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The Existence of a Meridional Curve in Closed Incompressible Surfaces in Fully Alternating Link Complements

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Abstract Menasco showed that a closed incompressible surface in the complement of a non-split prime alternating link in S^3 contains a circle isotopic in the link complement to a meridian of the links. Based on this result, he was able to argue the hyperbolicity of non-split prime alternating links in S^3. Adams et al. showed that if $F \subset S \times I \setminus L$ is an essential torus, then F contains a circle which is isotopic in $S \times I \setminus L$ to a meridian of L. The author generalizes his result as follows: Let S be a closed orientable surface, L be a fully alternating link in $S \times I$. If $F \subset S \times I \setminus L$ is a closed essential surface, then F contains a circle which is isotopic in $S \times I \setminus L$ to a meridian of L.

Keywords Fully alternating, Incompressible surfaces, Meridionally incompressible
2000 MR Subject Classification 57M50, 57N75

1 Introduction

A surface F in the complement of a link L in a 3-manifold M is called meridionally compressible (cf. [1]), if there is a disk $D \subset M$ meeting L such that D intersects the link once transversely in the interior of D and $D \cap F = \partial D$. Otherwise, F is called meridionally incompressible. One salient result from [9] is that any closed essential surface in a non-split alternating link exterior will contain a meridional curve of a link component and, thus, studying such essential surfaces can be reduced to studying essential surfaces with meridional boundary curves that are meridionally incompressible, or equally in some literature, pairwise incompressible. The importance of studying meridionally incompressible surfaces has been reflected in the work of numerous scholars. To name a few, Bonahon and Seibenmann's work on arborescent knots (cf. [5]), Oertel's work on star links (cf. [10]), Adams' work on generalized augmented alternating links [2], and toroidally alternating links (cf. [1]), Adams' et al. work on almost alternating links (cf. [4]), Fa's initial cataloging of incompressible meridionally incompressible patterns (cf. [6]), Lozanoand-Przytycki work on 3-braid links (cf. [8]), and Hayashi's results on alternating diagrams on closed surfaces of positive genus (cf. [7]). Menasco, by using his crossing ball technique (cf. [9]), showed the existence of a meridional curve in a closed incompressible surface in the complement of a non-split prime alternating link in S^3.

The definition of fully alternating links in thickened surfaces is introduced in [3] by Adams et al (please see §2 for the formal definition). Let S be a closed orientable surface with positive

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genus, L be a fully alternating link in $S \times I$. Adams et al. [3] have shown that if $F \subset S \times I \setminus L$ is an essential torus, then F contains a circle which is isotopic in $S \times I \setminus L$ to a meridian of L.

A surface $F \subset S \times I \setminus L$ is essential if it is incompressible, boundary-incompressible, and not boundary-parallel. In the present paper, by expanding Menasco's idea (cf. [9]), we generalize the result in [3] as above to arbitrary closed essential surfaces embedded in $S \times I \setminus L$ in a simple way. Here is the main result.

Theorem 1.1 *Let S be a closed orientable surface, L be a fully alternating link in $S \times I$. If $F \subset S \times I \setminus L$ is a closed essential surface, then F contains a circle which is isotopic in $S \times I \setminus L$ to a meridian of L.*

Thus, the study of closed incompressible surfaces $F \subset S \times I \setminus L$ can be understood as analyzing the meridionally essential surfaces (please see Definition 2.2) in $S \times I \setminus L$ having all their boundary components meridians of L. Concerning this type of surfaces, we have the following theorem.

Theorem 1.2 *Let S be a closed orientable surface, L be a non-split prime fully alternating link in $S \times I$, and $F \subset S \times I \setminus L$ be a meridionally essential surface having $n > 0$ boundary components, all of which are meridians of L. Then*

(a) *If $n = 2$, S is an annulus, necessarily peripheral since L is prime.*

(b) *If $n = 4$, 6 or 8, F has genus zero.*

(c) *For a fixed n, there are only finitely many such surfaces F, up to isotopy.*

2 Normal Position of Surfaces

In this paper, when we apply the notation alternating, it is in the regular sense, where the crossings alternate under, over, under, over, as one travels along each component of the link. We project the link L on a surface S, $\pi : S \times I \to S$, where $\pi(L)$ is considered as a link diagram on S with each vertex assigned a positive or a negative state. Then each connected component of $S - \pi(L)$ is called a complementary region.

Definition 2.1 (cf. [3]) *Let L be a link in a thickened surface $S \times I$, orientable or not, with the exception of the sphere and the projective plane. A link L in the thickened surface $S \times I$ is fully alternating if it satisfies the following two conditions*:

(i) *There exists a projection onto an embedded surface S, $\pi : S \times I \to S$, so that $\pi(L)$ is alternating on S.*

(ii) *The interior of the closure of every complementary region $S - \pi(L)$ is an open disk.*

Note that, condition (ii) is necessary, as the complementary regions are not necessarily disks. And the assumption of alternatingness implies that in the case S is orientable, each component must have at least two crossings. In the case S is non-orientable, there could be a component with Möbius band neighborhood with one crossing (cf. [3]).

Let S be a closed orientable surface, $M = S \times I$ be a 3-manifold as the thickened surface, and $L \subset M$ be a link. We identify S with $S \times \left\{\frac{1}{2}\right\}$, then it has a projection $\pi : S \times I \to S$. Using the model in [9], we place a 3-ball at each crossing of $\pi(L)$, which we refer to as a bubble B. See Figure 1(a). At each crossing, both the overstrand and the understrand are in ∂B. We define S_+ to be the surface S where the equatorial disk in each bubble is replaced by the upper hemisphere ∂B_+ of the bubble, and M_+ to be the 3-manifold bounded by S_+ and $S \times \{1\}$, so that M_+ does not intersect with the interior of any bubbles. Similarly, we can define S_- and M_- when replacing the equatorial disk in each bubble by a lower hemisphere ∂B_-.

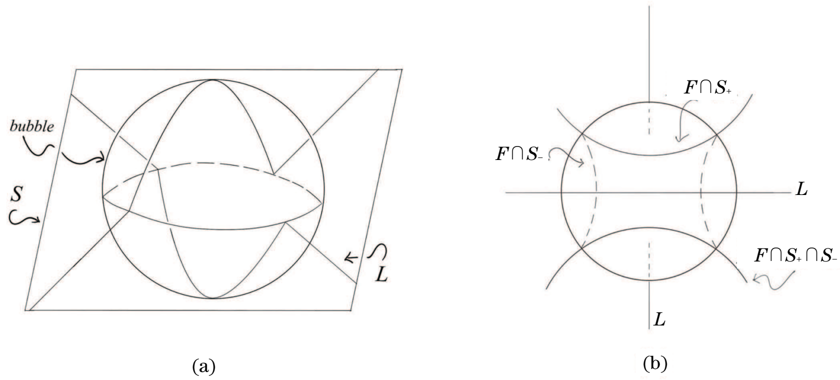


Figure 1

Let $F \subset M \setminus L$ be a closed incompressible surface. F intersects $S_\pm$ transversely in loops, and we can study each component C of $F \cap S_\pm$. The following is an important property for the class of alternating links, or fully alternating links:

(*) If B_1 and B_2 are two bubbles crossed in succession by a loop C of $F \cap S_\pm$, then:

(i) If the two arcs of $L \cap S_\pm$ in B_1 and B_2 lie on opposite sides of C, then C crosses L (at punctures) an even number of times between crossing B_1 and B_2, see Figure 2(a).

(ii) If the two arcs of $L \cap S_\pm$ in B_1 and B_2 lie on the same side of C, then C crosses L an odd number of times between crossing B_1 and B_2, see Figure 2(b).

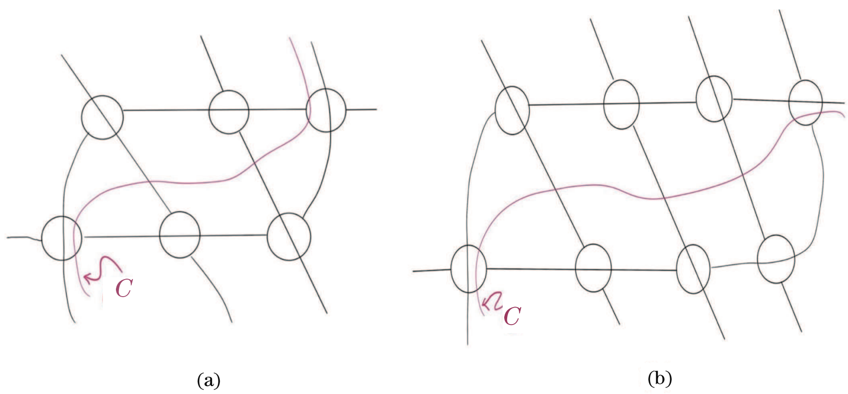


Figure 2

Definition 2.2 *Let M be a 3-manifold, $F \subset M \setminus L$ be a properly embedded closed surface, (or a surface with meridional punctured points). We say F is meridionally essential if it is incompressible, meridionally incompressible and not boundary-parallel.*

It is natural in the study of closed incompressible surfaces $F \subset M \setminus L$ to consider the operation of meridian surgery, indicated in Figure 3. It is easy to see, such surgeries can always preserve incompressibility of F. On the contrary, a closed incompressible surface $F \subset M \setminus L$ can be considered as the result of tubing (the "inverse" operation shown in Figure 3) meridionally essential surfaces.

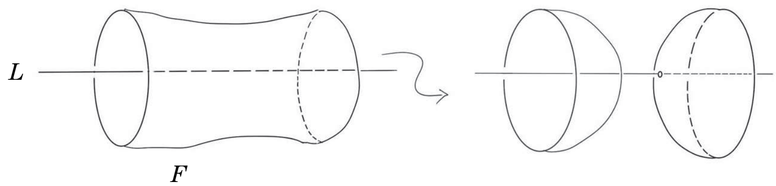


Figure 3

Let F be a closed essential surface embedded in $S \times I \setminus L$. We consider the following cases:

(1) F is a sphere.

(2) There are no essential spheres in $S \times I \setminus L$, and F is a closed, connected, incompressible surface.

F can be isotoped to intersect each bubble in a set of saddles. We will work on $F \cap S_+$, but everything we say also applies to $F \cap S_-$. We associate an order pair (s, i) to each embedding of F prior to isotopy, in which s is the number of saddles in F and $i = |F \cap S_+|$ is the number of intersection curves in $F \cap S_+$ (or equally, $F \cap S_-$). For the rest of this paper, we assume the surface F is chosen to minimize (s, i) in lexicographical ordering. We say F is in normal position if the above assumptions are satisfied. We notice that, given a diagram of $F \cap S_+$, we can obtain its dual diagram of $F \cap S_-$ as shown in Figure 1(b). Given a representative in the isotopy class of F so that the diagram $F \cap S_+$ is fixed, $F \cap S_+$ and $F \cap S_-$ have a one-to-one correspondence. Therefore, we can either require $|F \cap S_+|$ or $|F \cap S_-|$ to be minimized, and the intersection curve number in the dual diagram $|F \cap S_-|$ or $|F \cap S_+|$, respectively, will be naturally determined.

3 Proof of Main Result

Denote each connected component of $F \cap M_\pm$ as $F_i^\pm$. Hereafter we refer to S_+ and $F \cap S_+$, but all arguments and constructions applies to both $F \cap S_+$ and $F \cap S_-$. Similarly, we refer to M_+ and each connected component $F_i^+ \subset F \cap M_+$, but all arguments and constructions applies to M_- and $F_i^- \subset F \cap M_-$.

Proposition 3.1 *Let $F \subset M \setminus L$ be a closed essential surface in normal position. Then each connected component $F_i^+ \subset F \cap M_+$ is incompressible, and boundary-parallel to S_+ in M_+.*

Proof We first show that each connected component $F_i^+ \subset F \cap M_+$ is incompressible in M_+. Suppose F_i^+ compresses in M_+. Then there exists a properly embedded compressing disk $D \subset M_+$ of F_i^+, with $\partial D \subset F_i^+$. Since $F \subset M$ is either a sphere or an incompressible surface,

∂D must bound a disk D' in F, with $\partial D' = \partial D$. By the assumption that, in addition, there exist no essential spheres in $S \times I \setminus L$, we can isotope D' in $S \times I \setminus L$ to D while fixing its boundary. Suppose $D' \cap S_+ = \emptyset$, then we could isotope D to D' in M_+ with ∂D fixed in F_i^+. This would imply $D' \subset F_i^+$, contradicting the assumption that D is a compressing disk of F_i^+. Therefore, D' must intersect S_+ in a non-empty set of loop(s). Through a standard innermost loop argument, we can isotope F to eliminate the loop(s) of $D' \cap S_+$. This means, we can reduce $i = |F \cap S_+|$ by at least one, contradicting the assumption F is in normal position.

Now we have M_+ is a compression body, $\partial M_+ = S_+ \cup S \times \{1\}$, $(F_i^+, \partial F_i^+) \subset (M_+, \partial M_+)$, F_i^+ is incompressible, and $\partial F_i^+ \cap S \times \{1\} = \emptyset$. Therefore, by [11, Proposition 3.1], F_i^+ is boundary-parallel to S_+ in M_+.

We call each $F_i^\pm$ a dome. As a result of the above proposition, it is reasonable to define F_i^+ (or F_i^-) as a lowest dome if there are no component(s) of $F \cap M_+$ (resp., $F \cap M_-$) embedded in between the cobordism bounded by F_i^+ and S_+ (resp., F_i^- and S_-). To compare this generalization with the original technique in [9], we notice if we take the thickened surface $S \times I$ as $S^2 \times I$, and each simple closed curve of $F \cap S^2_\pm$ bounds a dome of disk.

The following lemma guarantees the existence of nontrivial intersections $F \cap S_+$.

Lemma 3.1 *Let $F \subset M \backslash L$ be a closed essential surface in normal position. There exists an isotopy of F such that the set of intersection curves $F \cap S_+$ is nonempty, and every intersection curve in $F \cap S_+$ intersects at least one bubble.*

Proof This is true because the proof of [3, Lemma 5(i)–(ii)] applies.

Let F_i denote a lowest dome of $F \cap M_+$, M_i^+ denote the cobordism bounded by F_i^+ and S_+, and ∂B_+ denote the upper hemisphere of a bubble B. Notice that as F passes through a bubble B, the saddle corresponds to two intersection curves on S_+ that run parallel to the overstrand of B. ∂B_+ is divided by an overstrand of L into two sides. We proof the following technical lemma.

Lemma 3.2 *Let $F \subset M \setminus L$ be a closed essential surface in normal position. Suppose B is a bubble that intersects with a lowest dome F_i^+, then $M_i^+ \cap \partial B_+$ does not consists of any rectangle, whose boundary consists of two arcs of $F_i^+ \cap \partial B_+$, and two arcs on the boundary of ∂B_+.*

Proof If $M_i^+ \cap \partial B_+$ consists of a rectangle R shown in the below Figure 4(a), which we call a sided rectangle, as R does not contain the link strand on ∂B_+. Then we can pull a neighborhood of an arc on F_i^+ to the bubble to form a band connecting the pair of saddles intersecting R. And since F_i^+ is a lowest dome, there is no arc intersection of $F \cap S_+$ in the interior of R. So we can pull the two saddles and the band through the bubble and out the other side of ∂B_+, as shown in Figure 5. This would decrease the number of saddles, contradicting F is in normal position.

If $M_i^+ \cap \partial B_+$ consists of a rectangle R shown in Figure 4(b), which we call a middle rectangle, as it contains the link strand on ∂B. This would contradict the assumption F being meridionally incompressible. Because F_i^+ is a lowest dome, there is no arc intersection of $F \cap S_+$ in the interior of R. And we can find an arc μ on F_i^+ so that $\partial \mu$ is contained in a saddle σ in B, which means $\sigma \cup \mu$ contains a meridional curve of F.

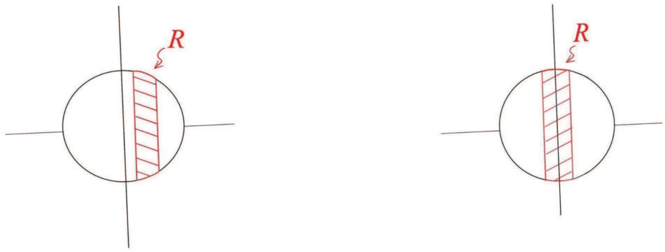


Figure 4

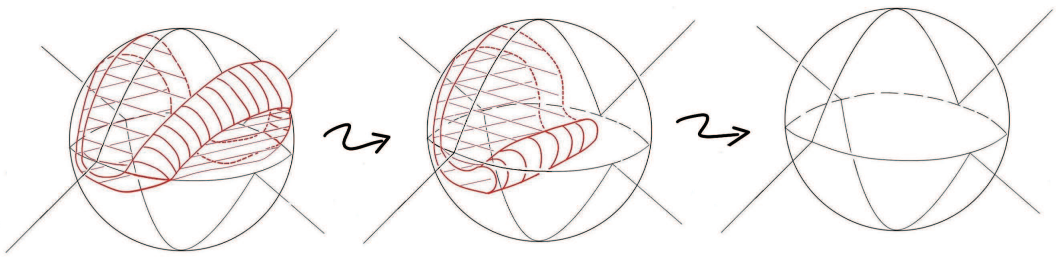


Figure 5

Proof of Theorem 1.1 Assume by contradiction that the projection $\pi(L)$ of L on S is fully alternating, and F is a closed meridionally essential surface embedded in $S \times I \setminus L$. We put F in normal position. According to Lemma 3.1, $F \cap M_+$ is nonempty. And, by Proposition 3.1, each connected component of $F \cap M_+$ is incompressible and boundary-parallel to S_+ in M_+. Hence, we can consider a lowest dome F_i^+ and the cobordism M_i^+ bounded by F_i^+ with S_+. By Lemma 3.1, every intersection curve in $F \cap S_+$ intersects at least one bubble. So we can assume that there is a simple closed curve $C \subset F_i^+ \cap S_+$ that intersects with a bubble B.

We now claim that $M_i^+ \cap \partial B_+$ does not contain the overstrand on ∂B_+. Because we suppose that the overstrand of ∂B_+ is contained in $M_i^+ \cap \partial B_+$, by the assumption that F_i^+ is a lowest dome, F_i^+ would have to meet both sides of ∂B_+. Thus $M_i^+ \cap \partial B_+$ would consist of a middle rectangle, which would contradict Lemma 3.2. But by alternating property (*), it follows that there must exist an arc $\alpha \subset C$ passing through one side of ∂B_+ such that the overstrand is on the right (similarly left), and then passing through the same side of ∂B_+ such that the overstrand is on the left (similarly right), with $M_i^+ \cap \partial B_+$ consisting of a sided rectangle between the two passes. By Lemma 3.2, this would contradict the assumption that F is in normal position. We note that the above proof also implies that there does not exist any essential sphere in $S \times I \setminus L$.

In addition, the authors of [3] give a definition for prime links in thickened surfaces in comparison with the prime links in S^3. A link L is prime in $S \times I$ if there does not exist an essential twice punctured sphere in $S \times I \setminus L$ such that both punctures are created by L.

It was also mentioned in [9], that each component C of $F \cap S_\pm$ can be associated a cyclic word $\omega_\pm(C)$ in the letter P (=puncture) and S_d (=saddle), which records, in order, the intersections of C with L and with bubbles, respectively. (Strictly, $\omega_\pm(C)$ depends on an orientation for C.)

$\omega_\pm(C)$ must have even length, and the number of boundary components of F equals the total number of P's in all the $\omega_+(C)$'s (or in all the $\omega_-(C)$'s). The following lemma is essentially [9, Lemmas 2–3].

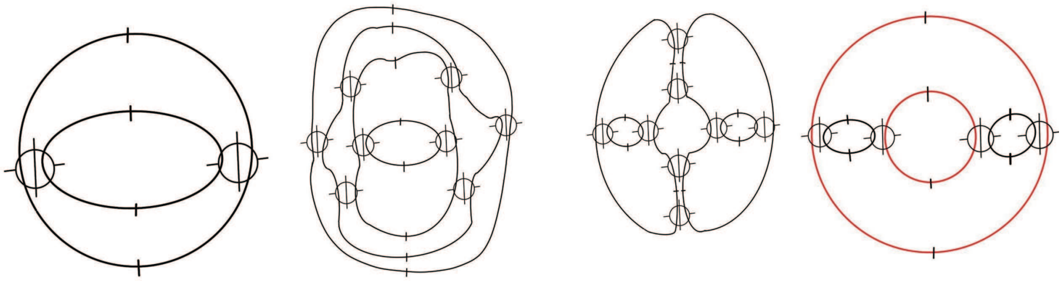
Lemma 3.3 *Let $F \subset S \times I \setminus L$ be a meridionally essential surface in normal position, and C be a component of $F \cap S_+$. If no word $\omega_+(C)$ associated to a loop C of $F \cap S_+$ is empty, no loop of $F \cap S_+$ crosses the same bubble more than once, and F has n meridian boundary components. Then we have*

(1) *each word $\omega_+(C)$ has at most $n-2$ S_d's,*

(2) *each word $\omega_\pm(C)$ contains at least two P's, and*

(3) *no word has the form $P^i S_d^j$ with $j > 0$.*

Proof of Theorem 1.2 **Claim 1** By Lemma 3.3(2), each word $\omega_+(C)$ contains at least two P's. Therefore $F \cap S_+$ consists of only one loop C. By primeness of the link, we know that C is a trivial loop that is the intersection between S_+ and a peripheral annulus.



(a) (b) (c) (d)

Figure 6

Claim 2 Similarly, by Lemma 3.3(2), when $n = 4$, there are at most two loops in $F \cap S_+$. If $F \cap S_+$ has only one loop C, its word $\omega(C)$ must be $PPPP = P^4$. Because if C crosses any bubbles, then a saddle which C intersects would also intersect another loop C' of $F \cap S_+$. If $F \cap S_+$ has two loops C_1 and C_2, then $\omega(C_1)$ and $\omega(C_2)$ contain at least two P's. If $\omega(C_1)$ and $\omega(C_1)$ are both P^2, then F is either a disjoint union of two annuli, or F is a two punctured torus. Otherwise we have $\omega(C_1) = \omega(C_2) = PS_dPS_d$. Since by Lemma 3.3(3), $\omega(C_1)$ and $\omega(C_2)$ contain at least two S_d's. This implies $F \cap S_-$ must have two loops C_1' and C_2', so that we also have $\omega(C_1') = \omega(C_2') = PS_dPS_d$. We can compute the Euler characteristic of the surface $\overline{F}$ obtained by capping with disks the boundaries of F. And we have $\chi(\overline{F}) = \chi(F) = 2$, therefore F has genus zero. See Figure 6(a) for this pattern. Similarly, we can analyze the case $n = 6$ and $n = 8$. We give the three patterns when $n = 8$ and F has genus one in Figure 6(b), (c) and (d). Notice that in Figure 6(d), the two red loops cobound an annulus in M_+. The other cases we have examined only consist of loops that bound disks as the domes in M_+.

Claim 3 Each word $\omega_+(C)$ contains at least two P's by Lemma 3.3(2), so the number of loops in $F \cap S_+$ is at most $\frac{n}{2}$. The number of possible words $\omega_+(C)$ is bounded, since by Lemma 3.3(1), each word $\omega_+(C)$ has at most $n-2$ S_d's. Each word $\omega_+(C)$ can be realized by only finitely many loops in S_+. Therefore, we can only have finitely many possibilities for $F \cap S_+$ (or F up to isotopy) for a fixed n.

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Declarations

Conflicsts of interest The authors declare no conflicts of interest.

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