n+\int_t^T[f_s(Y_s^n,\mu_s^{n-1},Z_s^n)-f_s(0,\delta_0,Z_s^n)+f_s(0,\delta_0,Z_s^n)]\,\mathrm{d}s-\int_t^T Z_s^n\,\mathrm{d}W_s^n\\&=\xi^n+\int_t^T[f_s(Y_s^n,\mu_s^{n-1},Z_s^n)-f_s(0,\delta_0,Z_s^n)]\,\mathrm{d}s-\int_t^T Z_s^n\,\mathrm{d}\widetilde{W}_s^n,\end{aligned}$$

where $\widetilde{W}_t^n:=W_t^n-\int_0^t\Gamma_s\,\mathrm{d}s$ is a Brownian motion under $\mathbb{P}^n$ with

$$\frac{\mathrm{d}\mathbb{P}^n}{\mathrm{d}\mathbb{P}}=\mathscr{E}(\Gamma\cdot W^n)_0^T,$$

and $f_s(0,\delta_0,Z_s^n)=\Gamma_sZ_s^n$ with $\Gamma_s\le\phi(0)(1+|Z_s^n|)$. Then we apply Itô's formula to $\mathrm{e}^{2\lambda t}|Y_t^n|^2$ to get

$$\begin{aligned}&\mathrm{e}^{2\lambda t}|Y_t^n|^2+\mathbb{E}_t^{\mathbb{P}^n}\Big[\int_t^T\mathrm{e}^{2\lambda s}|Z_s^n|^2\,\mathrm{d}s\Big]\\&=\mathbb{E}_t^{\mathbb{P}^n}[\mathrm{e}^{2\lambda t}|\xi^n|^2]+\mathbb{E}_t^{\mathbb{P}^n}\Big[\int_t^T 2\mathrm{e}^{2\lambda s}Y_s^n[f_s(Y_s^n,\mu_s^{n-1},Z_s^n)-f_s(0,\delta_0,Z_s^n)]\,\mathrm{d}s\Big]\\&\quad-2\lambda\int_t^T\mathrm{e}^{2\lambda s}|\delta Y_s^n|^2\,\mathrm{d}s.\end{aligned}$$

Using Assumption 2.2 again, we can get

$$\begin{aligned}\mathrm{e}^{2\lambda t}|Y_t^n|^2&\le\mathbb{E}_t^{\mathbb{P}^n}[\mathrm{e}^{2\lambda t}|\xi^n|^2]+\mathbb{E}_t^{\mathbb{P}^n}\Big[\int_t^T 2L_1\mathrm{e}^{2\lambda s}Y_s^n(|Y_s^n|+\|\mu_s^{n-1}\|_2)\,\mathrm{d}s\\&\qquad-2\lambda\int_t^T\mathrm{e}^{2\lambda s}|\delta Y_s^n|^2\,\mathrm{d}s\Big]\\&\le\mathrm{e}^{2\lambda t}K^2+\mathbb{E}_t^{\mathbb{P}^n}\Big[\int_t^T\Big(-\Big(2\lambda-2L_1-\frac{L_1^2}{\varepsilon}\Big)\mathrm{e}^{2\lambda s}|Y_s^n|^2+\frac{\varepsilon}{n-1}\sum_{k=1}^{n-1}\mathrm{e}^{2\lambda s}|Y_s^k|^2\Big)\,\mathrm{d}s\Big].\end{aligned}$$

We denote $y^n:=\sup\limits_{t\in[0,T]}\mathrm{e}^{2\lambda t}|Y_t^n|^2$. By definition, it is not difficult to know $y^0=0,y^1\le\mathrm{e}^{2\lambda T}K^2$. To use mathematical induction, we assume $y^n\le 2\mathrm{e}^{2\lambda t}K^2$. We choose ε such that $\varepsilon T<\frac{1}{2}$ and choose λ such that $2\lambda - 2L_1 - \frac{L_1^2}{\varepsilon} > 0$. Thus

$$\begin{aligned}
e^{2\lambda t}|Y_t^n|^2 &\le e^{2\lambda t}K^2 + \frac{\varepsilon}{n-1}\sum_{k=1}^{n-1}\mathbb{E}_t^{\mathbb{P}^n}\Big[\int_t^T e^{2\lambda s}|Y_s^k|^2\,ds\Big]\\
&\le e^{2\lambda t}K^2 + \frac{\varepsilon}{n-1}\sum_{k=1}^{n-1} y^k(T-t)\\
&\le e^{2\lambda t}K^2 + \frac{1}{2}\cdot 2e^{2\lambda T}K^2 \le 2e^{2\lambda T}K^2.
\end{aligned}$$

Hence, there exists a positive constant C depending on T, K, L_1, such that

$$\|Y^n\|_{\mathcal{S}_{\mathbb{F}}^\infty(0,T)} \le C.$$

To prove $Z^n \cdot W^n$ is a BMO martingale, we denote

$$\Phi(x) = \frac{1}{\gamma^2}(e^{\gamma|x|} - \gamma|x| - 1).$$

We apply Itô's formula to $\Phi(Y_t^n)$ to get

$$\begin{aligned}
\Phi(Y_t^n) &= \Phi(Y_T^n) + \int_t^T \Phi'(Y_s^n) f_s(Y_s^n, \mu_s^{n-1}, Z_s^n)\,ds\\
&\quad - \int_t^T \Phi'(Y_s^n) Z_s^n\,dW_s^n - \frac{1}{2}\int_t^T \Phi''(Y_s^n)|Z_s^n|^2\,ds\\
&\le \Phi(\xi^n) + \int_t^T L_1|\Phi'(Y_s^n)|(|Y_s^n| + \|\mu_s^{n-1}\|)\,ds\\
&\quad - \int_t^T \Phi'(Y_s^n) Z_s^n\,dW_s^n + \frac{1}{2}\int_t^T (\gamma|\Phi'(Y_s^n)| - \Phi''(Y_s^n))|Z_s^n|^2\,ds.
\end{aligned}$$

Taking conditional expectation, we get

$$\begin{aligned}
\Phi(Y_t^n) + \frac{1}{2}\mathbb{E}_t\int_t^T |Z_s^n|^2\,ds &\le \Phi(K) + \mathbb{E}_t\int_t^T L_1|\Phi'(Y_s^n)|(|Y_s^n| + \|\mu_s^{n-1}\|)\,ds\\
&\le \Phi(K) + L_1|\Phi'(C)|\mathbb{E}_t\int_t^T (|Y_s^n| + \|\mu_s^{n-1}\|)\,ds\\
&\le \Phi(K) + 2CTL_1|\Phi'(C)|.
\end{aligned}$$

Thus we know that $Z^n \cdot W^n$ is a BMO martingale.

Finally, we give the proof to Theorem 2.2.

Proof of Theorem 2.2 We use notations similar to Section 2.1 that $\delta Y_t^n := Y_t^n - \overline{Y}_t^n$, $\delta Z_t^n := Z_t^n - \overline{Z}_t^n$. From BSDE (1.1) and (2.2), we have

$$\begin{cases} -\delta Y_t^n = [f_t(Y_t^n, \mu_t^{n-1}, Z_t^n) - f_t(\overline{Y}_t^n, \mu_t, \overline{Z}_t^n)]\,dt - \delta Z_t^n\,dW_t^n,\\ \delta Y_T^n = 0. \end{cases} \tag{4.1}$$

Similar to Proposition 4.3, we use Girsonov transform to deal with the quadratic term of Z_t^n. To be specific, we write

$$\begin{aligned}
&f_t(Y_t^n, \mu_t^{n-1}, Z_t^n) - f_t(\overline{Y}_t^n, \mu_t, \overline{Z}_t^n)\\
&= f_t(Y_t^n, \mu_t^{n-1}, Z_t^n) - f_t(\overline{Y}_t^n, \mu_t, Z_t^n)\\
&\quad + f_t(\overline{Y}_t^n, \mu_t, Z_t^n) - f_t(\overline{Y}_t^n, \mu_t, \overline{Z}_t^n).
\end{aligned}$$

For the first two term, we know by Assumption 2.2 that there exists a process Γ_t^n such that

$$f_t(Y_t^n, \mu_t^{n-1}, Z_t^n) - f_t(Y_t^n, \mu_t^{n-1}, \overline{Z}_t^n) = \Gamma_t^n \delta Z_t^n,$$

where

$$|\Gamma_t^n| \leq \phi(|\overline{Y}_t^n| \vee \|\overline{Y}_t^n\|_{\mathcal{L}_2})(1 + \|Z_t^n\| + \|\overline{Z}_t^n\|).$$

Thanks to [48, Theorem 2.3], $\|\overline{Z}^n\|$ belongs to $\mathscr{Z}^2[0,T]$. Hence, we know by Proposition 4.3 that $|\Gamma_t^n|$ belongs to $\mathscr{Z}^2[0,T]$. We define $\widetilde{W}_t^n := W_t^n - \int_0^t \Gamma_s \, \mathrm{d}s$ and $\widetilde{W}_t^n$ is a Brownian motion under $\mathbb{P}^n$ with

$$\frac{\mathrm{d}\mathbb{P}^n}{\mathrm{d}\mathbb{P}} = \mathscr{E}(\Gamma^n \cdot W^n)_0^T.$$

Thus, we have

$$Y_t^n = \int_t^T [f_s(Y_s^n, \mu_s^{n-1}, Z_s^n) - f_s(\overline{Y}_s^n, \mu_s, Z_s^n)] \, \mathrm{d}s - \int_t^T Z_s^n \, \mathrm{d}\widetilde{W}_s^n$$

By Proposition 3.1, there exists a constant $C(p,\varepsilon)$ which depends only on p, ε, L_1 such that

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{t\in[0,T]} \mathrm{e}^{2\lambda pt}|\delta Y_t^n|^{2p} + \Big(\int_0^T \mathrm{e}^{2\lambda t}\|\delta Z_t^n\|^2 \, \mathrm{d}t\Big)^p\Big] \\ &\leq C_{p,\varepsilon}\mathbb{E}^{\mathbb{P}^n}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1}, \mu_t) \, \mathrm{d}t\Big)^{2p}\Big]. \end{aligned}$$

Moreover, if $C_\lambda := 2\lambda - \frac{L_1^2}{\varepsilon} - 2L_1 > 0$, we have

$$\mathbb{E}^{\mathbb{P}^n}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t}|\delta Y_t^n|^2 \, \mathrm{d}t\Big)^p\Big] \leq \frac{C_{p,\varepsilon}}{C_\lambda^p}\mathbb{E}^{\mathbb{P}^n}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\mu_t^{n-1}, \mu_t) \, \mathrm{d}t\Big)^{2p}\Big]. \tag{4.2}$$

Since (4.2) has the same form as (3.4), we can use the same method to get similar estimates. The only difference is that when we estimate (4.2) in a similar way, we need to estimate $\mathbb{E}^{\mathbb{P}^n}[\mathcal{W}_2(\widehat{\mu}_t^n, \mu_t)^{2p}]$ instead of $\mathbb{E}[\mathcal{W}_2(\widehat{\mu}_t^n, \mu_t)^{2p}] \lesssim \theta_n^{\gamma p}$ (see (3.5)).

Notice that $\|\Gamma^n \cdot W^n\|_{BMO}$ and $\|\Gamma^n \cdot W^n\|_{BMO_q}$ are equivalent for $q > 2$. So there exists a constant $q_1 > 1$ such that

$$\mathbb{E}[(\mathscr{E}(\Gamma^n \cdot W^n)_0^T)^{q_1}] \leq C_{q_1}.$$

So we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^n}\Big[\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\widehat{\mu}_t^{n-1}, \mu_t) \, \mathrm{d}t\Big)^{2p}\Big] \\ &\leq \mathbb{E}\Big[\mathscr{E}(\Gamma^n \cdot W^n)_0^T \cdot \Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\widehat{\mu}_t^{n-1}, \mu_t) \, \mathrm{d}t\Big)^{2p}\Big] \\ &\leq [\mathbb{E}(\mathscr{E}(\Gamma^n \cdot W^n)_0^T)^{q_1}]^{\frac{1}{q_1}}\Big[\mathbb{E}\Big(\int_0^T \mathrm{e}^{\lambda t}\mathcal{W}_2(\widehat{\mu}_t^{n-1}, \mu_t) \, \mathrm{d}t\Big)^{2pp_1}\Big]^{\frac{1}{p_1}}, \end{aligned} \tag{4.3}$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$. If we denote $\gamma_1 = \frac{1}{2+(1-\frac{1}{pp_1})d}$, then

(1) if $\overline{\alpha} < (2 - \alpha_\infty)$, we have

$$\sup_{0\leq t\leq T} \mathbb{E}^{\mathbb{P}^n}[\mathcal{W}_2(\mu_t^n, \mu_t)^{2p} + \mathcal{W}_2(\mathcal{L}(Y_t^n), \mu_t)^{2p}] \leq C\,\mathrm{e}^{CT}\alpha_n^{\gamma_1 p};$$

(2) if $\overline{\alpha} \geq 2$, we have, for any $\delta < 1 \wedge \gamma_1 \underline{\alpha} \wedge 2\gamma_1$,

$$\sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{P}^n}[\mathcal{W}_2(\mu_t^n, \mu_t)^{2p} + \mathcal{W}_2(\mathcal{L}(Y_t^n), \mu_t)^{2p}] \leq C\, \mathrm{e}^{CT} \mathrm{e}^{-p\delta \sum\limits_{i=1}^{n} \alpha_i}.$$

On the other side, since $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^n} = \mathscr{E}(-\Gamma^n \cdot W^n)_0^T$, we can know that there exists $q_2 > 1$ such that

$$\mathbb{E}^{\mathbb{P}^n}[(\mathscr{E}(-\Gamma^n \cdot W^n)_0^T)^{q_2}] \leq C_{q_2}.$$

Similarly, we have

$$\mathbb{E}[\mathcal{W}_2^{2p}(\mu_t^n, \mu_t)] \leq C_{q_2} [\mathbb{E}^{\mathbb{P}^n}(\mathcal{W}_2^{2p}(\mu_t^n, \mu_t))^{p_2}]^{\frac{1}{p_2}}. \tag{4.4}$$

Combining (4.3) and (4.4), we have

$$\mathbb{E}\Big[\Big(\int_0^T \mathrm{e}^{2\lambda t} |\delta Y_t^n|^2 \,\mathrm{d}t\Big)^p\Big] \leq \frac{C_{p,\varepsilon}}{C_\lambda^p} \Big[\mathbb{E}\Big(\int_0^T \mathrm{e}^{\lambda t} \mathcal{W}_2(\mu_t^{n-1}, \mu_t) \,\mathrm{d}t\Big)^{2pp_1p_2}\Big]^{\frac{1}{p_1p_2}}.$$

We denote $\gamma_2 = \frac{1}{2 + (1 - \frac{1}{pp_1p_2})d}$, then

(1) if $\overline{\alpha} < (2 - \alpha_\infty)$, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}[\mathcal{W}_2(\mu_t^n, \mu_t)^{2p} + \mathcal{W}_2(\mathcal{L}(Y_t^n), \mu_t)^{2p}] \leq C\, \mathrm{e}^{CT} \alpha_n^{\gamma_2 p};$$

(2) if $\overline{\alpha} \geq 2$, we have, for any $\delta < 1 \wedge \gamma_2 \underline{\alpha} \wedge 2\gamma_2$,

$$\sup_{0 \leq t \leq T} \mathbb{E}[\mathcal{W}_2(\mu_t^n, \mu_t)^{2p} + \mathcal{W}_2(\mathcal{L}(Y_t^n), \mu_t)^{2p}] \leq C\, \mathrm{e}^{CT} \mathrm{e}^{-p\delta \sum\limits_{i=1}^{n} \alpha_i}.$$

5 Numerical Experiments

In this section, we provide numerical examples to show the behavior of our SPoC algorithm. For simplicity, we uniformly divide the time interval $[0, T]$ into M parts with time step $\Delta t = \frac{T}{M}$. We define a sequential update scheme Υ which takes the $(n-1)$-th SPoC iterate and produces the n-th SPoC result, i.e.,

$$\Upsilon : \mu^{n-1} \to \mu^n.$$

The update scheme is defined as follows:

(1) Given the $n-1$ iterate distribution μ_t^{n-1}, we solve the following SDE to obtain Y^n and Z^n,

$$\begin{cases} \mathrm{d}Y_t^n = -f(t, Y_t^n, Z_t^n, \mu_t^{n-1}) \,\mathrm{d}t + Z_t^n \,\mathrm{d}W_t^n, \quad t \in [0, T], \\ Y_T^n = \xi^n. \end{cases} \tag{5.1}$$

(2) We then update the distribution term by

$$\mu_t^n = \mu_t^{n-1} + \alpha_n(\delta_{Y_t^n} - \mu_t^{n-1}),$$

which gives us μ_t^n.

By iterating this update scheme, we obtain a sequence of iterates $\{\mu_t^n\}_{n=1}^N$ that converges to the true solution μ_t as $N \to \infty$. The behavior of the algorithm depends on the choice of the step size α_n, which we will demonstrate with numerical examples in the following.

Algorithm 1 Framework of SPoC algorithm of McKean-Vlasov BSDE systems

Require: The terminal condition ξ, the update rate $\{\alpha_n\}$;
Initialize $n = 1$, $\mu^1_{t_m} = \delta_0$, $Y^1_T = \xi^1$ and $Z^1_T = 0$;
repeat
  $n = n + 1$;
  $Y^n_T = \xi^n$ and $Z^n_T = 0$;
  $\mu^n_T = \mu^{n-1}_T + \alpha_n(\delta_{Y^n_T} - \mu^{n-1}_T)$;
  Use backward Euler schemes to generate a path of Y^n by
  for $m = M - 1$ **to** 1 **do**
    $Y^n_{t_m} = \mathbb{E}[Y^n_{t_{m+1}}|\mathscr{F}^n_{t_m}] + f(t, Y^n_{t_{m+1}}, Z^n_{t_{m+1}}, \mu^{n-1}_{t_{m+1}})\Delta t$;
    $Z^n_{t_m} = \mathbb{E}[(\Delta W^n_{t_{m+1}})^{-1}(Y^n_{t_{m+1}} + f(t, Y^n_{t_{m+1}}, Z^n_{t_{m+1}}, \mu^{n-1}_{t_{m+1}})\Delta t - Y^n_{t_m})|\mathscr{F}^n_{t_m}]$;
   Update the empirical measure by
    $\mu^n_{t_m} = \mu^{n-1}_{t_m} + \alpha_n(\delta_{Y^n_{t_m}} - \mu^{n-1}_{t_m})$;
   $m = m - 1$;
  end for
until the end condition.

Remark 5.1 In the first step of the SPoC algorithm, since μ^{n-1} is fixed, the numerical solution of (5.1) can be obtained by classical methods, for example, tree algorithm (see [28]), Least-Squares Monte Carlo method (see [54–55]), θ method, multi-step schemes (see [24, 56–57]) or Deep BSDE method (see [32–33]). We give the pseudo-code in Algorithm 1. Since there are different methods for solving BSDEs numerically, we chose the simplest tree method (Algorithm 2) to clearly demonstrate the effectiveness of SPoC algorithm. It's worth noting that more complex and accurate numerical methods can be used in practice to solve the BSDE in the first step.

Remark 5.2 Since mean-field BSDE has PoC properties, we can also approximate it directly by solving a high-dimensional BSDE, i.e.,

$$\begin{cases} \mathrm{d}Y^{i,N}_t = -f(t, Y^{i,N}_t, Z^{i,i,N}_t, \mu^N_t)\,\mathrm{d}t + \displaystyle\sum_{j=1}^{N} Z^{i,j,N}_t\,\mathrm{d}W^{j,N}, \quad t \in [0,T], \\ Y^{i,N}_T = \xi^{i,N}, \\ \mu^N_t = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \delta_{Y^{j,N}_t}. \end{cases} \tag{5.2}$$

This is a system of equations for N coupled d-dimensional BSDE, which can be seen as a BSDE in $(\mathbb{R}^d)^N$. While the SPoC method only needs to solve a d-dimensional BSDE (5.1) at a time. To achieve the same accuracy as the PoC method (5.2), the SPoC method requires about N updates. In summary, solving mean field BSDE based on the PoC method requires solving BSDE in $(\mathbb{R}^d)^N$ once, while the SPoC method requires solving a d-dimensional BSDE N times. Moreover, the SPoC method provides a sustainable iterative way to continuously approach the mean-field limit, whereas the PoC method needs to determines the total number of particles before calculation. As more and more particles are added (without affecting existing particles), the approximation accuracy is continuously improved until a desired level is reached. That is to say, the required number of particles no longer needs to be specified in advance.

Algorithm 2 A tree-based approach of SPoC algorithm

Require: The terminal condition ξ, the update rate $\{\alpha_n\}$;
Initialize $n = 1$, $\mu^1_{t_m} = \delta_0$, $Y^1_T = \xi^1$ and $Z^1_T = 0$;
repeat
  $n = n + 1$;
  Use tree-based approach to solve classical BSDE
    $\mathrm{d}Y^n_t = -f(t, Y^n_t, Z^n_t, \mu^{n-1}_t)\,\mathrm{d}t + Z^n_t\,\mathrm{d}W_t$,
  for $m = M - 1$ **to** 1 **do**
    Update the empirical measure by
      $\mu^n_{t_m} = \mu^{n-1}_{t_m} + \alpha_n(\delta_{Y^n_{t_m}} - \mu^{n-1}_{t_m})$;
  end for
until the end condition.

In the following we show the performance of the SPoC algorithm with some examples.

Example 5.1 Consider the mean-field BSDE

$$\begin{cases} \mathrm{d}Y_t = -\Big(\dfrac{1}{2}Y_t - Z_t + \mathbb{E}Y_t - \mathrm{e}^{-\frac{1}{2}t}\sin t\Big)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad t \in [0, T], \\ Y_T = \sin(W_T + T), \end{cases} \tag{5.3}$$

which has an analytical solution of $(Y_t, Z_t) = (\sin(W_t + t), \cos(W_t + t))$.

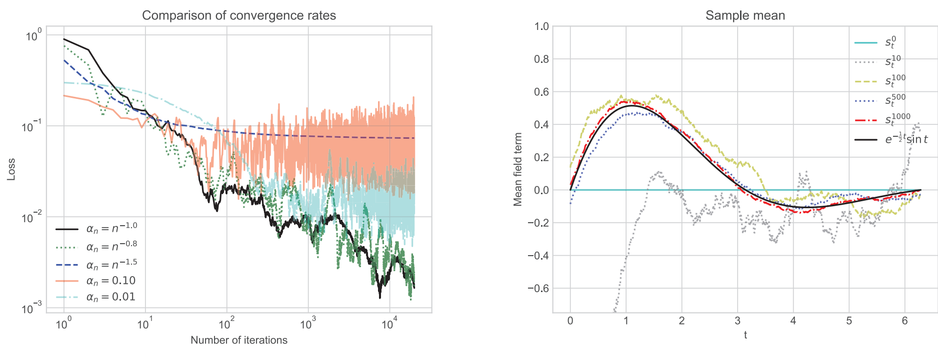

Figure 1 Comparison of convergence rates in Example 5.1.

In our numerical experiments, we set $T = 2\pi$ and divide the interval $[0, T]$ into $M = 100$ uniform parts. For a fixed empirical distribution, we use the tree method to solve a standard BSDE and update the empirical distribution with the results of the solution. In this example, the empirical distribution interacts in the form of $\mathbb{E}Y$, so we can solve (5.3) by replacing $\mathbb{E}Y_t$ with s^{n-1}_t, and then update $s^n_t = s^{n-1}_t + \alpha_n(Y^n_t - s^{n-1}_t)$. We iterate 10^5 times and generate 10 particle orbits each time. We record the sample mean of the empirical measures.

In Figure 1, we compare the convergence rates of different update steps α_n (left) as well as the sample mean s^n_t (right) with the analytical solution $\mathbb{E}Y_t = \mathbb{E}\sin(W_t + t) = \mathrm{e}^{-\frac{1}{2}t}\sin t$. In the left figure, we let the loss function be $\mathrm{Loss} = \frac{1}{T}\int |s^n_t - \mathbb{E}Y_t|\,\mathrm{d}t$. With steps size $\alpha_n = n^{-1}$ or $\alpha_n = n^{-0.8}$, the empirical distribution gradually converges to the true distribution. When the step size $\alpha_n = n^{-1.5}$, the error between the empirical distribution and the true distribution converges to a fixed constant because $\sum n^{-1.5} < \infty$. When the update step is a constant, i.e., $\alpha_n = 0.1$ or $\alpha_n = 0.01$, then the Loss function oscillates at a fixed error. In the right figure, we draw the image of s_t^n to see the process of the SPoC system approaching the mean-field system intuitively. At the beginning, s_t^0 initializes to 0. At the 50th iteration, the difference between s_t^{50} and the true value is large. By the 500th iteration, s_t^{500} is close to $\mathrm{e}^{-\frac{1}{2}t}\sin t$. To sum up, as the number of particles increases, s_t^n gradually converges to $\mathbb{E}Y_t$.

In Figure 2, we take $T = \pi$ and sample 5 Brownian motions and show the corresponding paths of Y_t and Z_t. From the two graphs below, we observe that both Y_t and Z_t are contained in $[-1, 1]$. In the upper right figure, we further calculate the value of $Y_t^2 + Z_t^2$ in each path, which is consistent with the analytical solution $\sin^2(W_t + t) + \cos^2(W_t + t) = 1$.

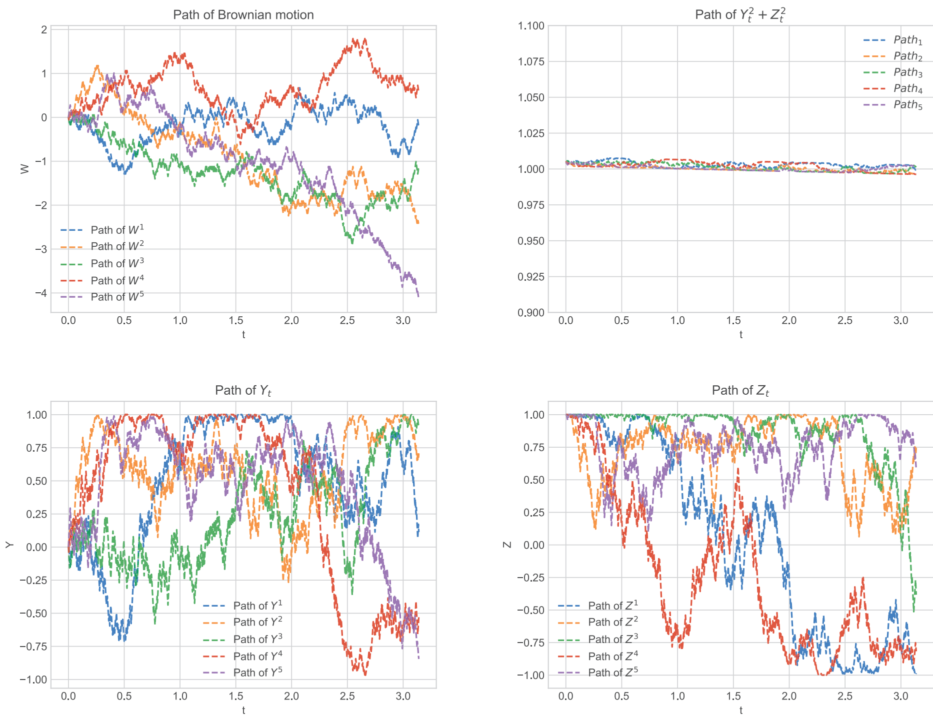

Figure 2 Path sampling for SPoC system.

Example 5.2 Consider the mean-field BSDE,

$$\begin{cases} \mathrm{d}Y_t = -\Big(Y_t - Z_t - \sqrt{\mathbb{E}Y_t^2} + \sqrt{\frac{1}{2}\mathrm{e}^{2t} - \cos(4t)}\Big)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad t \in [0, T], \\ Y_T = \mathrm{e}^{W_T}\sin W_T, \end{cases} \tag{5.4}$$

which has an analytical solution of $(Y_t, Z_t) = (\mathrm{e}^{W_t}\sin W_t, \mathrm{e}^{W_t}(\sin W_t + \cos W_t))$.

In this numerical experiment, we take $T = \pi$, with all other settings the same as before. It is worth noting that for a normal distribution $\xi \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$\mathbb{E}(\mathrm{e}^{a\xi}\cos(b\xi)) = \cos(b(\mu + a\sigma^2))\,\mathrm{e}^{a\mu + \frac{\sigma^2}{2}(a^2 - b^2)}.$$

In particular, for Brownian motion W_t we have $\mathbb{E}(\mathrm{e}^{aW_t}\cos(bW_t)) = \cos(abt)\,\mathrm{e}^{\frac{a^2-b^2}{2}t}$.

In Figure 3, we compare the convergence rates of different update steps α_n (left) as well as the sample mean s_t^n (right) compared with the analytical solution $\mathbb{E}Y_t = \mathbb{E}\,\mathrm{e}^{W_t}\sin W_t = \sin t$. As in Example 5.1, the comparison of convergence rates is displayed in the left figure. In the right figure, we plot the image of s_t^n to show that as the number of particles increases, s_t^n gradually converges to $\mathbb{E}Y_t$.

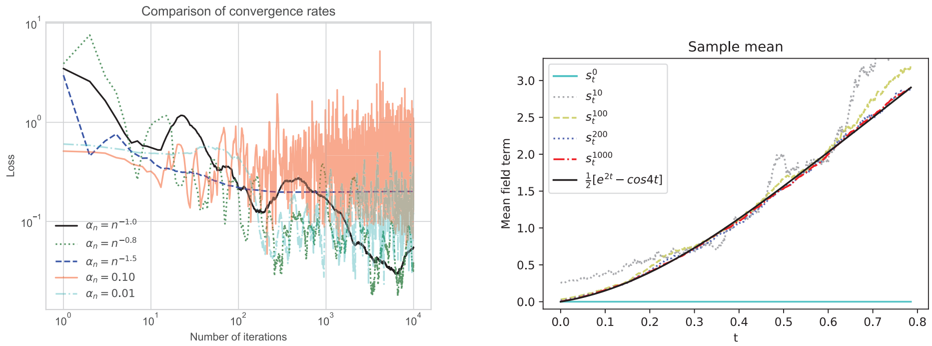

Figure 3 Comparison of convergence rates in Example 5.2.

Example 5.3 Consider the mean-field BSDE

$$\begin{cases} \mathrm{d}Y_t = \left[\left(\dfrac{1}{2} - Y_t\right)Z_t + \mathbb{E}Y_t - \dfrac{1}{2}\right]\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad t \in [0,T], \\ Y_T = \dfrac{1}{1+\mathrm{e}^{-W_T}}, \end{cases} \tag{5.5}$$

which has the analytical solution $(Y_t, Z_t) = \big(\frac{1}{1+\mathrm{e}^{-W_t}}, \frac{\mathrm{e}^{-W_t}}{(1+\mathrm{e}^{-W_t})^2}\big)$.

In this numerical experiment, we take $T = 1$, with all other settings the same as before. Noting that $f(x) := \frac{1}{1+\mathrm{e}^{-x}}$ is symmetric with respect to $\big(0, \frac{1}{2}\big)$, i.e.,

$$f(x) + f(-x) = 1.$$

In particular, for Brownian motion W_t we have $\mathbb{E}f(W_t) = \frac{1}{2}$. In Figure 4, we compare the convergence rates of different update steps α_n (left) as well as the sample mean s_t^n (right) compared with the analytical solution $\mathbb{E}Y_t = \frac{1}{2}$.

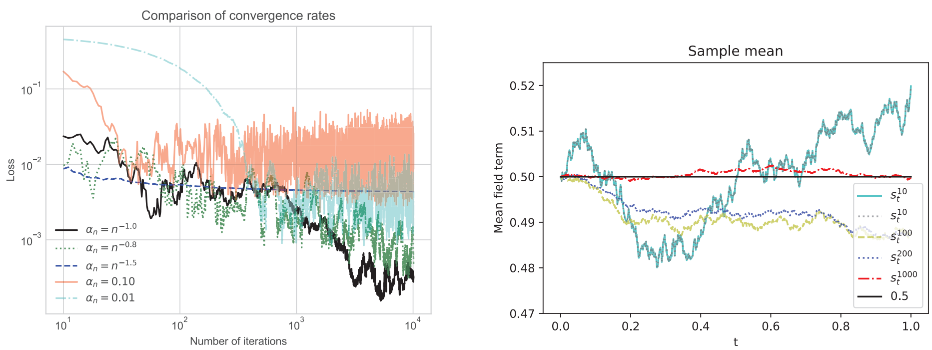

Figure 4 Comparison of convergence rates in Example 5.3.

A Auxiliary Lemma

In this section we apply the ODE method to prove several estimates for sequences that satisfy certain recursive relations.

Lemma A.1 *Let $\alpha(\cdot), \beta(\cdot) : C^1([1,\infty) \to (0,\infty))$ be decreasing, and*

$$\underline{c} := \liminf_{t\to\infty} \frac{-\beta'(t)}{\alpha(t)\beta(t)}, \quad \overline{c} := \limsup_{t\to\infty} \frac{-\beta'(t)}{\alpha(t)\beta(t)}. \tag{A.1}$$

Let $\varepsilon > 0$. Then, the function $y(\cdot) : [1,\infty) \to (0,\infty)$ satisfying

$$y'(t) = -\varepsilon\alpha(t)y(t) + \alpha(t)\beta(t)$$

enjoys the estimate

(1) *if $\varepsilon < \underline{c}$, we have*

$$y(t) \le C\,\mathrm{e}^{-\varepsilon\int_1^t \alpha(s)\,\mathrm{d}s}, \quad \forall t > 1.$$

(2) *If $\underline{c} \le \varepsilon \le \overline{c}$, we have, for any $\delta < \underline{c}$,*

$$y(t) \le C\,\mathrm{e}^{-\delta\int_1^t \alpha(s)\,\mathrm{d}s}, \quad \forall t > 1.$$

(3) *If $\varepsilon > \overline{c}$ we have*

$$y(t) \le C\beta(t), \quad \forall t > 1.$$

Comprehensively, we can combine the above three estimates as

$$y(t) \le C[\mathrm{e}^{-\delta\int_1^t \alpha(s)\,\mathrm{d}s} + \beta(t)], \quad \forall t > 1, \tag{A.2}$$

as long as $\delta \le \varepsilon$ and $\delta \notin [\underline{c}, \overline{c}]$, where $C = C(\alpha, \beta, \delta)$. Moreover, if $y(1) \ge \varepsilon^{-1}\beta(1)$, then

$$y(t) \ge \varepsilon^{-1}\beta(t), \quad \forall\, t \ge 1,$$

and consequently, $y(\cdot)$ is decreasing on $[1,\infty)$.

Remark A.1 The number δ cannot always be taken to equal ε. For example, if $\varepsilon = 1$ and $\alpha(t) = \beta(t) = t^{-1}$, then one can derive that $y(t) = Ct^{-1} + t^{-1}\ln t \sim O(t^{-1}\ln t)$, which is greater than $O(t^{-1})$.

If $\delta < \underline{c}$, without loss of generality, we may assume that $\delta < \frac{-\beta'(t)}{\alpha(t)\beta(t)}, \forall\, t \ge 1$. Then we have $-\delta\alpha(t) > [\ln\beta(t)]'$ and $\beta(t) \lesssim \mathrm{e}^{-\delta\int_1^t \alpha(s)\,\mathrm{d}s}$. On the other hand, if $\delta > \overline{c}$, we have $\mathrm{e}^{-\delta\int_1^t \alpha(s)\,\mathrm{d}s} \lesssim \beta(t)$.

Proof Let us prove the second part first. We take $y(1) \ge \varepsilon^{-1}\beta(1)$. Suppose $t_0 = \inf\{t > 1 : y(t) < \varepsilon^{-1}\beta(t)\} < \infty$. Then $y(t_0) = \varepsilon^{-1}\beta(t_0)$ and there is a number $\eta > 0$ such that $y(t) - \varepsilon^{-1}\beta(t) < 0$ for all $t \in (t_0, t_0+\eta]$. From the mean value theorem, there is a $t_1 \in (t_0, t_0+\eta)$ such that

$$(y - \varepsilon^{-1}\beta)'(t_1) = \frac{1}{\eta}[(y - \varepsilon^{-1}\beta)(t_0+\eta) - (y - \varepsilon^{-1}\beta)(t_0)] < 0,$$

so $y'(t_1) < \varepsilon^{-1}\beta'(t_1) \le 0$; meanwhile, the equation tells

$$y'(t_1) = -\varepsilon\alpha(t_1)(y(t_1) - \varepsilon^{-1}\beta(t_1)) > 0,$$

which leads to a contradiction.

Now let $w(t) = \mathrm{e}^{\int_1^t \alpha(s)\,\mathrm{d}s}$. Since $\delta \le \varepsilon$, it follows from comparison (a similar argument as above) that $y(t) \le z(t)$ for all $t \ge 1$, where $z(\cdot)$ satisfies

$$z'(t) = -\delta\alpha(t)z(t) + \alpha(t)\beta(t), \quad z(1) = y(1),$$

which has the explicit solution

$$z(t) = y(1)w(t)^{-\delta} + w(t)^{-\delta}\int_1^t \alpha(s)w(s)^\delta\beta(s)\,\mathrm{d}s, \quad t \ge 1. \tag{A.3}$$

We compute

$$\begin{aligned}\int_1^t \alpha(s)w(s)^\delta\beta(s)\,\mathrm{d}s &= \int_1^t w(s)^{\delta-1}\beta(s)\,\mathrm{d}w(s)\\ &= \frac{1}{\delta}w(t)^\delta\beta(t) - \frac{1}{\delta}\beta(1) - \int_0^t \frac{1}{\delta}w(s)^\delta\beta'(s)\,\mathrm{d}s. \end{aligned}\tag{A.4}$$

In the case $\varepsilon_1 := \underline{c} - \delta > 0$, there is $t_1 > 1$ such that for all $s > t_1$,

$$-\frac{1}{\delta}\beta'(s) \ge \frac{1}{\delta}\Big(\underline{c} - \frac{\varepsilon_1}{2}\Big)\alpha(s)\beta(s) = \Big(\frac{\underline{c}}{2\delta} - \frac{1}{2}\Big)\alpha(s)\beta(s) + \alpha(s)\beta(s),$$

which along with (A.4) implies

$$\Big(\frac{\underline{c}}{2\delta} - \frac{1}{2}\Big)\int_1^t \alpha(s)w(s)^\delta\beta(s)\,\mathrm{d}s \le \frac{1}{\delta}\beta(1) - \frac{1}{\delta}w(t)^\delta\beta(t) \le \frac{1}{\delta}\beta(1),$$

and from (A.3) we have

$$z(t) \le \Big[y(t_1) + \frac{2}{\underline{c} - \delta}\beta(t_1)\Big]w(t)^{-\delta}, \quad t > t_1.$$

In the case $\varepsilon_2 := \delta - \overline{c} > 0$, there is $t_2 > 1$ such that for all $s > t_2$,

$$-\frac{1}{\delta}\beta'(s) \le \frac{1}{\delta}\Big(\overline{c} + \frac{\varepsilon_2}{2}\Big)\alpha(s)\beta(s) = -\Big(\frac{1}{2} - \frac{\overline{c}}{2\delta}\Big)\alpha(s)\beta(s) + \alpha(s)\beta(s).$$

Then one can similarly obtain

$$\Big(\frac{1}{2} - \frac{\underline{c}}{2\delta}\Big)\int_1^t \alpha(s)w(s)^\delta\beta(s)\,\mathrm{d}s \le \frac{1}{\delta}w(t)^\delta\beta(t),$$

which yields the estimate

$$z(t) \le y(t_2)w(t)^{-\delta} + \frac{2}{\delta - \underline{c}}\beta(t), \quad t > t_2.$$

So the estimate (A.2) is proved. The proof is complete.

Corollary A.1 *Under the setting of Lemma* A.1, *we define* $\alpha_n = \alpha(n)$ *and* $\beta_n = \beta(n)$, *and assume that*

$$K_{\alpha,\beta} := \sup_{n>1} \frac{\alpha_n \beta_n}{\alpha_{n+1}\beta_{n+1}} < \infty.$$

Then, if $x_n, s_n > 0$ *(set* $s_0 = 0$*) satisfy*

$$\begin{aligned} &x_n \le (1-\varepsilon)s_{n-1} + \beta_n, \\ &s_n = s_{n-1} + \alpha_n(x_n - s_{n-1}), \quad n \ge 1, \end{aligned}$$

one has that

(1) *if* $\varepsilon < \underline{c}$, *we have*

$$s_n \le C\,\mathrm{e}^{-\varepsilon \sum\limits_{i=1}^{n} \alpha_i}.$$

(2) *If* $\underline{c} \le \varepsilon \le \overline{c}$, *we have, for any* $\delta < \underline{c}$,

$$s_n \le C\,\mathrm{e}^{-\delta \sum\limits_{i=1}^{n} \alpha_i}.$$

(3) *If* $\varepsilon > \overline{c}$, *we have*

$$s_n \le C\beta_n.$$

In short, we have

$$s_n \le C\Big[\mathrm{e}^{-\delta \sum\limits_{i=1}^{n} \alpha_i} + \beta_n\Big],$$

as long as $\delta \le \varepsilon$ *and* $\delta \notin [\underline{c}, \overline{c}]$.

Proof From the recursive relation one has that

$$s_n \le (1 - \varepsilon\alpha_n)s_{n-1} + \alpha_n\beta_n, \quad n \ge 1.$$

Let $y(\cdot)$ be the function satisfying $y(1) = \max\{s_1, \varepsilon^{-1}\beta(1)\}$ and

$$y'(t) = -\varepsilon\alpha(t+1)y(t) + K_{\alpha,\beta}\alpha(t+1)\beta(t+1).$$

By assuming $s_n \le y(n)$ for an $n \ge 1$, one computes

$$\begin{aligned} s_{n+1} &\le (1 - \varepsilon\alpha_{n+1})s_n + \alpha_{n+1}\beta_{n+1} \\ &\le (1 - \varepsilon\alpha_{n+1})y(n) + \alpha_{n+1}\beta_{n+1} \\ &= y(n) - \varepsilon\alpha(n+1)y(n) + \alpha(n+1)\beta(n+1) \\ &\le y(n) - \varepsilon\alpha(n+1)y(n) + K_{\alpha,\beta}\alpha(n+2)\beta(n+2) \\ &\le y(n) + \int_n^{n+1} [-\varepsilon\alpha(t+1)y(t) + K_{\alpha,\beta}\alpha(t+1)\beta(t+1)]\,\mathrm{d}t \\ &= y(n+1). \end{aligned}$$

By induction and Lemma A.1, one has

$$
\begin{aligned}
s_n \le y(n) &\le C\,\mathrm{e}^{-\varepsilon\int_1^n \alpha(s+1)\,\mathrm{d}s} + C\beta(n+1) \\
&\le C\,\mathrm{e}^{-\varepsilon\sum\limits_{i=3}^{n+1}\alpha_i} + C\beta_n.
\end{aligned}
$$

The proof is complete.

Corollary A.2 *Let α_n be the sequence defined at the beginning of Section 2. Define*

$$
w_1 = 1; \quad w_n = \frac{\alpha_n}{1-\alpha_n}\sum_{i=1}^{n-1} w_i, \quad n \ge 2. \tag{A.5}
$$

Then it holds that

(1) *if $\overline{\alpha} \ge 2$, we have, for any $\delta < \underline{\alpha} \wedge 2$,*

$$
\frac{\sum\limits_{i=1}^{n} w_i^2}{\big(\sum\limits_{i=1}^{n} w_i\big)^2} \le C\,\mathrm{e}^{-\delta\sum\limits_{i=1}^{n}\alpha_i}.
$$

(2) *If $2-\alpha_\infty > \overline{\alpha}$, we have*

$$
\frac{\sum\limits_{i=1}^{n} w_i^2}{\big(\sum\limits_{i=1}^{n} w_i\big)^2} \le C\alpha_n.
$$

Remark A.2 For the first case, we should have considered the case $2-\alpha_\infty \le \overline{\alpha}$, and then taken $\delta < \underline{\alpha} \wedge (2-\alpha_\infty)$. But the case $2-\alpha_\infty \le \overline{\alpha} < 2$ never appears because $\overline{\alpha} = 0$ as long as $\alpha_\infty > 0$. So $2-\alpha_\infty \le \overline{\alpha}$ means $\alpha_\infty = 0$ and $\overline{\alpha} \ge 2$.

Proof Notice that $w_n = \alpha_n \sum\limits_{i=1}^{n} w_i$. Letting

$$
s_n := \frac{\sum\limits_{i=1}^{n} w_i^2}{\big(\sum\limits_{i=1}^{n} w_i\big)^2},
$$

one has that

$$
\begin{aligned}
s_n &= \frac{(\sum\limits_{i=1}^{n-1} w_i)^2}{(\sum\limits_{i=1}^{n} w_i)^2} \cdot \frac{\sum\limits_{i=1}^{n-1} w_i^2}{(\sum\limits_{i=1}^{n-1} w_i)^2} + \frac{w_n^2}{(\sum\limits_{i=1}^{n} w_i)^2} \\
&= \Bigg(1 - \frac{w_n}{\sum\limits_{i=1}^{n} w_i}\Bigg)^2 s_{n-1} + \alpha_n^2 = (1-2\alpha_n+\alpha_n^2)s_{n-1} + \alpha_n^2 \\
&= [1-(2-\alpha_n)\alpha_n]s_{n-1} + \alpha_n^2.
\end{aligned}
$$

Then from Corollary A.1, it follows that for $\delta < 2 - \alpha_\infty$,

$$s_n \leq C\big[\,\mathrm{e}^{-\delta \sum\limits_{i=1}^{n} \alpha_i} + \alpha_n\big]$$

as long as $\delta \notin [\underline{\alpha}, \overline{\alpha}]$.

The convergence of (unweighted) empirical measures in the Wasserstein distance has been extensively investigated in the literature (see [40–42] and references therein). Here we prove an estimate for weighted empirical measures, based on the following density coupling lemma that can be proved analogously as [40, Lemma 2.2].

Lemma A.2 *Let f and g be probability density functions on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} |x|^r [f(x) + g(x)]\,\mathrm{d}x < \infty, \quad r \geq 1,$$

and define $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ and $\nu(\mathrm{d}x) = g(x)\,\mathrm{d}x$. Then one has

$$\mathcal{W}_r^r(\mu, \nu) \leq C_r \int_{\mathbb{R}^d} |x|^r |f(x) - g(x)|\,\mathrm{d}x.$$

Then we have the following convergence result.

Lemma A.3 *Let $r \in [1, \infty)$ and $p \in [2, \infty)$ and ξ_n be i.i.d. $\mathbb{R}^d$-valued random variables and $m := \mathcal{L}(\xi_1) \in \mathscr{P}_q$ with $q > rp + (p-1)d$, and let*

$$m_n := \frac{\sum\limits_{i=1}^{n} w_i \delta_{\xi_i}}{\sum\limits_{i=1}^{n} w_i} \quad \textit{with } w_n > 0,\ n = 1, 2, \cdots.$$

Then there is a constant C depending only on r, p, d and $\|\xi_1\|_{L^q}$, such that

$$\mathbb{E}\mathcal{W}_r^{rp}(m_n, m) \leq C\theta_n^{\gamma_{r,p}}$$

with

$$\theta_n = \frac{\sum\limits_{i=1}^{n} w_i^2}{\big(\sum\limits_{i=1}^{n} w_i\big)^2}, \quad \gamma_{r,p} = \frac{rp}{2r + 2\big(1 - \frac{1}{p}\big)d}.$$

Remark A.3 If α_n is the sequence defined at the beginning of Section 2 and w_n is defined by (A.5), then the estimate in Lemma A.3 further implies that

(1) if $\overline{\alpha} \geq 2$, we have, for any $\delta < \underline{\alpha} \wedge 2$,

$$\mathbb{E}\mathcal{W}_r^{rp}(m_n, m) \leq C\,\mathrm{e}^{-\delta\gamma_{r,p} \sum\limits_{i=1}^{n} \alpha_i}.$$

(2) If $2 - \alpha_\infty > \overline{\alpha}$, we have

$$\mathbb{E}\mathcal{W}_r^{rp}(m_n, m) \leq C\alpha_n^{\gamma_{r,p}}.$$

This form will be used often in what follows.

Proof Let ϕ_σ be the p.d.f. of the normal distribution $\Phi_\sigma = N(0, \sigma^2 I_d)$, and define

$$m_n^\sigma := \Phi_\sigma * m_n, \quad m^\sigma := \Phi_\sigma * m.$$

Denote $\pi^\sigma(\cdot)$ the p.d.f of m^σ. The p.d.f. $\pi_n^\sigma(\cdot)$ of m_n^σ is given by

$$\pi_n^\sigma(x) = \frac{\sum\limits_{i=1}^n w_i \phi_\sigma(x - \xi^i)}{\sum\limits_{i=1}^n w_i} =: \mathcal{K}_n(\phi_\sigma(x - \xi^i)).$$

From Lemma A.2 and Hölder's inequality, it follows that

$$\begin{aligned}\mathcal{W}_r^{rp}(m_n^\sigma, m^\sigma) &\leq \Big(C_r \int |x|^r |\pi_n^\sigma(x) - \pi^\sigma(x)| \,\mathrm{d}x \Big)^p \\ &\leq C_{r,p,q,d} \int (1 + |x|)^q |\pi_n^\sigma(x) - \pi^\sigma(x)|^p \,\mathrm{d}x.\end{aligned}$$

Define $\eta_i = \xi_i - \mathbb{E}\xi_i$. Since η^i is i.i.d with zero mean, we apply the discrete-time BDG inequality and Jenen's inequality to compute

$$\begin{aligned}\mathbb{E}|\mathcal{K}_n(\eta_i)|^p &= \frac{1}{(\sum w_i)^p} \mathbb{E}\Big| \sum w_i \eta_i \Big|^p \leq \frac{C_p}{(\sum w_i)^p} \mathbb{E}\Big| \sum w_i^2 |\eta_i|^2 \Big|^{\frac{p}{2}} \\ &\leq C_p \frac{(\sum w_i^2)^{\frac{p}{2}}}{(\sum w_i)^p} \mathbb{E}\Big| \frac{1}{\sum w_i^2} \sum w_i^2 |\eta_i|^2 \Big|^{\frac{p}{2}} \\ &\leq C_p \frac{(\sum w_i^2)^{\frac{p}{2}}}{(\sum w_i)^p} \mathbb{E}\Big[\frac{1}{\sum w_i^2} \sum w_i^2 |\eta_i|^p \Big] \\ &= C_p \theta_n^{\frac{p}{2}} \mathbb{E}|\eta_1|^p.\end{aligned}$$

So we have

$$\begin{aligned}\mathbb{E}|\pi_n^\sigma(x) - \pi^\sigma(x)|^p &= \mathbb{E}|\mathcal{K}_n(\phi_\sigma(x - \xi^i)) - \mathbb{E}\phi_\sigma(x - \xi))|^p \\ &\leq C_p \theta_n^{\frac{p}{2}} \mathbb{E}\phi_\sigma^p(x - \xi).\end{aligned}$$

We observe that $\phi_\sigma^p(x) = (2\pi)^{-\frac{pd}{2}} \sigma^{-pd} \mathrm{e}^{-\frac{px^2}{2\sigma^2}} = p^{-\frac{d}{2}} (2\pi)^{\frac{d}{2}(1-p)} \sigma^{d(1-p)} \phi_{\frac{\sigma}{\sqrt{p}}}(x)$. Thus

$$\begin{aligned}\mathbb{E}\phi_\sigma^p(x - \xi) &= \int \phi_\sigma^p(x - y) m(\mathrm{d}y) \\ &= p^{-\frac{d}{2}} (2\pi)^{\frac{d}{2}(1-p)} \sigma^{d(1-p)} \int \phi_{\frac{\sigma}{\sqrt{p}}}(x - y) m(\mathrm{d}y) \\ &= C_{p,d} \sigma^{d(1-p)} \int \phi_{\frac{\sigma}{\sqrt{p}}}(x - y) m(\mathrm{d}y).\end{aligned}$$

To sum up, one has that

$$\begin{aligned}\mathbb{E}\mathcal{W}_r^{rp}(m_n^\sigma, m^\sigma) &\leq C_{r,p,q,d} \int (1 + |x|)^q \mathbb{E}|\pi_n^\sigma(x) - \pi_t^\sigma(x)|^p \,\mathrm{d}x \\ &\leq C_{r,p,q,d} \sigma^{d(1-p)} \theta_n^{\frac{p}{2}} \iint (1 + |x|)^q \phi_{\frac{\sigma}{\sqrt{p}}}(x - y) \, m(\mathrm{d}y) \,\mathrm{d}x \\ &\leq C_{r,p,q,d} \sigma^{d(1-p)} \theta_n^{\frac{p}{2}} \mathbb{E}(1 + |\xi_1|)^q.\end{aligned}$$

On the other hand, it is easily seen that

$$\mathcal{W}_r^{rp}(m_n^\sigma, m_n) + \mathcal{W}_r^{rp}(m^\sigma, m) \le C\sigma^{rp}$$

The lemma is then proved by taking $\sigma = \theta_n^{-\frac{1}{[2r+2(1-p^{-1})d]}}$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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