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Time-Periodic Solutions to Quasilinear Hyperbolic Systems on General Networks*

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Abstract For quasilinear hyperbolic systems on general networks with time-periodic boundary-interface conditions with a dissipative structure, the existence and stability of the time-periodic classical solutions are discussed.

Keywords Time-periodic solution, Quasilinear hyperbolic system, General network, Classical solution, Asymptotic stability
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1 Introduction

In this paper, we consider a general network $\{\mathcal{V}_\beta, \mathcal{E}_\alpha\}_{\substack{\alpha=1,\cdots,N\\ \beta=1,\cdots,P}}$ and study the first order quasilinear hyperbolic systems on it. Here on each edge $\mathcal{E}_\alpha$, we can artificially assign two different real numbers $d_{\alpha 0} < d_{\alpha 1} \in \mathbb{R}$ to its two ends respectively, and we can then write down the corresponding hyperbolic PDEs as

$$\partial_t u^{(\alpha)} + A^{(\alpha)}(u^{(\alpha)})\partial_x u^{(\alpha)} = 0, \quad (t,x) \in \mathbb{R} \times [d_{\alpha 0}, d_{\alpha 1}], \quad \alpha = 1, \cdots, N, \tag{1.1}$$

where $u^{(\alpha)} = (u_1^{(\alpha)}, \cdots, u_{n^{(\alpha)}}^{(\alpha)})^{\mathrm{T}} \in \mathcal{U}^{(\alpha)}$ $(\alpha = 1, \cdots, N)$ is the unknown vector function of (t,x), $\mathcal{U}^{(\alpha)} \subset \mathbb{R}^{n^{(\alpha)}}$ is a small neighborhood of 0 for each α, the coefficient matrices $A^{(\alpha)}(u^{(\alpha)})$ satisfy several structure conditions of hyperbolicity, which would be listed in details later. In what follows, we would focus on the time-periodic solutions of the system. Time-periodicity is a common phenomenon in industrial applications and in the nature, and the study of time-periodic solutions to ODEs can date back to the work of Newton. For many types of important PDE systems, time-periodic and quasi-periodic solutions get comprehensive studies, such as [3, 15, 19] for wave equations, [4–5] for the Schrödinger equation, [2, 6] for incompressible Euler equations, [1, 8] for water wave equations, [7, 9, 17] for compressible Navier-Stokes equations. While for the first order quasilinear hyperbolic PDE systems, many progresses have been made recently. For Euler equations, a series of works of Temple and Young, such as [16], discuss the existence of time-periodic solutions with periodic boundary conditions for full Euler system, while [13] provides a family of time-periodic solutions to its approximate system, [21] constructs

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the time-periodic solutions in the supersonic case, and [18] shows the existence of time-periodic solutions with time-periodic outer force, [20] discusses the time-periodic solutions with source terms around supersonic Fanno flow. For general hyperbolic systems, [10–11] discuss the time-periodic and almost periodic solutions by the method of operator analysis. And [14] shows the well-posedness of the time-periodic solutions for general hyperbolic systems with dissipative time-periodic boundary conditions. In this paper, we would like to focus on the time-periodic solutions on a general network, which is a widely used model for transportation, irrigation, as well as gas and liquid supplies in the industry.

For hyperbolicity, we assume that $A^{(\alpha)}(u^{(\alpha)})$ are smooth $n^{(\alpha)}\times n^{(\alpha)}$ matrices, with $n^{(\alpha)}$ real eigenvalues $\lambda_k^{(\alpha)}(u^{(\alpha)})$ $(k=1,\cdots,n^{(\alpha)};\alpha=1,\cdots,N)$ satisfying

$$\begin{aligned}&\lambda_r^{(\alpha)}(u^{(\alpha)})<0<\lambda_s^{(\alpha)}(u^{(\alpha)}),\\&\forall\,u^{(\alpha)}\in\mathcal{U}^{(\alpha)},\ \forall\,r=1,\cdots,m^{(\alpha)},\ \forall\,s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,\end{aligned}\tag{1.2}$$

and a complete set of left and right eigenvectors $l_k^{(\alpha)}(u^{(\alpha)})=(l_{k1}^{(\alpha)}(u^{(\alpha)}),\cdots,l_{kn^{(\alpha)}}^{(\alpha)}(u^{(\alpha)}))$ and $r_k^{(\alpha)}(u^{(\alpha)})=(r_{1k}^{(\alpha)}(u^{(\alpha)}),\cdots,r_{n^{(\alpha)}k}^{(\alpha)}(u^{(\alpha)}))^{\mathrm{T}}$ $(k=1,\cdots,n^{(\alpha)};\alpha=1,\cdots,N)$, namely,

$$\begin{aligned}&l_k^{(\alpha)}(u^{(\alpha)})A^{(\alpha)}(u^{(\alpha)})=\lambda_k^{(\alpha)}(u^{(\alpha)})l_k^{(\alpha)}(u^{(\alpha)}),\\&\forall\,u^{(\alpha)}\in\mathcal{U}^{(\alpha)},\ \forall\,k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,\end{aligned}\tag{1.3}$$

$$\begin{aligned}&A^{(\alpha)}(u^{(\alpha)})r_k^{(\alpha)}(u^{(\alpha)})=\lambda_k^{(\alpha)}(u^{(\alpha)})r_k^{(\alpha)}(u^{(\alpha)}),\\&\forall\,u^{(\alpha)}\in\mathcal{U}^{(\alpha)},\ \forall\,k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N\end{aligned}\tag{1.4}$$

with

$$\det(l_{kj}^{(\alpha)}(u^{(\alpha)}))_{k,j=1}^{n^{(\alpha)}}\neq0,\quad\forall\,u^{(\alpha)}\in\mathcal{U}^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.\tag{1.5}$$

Without loss of generality, after a linear transformation $(\widetilde{u}_k^{(\alpha)}=l_k^{(\alpha)}(0)u^{(\alpha)})$ of the unknown variables if needed, we can assume

$$A^{(\alpha)}(0)=\mathrm{diag}\{\lambda_1^{(\alpha)}(0),\cdots,\lambda_{n^{(\alpha)}}^{(\alpha)}(0)\},\quad\forall\,\alpha=1,\cdots,N\tag{1.6}$$

and we can normalize our eigenvectors to satisfy

$$l_k^{(\alpha)}(u^{(\alpha)})r_j^{(\alpha)}(u^{(\alpha)})=\delta_{kj},\quad\forall\,u^{(\alpha)}\in\mathcal{U}^{(\alpha)},\ \forall\,k,j=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,\tag{1.7}$$

$$|r_k^{(\alpha)}(u^{(\alpha)})|=1,\quad\forall\,u^{(\alpha)}\in\mathcal{U}^{(\alpha)},\ \forall\,k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,\tag{1.8}$$

from which, we also get

$$l_{kj}^{(\alpha)}(0)=r_{kj}^{(\alpha)}(0)=\delta_{kj},\quad\forall\,k,j=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.\tag{1.9}$$

We further set

$$\mu_k^{(\alpha)}(u^{(\alpha)})=(\lambda_k^{(\alpha)}(u^{(\alpha)}))^{-1},\quad k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N\tag{1.10}$$

and

$$\mu_{\max}=\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}|\mu_k^{(\alpha)}(u^{(\alpha)})|.\tag{1.11}$$

After a rescaling of the temporal variable if needed, we can assume

$$\mu_{\max} \leq 1. \tag{1.12}$$

For this hyperbolic system on the network, we may set the initial condition

$$t = 0: \ u_k^{(\alpha)}(0, x) = \varphi_k^{(\alpha)}(x), \quad x \in [d_{\alpha 0}, d_{\alpha 1}]; \ k = 1, \cdots, n^{(\alpha)}; \ \alpha = 1, \cdots, N \tag{1.13}$$

with

$$\varphi_k^{(\alpha)} \in C^1[d_{\alpha 0}, d_{\alpha 1}], \quad \forall\, k = 1, \cdots, n^{(\alpha)}, \ \forall\, \alpha = 1, \cdots, N. \tag{1.14}$$

We may also present suitable boundary-interface conditions according to the structure of the network as follows. In our general network $\{\mathcal{V}_\beta, \mathcal{E}_\alpha\}$, for each edge $\mathcal{E}_\alpha = [d_{\alpha 0}, d_{\alpha 1}]$ $(\alpha = 1, \cdots, N)$, it has two ends assigned by $d_{\alpha 0}$ and $d_{\alpha 1}$, and it is natural to denote them as $\mathcal{V}_{\alpha 0}$ and $\mathcal{V}_{\alpha 1}$, respectively. Then we can denote the index sets for the edges sharing at least one node with $\mathcal{E}_\alpha$ as

$$\mathcal{J}_{\alpha 00} = \{\alpha^* = 1, \cdots, N \mid \mathcal{V}_{\alpha^* 0} = \mathcal{V}_{\alpha 0}, \alpha^* \neq \alpha\}, \tag{1.15}$$

$$\mathcal{J}_{\alpha 01} = \{\alpha^* = 1, \cdots, N \mid \mathcal{V}_{\alpha^* 1} = \mathcal{V}_{\alpha 0}\}, \tag{1.16}$$

$$\mathcal{J}_{\alpha 10} = \{\alpha^* = 1, \cdots, N \mid \mathcal{V}_{\alpha^* 0} = \mathcal{V}_{\alpha 1}\}, \tag{1.17}$$

$$\mathcal{J}_{\alpha 11} = \{\alpha^* = 1, \cdots, N \mid \mathcal{V}_{\alpha^* 1} = \mathcal{V}_{\alpha 1}, \alpha^* \neq \alpha\}. \tag{1.18}$$

Then we can denote our boundary-interface conditions as

$$\begin{aligned} u_r^{(\alpha)}(t, d_{\alpha 1}) = G_r^{(\alpha)}(&h_r^{(\alpha)}(t), u_{m^{(\alpha)}+1}^{(\alpha)}(t, d_{\alpha 1}), \cdots, u_{n^{(\alpha)}}^{(\alpha)}(t, d_{\alpha 1}), u_1^{(\alpha^*)}(t, d_{\alpha^* 0}), \cdots, \\ &u_{m^{(\alpha^*)}}^{(\alpha^*)}(t, d_{\alpha^* 0})\, (\alpha^* \in \mathcal{J}_{\alpha 10}), u_{m^{(\alpha^\sharp)}+1}^{(\alpha^\sharp)}(t, d_{\alpha^\sharp 1}), \cdots, u_{n^{(\alpha^\sharp)}}^{(\alpha^\sharp)}(t, d_{\alpha^\sharp 1})\, (\alpha^\sharp \in \mathcal{J}_{\alpha 11})), \\ &r = 1, \cdots, m^{(\alpha)}; \ \alpha = 1, \cdots, N \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} u_s^{(\alpha)}(t, d_{\alpha 0}) = G_s^{(\alpha)}(&h_s^{(\alpha)}(t), u_1^{(\alpha)}(t, d_{\alpha 0}), \cdots, u_{m^{(\alpha)}}^{(\alpha)}(t, d_{\alpha 0}), u_1^{(\alpha^*)}(t, d_{\alpha^* 0}), \cdots, \\ &u_{m^{(\alpha^*)}}^{(\alpha^*)}(t, d_{\alpha^* 0})\, (\alpha^* \in \mathcal{J}_{\alpha 00}), u_{m^{(\alpha^\sharp)}+1}^{(\alpha^\sharp)}(t, d_{\alpha^\sharp 1}), \cdots, u_{n^{(\alpha^\sharp)}}^{(\alpha^\sharp)}(t, d_{\alpha^\sharp 1})\, (\alpha^\sharp \in \mathcal{J}_{\alpha 01})), \\ &s = m^{(\alpha)} + 1, \cdots, n^{(\alpha)}; \ \alpha = 1, \cdots, N. \end{aligned} \tag{1.20}$$

For simplicity, we would shorthand these conditions as

$$x = d_{\alpha 1}: \ u_r^{(\alpha)} = G_r^{(\alpha)}(h_r^{(\alpha)}, u_s^{(\alpha)}, u_{r^*}^{(\mathcal{J}_{\alpha 10})}, u_{s^\sharp}^{(\mathcal{J}_{\alpha 11})})$$

and

$$x = d_{\alpha 0}: \ u_s^{(\alpha)} = G_s^{(\alpha)}(h_s^{(\alpha)}, u_r^{(\alpha)}, u_{r^*}^{(\mathcal{J}_{\alpha 00})}, u_{s^\sharp}^{(\mathcal{J}_{\alpha 01})}).$$

We assume that

$$G_k^{(\alpha)}(0, 0, 0, 0) = 0, \quad \forall\, k = 1, \cdots, n^{(\alpha)}, \ \forall\, \alpha = 1, \cdots, N. \tag{1.21}$$

And without loss of generality, redenote $h_k^{(\alpha)}(t)$ if needed, we may assume

$$\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\Big|\frac{\partial G_k^{(\alpha)}}{\partial h_k^{(\alpha)}}(0,0,0,0)\Big|\le 1. \tag{1.22}$$

In this paper, we want to analyze the properties of time-periodic solutions when all the given functions $h_k^{(\alpha)}$ in the boundary-interface conditions share one constant time period $T_*>0$, namely,

$$h_k^{(\alpha)}(t+T_*)=h_k^{(\alpha)}(t),\quad \forall\, t\in\mathbb{R},\ \forall\, k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{1.23}$$

As the example given in [14], we need a dissipative structure on the boundary-interface conditions to prevent the potential blowup of the classical solutions. Denote

$$n^*=\sum_{\alpha=1}^{N}n^{(\alpha)},$$

the $n^*\times n^*$ matrix

$$\Theta=(\theta_{pq})_{p,q=1}^{n^*}=\Big(\frac{\partial G_k^{(\alpha)}}{\partial u_{k^*}^{(\alpha^*)}}(0,0,\cdots,0)\Big)_{\substack{k=1,\cdots,n^{(\alpha)};\\ k^*=1,\cdots,n^{(\alpha^*)};\\ \alpha^*,\alpha=1,\cdots,N}} \tag{1.24}$$

and its minimal characterizing number θ as

$$\theta=\inf_{\gamma_p\neq 0}\max_{p=1,\cdots,n^*}\sum_{q=1}^{n^*}|\gamma_p\theta_{pq}\gamma_q^{-1}|. \tag{1.25}$$

The dissipative structure we need is

$$\theta<1. \tag{1.26}$$

We note that after a linear transformation of $u^{(\alpha)}$, (1.25)–(1.26) imply that

$$\begin{aligned}
&\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}\Big|\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(0,0,0,0)\Big|+\sum_{\alpha^*\in\mathcal{J}_{\alpha 10}}\sum_{r^*=1}^{m^{(\alpha^*)}}\Big|\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(0,0,0,0)\Big|\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha 11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}\Big|\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(0,0,0,0)\Big|\le\theta<1,\\
&\forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,
\end{aligned} \tag{1.27}$$

$$\begin{aligned}
&\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}\Big|\frac{\partial G_s^{(\alpha)}}{\partial u_r^{(\alpha)}}(0,0,0,0)\Big|+\sum_{\alpha^*\in\mathcal{J}_{\alpha 00}}\sum_{r^*=1}^{m^{(\alpha^*)}}\Big|\frac{\partial G_s^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(0,0,0,0)\Big|\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha 01}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}\Big|\frac{\partial G_s^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(0,0,0,0)\Big|\le\theta<1,\\
&\forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned} \tag{1.28}$$

Now we want to know, under the condition (1.23), whether or not we have a stable time-periodic classical solution. In fact, we have positive answers as follows.

Theorem 1.1 (Existence of time-periodic solutions) *Under the hypothesis* (1.26)*, there exists a small constant* $\varepsilon_1 > 0$*, such that for any given* $\varepsilon \in (0, \varepsilon_1)$*, any given* $T_* \in \mathbb{R}_+$ *and any given functions* $h_k^{(\alpha)}(t)\,(k = 1, \cdots, n^{(\alpha)}; \alpha = 1, \cdots, N)$ *satisfying* (1.23) *and*

$$\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\|h_k^{(\alpha)}\|_{C^1([0,T])} \le \varepsilon, \tag{1.29}$$

there exists a constant $M_1 > 0$ *and a family of* C^1 *functions*

$$\varphi_k^{(\alpha)}(x) = \varphi_k^{(\alpha,P)}(x), \quad k = 1, \cdots, n^{(\alpha)};\ \alpha = 1, \cdots, N$$

satisfying

$$\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\|\varphi_k^{(\alpha,P)}\|_{C^1} \le M_1\varepsilon, \tag{1.30}$$

such that the system (1.1), (1.13), (1.19)–(1.20) *admits a* C^1 *classical solution* $u_k^{(\alpha,P)}(t,x)$ *for* $t \in \mathbb{R}; k = 1, \cdots, n^{(\alpha)}; \alpha = 1, \cdots, N$*, satisfying*

$$u_k^{(\alpha,P)}(t + T_*, x) = u_k^{(\alpha,P)}(t, x), \quad \forall\, x \in [d_{\alpha 0}, d_{\alpha 1}],\ \forall\, k = 1, \cdots, n^{(\alpha)},\ \forall\, \alpha = 1, \cdots, N. \tag{1.31}$$

Theorem 1.2 (Stability around the time-periodic solution) *Under the hypothesis* (1.26)*, there exists a small constant* $\varepsilon_2 \in (0, \varepsilon_1)$ *and a constant* $M_2 > 0$*, such that for any given* $\varepsilon \in (0, \varepsilon_2)$*, any given* $T_* \in \mathbb{R}_+$*, any given* $h_k^{(\alpha)}(t)$ *satisfying* (1.23) *and* (1.29) *and any given* $\varphi_k^{(\alpha)}(x)$ *satisfying*

$$\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\|\varphi_k^{(\alpha)}\|_{C^1} \le \varepsilon, \tag{1.32}$$

the classical solution $u_k^{(\alpha)}(t,x)$ *to the system* (1.1), (1.13), (1.19)–(1.20) *satisfies*

$$\lim_{t\to+\infty}\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\|u_k^{(\alpha)}(t,\cdot) - u_k^{(\alpha,P)}(t,\cdot)\|_{C^1} = 0, \tag{1.33}$$

and especially

$$\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\|u_k^{(\alpha)}(t,\cdot) - u_k^{(\alpha,P)}(t,\cdot)\|_{C^0} \le M_2 a^{\left\lfloor \frac{t}{T_0} \right\rfloor}\varepsilon, \tag{1.34}$$

where $u^{(\alpha,P)}$ *is the time-periodic solution given in Theorem* 1.1*, and*

$$a = \frac{1+\theta}{2} \in (0,1), \tag{1.35}$$

$$T_0 = L\mu_{\max}, \tag{1.36}$$

$$L = \max_{\alpha=1,\cdots,N}|d_{\alpha 0}, d_{\alpha 1}|. \tag{1.37}$$

Remark 1.1 Compared with the result of [14], for the stability result, we need only C^1 regularity of $h_k^{(\alpha)}$ to get the stability not only in C^0 topology, but also in C^1 topology.

Remark 1.2 By taking $t \to +\infty$, Theorem 1.2 directly implies the uniqueness near 0 of the time-periodic solution. In fact, suppose for given functions $h_k^{(\alpha)}(t)$, we have two time-periodic solutions $u_*^{(\alpha,P)}$ and $u_{**}^{(\alpha,P)}$ with $\|u_*^{(\alpha,P)}\|_{C^1} \le \varepsilon_2$ and $\|u_{**}^{(\alpha,P)}\|_{C^1} \le \varepsilon_2$, then by Theorem 1.2, we have $\lim\limits_{t\to+\infty} \|u_*^{(\alpha,P)}(t,\cdot) - u_{**}^{(\alpha,P)}(t,\cdot)\|_{C^1} = 0$, but we know that $\|u_*^{(\alpha,P)}(t,\cdot) - u_{**}^{(\alpha,P)}(t,\cdot)\|_{C^1}$ is a periodic function, thus the only possibility is that $\|u_*^{(\alpha,P)}(t,\cdot) - u_{**}^{(\alpha,P)}(t,\cdot)\|_{C^1} \equiv 0$, namely, $u_*^{(\alpha,P)} \equiv u_{**}^{(\alpha,P)}$.

Remark 1.3 Theorem 1.2 also shows the global existence for small C^1 classical solutions on the general networks under the dissipative boundary structure (1.26).

2 Existence of the Time-Periodic Solutions

In this section, we will apply the iteration scheme as in [12] (see also [14]) to show the existence of the time-periodic solutions.

Since we need the estimates on the modulus of continuity for the time-periodic solutions later in Section 3, in this section we would prove the following result, which is a little bit stronger than Theorem 1.1.

Proposition 2.1 *The result of Theorem* 1.1 *holds. Moreover, there exists a continuous function $\Omega_P(\eta)$ on $\eta \in [0,1]$ with*

$$\Omega_P(0) = 0, \tag{2.1}$$

such that

$$\max_{\alpha=1,\cdots,N} \max_{k=1,\cdots,n^{(\alpha)}} \{\omega(\eta \mid \partial_t u_k^{(\alpha,P)}) + \omega(\eta \mid \partial_x u_k^{(\alpha,P)})\} \le \Omega_P(\eta), \tag{2.2}$$

where $\omega(\eta \mid \cdot)$ is the modulus of continuity

$$\omega(\eta \mid f) = \sup_{\substack{|t_1-t_2|<a \\ |x_1-x_2|<a}} |f(t_1,x_1) - f(t_2,x_2)|. \tag{2.3}$$

In order to show this result, we first linearize the system as follows. By multiplying the left eigenvectors $l_k^{(\alpha)}(u^{(\alpha)})$ to (1.1), we have

$$\begin{aligned} &\partial_t u_k^{(\alpha)} + \lambda_k^{(\alpha)}(u^{(\alpha)})\partial_x u_k^{(\alpha)} = \sum_{j=1}^{n^{(\alpha)}} B_{kj}^{(\alpha)}(u^{(\alpha)})(\partial_t u_j^{(\alpha)} + \lambda_k^{(\alpha)}(u^{(\alpha)})\partial_x u_j^{(\alpha)}), \\ &k = 1,\cdots,n^{(\alpha)};\ \alpha = 1,\cdots,N, \end{aligned} \tag{2.4}$$

where

$$B_{kj}^{(\alpha)}(u^{(\alpha)}) = \begin{cases} -\dfrac{l_{kj}^{(\alpha)}(u^{(\alpha)})}{l_{kk}^{(\alpha)}(u^{(\alpha)})}, & j \ne k; \quad \alpha = 1,\cdots,N, \\ 0, & j = k; \quad \alpha = 1,\cdots,N, \end{cases} \tag{2.5}$$

and by (1.9), this also leads to

$$B_{kj}^{(\alpha)}(0) = 0, \quad \forall\, k,j = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N. \tag{2.6}$$

Now we can set our linearized system as

$$\partial_t u_k^{(\alpha,\ell)} + \lambda_k^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_x u_k^{(\alpha,\ell)} = \sum_{j=1}^{n^{(\alpha)}} B_{kj}^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_t u_j^{(\alpha,\ell-1)} + \lambda_k^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_x u_j^{(\alpha,\ell-1)}),$$
$$k = 1,\cdots,n^{(\alpha)};\ \alpha = 1,\cdots,N;\ \ell \in \mathbb{Z}_+, \tag{2.7}$$
$$x = d_{\alpha 1}:\ u_r^{(\alpha,\ell)} = G_r^{(\alpha)}(h_r^{(\alpha)}(t), u_s^{(\alpha,\ell-1)}, u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}, u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}),$$
$$r = 1,\cdots,m^{(\alpha)};\ \alpha = 1,\cdots,N;\ \ell \in \mathbb{Z}_+, \tag{2.8}$$
$$x = d_{\alpha 0}:\ u_s^{(\alpha,\ell)} = G_s^{(\alpha)}(h_s^{(\alpha)}(t), u_r^{(\alpha,\ell-1)}, u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-1)}, u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-1)}),$$
$$s = m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha = 1,\cdots,N;\ \ell \in \mathbb{Z}_+, \tag{2.9}$$

and start the iteration from

$$u_k^{(\alpha,0)}(t,x) \equiv 0, \quad k = 1,\cdots,n^{(\alpha)};\ \alpha = 1,\cdots,N. \tag{2.10}$$

The a priori estimates we need are in the following.

Proposition 2.2 *For the above iteration scheme, under the hypotheses of Theorem* 1.1*, the sequence of* C^1 *approximate solutions* $\{u_k^{(\alpha,\ell)}(t,x) \mid \ell \in \mathbb{Z}_+\}$ *satisfy*

$$u_k^{(\alpha,\ell)}(t+T_*,x) = u_k^{(\alpha,\ell)}(t,x), \quad \forall\, t,x,\ \forall\, k = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N,\ \forall\, \ell \in \mathbb{Z}_+, \tag{2.11}$$
$$\|u_k^{(\alpha,\ell)}\|_{C^1} \overset{\text{def.}}{=} \max\{\|u_k^{(\alpha,\ell)}\|_{C^0}, \|\partial_t u_k^{(\alpha,\ell)}\|_{C^0}, \|\partial_x u_k^{(\alpha,\ell)}\|_{C^0}\} \le M_1\varepsilon,$$
$$\forall\, k = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N,\ \forall\, \ell \in \mathbb{Z}_+, \tag{2.12}$$
$$\|u_k^{(\alpha,\ell)} - u_k^{(\alpha,\ell-1)}\|_{C^0} \le M_1\varepsilon a^\ell, \quad \forall\, k = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N,\ \forall\, \ell \in \mathbb{Z}_+, \tag{2.13}$$

and there exists the function $\Omega_P(\eta)$ *as in Proposition* 2.1 *independent of* ℓ*, such that*

$$\omega(\eta \mid \partial_t u_k^{(\alpha,\ell)}) + \omega(\eta \mid \partial_x u_k^{(\alpha,\ell)}) \le \Omega_P(\eta),$$
$$\forall\, k = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N,\ \forall\, \ell \in \mathbb{Z}_+. \tag{2.14}$$

As long as we show Proposition 2.2, it is easy to check Proposition 2.1 and thus Theorem 1.1. In fact, (2.13) shows the convergence in C^0 space for the sequence $u^{(\alpha,\ell)} \to u^{(\alpha,P)}$, while (2.12) and (2.14) show a subsequence convergence in C^1, which provides the C^1 smoothness to $u^{(\alpha,P)}$. Meanwhile the iteration scheme guarantees $u^{(\alpha,P)}$ to be the solution to our original system (1.1), and (2.11) shows its periodicity.

Now, we should prove (2.11)–(2.14) inductively, namely, for each $\ell \in \mathbb{Z}_+$, we should show (2.11)–(2.14) and one more assistant estimate

$$\omega(\eta \mid \partial_t u_k^{(\alpha,\ell)}(\cdot,x)) \le \frac{1}{8}\Omega_P(\eta), \quad \forall\, x \in [d_{\alpha 0}, d_{\alpha 1}],\ \forall\, k = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N, \tag{2.15}$$

under the assumptions

$$u_k^{(\alpha,\tilde{\ell})}(t+T_*,x) = u_k^{(\alpha,\tilde{\ell})}(t,x),$$
$$\forall\, t,x,\ \forall\, k = 1,\cdots,n^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N,\ \forall\, \tilde{\ell} = 0,\cdots,\ell-1, \tag{2.16}$$

$$\|u_k^{(\alpha,\tilde{\ell})}\|_{C^1} \le M_1\varepsilon, \quad \forall\, k=1,\cdots,n^{(\alpha)}, \quad \forall\, \alpha=1,\cdots,N, \quad \forall\, \tilde{\ell}=0,\cdots,\ell-1, \tag{2.17}$$

$$\|u_k^{(\alpha,\tilde{\ell})}-u_k^{(\alpha,\tilde{\ell}-1)}\|_{C^0} \le M_1\varepsilon a^{\tilde{\ell}}, \quad \forall\, k=1,\cdots,n^{(\alpha)},\ \forall\, \alpha=1,\cdots,N,\ \forall\, \tilde{\ell}=1,\cdots,\ell-1, \tag{2.18}$$

$$\begin{aligned}&\omega(\eta\mid\partial_t u_k^{(\alpha,\tilde{\ell})})+\omega(\eta\mid\partial_x u_k^{(\alpha,\tilde{\ell})}) \le \Omega_P(\eta),\\ &\forall\, k=1,\cdots,n^{(\alpha)},\ \forall\, \alpha=1,\cdots,N,\ \forall\, \tilde{\ell}=0,\cdots,\ell-1,\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\omega(\eta\mid\partial_t u_k^{(\alpha,\tilde{\ell})}(\cdot,x)) \le \frac{1}{8}\Omega_P(\eta),\\ &\forall\, x,\ \forall\, k=1,\cdots,n^{(\alpha)},\ \forall\, \alpha=1,\cdots,N,\ \forall\, \tilde{\ell}=0,\cdots,\ell-1,\end{aligned} \tag{2.20}$$

where $\omega(\eta\mid f(\cdot,x))$ denotes the modulus of continuity with respect to the temporal variable as

$$\omega(\eta\mid f(\cdot,x)) = \sup_{|t_1-t_2|\le\eta}|f(t_1,x)-f(t_2,x)|. \tag{2.21}$$

In order to illustrate the proof in a clearer manner, we introduce a series of positive constants between θ and a as

$$0<\theta<a_0<a_1<a_2<a_3<a<1. \tag{2.22}$$

Then choosing ε_1 small enough, by (1.27)–(1.28), we have

$$\begin{aligned}&\sup_{|h_r^{(\alpha)}|\le\varepsilon_1}\ \sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}\ \sup_{u^{(\alpha^*)}\in\mathcal{U}^{(\alpha^*)}}\Big\{\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}\Big|\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})})\Big|\\ &+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}\Big|\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})})\Big|\\ &+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}\Big|\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})})\Big|\Big\}\le a_0,\\ &\forall\, r=1,\cdots,m^{(\alpha)},\ \forall\, \alpha=1,\cdots,N,\end{aligned} \tag{2.23}$$

$$\begin{aligned}&\sup_{|h_s^{(\hat{\alpha})}|\le\varepsilon_1}\ \sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}\ \sup_{u^{(\alpha^*)}\in\mathcal{U}^{(\alpha^*)}}\Big\{\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}\Big|\frac{\partial G_s^{(\alpha)}}{\partial u_r^{(\alpha)}}(h_s^{(\alpha)},u_r^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha00})},u_{s^\sharp}^{(\mathcal{J}_{\alpha01})})\Big|\\ &+\sum_{\alpha^*\in\mathcal{J}_{\alpha00}}\sum_{r^*=1}^{m^{(\alpha^*)}}\Big|\frac{\partial G_s^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_s^{(\alpha)},u_r^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha00})},u_{s^\sharp}^{(\mathcal{J}_{\alpha01})})\Big|\\ &+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha01}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}\Big|\frac{\partial G_s^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_s^{(\alpha)},u_r^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha00})},u_{s^\sharp}^{(\mathcal{J}_{\alpha01})})\Big|\Big\}\le a_0,\\ &\forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\, \alpha=1,\cdots,N,\end{aligned} \tag{2.24}$$

And to simplify the analysis, we would formally exchange the role of the variable t and x on each edge $\mathcal{E}_\alpha$ of the network, namely, by multiplying $\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)})=\frac{1}{\lambda_k^{(\alpha)}(u^{(\alpha,\ell-1)})}$ to (2.7), we can rewrite our system into

$$\partial_x u_r^{(\alpha,\ell)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t u_r^{(\alpha,\ell)}=\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_x u_j^{(\alpha,\ell-1)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t u_j^{(\alpha,\ell-1)}),$$

$$r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+, \tag{2.25}$$

$$x=d_{\alpha1}:\ u_r^{(\alpha,\ell)}=G_r^{(\alpha)}(h_r^{(\alpha)},u_s^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}),$$
$$r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+ \tag{2.26}$$

and

$$\begin{aligned}&\partial_x u_s^{(\alpha,\ell)}+\mu_s^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t u_s^{(\alpha,\ell)}\\&=\sum_{j=1}^{n^{(\alpha)}}B_{sj}^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_x u_j^{(\alpha,\ell-1)}+\mu_s^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t u_j^{(\alpha,\ell-1)}),\\&s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+,\end{aligned} \tag{2.27}$$

$$x=d_{\alpha0}:\ u_s^{(\alpha,\ell)}=G_s^{(\alpha)}(h_s^{(\alpha)},u_r^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha00},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha01},\ell-1)}),$$
$$s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+. \tag{2.28}$$

As long as we know $u^{(\alpha,\ell-1)}$, the system is transformed into a system of decoupled linear transport equations. Meanwhile, for $x\in[d_{\alpha0},d_{\alpha1}]$, we can denote the characteristic curve

$$t=t_k^{(\alpha,\ell)}(x;t_*,x_*),\quad k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+$$

by the ODEs

$$\begin{cases}\dfrac{\mathrm{d}t_k^{(\alpha,\ell)}}{\mathrm{d}x}(x;t_*,x_*)=\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}(t_k^{(\alpha,\ell)}(x;t_*,x_*),x)),\\ t_k^{(\alpha,\ell)}(x_*;t_*,x_*)=t_*.\end{cases} \tag{2.29}$$

First, we would like to get the C^0 estimate. On the boundary, by Hadamard's theorem, from (1.21) and (2.26), we can derive

$$\begin{aligned}&x=d_{\alpha1}:\\ &u_r^{(\alpha,\ell)}=h_r^{(\alpha)}(t)\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}}(\gamma h_r^{(\alpha)}(t),\gamma u_s^{(\alpha,\ell-1)},\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)})\mathrm{d}\gamma\\ &\quad+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}u_s^{(\alpha,\ell-1)}\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(\gamma h_r^{(\alpha)}(t),\gamma u_s^{(\alpha,\ell-1)},\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)})\mathrm{d}\gamma\\ &\quad+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}u_{r^*}^{(\alpha^*,\ell-1)}\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(\gamma h_r^{(\alpha)}(t),\gamma u_s^{(\alpha,\ell-1)},\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)})\mathrm{d}\gamma\\ &\quad+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}u_{s^\sharp}^{(\alpha^\sharp,\ell-1)}\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(\gamma h_r^{(\alpha)}(t),\gamma u_s^{(\alpha,\ell-1)},\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)})\mathrm{d}\gamma,\\ &r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+.\end{aligned}$$

Thus, using (1.22), (1.29), (2.17) and (2.23), we have

$$|u_r^{(\alpha,\ell)}(t,d_{\alpha1})|\le 2\varepsilon+M_1\varepsilon a_0\le M_1\varepsilon a_1,\quad \forall\, t\in\mathbb{R},\ \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{2.30}$$

Similarly, by (2.28), we have

$$|u_s^{(\alpha,\ell)}(t,d_{\alpha 0})| \le M_1\varepsilon a_1, \quad \forall\, t\in\mathbb{R},\ \forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{2.31}$$

Then on each edge $\mathcal{E}_\alpha$, by integrating (2.25) along the corresponding characteristic curve $t=t_r^{(\alpha,\ell)}(x;t_*,x_*)$, we can get

$$\begin{aligned}
&u_r^{(\alpha,\ell)}(t_*,x_*)\\
=\,&u_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(d_{\alpha 1};t_*,x_*),d_{\alpha 1})\\
&+\int_{d_{\alpha 1}}^{x_*}\sum_{j=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_x u_j^{(\alpha,\ell-1)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t u_j^{(\alpha,\ell-1)})(t_r^{(\alpha,\ell)}(y;t_*,x_*),y)\mathrm{d}y,\\
&\forall\, t_*\in\mathbb{R},\ \forall\, x_*\in[d_{\alpha 0},d_{\alpha 1}],\ \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}\tag{2.32}$$

Thus, by (2.30) and (2.6), (2.17), we have

$$\|u_r^{(\alpha,\ell)}\|_{C^0}\le M_1\varepsilon a_1+C\varepsilon^2\le M_1\varepsilon a_2, \quad \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{2.33}$$

Similarly, by (2.27), we have

$$\|u_s^{(\alpha,\ell)}\|_{C^0}\le M_1\varepsilon a_2, \quad \forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{2.34}$$

Then, for the estimates on derivatives of $u^{(\alpha)}$, we first take the temporal derivative on (2.26) and (2.28) to get the boundary value for

$$z_k^{(\alpha,\ell)}=\partial_t u_k^{(\alpha,\ell)}, \quad k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+, \tag{2.35}$$

which leads to

$$\begin{aligned}
x=d_{\alpha 1}:\ z_r^{(\alpha,\ell)}=\,&\dot h_r^{(\alpha)}(t)\frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)})\\
&+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}} z_s^{(\alpha,\ell-1)}\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)})\\
&+\sum_{\alpha^*\in\mathcal{J}_{\alpha 10}}\sum_{r^*=1}^{m^{(\alpha^*)}} z_{r^*}^{(\alpha^*,\ell-1)}\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)},u_s^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)})\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha 11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}} z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)},u_s^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}),\\
&r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N
\end{aligned}\tag{2.36}$$

and

$$\begin{aligned}
x=d_{\alpha 0}:\ z_s^{(\alpha,\ell)}=\,&\dot h^{(\alpha)}(t)\frac{\partial G_s^{(\alpha)}}{\partial h_s^{(\alpha)}}(h_s^{(\alpha)},u_r^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-1)})\\
&+\sum_{r=1}^{m^{(\alpha)}} z_r^{(\alpha,\ell-1)}\frac{\partial G_s^{(\alpha)}}{\partial u_r^{(\alpha)}}(h_s^{(\alpha)},u_r^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-1)})
\end{aligned}$$

$$+\sum_{\alpha^*\in\mathcal{J}_{\alpha00}}\sum_{r^*=1}^{m^{(\alpha^*)}} z_{r^*}^{(\alpha^*,\ell-1)}\frac{\partial G_s^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_s^{(\alpha)},u_r^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha00},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha01},\ell-1)})$$
$$+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha01}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}} z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_s^{(\alpha)},u_r^{(\alpha,\ell-1)},u_{r^*}^{(\mathcal{J}_{\alpha00},\ell-1)},u_{s^\sharp}^{(\mathcal{J}_{\alpha01},\ell-1)}),$$
$$s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N. \tag{2.37}$$

Thus, by (1.22), (1.29), (2.17) and (2.24), we have

$$|z_r^{(\alpha,\ell)}(t,d_{\alpha1})|\le 2\varepsilon+M_1\varepsilon a_0\le M_1a_1\varepsilon,\quad \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N \tag{2.38}$$

and

$$|z_s^{(\alpha,\ell)}(t,d_{\alpha0})|\le M_1a_1\varepsilon,\quad \forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{2.39}$$

Then we can take the temporal derivatives to (2.25), (2.27) to get

$$\begin{aligned}
&\partial_x z_r^{(\alpha,\ell)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t z_r^{(\alpha,\ell)}\\
=&-\sum_{k=1}^{n^{(\alpha)}}\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_r^{(\alpha,\ell)}\\
&+\sum_{j,k=1}^{n^{(\alpha)}}\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}(\partial_x u_j^{(\alpha,\ell-1)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)})\\
&+\sum_{j,k=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_j^{(\alpha,\ell-1)}\\
&+\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_x z_j^{(\alpha,\ell-1)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t z_j^{(\alpha,\ell-1)})\\
=&-\sum_{k=1}^{n^{(\alpha)}}\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_r^{(\alpha,\ell)}+\sum_{j,k=1}^{n^{(\alpha)}}\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}\partial_x u_j^{(\alpha,\ell-1)}\\
&+\sum_{j,k=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_j^{(\alpha,\ell-1)}\\
&+\sum_{j=1}^{n^{(\alpha)}}(\partial_x+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t)(B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)})\\
&-\sum_{j,k=1}^{n^{(\alpha)}}z_j^{(\alpha,\ell-1)}\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})\partial_x u_k^{(\alpha,\ell-1)},\quad r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N
\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}
&\partial_x z_s^{(\alpha,\ell)}+\mu_s^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t z_s^{(\alpha,\ell)}\\
=&-\sum_{k=1}^{n^{(\alpha)}}\frac{\partial\mu_s^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_s^{(\alpha,\ell)}+\sum_{j,k=1}^{n^{(\alpha)}}\frac{\partial B_{sj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}\partial_x u_j^{(\alpha,\ell-1)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j,k=1}^{n^{(\alpha)}} B_{sj}^{(\alpha)}(u^{(\alpha,\ell-1)})\frac{\partial\mu_s^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_j^{(\alpha,\ell-1)}\\
&+\sum_{j=1}^{n^{(\alpha)}}(\partial_x+\mu_s^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t)(B_{sj}^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)})\\
&-\sum_{j,k=1}^{n^{(\alpha)}} z_j^{(\alpha,\ell-1)}\frac{\partial B_{sj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})\partial_x u_k^{(\alpha,\ell-1)},\quad s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N. \qquad (2.41)
\end{aligned}
$$

Integrating (2.40) along the corresponding characteristic curve $t=t_r^{(\alpha,\ell)}(x;t_*,x_*)$ and using (2.6) we can get

$$
\begin{aligned}
&|z_r^{(\alpha,\ell)}(t_*,x_*)|\\
\le\ & |z_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(d_{\alpha 1};t_*,x_*),d_{\alpha 1})|\\
&+\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}|\nabla\mu_r^{(\alpha)}(u^{(\alpha)})|\|u^{(\alpha,\ell-1)}\|_{C^1}\int_{x_*}^{d_{\alpha 1}}|z_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(y;t_*,x_*),y)|\mathrm{d}y\\
&+|d_{\alpha 1}-d_{\alpha 0}|\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}\sum_{j,k=1}^{n^{(\alpha)}}\Big(2\Big|\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha)})\Big|+|B_{rj}^{(\alpha)}(u^{(\alpha)})|+\Big|\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha)})\Big|\Big)\|u^{(\alpha,\ell-1)}\|_{C^1}^2\\
&+2\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}\sum_{j,k=1}^{n^{(\alpha)}}\Big|\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha)})\Big|\|u^{(\alpha,\ell-1)}\|_{C^1}^2,\quad \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}
$$

By (2.17) and (2.38), we can apply Gronwall's inequality to get

$$
\|z_r^{(\alpha,\ell)}\|_{C^0}\le \mathrm{e}^{CM_1\varepsilon}(M_1\varepsilon a_1+CM_1^2\varepsilon^2)\le M_1\varepsilon a_2,\quad \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.42)
$$

Similarly, by integrating (2.41) along $t=t_s^{(\alpha,\ell)}(x;t_*,x_*)$, we have

$$
\|z_s^{(\alpha,\ell)}\|_{C^0}\le M_1\varepsilon a_2,\quad \forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.43)
$$

Then using (2.25), (2.27) and (1.12), (2.6), (2.17), we have

$$
\|\partial_x u_k^{(\alpha,\ell)}\|_{C^0}\le M_1\varepsilon a_2+CM_1^2\varepsilon^2\le M_1\varepsilon a_3,\quad \forall\, k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.44)
$$

Combining (2.33)–(2.34) and (2.42)–(2.44), we have (2.12).

In order to show (2.13), for $\ell=1$, we can get it directly from (2.12). While for $\ell\ge 2$, we first use the boundary-interface conditions (2.26), (2.28) and apply Hadamard's theorem to get

$$
\begin{aligned}
x=d_{\alpha 1}:\ & u_r^{(\alpha,\ell)}-u_r^{(\alpha,\ell-1)}\\
=&\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}(u_s^{(\alpha,\ell-1)}-u_s^{(\alpha,\ell-2)})\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)},\gamma u_s^{(\alpha,\ell-1)}+(1-\gamma)u_s^{(\alpha,\ell-2)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-2)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-2)})\mathrm{d}\gamma\\
&+\sum_{\alpha^*\in\mathcal{J}_{\alpha 10}}\sum_{r^*=1}^{m^{(\alpha^*)}}(u_{r^*}^{(\alpha^*,\ell-1)}-u_{r^*}^{(\alpha^*,\ell-2)})\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)},\gamma u_s^{(\alpha,\ell-1)}+(1-\gamma)u_s^{(\alpha,\ell-2)},
\end{aligned}
$$

$$
\begin{aligned}
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-2)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-2)})\mathrm{d}\gamma\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha 11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}(u_{s^\sharp}^{(\alpha^\sharp,\ell-1)}-u_{s^\sharp}^{(\alpha^\sharp,\ell-2)})\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)},\gamma u_s^{(\alpha,\ell-1)}+(1-\gamma)u_s^{(\alpha,\ell-2)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-2)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-2)})\mathrm{d}\gamma,\\
&r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+\setminus\{1\},
\end{aligned}
\tag{2.45}
$$

$$
\begin{aligned}
x=d_{\alpha 0}:\ & u_s^{(\alpha,\ell)}-u_s^{(\alpha,\ell-1)}\\
=&\sum_{r=1}^{m^{(\alpha)}}(u_r^{(\alpha,\ell-1)}-u_r^{(\alpha,\ell-2)})\int_0^1\frac{\partial G_s^{(\alpha)}}{\partial u_r^{(\alpha)}}(h_s^{(\alpha)},\gamma u_r^{(\alpha,\ell-1)}+(1-\gamma)u_r^{(\alpha,\ell-2)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-1)}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-2)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-1)}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-2)})\mathrm{d}\gamma\\
&+\sum_{\alpha^*\in\mathcal{J}_{\alpha 00}}\sum_{r^*=1}^{m^{(\alpha^*)}}(u_{r^*}^{(\alpha^*,\ell-1)}-u_{r^*}^{(\alpha^*,\ell-2)})\int_0^1\frac{\partial G_s^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_s^{(\alpha)},\gamma u_r^{(\alpha,\ell-1)}+(1-\gamma)u_r^{(\alpha,\ell-2)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-1)}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-2)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-1)}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-2)})\mathrm{d}\gamma\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha 01}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}(u_{s^\sharp}^{(\alpha^\sharp,\ell-1)}-u_{s^\sharp}^{(\alpha^\sharp,\ell-2)})\int_0^1\frac{\partial G_s^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_s^{(\alpha)},\gamma u_r^{(\alpha,\ell-1)}+(1-\gamma)u_r^{(\alpha,\ell-2)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-1)}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 00},\ell-2)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-1)}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 01},\ell-2)})\mathrm{d}\gamma,\\
&s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N;\ \ell\in\mathbb{Z}_+\setminus\{1\}.
\end{aligned}
\tag{2.46}
$$

Then by (2.18) and (2.23)–(2.24), we can get

$$
\max_{\alpha=1,\cdots,N}\max_{r=1,\cdots,m^{(\alpha)}}\sup_{t\in\mathbb{R}}|u_r^{(\alpha,\ell)}(t,d_{\alpha 1})-u_r^{(\alpha,\ell-1)}(t,d_{\alpha 1})|\le M_1a^{\ell-1}a_0\varepsilon \tag{2.47}
$$

and

$$
\max_{\alpha=1,\cdots,N}\max_{s=m^{(\alpha)}+1,\cdots,n^{(\alpha)}}\sup_{t\in\mathbb{R}}|u_s^{(\alpha,\ell)}(t,d_{\alpha 0})-u_s^{(\alpha,\ell-1)}(t,d_{\alpha 0})|\le M_1a^{\ell-1}a_0\varepsilon. \tag{2.48}
$$

While, on each edge $\mathcal{E}_\alpha$, we can use (2.25) to get

$$
\begin{aligned}
&\partial_x(u_r^{(\alpha,\ell)}-u_r^{(\alpha,\ell-1)})+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})\partial_t(u_r^{(\alpha,\ell)}-u_r^{(\alpha,\ell-1)})\\
=&-(\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})-\mu_r^{(\alpha)}(u^{(\alpha,\ell-2)}))\partial_t u_r^{(\alpha,\ell-1)}\\
&+\sum_{j=1}^{n^{(\alpha)}}(B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})-B_{rj}^{(\alpha)}(u^{(\alpha,\ell-2)}))\partial_x u_j^{(\alpha,\ell-2)}\\
&+\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_x u_j^{(\alpha,\ell-1)}-\partial_x u_j^{(\alpha,\ell-2)})\\
&+\sum_{j=1}^{n^{(\alpha)}}(B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})-B_{rj}^{(\alpha)}(u^{(\alpha,\ell-2)})\mu_r^{(\alpha)}(u^{(\alpha,\ell-2)}))\partial_t u_j^{(\alpha,\ell-2)}\\
&+\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})(\partial_t u_j^{(\alpha,\ell-1)}-\partial_t u_j^{(\alpha,\ell-2)})
\end{aligned}
$$

$$
\begin{aligned}
&= -\partial_t u_r^{(\alpha,\ell-1)} \sum_{k=1}^{n^{(\alpha)}} (u_k^{(\alpha,\ell-1)} - u_k^{(\alpha,\ell-2)}) \int_0^1 \frac{\partial \mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}} (\gamma u^{(\alpha,\ell-1)} + (1-\gamma) u^{(\alpha,\ell-2)}) \mathrm{d}\gamma \\
&\quad + \sum_{j,k=1}^{n^{(\alpha)}} \partial_x u_j^{(\alpha,\ell-2)} (u_k^{(\alpha,\ell-1)} - u_k^{(\alpha,\ell-2)}) \int_0^1 \frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}} (\gamma u^{(\alpha,\ell-1)} + (1-\gamma) u^{(\alpha,\ell-2)}) \mathrm{d}\gamma \\
&\quad + \sum_{j,k=1}^{n^{(\alpha)}} \partial_t u_j^{(\alpha,\ell-2)} (u_k^{(\alpha,\ell-1)} - u_k^{(\alpha,\ell-2)}) \int_0^1 \frac{\partial (B_{rj}^{(\alpha)} \mu_r^{(\alpha)})}{\partial u_k^{(\alpha)}} (\gamma u^{(\alpha,\ell-1)} + (1-\gamma) u^{(\alpha,\ell-2)}) \mathrm{d}\gamma \\
&\quad + \sum_{j=1}^{n^{(\alpha)}} (\partial_x + \mu_r^{(\alpha)}(u^{(\alpha,\ell-1)}) \partial_t) (B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)}) (u_j^{(\alpha,\ell-1)} - u_j^{(\alpha,\ell-2)})) \\
&\quad - \sum_{j,k=1}^{n^{(\alpha)}} (u_j^{(\alpha,\ell-1)} - u_j^{(\alpha,\ell-2)}) \frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}} (u^{(\alpha,\ell-1)}) (\partial_x u_k^{(\alpha,\ell-1)} + \mu_r^{(\alpha)}(u^{(\alpha,\ell-1)}) \partial_t u_k^{(\alpha,\ell-1)}), \\
&\quad r = 1, \cdots, m^{(\alpha)};\ \alpha = 1, \cdots, N;\ \ell \in \mathbb{Z}_+ \setminus \{1\}.
\end{aligned}
$$

Then we can integrate this equation along the corresponding characteristic curve

$$
t = t_r^{(\alpha,\ell)}(x; t_*, x_*)
$$

to get

$$
\begin{aligned}
&|u_r^{(\alpha,\ell)}(t_*, x_*) - u_r^{(\alpha,\ell-1)}(t_*, x_*)| \\
\le\ & |u_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}) - u_r^{(\alpha,\ell-1)}(t_r^{(\alpha,\ell)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})| \\
&+ \max_{k=1,\cdots,n^{(\alpha)}} \|u_k^{(\alpha,\ell-1)} - u_k^{(\alpha,\ell-2)}\|_{C^0} |d_{\alpha 1} - d_{\alpha 0}| \\
&\cdot \Big(\sup_{u^{(\alpha)} \in \mathcal{U}^{(\alpha)}} |\nabla \mu_r^{(\alpha)}(u^{(\alpha)})| \|u^{(\alpha,\ell-1)}\|_{C^1} + \sum_{j=1}^{n^{(\alpha)}} \sup_{u^{(\alpha)} \in \mathcal{U}^{(\alpha)}} |\nabla B_{rj}^{(\alpha)}(u^{(\alpha)})| \|u^{(\alpha,\ell-2)}\|_{C^1} \\
&+ \sum_{j,k=1}^{n^{(\alpha)}} \sup_{u^{(\alpha)} \in \mathcal{U}^{(\alpha)}} |\nabla (B_{rj}^{(\alpha)}(u^{(\alpha)}) \mu_r^{(\alpha)}(u^{(\alpha)}))| \|u^{(\alpha,\ell-2)}\|_{C^1} \\
&+ \sum_{j,k=1}^{n^{(\alpha)}} 2 \sup_{u^{(\alpha)} \in \mathcal{U}^{(\alpha)}} |\nabla B_{rj}^{(\alpha)}(u^{(\alpha)})| \|u^{(\alpha,\ell-1)}\|_{C^1} \Big) \\
&+ 2 \max_{\alpha=1,\cdots,N} \sum_{j=1}^{n^{(\alpha)}} \|u_j^{(\alpha,\ell-1)} - u_j^{(\alpha,\ell-2)}\|_{C^0} \sup_{u^{(\alpha)} \in \mathcal{U}^{(\alpha)}} |\nabla B_{rj}^{(\alpha)}(u^{(\alpha)})| \|u^{(\alpha,\ell-1)}\|_{C^0}, \\
&\forall\, r = 1, \cdots, m^{(\alpha)},\ \forall\, \alpha = 1, \cdots, N.
\end{aligned}
$$

Then, by (2.17)–(2.18) and (2.47), we can get

$$
\begin{aligned}
&\|u_r^{(\alpha,\ell)} - u_r^{(\alpha,\ell-1)}\|_{C^0} \le M_1 a^{\ell-1} a_0 \varepsilon + M_1 a^{\ell-1} \varepsilon \cdot C M_1 \varepsilon \le M_1 a^\ell \varepsilon, \\
&\forall\, r = 1, \cdots, m^{(\alpha)},\ \forall\, \alpha = 1, \cdots, N,
\end{aligned} \tag{2.49}
$$

and similarly, from (2.27),

$$
\|u_s^{(\alpha,\ell)} - u_s^{(\alpha,\ell-1)}\|_{C^0} \le M_1 a^\ell \varepsilon, \quad \forall\, s = m^{(\alpha)} + 1, \cdots, n^{(\alpha)},\ \forall\, \alpha = 1, \cdots, N, \tag{2.50}
$$

which leads to (2.13).

Next, we show the validity of (2.15). In fact, we can specifically choose $\Omega_P(\eta)$ as

$$\Omega_P(\eta) = 16(a_1 - a_0)^{-1} \sum_{\alpha=1}^{N} \sum_{k=1}^{n^{(\alpha)}} \omega(\eta \mid \dot{h}_k^{(\alpha)}) + 16\eta, \tag{2.51}$$

We shall note that, by the concavity, for any given $\gamma, \delta \in (0,1)$, we have

$$\Omega_P(\gamma\delta) = \Omega_P(\gamma\delta + (1-\gamma)0) \geq \gamma\Omega_P(\delta) + (1-\gamma)\Omega_P(0) = \gamma\Omega_P(\delta), \tag{2.52}$$

and thus,

$$\Omega_P(\delta) \leq \frac{1}{\gamma}\Omega_P(\gamma\delta), \quad \forall\, \gamma, \delta \in (0,1). \tag{2.53}$$

By the assumption $h_k^{(\alpha)} \in C^1(\mathbb{R})$, we know $\Omega_P(\eta)$ is continuous on $[0, +\infty)$ with

$$\Omega_P(0) = 0.$$

Now by (2.36), we have

$$\begin{aligned}
& z_r^{(\alpha,\ell)}(t_1, d_{\alpha 1}) - z_r^{(\alpha,\ell)}(t_2, d_{\alpha 1}) \\
= & (\dot{h}_r^{(\alpha)}(t_1) - \dot{h}_r^{(\alpha)}(t_2)) \frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}} (h_r^{(\alpha)}(t_1), u_s^{(\alpha,\ell-1)}(t_1, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1, d_{\alpha 1})) \\
& + \dot{h}_r^{(\alpha)}(t_2) \Big(\frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}} (h_r^{(\alpha)}(t_1), u_s^{(\alpha,\ell-1)}(t_1, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1, d_{\alpha 1})) \\
& - \frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}} (h_r^{(\alpha)}(t_2), u_s^{(\alpha,\ell-1)}(t_2, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2, d_{\alpha 1})) \Big) \\
& + \sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}} (z_s^{(\alpha,\ell-1)}(t_1, d_{\alpha 1}) - z_s^{(\alpha,\ell-1)}(t_2, d_{\alpha 1})) \\
& \cdot \frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}} (h_r^{(\alpha)}(t_1), u_s^{(\alpha,\ell-1)}(t_1, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1, d_{\alpha 1})) \\
& + \sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}} z_s^{(\alpha,\ell-1)}(t_2, d_{\alpha 1}) \\
& \cdot \Big(\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}} (h_r^{(\alpha)}(t_1), u_s^{(\alpha,\ell-1)}(t_1, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1, d_{\alpha 1})) \\
& - \frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}} (h_r^{(\alpha)}(t_2), u_s^{(\alpha,\ell-1)}(t_2, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2, d_{\alpha 1})) \Big) \\
& + \sum_{\alpha^* \in \mathcal{J}_{\alpha 10}} \sum_{r^*=1}^{m^{(\alpha^*)}} (z_{r^*}^{(\alpha^*,\ell-1)}(t_1, d_{\alpha 1}) - z_{r^*}^{(\alpha^*,\ell-1)}(t_2, d_{\alpha 1})) \\
& \cdot \frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}} (h_r^{(\alpha)}(t_1), u_s^{(\alpha,\ell-1)}(t_1, d_{\alpha 1}), u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1, d_{\alpha 1}), u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1, d_{\alpha 1}))
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\alpha^*\in\mathcal{J}_{\alpha10}} \sum_{r^*=1}^{m^{(\alpha^*)}} z_{r^*}^{(\alpha^*,\ell-1)}(t_2,d_{\alpha1})\\
&\cdot \Big(\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1}))\\
&- \frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)}(t_2),u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\Big)\\
&+ \sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}} \sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}} (z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}(t_1,d_{\alpha1})-z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}(t_2,d_{\alpha1}))\\
&\cdot \frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1}))\\
&+ \sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}} \sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}} z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}(t_2,d_{\alpha1})\\
&\cdot \Big(\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1}))\\
&- \frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)}(t_2),u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\Big)\\
=&\,(\dot h_r^{(\alpha)}(t_1)-\dot h_r^{(\alpha)}(t_2))\frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1}))\\
&+ \dot h_r^{(\alpha)}(t_2)\cdot\Big\{\int_{t_2}^{t_1}\dot h_r^{(\alpha)}(\tau)\mathrm{d}\tau\int_1^0\frac{\partial^2 G_r^{(\alpha)}}{\partial {h_r^{(\alpha)}}^2}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),\\
&\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),\\
&\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma\\
&+ \sum_{\breve{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}\int_{t_2}^{t_1} z_{\breve{s}}^{(\alpha,\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1\frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\alpha)}\partial h_r^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),\\
&\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),\\
&\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma\\
&+ \sum_{\breve{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\breve{r}=1}^{m^{(\breve{\alpha})}}\int_{t_2}^{t_1} z_{\breve{r}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1\frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{r}}^{(\breve{\alpha})}\partial h_r^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),\\
&\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),\\
&\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma\\
&+ \sum_{\breve{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\breve{s}=m^{(\breve{\alpha})}+1}^{n^{(\breve{\alpha})}}\int_{t_2}^{t_1} z_{\breve{s}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1\frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\breve{\alpha})}\partial h_r^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),\\
&\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),
\end{aligned}
$$

$$
\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma\Big\}
$$

$$
+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}(z_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})-z_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}))
$$

$$
\cdot\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1}),u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1}))
$$

$$
+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}z_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1})
$$

$$
\cdot\Big\{\int_{t_2}^{t_1}\dot{h}_r^{(\alpha)}(\tau)\mathrm{d}\tau\int_1^0\frac{\partial^2 G_r^{(\alpha)}}{\partial h_r^{(\alpha)}\partial u_s^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),
$$

$$
\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2,d_{\alpha 1}),
$$

$$
\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma
$$

$$
+\sum_{\breve{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}\int_{t_2}^{t_1}z_{\breve{s}}^{(\alpha,\ell-1)}(\tau,d_{\alpha 1})\mathrm{d}\tau\int_0^1\frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\alpha)}\partial u_s^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),
$$

$$
\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2,d_{\alpha 1}),
$$

$$
\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma
$$

$$
+\sum_{\breve{\alpha}\in\mathcal{J}_{\alpha 10}}\sum_{\breve{r}=1}^{m^{(\breve{\alpha})}}\int_{t_2}^{t_1}z_{\breve{r}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha 1})\mathrm{d}\tau\int_0^1\frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{r}}^{(\breve{\alpha})}\partial u_s^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),
$$

$$
\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2,d_{\alpha 1}),
$$

$$
\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma
$$

$$
+\sum_{\breve{\alpha}\in\mathcal{J}_{\alpha 11}}\sum_{\breve{s}=m^{(\breve{\alpha})}+1}^{n^{(\breve{\alpha})}}\int_{t_2}^{t_1}z_{\breve{s}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha 1})\mathrm{d}\tau\int_0^1\frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\breve{\alpha})}\partial u_s^{(\alpha)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),
$$

$$
\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2,d_{\alpha 1}),
$$

$$
\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma\Big\}
$$

$$
+\sum_{\alpha^*\in\mathcal{J}_{\alpha 10}}\sum_{r^*=1}^{m^{(\alpha^*)}}(z_{r^*}^{(\alpha^*,\ell-1)}(t_1,d_{\alpha 1})-z_{r^*}^{(\alpha^*,\ell-1)}(t_2,d_{\alpha 1}))
$$

$$
\cdot\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1}),u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1}))
$$

$$
+\sum_{\alpha^*\in\mathcal{J}_{\alpha 10}}\sum_{r^*=1}^{m^{(\alpha^*)}}z_{r^*}^{(\alpha^*,\ell-1)}(t_2,d_{\alpha 1})
$$

$$
\cdot\Big\{\int_{t_2}^{t_1}\dot{h}_r^{(\alpha)}(\tau)\mathrm{d}\tau\int_1^0\frac{\partial^2 G_r^{(\alpha)}}{\partial h_r^{(\alpha)}\partial u_{r^*}^{(\alpha^*)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),
$$

$$
\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2,d_{\alpha 1}),
$$

$$
\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma
$$

$$+\sum_{\breve{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}\int_{t_2}^{t_1} z_{\breve{s}}^{(\alpha,\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1 \frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\alpha)}\partial u_{r^*}^{(\alpha^*)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$
$$\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),$$
$$\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma$$
$$+\sum_{\breve{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\breve{r}=1}^{m^{(\breve{\alpha})}}\int_{t_2}^{t_1} z_{\breve{r}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1 \frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{r}}^{(\breve{\alpha})}\partial u_{r^*}^{(\alpha^*)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$
$$\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),$$
$$\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma$$
$$+\sum_{\breve{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\breve{s}=m^{(\breve{\alpha})}+1}^{n^{(\breve{\alpha})}}\int_{t_2}^{t_1} z_{\breve{s}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1 \frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\breve{\alpha})}\partial u_{r^*}^{(\alpha^*)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$
$$\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),$$
$$\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma\Big\}$$
$$+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}(z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}(t_1,d_{\alpha1})-z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}(t_2,d_{\alpha1}))$$
$$\cdot\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)}(t_1),u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1}),u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1}),u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1}))$$
$$+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}} z_{s^\sharp}^{(\alpha^\sharp,\ell-1)}(t_2,d_{\alpha1})$$
$$\cdot\Big\{\int_{t_2}^{t_1}\dot{h}_r^{(\alpha)}(\tau)\mathrm{d}\tau\int_1^0 \frac{\partial^2 G_r^{(\alpha)}}{\partial h_r^{(\alpha)}\partial u_{s^\sharp}^{(\alpha^\sharp)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$
$$\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),$$
$$\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma$$
$$+\sum_{\breve{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}\int_{t_2}^{t_1} z_{\breve{s}}^{(\alpha,\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1 \frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\alpha)}\partial u_{s^\sharp}^{(\alpha^\sharp)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$
$$\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),$$
$$\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma$$
$$+\sum_{\breve{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\breve{r}=1}^{m^{(\breve{\alpha})}}\int_{t_2}^{t_1} z_{\breve{r}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1 \frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{r}}^{(\breve{\alpha})}\partial u_{s^\sharp}^{(\alpha^\sharp)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$
$$\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},\ell-1)}(t_2,d_{\alpha1}),$$
$$\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_1,d_{\alpha1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},\ell-1)}(t_2,d_{\alpha1}))\mathrm{d}\gamma$$
$$+\sum_{\breve{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\breve{s}=m^{(\breve{\alpha})}+1}^{n^{(\breve{\alpha})}}\int_{t_2}^{t_1} z_{\breve{s}}^{(\breve{\alpha},\ell-1)}(\tau,d_{\alpha1})\mathrm{d}\tau\int_0^1 \frac{\partial^2 G_r^{(\alpha)}}{\partial u_{\breve{s}}^{(\breve{\alpha})}\partial u_{s^\sharp}^{(\alpha^\sharp)}}(\gamma h_r^{(\alpha)}(t_1)+(1-\gamma)h_r^{(\alpha)}(t_2),$$

$$
\begin{aligned}
&\gamma u_s^{(\alpha,\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_s^{(\alpha,\ell-1)}(t_2,d_{\alpha 1}),\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha 10},\ell-1)}(t_2,d_{\alpha 1}),\\
&\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_1,d_{\alpha 1})+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},\ell-1)}(t_2,d_{\alpha 1}))\mathrm{d}\gamma\Big\},\\
&r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N.
\end{aligned}
$$

Thus, by (2.51) and (1.29), (2.17), (2.20), we can get

$$
\begin{aligned}
\omega(\eta\mid z_r^{(\alpha,\ell)}(\cdot,d_{\alpha 1}))&\le 2\omega(\eta\mid \dot h_r^{(\alpha)})+C\varepsilon^2|t_1-t_2|\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}|\nabla^2G_r^{(\alpha)}(u^{(\alpha)})|+\frac18\Omega_P(\omega)a_0\\
&\le\frac{a_1-a_0}{8}\Omega_P(\eta)+\frac{a_0}{8}\Omega_P(\eta)+C\varepsilon^2\eta\\
&\le\frac{a_2}{8}\Omega_P(\eta),\quad\forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.54)
\end{aligned}
$$

Similarly, by (2.37), we have

$$
\omega(\eta\mid z_s^{(\alpha,\ell)}(\cdot,d_{\alpha 0}))\le\frac{a_2}{8}\Omega_P(\eta),\quad\forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.55)
$$

While, on each edge $\mathcal{E}_\alpha$, we could use (2.29) to get

$$
\begin{aligned}
&|t_k^{(\alpha,\ell)}(x;t_1,x_*)-t_k^{(\alpha,\ell)}(x;t_2,x_*)|\\
\le{}&|t_k^{(\alpha,\ell)}(x_*;t_1,x_*)-t_k^{(\alpha,\ell)}(x_*;t_2,x_*)|\\
&+\Big|\int_{x_*}^{x}\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}(t_k^{(\alpha,\ell)}(y;t_1,x_*),y))-\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}(t_k^{(\alpha,\ell)}(y;t_2,x_*),y))\mathrm{d}y\Big|\\
\le{}&|t_1-t_2|+\Big|\int_{x_*}^{x}(t_k^{(\alpha,\ell)}(y;t_1,x_*)-t_k^{(\alpha,\ell)}(y;t_2,x_*))\\
&\cdot\int_0^1\sum_{j=1}^{n^{(\alpha)}}\Big(\frac{\partial\mu_k^{(\alpha)}}{\partial u_j^{(\alpha)}}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)}\Big)(\gamma t_k^{(\alpha,\ell)}(y;t_1,x_*)+(1-\gamma)t_k^{(\alpha,\ell)}(y;t_2,x_*),y)\mathrm{d}\gamma\mathrm{d}y\Big|,\\
&\forall\, k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}
$$

Thus, by Gronwall's inequality,

$$
\begin{aligned}
&|t_k^{(\alpha,\ell)}(x;t_1,x_*)-t_k^{(\alpha,\ell)}(x;t_2,x_*)|\le|t_1-t_2|\mathrm{e}^{C\varepsilon}\le|t_1-t_2|(1+C\varepsilon),\\
&\forall\, x\in[d_{\alpha 0},d_{\alpha 1}],\ \forall\, k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.56)
\end{aligned}
$$

Now we can integrate (2.40) along $t=t_r^{(\alpha,\ell)}(x;t_1,x_*)$ and $t=t_r^{(\alpha,\ell)}(x;t_2,x_*)$, respectively, to get

$$
\begin{aligned}
&z_r^{(\alpha,\ell)}(t_1,x_*)-z_r^{(\alpha,\ell)}(t_2,x_*)\\
={}&z_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(d_{\alpha 1};t_1,x_*),d_{\alpha 1})-z_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(d_{\alpha 1};t_2,x_*),d_{\alpha 1})\\
&+\int_{d_{\alpha 1}}^{x_*}-\sum_{k=1}^{n^{(\alpha)}}\Big(\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}\Big)\Big|_{(t_r^{(\alpha,\ell)}(y;t_1,x_*),y)}z_r^{(\alpha,\ell)}\Big|_{(t_r^{(\alpha,\ell)}(y;t_2,x_*),y)}^{(t_r^{(\alpha,\ell)}(y;t_1,x_*),y)}\mathrm{d}y\\
&+\int_{d_{\alpha 1}}^{x_*}-\sum_{k=1}^{n^{(\alpha)}}\Big(\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}\Big)\Big|_{(t_r^{(\alpha,\ell)}(y;t_2,x_*),y)}^{(t_r^{(\alpha,\ell)}(y;t_1,x_*),y)}z_r^{(\alpha,\ell)}((t_r^{(\alpha,\ell)}(y;t_2,x_*),y))\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{d_{\alpha 1}}^{x_*}\sum_{j,k=1}^{n^{(\alpha)}}\Big(\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}\partial_x u_j^{(\alpha,\ell-1)}\\
&+B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})\frac{\partial \mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_j^{(\alpha,\ell-1)}\\
&-\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}z_j^{(\alpha,\ell-1)}\partial_x u_k^{(\alpha,\ell-1)}\Big)\Big|_{(t_r^{(\alpha,\ell)}(y;t_2,x_*),y)}^{(t_r^{(\alpha,\ell)}(y;t_1,x_*),y)}\mathrm{d}y\\
&+\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)}\Big|_{(t_2,x_*)}^{(t_1,x_*)}-\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)}\Big|_{(t_r^{(\alpha,\ell)}(d_{\alpha 1};t_2,x_*),d_{\alpha 1})}^{(t_r^{(\alpha,\ell)}(d_{\alpha 1};t_1,x_*),d_{\alpha 1})},\\
&\forall\, x_*\in[d_{\alpha 0},d_{\alpha 1}],\ \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}
$$

Thus, applying Gronwall's inequality and noting (2.54), (2.56) and (2.17), (2.19), we have that for any given $|t_1-t_2|<\eta$ and $x_*\in[d_{\alpha 0},d_{\alpha 1}]$,

$$
\begin{aligned}
&|z_r^{(\alpha,\ell)}(t_1,x_*)-z_r^{(\alpha,\ell)}(t_2,x_*)|\le \mathrm{e}^{C\varepsilon}\Big(\frac{a_2}{8}\Omega_P((1+C\varepsilon)\eta)+C\varepsilon(\varepsilon a+\Omega_P((1+C\varepsilon)\eta))\Big),\\
&\forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.57)
\end{aligned}
$$

By (2.53), we have

$$
\Omega_P((1+C\varepsilon)\eta)\le(1+C\varepsilon)\Omega_P(\eta),
$$

and therefore,

$$
\omega(\eta\mid z_r^{(\alpha,\ell)}(\cdot,x_*))\le\frac{1}{8}\Omega_P(\eta),\quad \forall\, x_*\in[d_{\alpha 0},d_{\alpha 1}],\ \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.58)
$$

Similarly, by integrating (2.41), it holds

$$
\begin{aligned}
&\omega(\eta\mid z_s^{(\alpha,\ell)}(\cdot,x_*))\le\frac{1}{8}\Omega_P(\eta),\\
&\forall\, x_*\in[d_{\alpha 0},d_{\alpha 1}],\ \forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \qquad (2.59)
\end{aligned}
$$

At last, we should check (2.14) as follows. For the special case that (t_2,x_2) locates on the r-th characteristic curve passing through (t_1,x_1), namely,

$$
t_r^{(\alpha,\ell)}(x_2;t_1,x_1)=t_2
$$

with $|x_1-x_2|<\eta$, we can integrate (2.40) along $t=t_r^{(\alpha,\ell)}(x;t_1,x_1)$ to get

$$
\begin{aligned}
&z_r^{(\alpha,\ell)}(t_1,x_1)-z_r^{(\alpha,\ell)}((t_r^{(\alpha,\ell)}(x_2;t_1,x_1)),x_2)\\
=&\int_{x_1}^{x_2}\Big(-\sum_{k=1}^{n^{(\alpha)}}\frac{\partial\mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_r^{(\alpha,\ell)}\\
&+\sum_{j,k=1}^{n^{(\alpha)}}\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}(\partial_x u_j^{(\alpha,\ell-1)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j,k=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})\frac{\partial \mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)}z_j^{(\alpha,\ell-1)}\\
&-\sum_{j,k=1}^{n^{(\alpha)}} z_j^{(\alpha,\ell-1)}\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(\partial_x u_k^{(\alpha,\ell-1)}+\mu_r^{(\alpha)}(u^{(\alpha,\ell-1)})z_k^{(\alpha,\ell-1)})\Big)\Big|_{(t_r^{(\alpha,\ell)}(y;t_1,x_*),y)}\mathrm{d}y\\
&+\sum_{j=1}^{n^{(\alpha)}}(B_{rj}^{(\alpha)}(u^{(\alpha,\ell-1)})z_j^{(\alpha,\ell-1)})\Big|_{(t_2,x_2)}^{(t_1,x_1)}.
\end{aligned}
$$

Thus, by (2.6), (2.12), (2.17) and (2.19), we have

$$
\begin{aligned}
&|z_r^{(\alpha,\ell)}(t_1,x_1)-z_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(x_2;t_1,x_1),x_2)|\le C\varepsilon^2\eta+C\varepsilon\Omega_P(\eta),\\
&\forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}\tag{2.60}
$$

Then for the general two given points (t_1,x_1) and (t_2,x_2) with $|t_1-t_2|<\eta, |x_1-x_2|<\eta$, we want to estimate $|z_r^{(\alpha,\ell)}(t_1,x_1)-z_r^{(\alpha,\ell)}(t_2,x_2)|$. Without loss of generality, we assume $x_1<x_2$, then we can set $t_3=t_r^{(\alpha,\ell)}(x_2;t_1,x_1)$, noting our assumption (1.12), it holds

$$|t_3-t_1|\le\mu_{\max}|x_2-x_1|\le|x_2-x_1|\le\eta$$

and thus

$$|t_3-t_2|\le|t_3-t_1|+|t_1-t_2|<2\eta.$$

Now, by our estimates (2.58) and (2.60), it is direct to get

$$
\begin{aligned}
&\quad|z_r^{(\alpha,\ell)}(t_1,x_1)-z_r^{(\alpha,\ell)}(t_2,x_2)|\\
&\le|z_r^{(\alpha,\ell)}(t_1,x_1)-z_r^{(\alpha,\ell)}(t_3,x_2)|+|z_r^{(\alpha,\ell)}(t_3,x_2)-z_r^{(\alpha,\ell)}(t_2,x_2)|\\
&=|z_r^{(\alpha,\ell)}(t_1,x_1)-z_r^{(\alpha,\ell)}(t_r^{(\alpha,\ell)}(x_2;t_1,x_1),x_2)|+|z_r^{(\alpha,\ell)}(t_2,x_2)-z_r^{(\alpha,\ell)}(t_2,x_2)|\\
&\le C\varepsilon^2\eta+C\varepsilon\Omega_P(\eta)+\frac{1}{8}\Omega_P(\eta)\\
&\le\frac{1}{7}\Omega_P(\eta),\quad\forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}\tag{2.61}
$$

Similarly, we can get

$$|z_s^{(\alpha,\ell)}(t_1,x_1)-z_s^{(\alpha,\ell)}(t_2,x_2)|\le\frac{1}{7}\Omega_P(\eta),\quad\forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\quad\forall\,\alpha=1,\cdots,N.\tag{2.62}$$

Then by (2.25) and (2.27), we have that

$$
\begin{aligned}
&\partial_x u_k^{(\alpha,\ell)}(t_1,x_1)-\partial_x u_k^{(\alpha,\ell)}(t_2,x_2)\\
&\le(-\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}))|_{(t_2,x_2)}^{(t_1,x_1)}z_k^{(\alpha,\ell)}(t_1,x_1)+(-\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}(t_2,x_2)))z_k^{(\alpha,\ell)}|_{(t_2,x_2)}^{(t_1,x_1)}\\
&\quad+\sum_{j=1}^{n^{(\alpha)}}B_{kj}^{(\alpha)}(u_k^{(\alpha,\ell)})|_{(t_2,x_2)}^{(t_1,x_1)}\partial_x u_j^{(\alpha,\ell-1)}(t_1,x_1)+\sum_{j=1}^{n^{(\alpha)}}B_{kj}^{(\alpha)}(u_k^{(\alpha,\ell)}(t_2,x_2))\cdot\partial_x u_j^{(\alpha,\ell-1)}|_{(t_2,x_2)}^{(t_1,x_1)}\\
&\quad+\sum_{j=1}^{n^{(\alpha)}}(B_{kj}^{(\alpha)}(u_k^{(\alpha,\ell)})\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}))|_{(t_2,x_2)}^{(t_1,x_1)}z_j^{(\alpha,\ell-1)}(t_1,x_1)
\end{aligned}
$$

$$+\sum_{j=1}^{n^{(\alpha)}}(B_{kj}^{(\alpha)}(u_k^{(\alpha,\ell)}(t_2,x_2))\mu_k^{(\alpha)}(u^{(\alpha,\ell-1)}(t_2,x_2)))\cdot z_j^{(\alpha,\ell-1)}|_{(t_2,x_2)}^{(t_1,x_1)},$$
$$k=1,\cdots,n^{(\alpha)};\quad \alpha=1,\cdots,N.$$

By (2.6), (2.17) and (2.19), this directly leads to

$$|\partial_x u_k^{(\alpha,\ell)}(t_1,x_1)-\partial_x u_k^{(\alpha,\ell)}(t_2,x_2)|\le\frac{1}{7}\Omega_P(\eta)+C\varepsilon^2 a+C\varepsilon\Omega_P(\eta)\le\frac{1}{6}\Omega_P(\eta),$$
$$\forall\, k=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{2.63}$$

Combining (2.61)–(2.63), we can get (2.14) and complete our proof of Proposition 2.2 and thus of Theorem 1.1 and Proposition 2.1.

3 Stability of the Time-Periodic Solution

In order to prove Theorem 1.2, as (2.4)–(2.6), we can multiply the left eigenvector $l_k^{(\alpha)}(u^{(\alpha)})$ to (1.1) to get

$$\partial_x u_k^{(\alpha)}+\mu_k^{(\alpha)}(u^{(\alpha)})\partial_t u_k^{(\alpha)}=\sum_{j=1}^{n^{(\alpha)}}B_{kj}^{(\alpha)}(u^{(\alpha)})(\partial_x u_j^{(\alpha)}+\mu_k^{(\alpha)}(u^{(\alpha)})\partial_t u_j^{(\alpha)}),$$
$$k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N \tag{3.1}$$

and

$$\partial_x u_k^{(\alpha,P)}+\mu_k^{(\alpha)}(u^{(\alpha,P)})\partial_t u_k^{(\alpha,P)}=\sum_{j=1}^{n^{(\alpha)}}B_{kj}^{(\alpha)}(u^{(\alpha,P)})(\partial_x u_j^{(\alpha,P)}+\mu_k^{(\alpha)}(u^{(\alpha,P)})\partial_t u_j^{(\alpha,P)}),$$
$$k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N \tag{3.2}$$

with

$$B_{kj}^{(\alpha)}(0)=0,\quad\forall\,k,j=1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N. \tag{3.3}$$

While, on the boundary, we have (1.19)–(1.20) as

$$x=d_{\alpha1}:\ u_r^{(\alpha)}=G_r^{(\alpha)}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}),$$
$$r=1,\cdots,m^{(\alpha)};\quad\alpha=1,\cdots,N, \tag{3.4}$$

$$x=d_{\alpha0}:\ u_s^{(\alpha)}=G_s^{(\alpha)}(h_s^{(\alpha)},u_r^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha00})},u_{s^\sharp}^{(\mathcal{J}_{\alpha01})}),$$
$$s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\quad\alpha=1,\cdots,N \tag{3.5}$$

and

$$x=d_{\alpha1}:\ u_r^{(\alpha,P)}=G_r^{(\alpha)}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)}),$$
$$r=1,\cdots,m^{(\alpha)};\quad\alpha=1,\cdots,N, \tag{3.6}$$
$$x=d_{\alpha0}:\ u_s^{(\alpha,P)}=G_s^{(\alpha)}(h_s^{(\alpha)},u_r^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha00},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha01},P)}),$$

$$s=m^{(\alpha)}+1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N, \tag{3.7}$$

Different from the stability result of [14], besides the exponential convergence in the C^0 topology, we should also show the convergence in C^1 topology. Denote

$$\zeta(t)=\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\sup_{x\in[d_{\alpha0},d_{\alpha1}]}|u_k^{(\alpha)}(t,x)-u_k^{(\alpha,P)}(t,x)| \tag{3.8}$$

and

$$\beta(t)=\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\sup_{x\in[d_{\alpha0},d_{\alpha1}]}|z_k^{(\alpha)}(t,x)-z_k^{(\alpha,P)}(t,x)|, \tag{3.9}$$

where

$$z_k^{(\alpha)}(t,x)=\partial_t u_k^{(\alpha)}(t,x),\quad z_k^{(\alpha,P)}(t,x)=\partial_t u_k^{(\alpha,P)}(t,x).$$

Using the bootstrap argument, we need to show that for any given $p\in\mathbb{Z}_+$ and any given $t_0\in[pT_0,(p+1)T_0]$, we have

$$\zeta(t)\le M_2a^p\varepsilon,\quad \forall\, t\in[pT_0,t_0] \tag{3.10}$$

and

$$\beta(t)\le\Omega_P(M_3a^p\varepsilon)+M_3a^p\varepsilon,\quad \forall\, t\in[pT_0,t_0], \tag{3.11}$$

under the assumption

$$\zeta(t)\le M_2a^{p-1}\frac{a_1}{a_0}\varepsilon,\quad \forall\, t\in[(p-1)T_0,t_0] \tag{3.12}$$

and

$$\beta(t)\le\Omega_P\Big(M_3\frac{a_2}{a_1}a^{p-1}\varepsilon\Big)+M_3\frac{a_2}{a_1}a^{p-1}\varepsilon,\quad \forall\, t\in[(p-1)T_0,t_0] \tag{3.13}$$

for some positive constants M_2 and M_3.

In fact, on the boundary, by (3.4) and (3.6), we have that at the boundary $x=d_{\alpha1}$,

$$\begin{aligned}
&u_r^{(\alpha)}(t,d_{\alpha1})-u_r^{(\alpha,P)}(t,d_{\alpha1})\\
=&\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}(u_s^{(\alpha)}(t,d_{\alpha1})-u_s^{(\alpha,P)}(t,d_{\alpha1}))\cdot\int_0^1\frac{\partial G^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\\
&+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}(u_{r^*}^{(\alpha^*)}(t,d_{\alpha1})-u_{r^*}^{(\alpha^*,P)}(t,d_{\alpha1}))\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\
&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}(u_{s^\sharp}^{(\alpha^\sharp)}(t,d_{\alpha1})-u_{s^\sharp}^{(\alpha^\sharp,P)}(t,d_{\alpha1}))\int_0^1\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},
\end{aligned}$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha 10})} + (1-\gamma) u_{r^*}^{(\mathcal{J}_{\alpha 10},P)}, \gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha 11})} + (1-\gamma) u_{s^\sharp}^{(\mathcal{J}_{\alpha 11},P)})|_{(t,d_{\alpha 1})} \mathrm{d}\gamma,$$
$$\forall\, r = 1,\cdots,m^{(\alpha)},\ \forall\, \alpha = 1,\cdots,N.$$

By our a priori assumption (3.12) and our choice of constants (2.23), it holds that

$$\max_{\alpha=1,\cdots,N} \max_{r=1,\cdots,m^{(\alpha)}} |u_r^{(\alpha)}(t,d_{\alpha 1}) - u_r^{(\alpha,P)}(t,d_{\alpha 1})| \le M_2 a^{p-1} a_1 \varepsilon, \quad \forall\, t \in [(p-1)T_0, t_0]. \tag{3.14}$$

Similarly, by (3.5) and (3.7), we can get

$$\begin{aligned} &\max_{\alpha=1,\cdots,N} \max_{s=m^{(\alpha)}+1,\cdots,n^{(\alpha)}} |u_s^{(\alpha)}(t,d_{\alpha 0}) - u_s^{(\alpha,P)}(t,d_{\alpha 0})| \\ &\le M_2 a^{p-1} a_1 \varepsilon, \quad \forall\, t \in [(p-1)T_0, t_0]. \end{aligned} \tag{3.15}$$

To get the a priori estimates on each edge $\mathcal{E}_\alpha$, we introduce the characteristic curves $t_k^{(\alpha)}(x;t_*,x_*)$ and $t_k^{(\alpha,P)}(x;t_*,x_*)$ as before

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x} t_k^{(\alpha)}(x;t_*,x_*) = \mu_k^{(\alpha)}(u^{(\alpha)}(t_k^{(\alpha)}(x;t_*,x_*),x)), \\ t_k^{(\alpha)}(x_*;t_*,x_*) = t_*, \end{cases} \quad k = 1,\cdots,n^{(\alpha)};\ \alpha = 1,\cdots,N, \tag{3.16}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x} t_k^{(\alpha,P)}(x;t_*,x_*) = \mu_k^{(\alpha)}(u^{(\alpha,P)}(t_k^{(\alpha,P)}(x;t_*,x_*),x)), \\ t_k^{(\alpha,P)}(x_*;t_*,x_*) = t_*, \end{cases} \quad k = 1,\cdots,n^{(\alpha)};\ \alpha = 1,\cdots,N. \tag{3.17}$$

By (3.1)–(3.2), we have

$$\begin{aligned} &\partial_x(u_r^{(\alpha)} - u_r^{(\alpha,P)}) + \mu_r^{(\alpha)}(u^{(\alpha)})\partial_t(u_r^{(\alpha)} - u_r^{(\alpha,P)}) \\ = &(\mu_r^{(\alpha)}(u^{(\alpha,P)}) - \mu_r^{(\alpha)}(u^{(\alpha)}))\partial_t u_r^{(\alpha,P)} + \sum_{j=1}^{n^{(\alpha)}} (B_{rj}^{(\alpha)}(u^{(\alpha)})(\partial_x u_j^{(\alpha)} + \mu_r^{(\alpha)}(u^{(\alpha)})\partial_t u_j^{(\alpha)}) \\ &- B_{rj}^{(\alpha)}(u^{(\alpha,P)})(\partial_x u_j^{(\alpha,P)} + \mu_r^{(\alpha)}(u^{(\alpha,P)})\partial_t u_j^{(\alpha,P)})) \\ = &(\mu_r^{(\alpha)}(u^{(\alpha,P)}) - \mu_r^{(\alpha)}(u^{(\alpha)}))\partial_t u_r^{(\alpha,P)} + \sum_{j=1}^{n^{(\alpha)}} (B_{rj}^{(\alpha)}(u^{(\alpha)})(\partial_x u_j^{(\alpha)} + \mu_r^{(\alpha)}(u^{(\alpha)})\partial_t u_j^{(\alpha)}) \\ &- B_{rj}^{(\alpha)}(u^{(\alpha)})(\partial_x u_j^{(\alpha,P)} + \mu_r^{(\alpha)}(u^{(\alpha)})\partial_t u_j^{(\alpha,P)})) \\ &+ \sum_{j=1}^{n^{(\alpha)}} ((B_{rj}^{(\alpha)}(u^{(\alpha)}) - B_{rj}^{(\alpha)}(u^{(\alpha,P)}))\partial_x u_j^{(\alpha,P)} \\ &+ (B_{rj}^{(\alpha)}(u^{(\alpha)})\mu_r^{(\alpha)}(u^{(\alpha)}) - B_{rj}^{(\alpha)}(u^{(\alpha,P)})\mu_r^{(\alpha)}(u^{(\alpha,P)}))\partial_t u_j^{(\alpha,P)}) \\ = &-\partial_t u_r^{(\alpha,P)} \Big(\sum_{k=1}^{n^{(\alpha)}} (u_k^{(\alpha)} - u_k^{(\alpha,P)}) \int_0^1 \frac{\partial \mu_r^{(\alpha)}}{\partial u_k^{(\alpha)}} (\gamma u^{(\alpha)} + (1-\gamma) u^{(\alpha,P)}) \mathrm{d}\gamma \Big) \\ &+ \big(\partial_x + \mu_r^{(\alpha)}(u^{(\alpha)})\partial_t\big) \Big(\sum_{j=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha)})(u_j^{(\alpha)} - u_j^{(\alpha,P)}) \Big) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j,k=1}^{n^{(\alpha)}}(u_j^{(\alpha)}-u_j^{(\alpha,P)})\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha)})\big(\partial_x u_k^{(\alpha)}+\mu_r^{(\alpha)}(u^{(\alpha)})\partial_t u_k^{(\alpha)}\big)\\
&+\sum_{j,k=1}^{n^{(\alpha)}}\partial_x u_j^{(\alpha,P)}(u_k^{(\alpha)}-u_k^{(\alpha,P)})\int_0^1\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(\gamma u^{(\alpha)}+(1-\gamma)u^{(\alpha,P)})\mathrm{d}\gamma\\
&+\sum_{j,k=1}^{n^{(\alpha)}}\partial_t u_j^{(\alpha,P)}(u_k^{(\alpha)}-u_k^{(\alpha,P)})\int_0^1\frac{\partial (B_{rj}^{(\alpha)}\mu_r^{(\alpha)})}{\partial u_k^{(\alpha)}}(\gamma u^{(\alpha)}+(1-\gamma)u^{(\alpha,P)})\mathrm{d}\gamma,\\
&r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N.
\end{aligned}
$$

Integrating along the corresponding characteristic curve $t=t_r^{(\alpha)}(x;t_*,x_*)$, we have

$$
\begin{aligned}
&|u_r^{(\alpha)}(t_*,x_*)-u_r^{(\alpha,P)}(t_*,x_*)|\\
\le\ &|u_r^{(\alpha)}(t_r^{(\alpha)}(d_{\alpha1};t_*,x_*),d_{\alpha1})-u^{(\alpha,P)}(t_r^{(\alpha)}(d_{\alpha1};t_*,x_*),d_{\alpha1})|\\
&+\sum_{j=1}^{n^{(\alpha)}}B_{rj}^{(\alpha)}(u^{(\alpha)})(u_j^{(\alpha)}-u_j^{(\alpha,P)})\big|_{(t_r^{(\alpha)}(d_{\alpha1};t_*,x_*),d_{\alpha1})}^{(t_*,x_*)}\\
&+C(\|u^{(\alpha,P)}\|_{C^1}+\|u^{(\alpha)}\|_{C^1})\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}\{|\nabla\mu_r^{(\alpha)}|+|\nabla B_{rj}^{(\alpha)}|+|\nabla(B_{rj}^{(\alpha)}\mu_r^{(\alpha)})|\}\\
&\cdot\Big|\int_{d_{\alpha1}}^{x_*}|u^{(\alpha)}-u^{(\alpha,P)}||_{t_r^{(\alpha)}(y;t_*,x_*)}\mathrm{d}y\Big|.
\end{aligned}
$$

By (3.3), (3.12)–(3.14) and our a priori assumptions, we can get

$$
\begin{aligned}
&\max_{\alpha=1,\cdots,N}\max_{r=1,\cdots,m^{(\alpha)}}|u_r^{(\alpha)}(t,x)-u_r^{(\alpha,P)}(t,x)|\le M_2a^{p-1}a_1\varepsilon+C\varepsilon M_2a^{p-1}\frac{a_1}{a_0}\varepsilon\le M_2a^p\varepsilon,\\
&\forall\, t\in[pT_0,t_0],\ \forall\, x\in[d_{\alpha0},d_{\alpha1}],\ \forall\, r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}\tag{3.18}
$$

Similarly, one can get

$$
\begin{aligned}
&\max_{\alpha=1,\cdots,N}\max_{r=1,\cdots,m^{(\alpha)}}|u_s^{(\alpha)}(t,x)-u_s^{(\alpha,P)}(t,x)|\le M_2a^p\varepsilon,\\
&\forall\, t\in[pT_0,t_0],\ \forall\, x\in[d_{\alpha0},d_{\alpha1}],\ \forall\, s=m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,
\end{aligned}\tag{3.19}
$$

which yields (3.10) and completes the proof of the exponential convergence in the C^0 space.

In order to get the estimate for the convergence in the C^1 space, on the boundary, we can take temporal derivative to (3.4) and (3.6) to get

$$
\begin{aligned}
x=d_{\alpha1}:\ z_r^{(\alpha)}=\ &\dot h_r^{(\alpha)}(t)\frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})})\\
&+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}z_s^{(\alpha)}\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})})\\
&+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}z_{r^*}^{(\alpha^*)}\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})})\\
&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}z_{s^\sharp}^{(\alpha^\sharp)}\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)},u_s^{(\alpha)},u_{r^*}^{(\mathcal{J}_{\alpha10})},u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}),
\end{aligned}
$$

$$r=1,\cdots,m^{(\alpha)},\ \alpha=1,\cdots,N \tag{3.20}$$

and

$$\begin{aligned}x=d_{\alpha1}:\ z_r^{(\alpha,P)}=&\dot{h}_r^{(\alpha)}(t)\frac{\partial G_r^{(\alpha)}}{\partial h_r^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})\\&+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}z_s^{(\alpha,P)}\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})\\&+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}z_{r^*}^{(\alpha^*,P)}\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})\\&+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}z_{s^\sharp}^{(\alpha^\sharp,P)}\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)}),\\&r=1,\cdots,m^{(\alpha)},\ \alpha=1,\cdots,N.\end{aligned}\tag{3.21}$$

Thus, at $x=d_{\alpha1}$,

$$\begin{aligned}&z_r^{(\alpha)}(t,d_{\alpha1})-z_r^{(\alpha,P)}(t,d_{\alpha1})\\=&\dot{h}_r^{(\alpha)}(t)\Big(\sum_{\hat{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}(u_{\widehat{s}}^{(\alpha)}-u_{\widehat{s}}^{(\alpha,P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{s}}^{(\alpha)}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\\&+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\widehat{r}=1}^{m^{(\hat{\alpha})}}(u_{\widehat{r}}^{(\hat{\alpha})}-u_{\widehat{r}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{r}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\\&+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\widehat{s}=m^{(\hat{\alpha})}+1}^{n^{(\hat{\alpha})}}(u_{\widehat{s}}^{(\hat{\alpha})}-u_{\widehat{s}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{s}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\Big)\\&+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}(z_s^{(\alpha)}-z_s^{(\alpha,P)})\frac{\partial G_r^{(\alpha)}}{\partial u_s^{(\alpha)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})\\&+\sum_{s=m^{(\alpha)}+1}^{n^{(\alpha)}}z_s^{(\alpha)}\Big(\sum_{\hat{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}(u_{\widehat{s}}^{(\alpha)}-u_{\widehat{s}}^{(\alpha,P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{s}}^{(\alpha)}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\\&+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\widehat{r}=1}^{m^{(\hat{\alpha})}}(u_{\widehat{r}}^{(\hat{\alpha})}-u_{\widehat{r}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{r}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\\&\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\\&+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\widehat{s}=m^{(\hat{\alpha})}+1}^{n^{(\hat{\alpha})}}(u_{\widehat{s}}^{(\hat{\alpha})}-u_{\widehat{s}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{s}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},\end{aligned}$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\Big)$$

$$+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}(z_{r^*}^{(\alpha^*)}-z_{r^*}^{(\alpha^*,P)})\frac{\partial G_r^{(\alpha)}}{\partial u_{r^*}^{(\alpha^*)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})$$

$$+\sum_{\alpha^*\in\mathcal{J}_{\alpha10}}\sum_{r^*=1}^{m^{(\alpha^*)}}z_{r^*}^{(\alpha^*)}\Big(\sum_{\hat{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}(u_{\hat{s}}^{(\alpha)}-u_{\hat{s}}^{(\alpha,P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\hat{s}}^{(\alpha)}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma$$

$$+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\widehat{r}=1}^{m^{(\hat{\alpha})}}(u_{\widehat{r}}^{(\hat{\alpha})}-u_{\widehat{r}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{r}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma$$

$$+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\widehat{s}=m^{(\hat{\alpha})}+1}^{n^{(\hat{\alpha})}}(u_{\widehat{s}}^{(\hat{\alpha})}-u_{\widehat{s}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{s}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\Big)$$

$$+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}(z_{s^\sharp}^{(\alpha^\sharp)}-z_{s^\sharp}^{(\alpha^\sharp,P)})\frac{\partial G_r^{(\alpha)}}{\partial u_{s^\sharp}^{(\alpha^\sharp)}}(h_r^{(\alpha)},u_s^{(\alpha,P)},u_{r^*}^{(\mathcal{J}_{\alpha10},P)},u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})$$

$$+\sum_{\alpha^\sharp\in\mathcal{J}_{\alpha11}}\sum_{s^\sharp=m^{(\alpha^\sharp)}+1}^{n^{(\alpha^\sharp)}}z_{s^\sharp}^{(\alpha^\sharp)}\Big(\sum_{\hat{s}=m^{(\alpha)}+1}^{n^{(\alpha)}}(u_{\hat{s}}^{(\alpha)}-u_{\hat{s}}^{(\alpha,P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\hat{s}}^{(\alpha)}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}$$

$$+(1-\gamma)u_s^{(\alpha,P)},\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma$$

$$+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha10}}\sum_{\widehat{r}=1}^{m^{(\hat{\alpha})}}(u_{\widehat{r}}^{(\hat{\alpha})}-u_{\widehat{r}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{r}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma$$

$$+\sum_{\widehat{\alpha}\in\mathcal{J}_{\alpha11}}\sum_{\widehat{s}=m^{(\hat{\alpha})}+1}^{n^{(\hat{\alpha})}}(u_{\widehat{s}}^{(\hat{\alpha})}-u_{\widehat{s}}^{(\hat{\alpha},P)})\int_0^1\frac{\partial^2G_r^{(\alpha)}}{\partial u_{\widehat{s}}^{(\hat{\alpha})}\partial h_r^{(\alpha)}}(h_r^{(\alpha)}(t),\gamma u_s^{(\alpha)}+(1-\gamma)u_s^{(\alpha,P)},$$

$$\gamma u_{r^*}^{(\mathcal{J}_{\alpha10})}+(1-\gamma)u_{r^*}^{(\mathcal{J}_{\alpha10},P)},\gamma u_{s^\sharp}^{(\mathcal{J}_{\alpha11})}+(1-\gamma)u_{s^\sharp}^{(\mathcal{J}_{\alpha11},P)})|_{(t,d_{\alpha1})}\mathrm{d}\gamma\Big)$$

$$r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N.$$

Thus, by the dissipative structure (2.23), the C^0 convergence (3.10) and the a priori assumption (3.13), we have

$$\max_{\alpha=1,\cdots,N}\max_{r=1,\cdots,m^{(\alpha)}}|z_r^{(\alpha)}(t,d_{\alpha1})-z_r^{(\alpha,P)}(t,d_{\alpha1})|$$
$$\leq Ca^p\varepsilon^2+a_0\Omega_P\Big(M_3\frac{a_2}{a_1}a^{p-1}\varepsilon\Big)+M_3\frac{a_0a_2}{a_1}a^{p-1}\varepsilon,\quad\forall\,t\in[pT_0,t_0].\tag{3.22}$$

Similarly, from (3.5) and (3.7), we can get

$$\max_{\alpha=1,\cdots,N}\max_{s=m^{(\alpha)}+1,\cdots,n^{(\alpha)}}|z_s^{(\alpha)}(t,d_{\alpha0})-z_s^{(\alpha,P)}(t,d_{\alpha0})|$$

$$\leq Ca^p\varepsilon^2 + a_0\Omega_P\Big(M_3\frac{a_2}{a_1}a^{p-1}\varepsilon\Big) + M_3\frac{a_0a_2}{a_1}a^{p-1}\varepsilon, \quad \forall\, t\in[pT_0,t_0]. \tag{3.23}$$

On each edge $\mathcal{E}_\alpha$, similar as (2.40)–(2.41), we can take the temporal derivative to (3.1)–(3.2) to get

$$\begin{aligned}
&\partial_x z_k^{(\alpha)} + \mu_k^{(\alpha)}(u^{(\alpha)})\partial_t z_k^{(\alpha)}\\
&=\sum_{j=1}^{n^{(\alpha)}}(\partial_x+\mu_k^{(\alpha)}(u^{(\alpha)})\partial_t)(B_{kj}^{(\alpha)}(u^{(\alpha)})z_j^{(\alpha)}) + \sum_{j,\hat{j}=1}^{n^{(\alpha)}}\Gamma_{kj\hat{j}}^{(\alpha)}(u^{(\alpha)})z_j^{(\alpha)}z_{\hat{j}}^{(\alpha)},\\
&k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N,
\end{aligned}\tag{3.24}$$

$$\begin{aligned}
&\partial_x z_k^{(\alpha,P)} + \mu_k^{(\alpha)}(u^{(\alpha,P)})\partial_t z_k^{(\alpha,P)}\\
&=\sum_{j=1}^{n^{(\alpha)}}(\partial_x+\mu_k^{(\alpha)}(u^{(\alpha,P)})\partial_t)(B_{kj}^{(\alpha)}(u^{(\alpha,P)})z_j^{(\alpha,P)}) + \sum_{j,\hat{j}=1}^{n^{(\alpha)}}\Gamma_{kj\hat{j}}^{(\alpha)}(u^{(\alpha,P)})z_j^{(\alpha,P)}z_{\hat{j}}^{(\alpha,P)},\\
&k=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N,
\end{aligned}\tag{3.25}$$

where

$$\begin{aligned}
\Gamma_{kj\hat{j}}^{(\alpha)}(u^{(\alpha)}) &= B_{kj}^{(\alpha)}(u^{(\alpha)})\frac{\partial\mu_k^{(\alpha)}}{\partial u_{\hat{j}}^{(\alpha)}}(u^{(\alpha)}) - \frac{\partial\mu_k^{(\alpha)}}{\partial u_j^{(\alpha)}}(u^{(\alpha)})\delta_{k\hat{j}}\\
&\quad+\sum_{\tilde{j}=1}^{n^{(\alpha)}}\frac{\partial B_{k\tilde{j}}^{(\alpha)}}{\partial u_{\hat{j}}^{(\alpha)}}(u^{(\alpha)})\sigma_{\tilde{j}j}^{(\alpha)}(u^{(\alpha)}) + \sum_{\tilde{j}=1}^{n^{(\alpha)}}\frac{\partial B_{kj}^{(\alpha)}}{\partial u_{\tilde{j}}^{(\alpha)}}(u^{(\alpha)})\sigma_{\tilde{j}\hat{j}}^{(\alpha)}(u^{(\alpha)}),\\
&\quad k,j,\hat{j}=1,\cdots,n^{(\alpha)},
\end{aligned}\tag{3.26}$$

and $\sigma_{kj}^{(\alpha)}(u^{(\alpha)})$ is the kj-th element of the matrix $A^{-1}(u^{(\alpha)})$. Integrating along the corresponding characteristic curves $t=t_r^{(\alpha)}(x;t_*,x_*)$ and $t=t_r^{(\alpha,P)}(x;t_*,x_*)$, respectively for $r=1,\cdots,m^{(\alpha)}$, we have

$$\begin{aligned}
z_r^{(\alpha)}(t_*,x_*) &= z_r^{(\alpha)}(t_r^{(\alpha)}(d_{\alpha1};t_*,x_*),d_{\alpha1}) + \sum_{j=1}^{n^{(\alpha)}}(B_{rj}^{(\alpha)}(u^{(\alpha)})z_j^{(\alpha)})\Big|_{(t_r^{(\alpha)}(d_{\alpha1};t_*,x_*),d_{\alpha1})}^{(t_*,x_*)}\\
&\quad+\int_{d_{\alpha1}}^{x_*}\Big(\sum_{j,k=1}^{n^{(\alpha)}}\Gamma_{rjk}^{(\alpha)}(u^{(\alpha)})z_j^{(\alpha)}z_k^{(\alpha)}\Big)\Big|_{(t_r^{(\alpha)}(y;t_*,x_*),y)}\mathrm{d}y,\\
&\quad r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N
\end{aligned}$$

and

$$\begin{aligned}
z_r^{(\alpha,P)}(t_*,x_*) &= z_r^{(\alpha,P)}(t_r^{(\alpha,P)}(d_{\alpha1};t_*,x_*),d_{\alpha1}) + \sum_{j=1}^{n^{(\alpha)}}(B_{rj}^{(\alpha)}(u^{(\alpha,P)})z_j^{(\alpha,P)})\Big|_{(t_r^{(\alpha,P)}(d_{\alpha1};t_*,x_*),d_{\alpha1})}^{(t_*,x_*)}\\
&\quad+\int_{d_{\alpha1}}^{x_*}\Big(\sum_{j,k=1}^{n^{(\alpha)}}\Gamma_{rjk}^{(\alpha)}(u^{(\alpha,P)})z_j^{(\alpha,P)}z_k^{(\alpha,P)}\Big)\Big|_{(t_r^{(\alpha,P)}(y;t_*,x_*),y)}\mathrm{d}y,\\
&\quad r=1,\cdots,m^{(\alpha)};\ \alpha=1,\cdots,N.
\end{aligned}$$

Thus,

$$
\begin{aligned}
& z_r^{(\alpha)}(t_*, x_*) - z_r^{(\alpha,P)}(t_*, x_*) \\
={}& z_r^{(\alpha)}(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}) - z_r^{(\alpha,P)}(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}) \\
& + z_r^{(\alpha,P)}(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}) - z_r^{(\alpha,P)}(t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}) \\
& + \Big(\sum_{j=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha)})(z_j^{(\alpha)} - z_j^{(\alpha,P)}) \\
& + \sum_{j,k=1}^{n^{(\alpha)}} z_j^{(\alpha,P)}(u_k^{(\alpha)} - u_k^{(\alpha,P)}) \int_0^1 \frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(\gamma u^{(\alpha)} + (1-\gamma)u^{(\alpha,P)}) \mathrm{d}\gamma \Big)\Big|_{(t_*, x_*)} \\
& - \Big(\sum_{j=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha)})(z_j^{(\alpha)} - z_j^{(\alpha,P)}) \\
& + \sum_{j,k=1}^{n^{(\alpha)}} z_j^{(\alpha,P)}(u_k^{(\alpha)} - u_k^{(\alpha,P)}) \int_0^1 \frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(\gamma u^{(\alpha)} + (1-\gamma)u^{(\alpha,P)}) \mathrm{d}\gamma \Big)\Big|_{(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})} \\
& - \sum_{j=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha,P)})\Big|_{(t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})}^{(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})} z_j^{(\alpha,P)}((t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})) \\
& - \sum_{j=1}^{n^{(\alpha)}} B_{rj}^{(\alpha)}(u^{(\alpha,P)}((t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}))) \cdot z_j^{(\alpha,P)}\Big|_{(t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})}^{(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})} \\
& + \int_{d_{\alpha 1}}^{x_*} \Big(\sum_{j,k=1}^{n^{(\alpha)}} z_j^{(\alpha)} z_k^{(\alpha)} \Big(\sum_{\tilde{j}=1}^{n^{(\alpha)}} (u_{\tilde{j}}^{(\alpha)} - u_{\tilde{j}}^{(\alpha,P)}) \int_0^1 \frac{\partial \Gamma_{rjk}^{(\alpha)}}{\partial u_{\tilde{j}}^{(\alpha)}}(\gamma u^{(\alpha)} + (1-\gamma)u^{(\alpha,P)}) \mathrm{d}\gamma \Big) \\
& + \Gamma_{rjk}^{(\alpha)}(u^{(\alpha,P)})(z_j^{(\alpha)} - z_j^{(\alpha,P)}) z_k^{(\alpha)} + \Gamma_{rjk}^{(\alpha)}(u^{(\alpha,P)}) z_j^{(\alpha,P)}(z_k^{(\alpha)} - z_k^{(\alpha,P)}) \Big)\Big|_{(t_r^{(\alpha)}(x; t_*, x_*), x)} \mathrm{d}x \\
& + \int_{d_{\alpha 1}}^{x_*} \Big(\sum_{j,k=1}^{n^{(\alpha)}} \Gamma_{rjk}^{(\alpha)}(u^{(\alpha,P)})\Big|_{(t_r^{(\alpha,P)}(x; t_*, x_*), x)}^{(t_r^{(\alpha)}(x; t_*, x_*), x)} z_j^{(\alpha,P)} \\
& \cdot (t_r^{(\alpha)}(x; t_*, x_*), x) z_k^{(\alpha,P)}(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1}) \\
& + \Gamma_{rjk}^{(\alpha)}(u^{(\alpha,P)}(t_r^{(\alpha,P)}(x; t_*, x_*), x)) \cdot z^{(\alpha,P)}\Big|_{(t_r^{(\alpha,P)}(x; t_*, x_*), x)}^{(t_r^{(\alpha)}(x; t_*, x_*), x)} z_k^{(\alpha,P)}(t_r^{(\alpha)}(x; t_*, x_*), x) \\
& + \Gamma_{rjk}^{(\alpha)}(u^{(\alpha,P)}(t_r^{(\alpha,P)}(x; t_*, x_*), x)) z^{(\alpha,P)}(t_r^{(\alpha,P)}(x; t_*, x_*), x) \cdot z_k^{(\alpha,P)}\Big|_{(t_r^{(\alpha,P)}(x; t_*, x_*), x)}^{(t_r^{(\alpha)}(x; t_*, x_*), x)} \Big) \mathrm{d}x, \\
& r = 1, \cdots, m^{(\alpha)}; \ \alpha = 1, \cdots, N,
\end{aligned}
$$

where

$$
\begin{aligned}
& B_{rj}^{(\alpha)}(u^{(\alpha,P)})\Big|_{(t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})}^{(t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})} \\
={}& (t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*) - t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*)) \\
& \cdot \sum_{k=1}^{n^{(\alpha)}} \int_0^1 \Big(\frac{\partial B_{rj}^{(\alpha)}}{\partial u_k^{(\alpha)}}(u^{(\alpha,P)}) z_k^{(\alpha,P)} \Big)\Big|_{(\gamma t_r^{(\alpha)}(d_{\alpha 1}; t_*, x_*) + (1-\gamma) t_r^{(\alpha,P)}(d_{\alpha 1}; t_*, x_*), d_{\alpha 1})} \mathrm{d}\gamma, \\
& r = 1, \cdots, m^{(\alpha)}; \ j = 1, \cdots, n^{(\alpha)}; \ \alpha = 1, \cdots, N
\end{aligned}
$$

and

$$
\begin{aligned}
&\Gamma^{(\alpha)}_{rjk}(u^{(\alpha,P)})|^{(t^{(\alpha)}_r(x;t_*,x_*),x)}_{(t^{(\alpha,P)}_r(x;t_*,x_*),x)}\\
=&(t^{(\alpha)}_r(x;t_*,x_*)-t^{(\alpha,P)}_r(x;t_*,x_*))\\
&\cdot\sum_{\tilde{j}=1}^{n^{(\alpha)}}\int_0^1\Big(\frac{\partial\Gamma^{(\alpha)}_{rjk}}{\partial u^{(\alpha)}_{\tilde{j}}}(u^{(\alpha,P)})z^{(\alpha,P)}_{\tilde{j}}\Big)\Big|_{(\gamma t^{(\alpha)}_r(x;t_*,x_*)+(1-\gamma)t^{(\alpha,P)}_r(x;t_*,x_*),x)}\mathrm{d}\gamma,\\
&r=1,\cdots,m^{(\alpha)};\ k,j=1,\cdots,n^{(\alpha)};\ \alpha=1,\cdots,N.
\end{aligned}
$$

And one may estimate the distance of these two characteristics, when they pass through one same point (t_*,x_*). In fact,

$$
\begin{aligned}
&t^{(\alpha)}_r(x;t_*,x_*)-t^{(\alpha,P)}_r(x;t_*,x_*)\\
=&t^{(\alpha)}_r(x_*;t_*,x_*)-t^{(\alpha,P)}_r(x_*;t_*,x_*)\\
&+\int_{x_*}^x\mu^{(\alpha)}_r(u^{(\alpha)}(t^{(\alpha)}_r(y;t_*,x_*),y))-\mu^{(\alpha)}_r(u^{(\alpha,P)}(t^{(\alpha,P)}_r(y;t_*,x_*),y))\mathrm{d}y\\
=&\int_{x_*}^x\mu^{(\alpha)}_r(u^{(\alpha)}(t^{(\alpha)}_r(y;t_*,x_*),y))-\mu^{(\alpha)}_r(u^{(\alpha,P)}(t^{(\alpha)}_r(y;t_*,x_*),y))\mathrm{d}y\\
&+\int_{x_*}^x\mu^{(\alpha)}_r(u^{(\alpha,P)}_r(t^{(\alpha)}_r(y;t_*,x_*),y))-\mu^{(\alpha)}_r(u^{(\alpha,P)}(t^{(\alpha,P)}_r(y;t_*,x_*),y))\mathrm{d}y\\
=&\sum_{k=1}^{n^{(\alpha)}}\int_{x_*}^x(u^{(\alpha)}_r(t^{(\alpha)}_r(y;t_*,x_*),y)-u^{(\alpha,P)}(t^{(\alpha)}_r(y;t_*,x_*),y))\\
&\cdot\int_0^1\frac{\partial\mu^{(\alpha)}_r}{\partial u^{(\alpha)}_k}(\gamma u^{(\alpha)}_r(t^{(\alpha)}_r(y;t_*,x_*),y)+(1-\gamma)u^{(\alpha,P)}(t^{(\alpha)}_r(y;t_*,x_*),y))\mathrm{d}\gamma\mathrm{d}y\\
&+\int_{x_*}^x(t^{(\alpha)}_r(y;t_*,x_*)-t^{(\alpha,P)}_r(y;t_*,x_*))\\
&\cdot\sum_{k=1}^{n^{(\alpha)}}\int_0^1\Big(\frac{\partial\mu^{(\alpha)}_r}{\partial u^{(\alpha)}_k}(u^{(\alpha,P)})\partial_t u^{(\alpha,P)}_k\Big)\Big|_{(\gamma t^{(\alpha)}_r(y;t_*,x_*)+(1-\gamma)t^{(\alpha,P)}_r(y;t_*,x_*),y)}\mathrm{d}\gamma\mathrm{d}y.
\end{aligned}
$$

Noting our assumption (1.12) and the definition (1.36)–(1.37), for $(t_*,x_*)\in[pT_0,t_0]\times[d_{\alpha0},d_{\alpha1}]$, we have

$$
\begin{aligned}
t^{(\alpha)}_r(y;t_*,x_*)\in[(p-1)T_0,t_0],\quad\forall\,y\in[d_{\alpha0},d_{\alpha1}],\ \forall\,r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,\\
t^{(\alpha,P)}_r(y;t_*,x_*)\in[(p-1)T_0,t_0],\quad\forall\,y\in[d_{\alpha0},d_{\alpha1}],\ \forall\,r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N.
\end{aligned}
$$

Then by (3.10), using Gronwall's inequality, we have

$$
\begin{aligned}
&|t^{(\alpha)}_r(x;t_*,x_*)-t^{(\alpha,P)}_r(x;t_*,x_*)|\le M_4a^{p-1}\frac{a_1}{a_0}\varepsilon\cdot\mathrm{e}^{M_4\varepsilon}\le(1+C\varepsilon)M_4\frac{a_1}{a_0}a^{p-1}\varepsilon,\\
&\forall\,t_*\in[pT_0,t_0],\ \forall\,x_*\in[d_{\alpha0},d_{\alpha1}],\ \forall\,r=1,\cdots,m^{(\alpha)},\ \forall\,\alpha=1,\cdots,N,
\end{aligned}\tag{3.27}
$$

where

$$
M_4=L(M_2+M_0)\sup_{u^{(\alpha)}\in\mathcal{U}^{(\alpha)}}|\nabla\mu^{(\alpha)}_k(u^{(\alpha)})|.
$$

Similarly, we can get

$$|t_s^{(\alpha)}(x;t_*,x_*) - t_s^{(\alpha,P)}(x;t_*,x_*)| \le (1+C\varepsilon)M_4\frac{a_1}{a_0}a^{p-1}\varepsilon,$$
$$\forall\, t_* \in [pT_0, t_0],\ \forall\, x_* \in [d_{\alpha 0}, d_{\alpha 1}],\ \forall\, s = m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha = 1,\cdots,N. \tag{3.28}$$

By (2.53), (3.3), (3.13) and (3.22), we have

$$\begin{aligned}
&|z_r^{(\alpha)}(t,x) - z_r^{(\alpha,P)}(t,x)|\\
\le{}& a_0\Omega_P\Big(M_3\frac{a_2}{a_1}a^{p-2}\varepsilon\Big) + M_3\frac{a_0a_2}{a_1}a^{p-1}\varepsilon + Ca^p\varepsilon^2 + \Omega_P\Big((1+C\varepsilon)M_3a^{p-1}\frac{a_2}{a_1}\varepsilon\Big)\\
&+ C\varepsilon\Big(\Omega_P\Big(M_3\frac{a_2}{a_1}a^{p-2}\varepsilon\Big) + M_3\frac{a_0a_2}{a_1}a^{p-1}\varepsilon\Big) + Ca^p\varepsilon^2\\
\le{}& (1+C\varepsilon)a_0\Omega_P\Big((1+C\varepsilon)M_3\frac{a_2}{a_1}a^{p-2}\varepsilon\Big) + (1+C\varepsilon)M_3\frac{a_0a_2}{a_1}a^{p-1}\varepsilon\\
\le{}& (1+C\varepsilon)^2\frac{a_0}{a_1}\frac{a_2}{a}\Omega_P(M_3a^{p-1}\varepsilon) + (1+C\varepsilon)M_3\frac{a_0}{a_1}\frac{a_2}{a}a^p\varepsilon\\
\le{}& \Omega_P(M_3a^{p-1}\varepsilon) + M_3a^p\varepsilon, \quad \forall\, x \in [d_{\alpha 0}, d_{\alpha 1}],\ \forall\, t \in [pT_0, t_0],\ \forall\, r = 1,\cdots,m^{(\alpha)},\ \forall\,\alpha = 1,\cdots,N.
\end{aligned}$$

Similarly, we can get

$$|z_s^{(\alpha)}(t,x) - z_s^{(\alpha,P)}(t,x)| \le \Omega_P(M_3a^{p-1}\varepsilon) + M_3a^p\varepsilon,$$
$$\forall\, x \in [d_{\alpha 0}, d_{\alpha 1}],\ \forall\, t \in [pT_0, t_0],\ \forall\, s = m^{(\alpha)}+1,\cdots,n^{(\alpha)},\ \forall\,\alpha = 1,\cdots,N, \tag{3.29}$$

which leads to (3.11).

At last, by the original equations (1.1), we have

$$\begin{aligned}
&\partial_x u_k^{(\alpha)}(t,x) - \partial_x u_k^{(\alpha,P)}(t,x)\\
={}& \sum_{j=1}^{n^{(\alpha)}} \sigma_{kj}^{(\alpha)}(u^{(\alpha)})z_j^{(\alpha)} - \sigma_{kj}^{(\alpha)}(u^{(\alpha,P)})z_j^{(\alpha,P)}\\
={}& \sum_{j=1}^{n^{(\alpha)}} \sigma_{kj}^{(\alpha)}(u^{(\alpha)})(z_j^{(\alpha)} - z_j^{(\alpha,P)})\\
&+ \sum_{j,\hat{j}=1}^{n^{(\alpha)}} z_j^{(\alpha,P)}(u_{\tilde{j}}^{(\alpha)} - u_{\tilde{j}}^{(\alpha,P)})\int_0^1 \frac{\partial\sigma_{kj}^{(\alpha)}}{\partial u_{\tilde{j}}^{(\alpha)}}(\gamma u^{(\alpha)} + (1-\gamma)u^{(\alpha,P)})\mathrm{d}\gamma.
\end{aligned}$$

Now using (3.10)–(3.11), we can get

$$\lim_{t\to+\infty}\max_{\alpha=1,\cdots,N}\max_{k=1,\cdots,n^{(\alpha)}}\sup_{x\in[d_{\alpha 0},d_{\alpha 1}]}|\partial_x u_k^{(\alpha)}(t,x) - \partial_x u_k^{(\alpha,P)}(t,x)| = 0,$$

which completes the proof of Theorem 1.2.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Baldi, P., Berti, M., Haus, E. and Montalto, R., Time quasi-periodic gravity water waves in finite depth, *Invent. Math.*, **214**(2), 2018, 739–911.

[2] Baldi, P. and Montalto, R., Quasi-periodic incompressible Euler flows in 3D, *Adv. Math.*, **384**, 2021, 74 pp.

[3] Bourgain, J., Construction of periodic solutions of nonlinear wave equations in higher dimension, *Geom. Funct. Anal.*, **5**(4), 1995, 629–639.

[4] Bourgain, J., Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations, *Ann. of Math.*, **148**(2), 1998, 363–439.

[5] Bourgain, J. and Wang, W.-M., Quasi-periodic solutions of nonlinear random Schrödinger equations, *J. Eur. Math. Soc.*, **10**(1), 2008, 1–45.

[6] Crouseilles, N. and Faou, E., Quasi-periodic solutions of the 2D Euler equation, *Asymptot. Anal.*, **81**(1), 2013, 31–34.

[7] Feireisl, E., Mucha, P. B., Novotný, A. and Pokorný, M., Time-periodic solutions to the full Navier-Stokes-Fourier system, *Arch. Ration. Mech. Anal.*, **204**(3), 2012, 745–786.

[8] Iooss, G., Plotnikov, P. I. and Toland, J. F., Standing waves on an infinitely deep perfect fluid under gravity, *Arch. Ration. Mech. Anal.*, **177**(3), 2005, 367–478.

[9] Jin, C. H. and Yang, T., Time periodic solution for a 3-D compressible Navier-Stokes system with an external force in R^3, *J. Differential Equations*, **259**(7), 2015, 2576–2601.

[10] Kmit, I., Recke, L. and Tkachenko, V., Classical bounded and almost periodic solutions to quasilinear first-order hyperbolic systems in a strip, *J. Differential Equations*, **269**(3), 2020, 2532–2579.

[11] Kmit, I., Recke, L. and Tkachenko, V., Bounded and almost periodic solvability of nonautonomous quasilinear hyperbolic systems, *J. Evol. Equ.*, **21**(4), 2021, 4171–4212.

[12] Li, T.-T. and Yu, W. C., Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke University Mathematics Series, volume **V**, Duke University, 1985.

[13] Pego, R. L., Some explicit resonating waves in weakly nonlinear gas dynamics, *Stud. Appl. Math.*, **79**(3), 1988, 263–270.

[14] Qu, P., Time-periodic solutions to quasilinear hyperbolic systems with time-periodic boundary conditions, *J. Math. Pures Appl.*, **139**, 2020, 356–382.

[15] Rabinowitz, P. H., Free vibrations for a semilinear wave equation, *Comm. Pure Appl. Math.*, **31**(1), 1978, 31–68.

[16] Temple, B. and Young, R., Time-periodic linearized solutions of the compressible Euler equations and a problem of small divisors, *SIAM J. Math. Anal.*, **43**(1), 2011, 1–49.

[17] Tsuda, K., On the existence and stability of time periodic solution to the compressible Navier-Stokes equation on the whole space, *Arch. Ration. Mech. Anal.*, **219**(2), 2016, 637–678.

[18] Tsuge, N., Existence of a time periodic solution for the compressible Euler equation with a time periodic outer force, *Nonlinear Anal. Real World Appl.*, **53**, 2020, 22 pp.

[19] Wayne, C. E., Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Comm. Math. Phys.*, **127**(3), 1990, 479–528.

[20] Yu, H. M., Zhang, X. M. and Sun, J. W., Global existence and stability of time-periodic solution to isentropic compressible Euler equations with source term, 2022, arXiv: 2204.01939.

[21] Yuan, H. R., Time-periodic isentropic supersonic Euler flows in one-dimensional ducts driving by periodic boundary conditions, *Acta Math. Sci. Ser. B*, **39**(2), 2019, 1–10.